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Congestion-Dependent Pricing and Forward Contracts for Complementary Segments of a Communication Network

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***Abstract*—Congestion-dependent pricing is a form of traffic management that ensures the efficient allocation of bandwidth between users and applications. As the unpredictability of congestion prices creates revenue uncertainty for network providers and cost uncertainty for users, it has been suggested that forward contracts could be used to manage these risks. We develop a novel game-theoretic model of a multi-provider communication network with two complementary segments, and investigate whether forward contracts would be adopted by service providers. Service on the upstream segment is provided by a single Internet Service Provider (ISP) and priced dynamically to maximize profit, while several smaller ISPs sell connectivity on the downstream network segment, with the advance possibility of entering into forward contracts with their users for some of their capacity. We show that the equilibrium forward contracting volumes are necessarily asymmetric, with one downstream provider entering into fewer forward contracts than the other competitors, thus ensuring a high subsequent downstream price level. In practice, network providers will choose the extent of forward contracting strategically based not only on their risk tolerance, but also on the market structure in the interprovider network and their peers' actions.**

***Index Terms*—Internet, contracts, traffic control (communication), communication systems, communication system traffic, game theory, economics.**

I. Introduction

The pricing for Internet service is currently based on access bandwidth and usage. However, with the growing diversity of applications using the Internet, there is considerable interest in designing a future Internet architecture that would allow users to indicate the value they place on network service by purchasing end-to-end Quality of Service (QoS) from the service provider.

Congestion-dependent pricing for communication networks has been proposed [2]–[7] as a method of traffic management that can efficiently allocate bandwidth among users—e.g. households, small businesses, large service providers—who place different value on their applications. Congestion-dependent pricing ensures that users have an incentive to control congestion. The highly influential paper of Gibbens and Kelly [2] proposed a mechanism to implement usage-based charging. In that scheme, prices are set on the basis of aggregate traffic and communicated periodically to users, who can then decide for themselves how to best satisfy their requirements at the given price.

Financial contracts could be used to provide more predictable prices to both service providers and users in a network with congestion pricing. Semret and Lazar and their co-authors published a series of papers on bandwidth pricing and contracts. These include Semret and Lazar [8], which proposes a market for circuit switched calls, wherein calls are admitted or rejected at or soon after their arrival time and, if admitted, receive a fixed allocation of capacity and have the option of securing the resource at a guaranteed maximum price for a guaranteed minimum duration. The reservation fee is determined using the Black-Scholes option pricing approach. Semret, Liao, Campbell and Lazar [9] consider a game-theoretic model of capacity provisioning in a differentiated services Internet, where the players consist of one capacity seller per network, one broker per service per network, and a set of network users. The purchase of forward contracts by the network users is proposed by Anderson et al. as a "Contract and Balancing Mechanism" [10], which is shown to give users an incentive to control congestion, while avoiding the network provider's perverse incentive to cause congestion. On the other hand, Yuksel et al. [11] propose a "contract-switched" Internet, featuring a dynamic inter-provider pricing system to provide end-to-end QoS, in conjunction with longer-term financial contracts used for risk management.

In this paper, we ask whether long-term forward contracts would be offered to users in a future Internet with a dynamic inter-provider pricing system. Our analysis differs from the above papers by considering the fraction of a provider's capacity to be funded by long-term contracts as a strategic variable. While our analysis is motivated by contracts between Internet Service Providers and end-users, our model is sufficiently general to be applicable to contracting by large corporate customers, as considered in [10].

To study the dynamic interactions between multiple network providers in a tractable setting, we develop a two-stage model of bandwidth sold on two complementary[1] segments of a multi-provider communication network by means of dynamic pricing (a spot market). Specifically, the upstream segment is provided by a single large Internet Service Provider, denoted *UISP*, and the downstream segment is provided by several smaller ISPs, denoted ISP_1, ISP_2, $\ldots$, ISP_n. The upstream ISP connects the downstream ISPs to the Internet backbone. A schematic diagram of the business relationships is displayed in

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[1]Two services are said to be *complementary* if they are used together because they have little or no value when used separately.

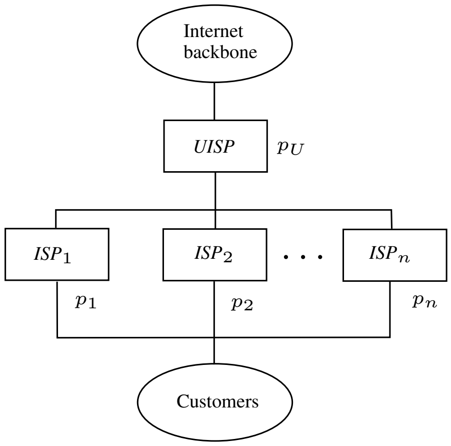

Figure 1. Network diagram

Figure 1. Foros, Kind, and Sørgard [12] point out that the question of whether backbone providers have incentives to abuse their potential market power has received much attention, both theoretically and in antirust cases. They analyze the interplay between firms and regulatory authorities in different countries by considering a scenario with a single backbone provider.

In the first stage of our model, the downstream ISPs choose the capacity to sell using forward contracts. In the second stage, all ISPs set prices to maximize their respective revenue. Customers must purchase the same amount of bandwidth upstream and downstream in order to use the network services.

This paper is organized as follows. Section II presents an overview of our two-stage model. In section III we describe a novel extension of the Bertrand-Edgeworth pricing game to model the second-stage interaction between the upstream ISP and the capacity-constrained downstream ISPs competing in prices with each other. We show in section IV that, for sufficiently low market potential, downstream prices are competed down to marginal cost, while for sufficiently high market potential, there may be multiple pure-strategy Nash equilibrium outcomes, with different divisions of the total industry profit between the upstream and downstream providers. We assume the large upstream ISP has all the bargaining power and can choose which equilibrium will arise. In the region of intermediate market potential, we find an equilibrium point using mixed strategies for the downstream ISPs (section V). With uncertain future demand for network service, providers have an incentive to enter into forward contracts in the first stage. However, the extent of forward contracting changes the dynamic price outcome in the second stage. In section VI we use the pricing analysis to investigate the downstream ISPs' incentives for using forward contracts to fund their bandwidth. We find that a downstream ISP choosing a low contracting volume is able to raise the general downstream price level, allowing its competitors to contract more. A pure-strategy Nash equilibrium of contracting volumes, if it exists, must have a unique lowest volume of contracting. We further prove that an increase in this lowest volume has a negative marginal externality[2] on other downstream ISPs' utility, whereas an increase in any other contracting volume creates positive marginal externalities. In section VII, we present conclusions. In order to aid readability, we have relegated the more technical aspects of the proofs of the first two theorems to three lemmas, which are proved in the appendix.

II. Model Overview

We consider the following two-stage contracting and pricing game played by *UISP* and $ISP_1, \ldots, ISP_n$. In the first stage, the ISP_i simultaneously choose to sell capacities $0 \le f_i \le k$ by means of forward contracts, where k is each ISP_i's total capacity. This bandwidth is sold at a price which is fixed in the first stage.

In the second stage, the providers *UISP* and $ISP_1, \ldots, ISP_n$ simultaneously set prices p_U and $p_1, \ldots, p_n$, to maximize profits π_U and $\pi_1, \ldots, \pi_n$ from their uncontracted capacity. The second-stage profits are functions of the prices and the forward contracts f_i chosen in the first stage.

The price sensitivity of bandwidth demand is not known at the time of contracting, but dynamic pricing allows the ISPs to choose their second-stage prices based on the realized price sensitivity. We therefore model the price sensitivity β as a random variable which is revealed between the two stages of the game. This means that a risk-averse ISP_i has an incentive to enter into forward contracts to hedge against demand uncertainty and maximize its total expected utility

$$\Pi_i(f_1, \ldots, f_n) \;=\; \mathbb{E}_\beta \mathscr{U}(I_i + \mathbb{E}_p \pi_i), \qquad (1)$$

where $\mathbb{E}_\beta$ denotes expectation over the random price sensitivity β, $\mathscr{U}$ is ISP_i's increasing and strictly concave utility function, I_i is ISP_i's income derived from forward contracting, $\mathbb{E}_p$ denotes expectation over the second-stage mixed strategy prices, and π_i is ISP_i's second stage profit. Although the income I_i from forward contracting is fixed during the first stage and so does not depend on the prices realized during the second stage, we will assume that it does depend on the expected second-stage prices $\mathbb{E}_\beta \mathbb{E}_p p_i$, as this is the fair-market level at which risk-neutral users are willing to enter into forward contracts.

Before we can fully define and analyze the first-stage contracting game in section VI, we first need to develop the second-stage pricing model.

III. Pricing Model

We model the second-stage behavior of the downstream ISPs as "Bertrand-Edgeworth" price competition with capacity constraints, first studied by Edgeworth who showed that the duopoly case might not have an equilibrium in prices [13]. The formulation of the problem with the "rationing rule" considered here is due to Levitan and Shubik [14]. They found that prices are competed down to the perfectly competitive level equal to marginal cost when demand is low;

[2]An *externality* of an economic transaction is an impact on a party that is not directly involved in the transaction. The *marginal externality* of an ISP's contracting strategy is the impact of a unit increase in that ISP's contracting volume.

and there is a pure-strategy Nash equilibrium, a pair of prices such that neither firm can increase its profit by unilaterally changing its price when demand is high. For the intermediate region of demand, they derived a Nash equilibrium in mixed (random) strategies. Vives [15] established the mixed-strategy equilibrium for the case of symmetric oligopoly with more than two competitors and proved convergence to the perfectly competitive price as the number of firms increases. For any fixed choice of upstream price p_U, our downstream pricing model differs by taking into account forward contracts previously sold by the *ISP*$_i$ for diverse fractions of their bandwidth. An important analytic contribution of this paper is the characterization of the mixed-strategy equilibrium for this more complicated asymmetric model. This result is used to find an equilibrium for the full second-stage pricing game where *UISP* and the *ISP*$_i$ choose prices simultaneously.

We assume that *UISP* is a large provider connecting the *ISP*$_i$ to the Internet backbone and has all the bargaining power. Thus, where the second-stage pricing game has multiple equilibria, the equilibrium with largest p_U arises. In the special case of $n = 1$, *ISP*$_1$ is another monopolist and our game describes a bilateral monopoly.

On the other hand, where the pricing game has no pure-strategy Nash equilibrium and prices fluctuate, a realistic analysis needs to take into account the timescales over which providers are likely to adjust their prices. This in turn depends on the technologies used for price updates. While the downstream providers can directly broadcast their prices to local users connected to their networks every few seconds, this approach does not scale to a large multi-provider network such as the Internet. The monopolistic transit provider is more likely to make use of a general pricing system. Proposals for implementing inter-provider pricing by extending the Border Gateway Protocol (BGP) [16] have been made by [17], [18]. Such a system would propagate price changes over the BGP convergence timescale of several minutes. For this reason, we assume that the downstream ISPs' prices are updated on a shorter timescale than the upstream ISP's price, and we model the downstream ISPs' behavior by mixed strategies and the upstream ISP's behavior by a pure strategy.

The bandwidth demand D_U on the upstream ISP's network is the sum of the bandwidth demands D_i served by each *ISP*$_i$ on the complementary network segment, i.e.,

$$D_U = \sum_{i=1}^{n} D_i. \tag{2}$$

We assume the costs of building the firms' infrastructure are sunk, and zero marginal costs are incurred during operation of the network. According to Odlyzko [19], "marginal costs are zero up to the point where congestion occurs and forces addition of new capacity." Of course, ISPs also incur non-bandwidth marginal costs, such as the costs of billing and customer support. However, any constant marginal costs can be normalized to zero by redefining the prices, provided the marginal costs incurred by the competing downstream ISPs are equal. Let the upstream ISP's payoff be

$$\pi_U \;=\; p_U D_U. \tag{3}$$

Suppose each *ISP*$_i$ has previously sold capacity f_i by means of forward contracts, so his (second-stage) payoff is[3]

$$\pi_i \;=\; p_i(D_i - f_i). \tag{4}$$

In order to obtain closed-form expressions for the equilibrium, we work with a linear demand function [14], which has been used in the network pricing literature, e.g., [20],

$$d_{\text{market}}(p) \;=\; \alpha - \beta p, \tag{5}$$

where the *total price* $p \equiv p_U + p_m$ and p_m is the price charged by the *marginal ISP*$_i$ with a positive market share, that is, the highest price charged by any *ISP*$_i$ with a positive market share. The downstream ISPs' incentives for choosing their *contracting volumes* f_i under demand uncertainty are to be discussed in section VI. For the first-stage pricing model, we suppose simply that the *market potential*[4] α and the *price sensitivity* β are given non-negative constants, and the contracting volumes are given constants with $0 < f_i < k$ for some k.

Assume the upstream ISP is not subject to any capacity constraint, other than the total capacity nk resulting from the capacity of the complementary network segment. To determine the market share of each ISP, we use the *rationing rule* maximizing consumer surplus chosen by [14], [21], which can intuitively be seen as a "water-filling" model: demand fills the downstream ISPs' capacities in increasing order of price, up to the point where the total demand at the next *ISP*$_i$'s price would be insufficient to leave any market share to that *ISP*$_i$. Demand is split equally between several ISPs with the same price where there is not enough demand to fill their networks completely. An implicit economic assumption in the "water-filling" model is that there is no income effect[5] on bandwidth consumption.

More formally, "water-filling" specifies the bandwidth D_i provided by *ISP*$_i$ and the total bandwidth provided by the upstream (and downstream) network D_U by the following four conditions. The bandwidth provided by *ISP*$_i$ must satisfy the capacity constraint

$$0 \le D_i \le k, \tag{6}$$

the capacity of *ISP*$_i$ must be exhausted if market demand at price p_i exceeds the total bandwidth used in the network

$$d_{\text{market}}(p_U + p_i) > D_U \;\Rightarrow\; D_i = k, \tag{7}$$

the bandwidth provided by *ISP*$_i$ must be zero if market demand at price p_i is less than the total bandwidth used in the network

$$d_{\text{market}}(p_U + p_i) < D_U \;\Rightarrow\; D_i = 0, \tag{8}$$

and, finally, demand splits equally between ISPs choosing the same price

$$p_i = p_j \;\Rightarrow\; D_i = D_j. \tag{9}$$

[3]A choice of price p_i such that $D_i < f_i$ can be interpreted as *ISP*$_i$ purchasing bandwidth from the customers.

[4]The *market potential* is the maximum achievable demand, which is given by the limit of the demand function as the price goes to zero.

[5]The *income effect* occurs when a decrease in the price for a good, other things remaining the same, will leave the consumer with more income left over, some of which will be spent on buying more of the good.

In the rest of this paper, we shall assume without loss of generality that the ISP_i are ordered by their contracting volumes as

$$0 < f_1 \leq f_2 \leq \cdots \leq f_n < k.$$

IV. Pure-Strategy Equilibrium Analysis

The equilibrium outcome of the pricing game depends on the available bandwidth capacity compared to the market potential. More precisely, the following definition partitions the range of market potential α into three regions by comparing it with the number n of ISP_is, the capacity k of each ISP_i, and the contracting volume f_1 of firm 1.

Definition 1 (high, low, intermediate market potential)**.** Let $0 \leq f_1 < k$. Consider the thresholds

$$\alpha_l(f_1) = 2(n-1)k + 2f_1, \tag{10}$$

$$\alpha_h(f_1) = (2n+1)k - f_1. \tag{11}$$

We say that market potential is f_1*-high* if

$$\alpha \geq \alpha_h(f_1); \tag{12}$$

that market potential is f_1*-low* if

$$\alpha \leq \alpha_l(f_1); \tag{13}$$

and that market potential is f_1*-intermediate* if

$$\alpha_l(f_1) < \alpha < \alpha_h(f_1). \tag{14}$$

As we will now show, in the region of f_1-high market potential network capacity is exhausted. Thus, the total upstream and downstream price $p_1 + p_U$ is the *congestion price*, the lowest price at which demand can be satisfied. In the region of f_1-low market potential, competition forces the downstream market price p_1 down to marginal cost, which is normalized to zero. In the region of f_1-intermediate market potential, oscillatory price behavior follows, as will be explored in the next section. The following theorem characterizes the pure-strategy Nash equilibria in the three regions.

Theorem 1. *Pure-strategy equilibria are characterized as follows:*

(i) *If market potential is f_1-high in the pricing game, then there is a range of pure-strategy equilibria given by*

$$p_1 = p_2 = \cdots = p_n \tag{15}$$

$$\beta(p_1 + p_U) = \alpha - kn \tag{16}$$

$$k - \beta p_1 \leq f_i \ \forall i \tag{17}$$

$$\beta p_U \geq kn, \tag{18}$$

moreover, any f_1-high pure-strategy equilibrium is of this form.

(ii) *If market potential is f_1-low, then there is a unique pure-strategy equilibrium such that every ISP_i sets a zero price ($p_i = 0$) and UISP sets $p_U = \frac{\alpha}{2\beta}$.*

(iii) *If market potential is f_1-intermediate and $n = 1$, then there is a unique pure-strategy equilibrium given by*

$$p_1 = \frac{\alpha - 2f_1}{3\beta}, \qquad p_U = \frac{\alpha + f_1}{3\beta}. \tag{19}$$

If market potential is f_1-intermediate and $n \geq 2$, then there is no pure-strategy equilibrium.

Some observations may be in order. To begin, note that the general form of the result only differs between the *bilateral monopoly* ($n = 1$) and the true downstream oligopoly case ($n \geq 2$) when market potential is f_1-intermediate and competition results in the non-existence of any pure-strategy equilibrium in the oligopoly case. However, the boundaries between the regions depend on the number n of downstream firms. In the bilateral monopoly case, for example, the equilibrium with $p_1 = 0$ arises only if $f_1 \geq \frac{\alpha}{2}$. In the absence of competition to force the downstream price to zero, this will only happen when market potential is so low that, given the contracting volume f_1, provider ISP_1 cannot obtain a positive profit by setting $p_1 > 0$.

On the other hand, when $n \geq 2$, the theorem says that a pure-strategy equilibrium where the ISP_i set positive prices $p_i > 0$ is necessarily of the form given by (15)–(18). It is easy to check that this system is inconsistent when market potential is not f_1-high, so an equilibrium of this form can only exist for f_1-high market potential.

Observe that none of the results stated in Theorem 1 depend on the contracting volumes $f_2, \ \ldots, \ f_n$, but only on the lowest contracting volume f_1. In general, any contracting weakens a downstream provider's incentive to set a high price in the pricing game, and the provider with the lowest contracting volume, ISP_1, will have the strongest incentive to do so. When *UISP* holds all the bargaining power and market potential is f_1-high, the equilibrium with the highest p_U arises, and the equilibrium price levels are determined by ISP_1 and *UISP*, the other downstream ISPs being able to follow ISP_1's price p_1.

When the downstream ISPs have some of the bargaining power, the prices they set increase with market potential. The competition between the downstream ISPs is more significant in this case, and the game is closer to the classical Bertrand-Edgeworth price competition with capacity constraints.

Proof of Theorem 1: If market potential is f_1-high, this allows the choice of p_1, p_U satisfying the outlined conditions. We verify that these choices of prices do indeed constitute a pure-strategy Nash equilibrium. Here, *UISP* serves a market of maximal size nk, and he can do no better by cutting his price. The effect on *UISP*'s profit of a rise in p_U is

$$\left.\frac{\partial \pi_U}{\partial p_U}\right|_+ = \alpha - \beta(p_U + p_1) - \beta p_U \ \leq \ nk - \beta p_U \ \leq \ 0,$$

at the chosen point as well as for any higher value of p_U. Therefore, *UISP* has no incentive to change his strategy.

Since firm ISP_1's market share $\alpha - \beta(p_U + p_1) - (n-1)k$ is equal to k at our chosen point, and $f_1 \leq k$, it follows that ISP_1 cannot gain by cutting his price. Moreover ISP_1 cannot increase his profit by raising his price either, since

$$\left.\frac{\partial \pi_1}{\partial p_1}\right|_+ = \alpha - \beta(p_U + p_1) - (n-1)k - \beta p_1 - f_1 \leq 0, \tag{20}$$

where the inequality follows from (17). We have shown that the chosen point is indeed a pure-strategy Nash equilibrium.

If market potential is f_1-low, consider the set of strategies $p_i = 0 \ \forall i$, $\beta p_U = \frac{\alpha}{2}$. The price p_U is clearly *UISP*'s best response to the zero strategy chosen by the ISP_i: it is the monopolistic price. Observe that the total market served is $D_U = \frac{\alpha}{2} \leq (n-1)k + f_1$. Therefore, if ISP_i were to choose any other price $p_i > 0$, his profit would be negative. We have established that this set of strategies is indeed a Nash equilibrium.

Conversely, consider any pure-strategy equilibrium given by the tuple of prices $(p_U;\ p_1,\ p_2,\ \ldots,\ p_n)$. We will start by showing that the equilibrium satisfies (15)–(18) for $n \geq 2$ provided some $p_i D_i > 0$. Let i be such that $p_i D_i > 0$ with p_i maximal. Suppose there was some j such that $p_j > p_i$. Then we would have $D_j = 0$ by the definition of i, so ISP_j would have an incentive to set p_j equal to p_i. Suppose now that there was some j such that $p_j < p_i$. It follows from assumptions (7)–(8) that $D_j = k$ and ISP_j would be able to increase his price to any $p_j < p_i$ while retaining a market share of k. Since $f_j < k$, he would increase his profit by doing so. Therefore, we have shown that all prices are equal in our equilibrium (15).

Suppose we had $D_1 < k$. Then, if $n \geq 2$, ISP_1 would have an incentive to increase his market share by cutting his price by any small amount. Hence we must have $D_1 = k$ at equilibrium and the total market served is nk (16).

Our previous argument shows that (17) and (18) must hold at equilibrium, so the ISP_1 and *UISP* respectively have no incentive to increase their price. We have therefore shown that every non-trivial pure-strategy equilibrium is of the given form.

To show the unique characterization for the equilibrium, consider any pure-strategy Nash equilibrium in prices $(p_U;\ p_1,\ p_2,\ \ldots,\ p_n)$. We use the following two results, which are direct consequences of the definitions of the downstream ISPs' demand and payoff functions (4)–(9) and of the thresholds for high and low market potential given in Definition 1.

- Suppose market potential is not f_1-low. Then there exists $1 \leq i \leq n$ such that ISP_i has $p_i D_i > 0$ in equilibrium.
- Suppose market potential is not f_1-high. If $n \geq 2$ then every ISP_i has $p_i = 0$ in equilibrium.

If market potential is f_1-high, the first result shows that some $p_i D_i > 0$ in equilibrium. For $n \geq 2$, we have shown that any such equilibrium must be of the form given by (15)–(18). For $n = 1$, the same argument shows (17)–(18), and it is easy to see that, if no provider has an incentive to cut his price, then we have (16).

If market potential is f_1-low, $n \geq 2$, the second result shows that every ISP_i has $p_i D_i = 0$. If market potential is f_1-low and $n = 1$, it is easy to see that the unique pure-strategy equilibrium is given by $p_1 = 0$, $p_U = \frac{\alpha}{2\beta}$.

If market potential is f_1-intermediate and $n \geq 2$, the two results are contradictory, so there is no pure-strategy equilibrium. Finally, if market potential is f_1-intermediate and $n = 1$, it is easy to see that the unique pure-strategy equilibrium is given by (19). This completes the proof of the theorem. ∎

V. Mixed-Strategy Equilibrium Analysis

From Theorem 1, we know that for f_1-intermediate market potential there is no pure-strategy Nash equilibrium when the downstream market is a true oligopoly ($n \geq 2$). Since the downstream ISPs set their prices on a shorter timescale than the upstream ISP, we assume they use mixed strategies, interpreted as distributions of fluctuating prices following [14]. The pricing game can be shown to have an equilibrium point.

Theorem 2. *Suppose $n \geq 2$ and market potential is f_1-intermediate in the pricing game. Then there exists a unique equilibrium point $(p_U;\ p_1,\ \ldots,\ p_n)$ where the price p_U is a pure strategy for UISP and the prices p_i are mixed strategies for each ISP_i, respectively, such that p_U is locally optimal and each p_i is optimal given the other ISPs' strategies.*

Local optimality of the upstream equilibrium price p_U means that *UISP* has no incentive to make small-scale deviations. The question of global optimality of p_U is of little importance, since the other ISPs can in any case not be expected to maintain their strategies if *UISP* makes large-scale deviations.[6] However, an interesting question that remains is whether allowing *UISP* to play a mixed strategy leads to a different equilibrium point. We will consider this in Theorem 3.

Proof of Theorem 2: The proof of this theorem makes use of a generalization of the solution of the Bertrand-Edgeworth oligopoly in [14], [15], taking forward contracting into account.

Preliminaries: Reduced Pricing Game: We start by considering the *reduced pricing game* arising between the ISP_i if *UISP* has precommitted to a fixed price p_U. In analogy with Definition 1, the following regions turn out to be useful.

Definition 2. Let $0 \leq f_1 \leq k$. We say that market potential is $(f_1,\, p_U)$-*high* if

$$\beta p_U \ \leq\ \alpha - k(n+1) + f_1; \tag{21}$$

that market potential is $(f_1,\ p_U)$-*low* if

$$\beta p_U \ \geq\ \alpha - k(n-1) - f_1; \tag{22}$$

and that market potential is $(f_1,\ p_U)$-*intermediate* if

$$\alpha - k(n+1) + f_1 \ <\ \beta p_U \ <\ \alpha - k(n-1) - f_1. \tag{23}$$

The form of the equilibrium depends on the level of market potential. The following lemma (proved in the appendix) shows that the reduced pricing game between the ISP_i has a unique pure-strategy equilibrium if market potential is $(f_1,\, p_U)$-high or low, and a unique mixed-strategy equilibrium if market potential is $(f_1,\, p_U)$-intermediate. For high market potential, every ISP_i sets the same positive price, while for low market potential, every ISP_i sets price zero. For intermediate market potential, each ISP_i sets a random price chosen from an interval whose upper bound is a decreasing function in its contracting volume f_i. ISP_1's strategy may include setting the price to the upper bound with a positive probability.

[6] This argument for the stability of local equilibria is made in [22].

Lemma 1. *The reduced pricing game has the following Nash equilibria.*

(i) *If market potential is* $(f_1,\, p_U)$*-high, then there is a unique pure-strategy equilibrium, in which each ISP*$_i$ *chooses almost surely (i.e., with probability one)*

$$p_i = \frac{\alpha - \beta p_U - kn}{\beta}. \tag{24}$$

(ii) *If market potential is* $(f_1,\, p_U)$*-low, then there is a unique pure-strategy equilibrium, in which each ISP*$_i$ *chooses almost surely*

$$p_i = 0. \tag{25}$$

(iii) *If market potential is* $(f_1,\, p_U)$*-intermediate, then the reduced pricing game has the following unique mixed-strategy equilibrium. Let*

$$p_1^1 \equiv \frac{\alpha - \beta p_U - k(n{-}1) - f_1}{2\beta} \tag{26}$$

$$p_0 \equiv \frac{\beta (p_1^1)^2}{k{-}f_1} \tag{27}$$

$$h(p) = \frac{p - p_0}{p(kn - \alpha + \beta(p + p_U))} \tag{28}$$

$$H_j(p) = (k{-}f_j)h(p). \tag{29}$$

Define $p_1^{i+1} \in [0,\ p_1^1]$ *to be the unique value satisfying*

$$h(p_1^{i+1}) \equiv \frac{(k{-}f_{i+1})^{i-1}}{\prod_{j=1}^{i}(k{-}f_j)} \quad \text{for } 2 \le (i{+}1) \le n \tag{30}$$

$$p_1^{n+1} \equiv p_0. \tag{31}$$

For each $1 \le j \le n$*, define the function* $G_j(p)$ *on* $[p_0,\ p_1^j]$ *piecewise for* $p \in [p_1^{i+1}, p_1^i]$*,* $i \ge j$*,* $i \ge 2$ *as*

$$G_j(p) \equiv \begin{cases} \left(\frac{\prod_{k \le i,\, k \ne j} H_k(p)}{(H_j(p))^{i-2}} \right)^{\frac{1}{i-1}} & \text{if } p > p_0, \\ 0 & \text{if } p = p_0. \end{cases} \tag{32}$$

Then the reduced pricing game has a unique mixed-strategy Nash equilibrium, in which each ISP$_j$ *plays a random* $p_j \in [p_0\ p_1^j)$ *according to the cumulative density function* G_j*, and ISP*$_1$ *chooses the value* $p_1 = p_1^1$ *with positive probability* $1 - \frac{k - f_2}{k - f_1}$*.*
The mixed strategies p_i *(as random variables) almost surely satisfy*

$$\max\left\{0,\ \frac{\alpha - kn}{\beta} - p_U\right\} < p_i < \frac{\alpha - k(n{-}1)}{\beta} - p_U, \tag{33}$$

and ISP$_i$*'s expected payoff over every mixed strategy* p_j *is*

$$\mathbb{E}_p \pi_i = p_0(k{-}f_i). \tag{34}$$

Moreover, $\mathbb{E}p_{max} \equiv \mathbb{E}\max_i\{p_i\}$ *is everywhere a continuous function of* p_U*. It is continuously differentiable in the region of* $(f_1,\, p_U)$*-intermediate market potential, but it is not differentiable at the boundary points* $\beta p_U = \alpha - k(n+1) + f_1$ *and* $\beta p_U = \alpha - k(n{-}1) - f_1$ *towards* $(f_1,\, p_U)$*-low and* $(f_1,\, p_U)$*-high market potential.*

Existence: This lemma allows us to complete the proof of Theorem 2 by showing the existence of the equilibrium point. Let p_U be such that

$$\max\{k(n{-}1),\ \alpha - k(n{+}1) + f_1\} \ \le\ \beta p_U \ \le\ \min\left\{kn,\, \frac{\alpha}{2}\right\}.$$

It follows that

$$\beta p_U \ \le\ \frac{\alpha}{2} \ =\ \alpha - \frac{\alpha}{2} \ <\ \alpha - k(n{-}1) - f_1,$$

since $\alpha > 2(n{-}1)k + 2f_1$.

Let $\{p_i\}_i$ be the mixed-strategy equilibrium of Lemma 1. Then the mixed strategy p_i maximizes *ISP*$_i$'s profit. To prove our theorem, we just need to show that *UISP*'s expected profit is at a local maximum at some p_U in this range.

First, suppose that $\beta p_U > \alpha - k(n+1) + f_1$. Then *UISP*'s expected profit is

$$\mathbb{E}\pi_U \ =\ p_U(\alpha - \beta(p_U + \mathbb{E}p_{max})),$$

which is locally maximized by p_U if and only if

$$p_U = \frac{\alpha - \beta\mathbb{E}p_{max}}{2\beta}. \tag{35}$$

At the upper bound of the allowed range for p_U, if $p_U = \min\left\{kn,\, \frac{\alpha}{2}\right\}$, then

$$\mathbb{E}p_{max} \ \ge\ \frac{\alpha}{\beta} - 2p_U.$$

At the lower bound of the allowed range, if $\beta p_U = k(n-1) > \alpha - k(n+1) + f_1$, then

$$\mathbb{E}p_{max} \ \le\ \frac{\alpha - k(n{-}1)}{\beta} - p_U \ =\ \frac{\alpha}{\beta} - 2p_U.$$

Since $\mathbb{E}p_{max}$ is continuous in p_U, the Intermediate Value Theorem shows that there exists a value $p_U^* \in \left[k(n{-}1),\ \min\left\{kn,\, \frac{\alpha}{2}\right\}\right]$ such that (35) holds.

On the other hand, at the lower bound $\beta p_U = \alpha - k(n+1) + f_1 \ge k(n-1)$ the mixed-strategy equilibrium of p_i turns out to be the pure-strategy equilibrium given by $p_i = \mathbb{E}p_{max} = \frac{\alpha - \beta p_U - kn}{\beta} < \frac{\alpha}{\beta} - 2p_U$ since $\beta p_U < kn$. Here, by the Intermediate Value Theorem, there exists a value $p_U^* \in \left(\alpha - k(n{+}1) + f_1,\ \min\left\{kn,\ \frac{\alpha}{2}\right\}\right]$ such that (35) holds. Since $\beta p_U^* > \alpha - k(n{+}1) + f_1$, the total demand served by *UISP* retains its functional form in some neighborhood of p_U^*, and p_U^* does indeed locally maximize *UISP*'s profit.

Uniqueness: To prove that there is only one equilibrium point with the given properties, we first need a technical lemma, proved in the appendix, on the variation with the constant price p_U of the expected maximum price chosen by an *ISP*$_i$.

Lemma 2. *Suppose market potential is* f_1*-intermediate. Let the expected maximum downstream price be* $\mathbb{E}p_{max} = \mathbb{E}p_{max}(p_U, (f_i)_{i=1}^n)$ *as specified in Lemma 1. Let* p_U^* *be the pure strategy followed by UISP at the equilibrium point constructed above. Then, at* p_U^**, the function* $\mathbb{E}p_{max}$ *satisfies*

$$\left.\frac{\partial \mathbb{E}p_{max}}{\partial p_U}\right|_{p_U = p_U^*} \ >\ -2. \tag{36}$$

Consider any equilibrium point $(p_U;\ p_1,\ \ldots,\ p_n)$, where p_U is a locally optimal pure strategy and each p_i is an optimal mixed strategy. It follows from the non-existence of a pure-strategy equilibrium, proved in Theorem 1, that market potential is (f_1, p_U)-intermediate.

Consider the function

$$f(p_U) = \alpha - 2\beta p_U - \beta \mathbb{E} p_{max}(p_U).$$

At any equilibrium point satisfying our assumptions, we have $f(p_U) = 0$. We have already shown the existence of such a point $p_U = p_U^{(1)}$. It follows from Lemma 1 that f is continuously differentiable. By Lemma 2 $f'(p_U^{(1)}) = -2\beta - \beta \frac{\partial \mathbb{E} p_{max}}{\partial p_U} < 0$.

Suppose, for a contradiction, that there exists $p_U^{(2)} \neq p_U^{(1)}$ with the same properties. Without loss of generality $p_U^{(1)} < p_U^{(2)}$. It follows from the sign of the derivative of f that we can find $0 < \epsilon_1,\ \epsilon_2 < \frac{1}{2}(p_U^{(2)} - p_U^{(1)})$ such that $f(p_U^{(1)} + \epsilon_1) < 0$ and $f(p_U^{(2)} - \epsilon_2) > 0$. Since f is a continuous function, the Intermediate Value Theorem gives $p_U^{(3)} \in (p_U^{(1)} + \epsilon_1,\ p_U^{(2)} - \epsilon_2)$ such that $f(p_U^{(3)}) = 0$.

Inductively, we obtain an infinite sequence $p_U^{(1)},\ p_U^{(2)},\ p_U^{(3)},\ \ldots$ of distinct points in $[p_U^{(1)},\ p_U^{(2)}]$ such that $f(p_U^{(1)}) = f(p_U^{(2)}) = \cdots = 0$. By the Bolzano-Weierstrass Theorem, this sequence must have an accumulation point $\overline{p_U}$. Clearly then $f(\overline{p_U}) = 0$ and $f'(\overline{p_U}) = 0$, which contradicts Lemma 2. We have therefore established uniqueness of *UISP*'s equilibrium price $p_U^{(1)}$. By Lemma 1, the equilibrium point is unique. ■

One remaining question is whether allowing the upstream ISP to play any mixed strategy gives rise to a different equilibrium. It turns out that this is not the case for mixed-strategy Nash equilibria where bandwidth demand can be served completely and is sufficient to fill all but one downstream ISPs' networks almost surely.

Theorem 3. *Let market potential be f_1-intermediate. Suppose there exists a mixed-strategy Nash equilibrium in the pricing game such that almost surely*

$$k(n-1) \ \leq\ \alpha - \beta(p_U + p_i)\ \leq\ kn. \qquad (37)$$

Then p_U is a pure strategy and the equilibrium is the equilibrium point given in Theorem 2.

Proof of Theorem 3: Let

$$\begin{aligned} \underline{p_U} &= \sup\{p : \mathbb{P}\{p_U < p\} = 0\}, \\ \overline{p_U} &= \inf\{p : \mathbb{P}\{p_U > p\} = 0\}. \end{aligned}$$

Inequality (37) must still hold almost surely if *UISP* plays any pure strategy $p_U \in [\underline{p_U},\ \overline{p_U}]$. For any such pure strategy, *UISP*'s expected profit is

$$\mathbb{E}\pi_U(p_U)\ =\ p_U(\alpha - \beta p_U - \beta \mathbb{E} p_{max}).$$

This is a quadratic function with a unique maximum on the domain $p_U \in [\underline{p_U},\ \overline{p_U}]$. Therefore, *UISP* plays a pure strategy. ■

Given the forward contracts entered into by the downstream providers, we have thus completely characterized the ISPs' pricing behavior. In general, the size of the market potential relative to the available capacity determines whether the game has a pure or mixed-strategy equilibrium.

When market potential is low, there is a pure-strategy Nash equilibrium with downstream prices equal to zero or marginal cost. The downstream ISPs compete the price down in this case, or, for a single downstream firm operating as part of a bilateral monopoly, the capacity sold by forward contracts absorbs all demand.

When market potential is high, there is a range of pure-strategy Nash equilibria with different divisions of the same total network price between the upstream and downstream industries. Bandwidth demand attains the level of available capacity. At this point the total price is equal to the value of a marginal unit of capacity. This price is commonly referred to as the *congestion price*. The balance of bargaining power between the firms determines which equilibrium arises. When the upstream ISP has all the bargaining power, the fraction of the total income obtained by the downstream industry is a decreasing function of the lowest contracting volume f_1, but is independent of all other contracting volumes.

For intermediate market potential, there is a pure-strategy Nash equilibrium only in the case of a bilateral monopoly (and capacity is not exhausted in this case). For a downstream oligopoly ($n \geq 2$), there exists an equilibrium point consisting of optimal mixed strategies for each downstream ISP and a locally optimal pure strategy for the upstream ISP.

Despite the different pricing outcome in the two non-trivial cases of intermediate and high market potential, the next section shows that the incentives for forward contracting can be analyzed in a uniform way over both regions.

VI. Forward Contracting

Having analyzed the second-stage pricing subgame in sections III through V, by backward induction we can turn our attention to the first stage choice of forward contracting in the game described in section II. In particular, we will analyze the network providers' choice of contracting under uncertain bandwidth demand. We will establish that the equilibrium contracting volumes are always asymmetric, with one provider choosing the unique lowest contracting volume, before deriving the form of the externalities within the oligopoly that are due to the choice of contracting volumes in equilibrium. The results of the previous sections show that the lowest contracting volume is an important factor in determining second-stage prices. In the case of the pure-strategy equilibrium outcome, the lowest contracting volume is the only contracting volume that determines the second-stage outcome. As the smallest contracting volume increases, downstream prices decline, hurting all downstream providers. However, the firm with the smallest contracting volume is clearly subject to more price risk than the other providers.

We now relax the assumption that $0 < f_i < k$ to allow capacities $0 \leq f_i \leq k$ sold by forward contracting. When some $f_i = 0$ or $f_i = k$, we assume the outcome of the second-stage pricing game is the continuous extension of the pure-strategy equilibrium of Theorem 1 or the equilibrium of Theorem 2, as appropriate.[7]

We formally define the first-stage income from forward contracts sold at the expected second-stage price as

$$I_i = f_i \mathbb{E}_\beta \mathbb{E}_p p_i. \tag{38}$$

Recall from Definition 1 that the market potential α is 0-*high* if $\alpha \geq (2n+1)k$ and 0-*low* if $\alpha \leq 2(n-1)k$. In the case of 0-high market potential, it is easy to show that a pure-strategy Nash equilibrium of contracting volumes exists and all but one contracting volumes are maximal $f_2 = f_3 = \cdots = f_n = k$ in equilibrium. In the more general case where we only know that market potential is not 0-low (so the downstream ISPs may not compete prices down to zero in the second stage), we do not know whether there is a pure-strategy Nash equilibrium in the first-stage choice of contracting volumes. However, any such equilibrium must satisfy the following result.

Theorem 4. *Suppose market potential is not* 0*-low and the ISPs' second-stage moves are the ones predicted by Theorems 1 and 2, assuming the greatest* p_U *when there are multiple equilibria. Suppose there is a pure-strategy equilibrium of positive first-stage contracting volumes, so without loss of generality*

$$0 < f_1 \leq f_i \quad \textit{for every } i. \tag{39}$$

Then the lowest contracting volume is unique, i.e.,

$$f_1 < f_i \quad \textit{for every } i > 1. \tag{40}$$

Thus when market potential is not 0-low, any contracting equilibrium where the downstream ISPs obtain positive payoffs must be asymmetric. A risk-averse provider would seek to set a high contracting volume as insurance against price risk, hoping that some other provider will choose a low contracting volume and thereby raise the downstream second-stage price level.

How would the lowest contracting *ISP*$_i$ be chosen in practice? Although no provider would want to be the one choosing the lowest contracting volume, such a provider may arise naturally in practice, for example, due to asymmetries in information, risk aversion, or timing. Nevertheless, the lack of symmetric equilibrium may be a source of uncertainty for network providers considering investment into bandwidth.

Proof of Theorem 4: Clearly market potential is not f_1-low, since *ISP*$_1$ can achieve a positive profit by choosing a sufficiently low contracting volume $f_1 > 0$, subject to market potential not being 0-low.

Suppose first that market potential is f_1-high. The second-stage subgame has a pure-strategy equilibrium, which is independent of f_j, for $j > 1$. Since *ISP*$_j$, $j > 1$, is strictly risk-averse, he has an incentive to choose $f_j > f_1$.

Suppose that market potential is f_1-intermediate instead. Suppose, for a contradiction, that $f_2 = f_1$. We will show that, if *ISP*$_1$ has no incentive to choose a lower contracting volume, then he must have an incentive to choose a higher one. For each β, *ISP*$_1$'s profit varies with $f_1 = f_2$ according to

$$\begin{aligned} \frac{d}{df_1}\bigg|_\pm (\mathbb{E}_p \pi_1 + I_1) &= (p_0(\beta)(k - f_1) + f_1 \mathbb{E}_\beta \mathbb{E} p_1) \\ &= -p_0 + (k - f_1) \left(\frac{\partial p_0}{\partial f_1}\bigg|_\pm + \frac{\partial p_0}{\partial p_U} \frac{d\mathbb{E}p_U^*}{df_1}\bigg|_\pm \right) \\ &\quad + \mathbb{E}_\beta \mathbb{E} p_1 + f_1 \mathbb{E}_\beta \left(\frac{\partial \mathbb{E} p_1}{\partial f_1}\bigg|_\pm + \frac{\partial \mathbb{E} p_1}{\partial p_U} \frac{d\mathbb{E}p_U^*}{df_1}\bigg|_\pm \right), \end{aligned}$$

where

$$\frac{d\mathbb{E}p_U^*}{df_1}\bigg|_\pm = -\frac{\partial \mathbb{E}p_{max}}{\partial f_1}\bigg|_\pm \left(2 + \frac{\partial \mathbb{E}p_{max}}{\partial p_U}\right)^{-1}.$$

It is easy to check that

$$\frac{\partial \mathbb{E}p_1}{\partial f_1}\bigg|_- \leq \frac{\partial \mathbb{E}p_1}{\partial f_1}\bigg|_+ \text{ and } \frac{\partial \mathbb{E}p_{max}}{\partial f_1}\bigg|_- \leq \frac{\partial \mathbb{E}p_{max}}{\partial f_1}\bigg|_+.$$

Trivially

$$\frac{\partial p_0}{\partial f_1}\bigg|_- < 0 = \frac{\partial p_0}{\partial f_1}\bigg|_+.$$

Since $\frac{\partial \mathbb{E}p_1}{\partial p_U} < 0$ and $\frac{\partial p_0}{\partial p_U} < 0$, clearly

$$\frac{d}{df_1}\bigg|_- (\mathbb{E}_p \pi_1 + I_1) < \frac{d}{df_1}\bigg|_+ (\mathbb{E}_p \pi_1 + I_1),$$

so

$$\frac{\partial}{\partial f_1}\bigg|_+ \mathbb{E}_\beta \mathscr{U}(\mathbb{E}_p \pi_1 + I_1) > \frac{\partial}{\partial f_1}\bigg|_- \mathbb{E}_\beta \mathscr{U}(\mathbb{E}_p \pi_1 + I_1).$$

The right-hand side must be non-negative since *ISP*$_1$ has no incentive to decrease his contracting volume. Hence the left-hand side is positive, and *ISP*$_1$ can increase his expected utility by raising his contracting volume slightly. This is a contradiction, so $f_2 \neq f_1$ as required. ∎

We now quantify the impact of one downstream provider's choice of contracting volume on its competitors' utility.

Theorem 5. *Suppose*

$$0 \leq f_1 < f_2 \leq \cdots \leq f_n < k, \tag{41}$$

and the ISPs' second-stage moves are the ones predicted by Theorems 1 and 2, assuming the greatest p_U *when there are multiple equilibria.*

If market potential is f_1*-intermediate, an increase of* f_1 *by* *ISP*$_1$ *results in a negative marginal externality on the other downstream ISPs' payoffs; and an increase of* f_j *by* *ISP*$_j$*, for any* $j > 1$*, results in a positive marginal externality on the other downstream ISPs' payoffs.*

[7]It is easy to show that this extension is well-defined and constitutes an equilibrium. However, when $f_i = k$ the equilibrium may no longer be uniquely characterized as above. In this case *ISP*$_i$ is indifferent between two prices if even the higher one guarantees full network utilization. This leads to the emergence of equilibria where *ISP*$_i$ can raise his price without any loss of second-stage income, violating the law of one price. To exclude such unrealistic equilibria, our construction explicitly restricts attention to equilibria that are the limit of equilibria arising when every $f_j < k$.

When $f_i = 0$, additional equilibria exist where prices are set so high that zero demand is served by the network $D_U = 0$. Such equilibria, where the ISPs effectively refuse to interconnect are unrealistic if another equilibrium exists where all ISPs have positive profits, as is the case in the regions of high and intermediate market potential. These equilibria disappear if at least one *ISP*$_i$ has $f_i > 0$, as *ISP*$_i$ then has an incentive to set $p_i = 0$. In the region of low market potential, $f_i = 0$ again leads to additional equilibria, where the downstream ISPs have zero profit.

If market potential is f_1-high, an increase of f_1 by ISP$_1$ results in a negative marginal externality on the other downstream ISPs' payoffs; and an increase of f_j by ISP$_j$, for any $j > 1$, results in zero marginal externality on the other downstream ISPs' payoffs.

Choosing a low contracting volume f_1 is like providing a "public good"[8] to the oligopoly, by raising the general price level, but doing so is privately costly to *ISP*$_1$, as it implies a low level of insurance against demand uncertainty. In the case of f_1-intermediate market potential, the choices of the contracting volumes $f_2, \ldots, f_n$ result in externalities with the opposite sign, so greater contracting volumes benefit other ISPs. The presence of externalities means that downstream providers have an incentive to coordinate their actions by collusion. In this case, there is a particular incentive for a provider to make side-payments to a competitor in return for this provider agreeing to refrain from entering into forward contracts.

Proof of Theorem 5: If market potential is f_1-high, every *ISP*$_i$ charges price $p_1 = \frac{k-f_1}{\beta}$ in the second stage. The theorem is trivial in this case.

If market potential is f_1-intermediate, let $p_0^* = p_0(p_U^*)$ and $\mathbb{E}_p p_i^* = (\mathbb{E}_p p_i)(p_U^*)$. Then:

$$\frac{dp_0^*}{df_i} = \frac{\partial p_0}{\partial f_i} - \frac{\partial p_0}{\partial p_U}\frac{\partial \mathbb{E}p_{max}}{\partial f_i}\left(2 + \frac{\partial \mathbb{E}p_{max}}{\partial p_U}\right)^{-1},$$
$$\frac{d\mathbb{E}p_j^*}{df_i} = \frac{\partial \mathbb{E}p_j}{\partial f_i} - \frac{\partial \mathbb{E}p_j}{\partial p_U}\frac{\partial \mathbb{E}p_{max}}{\partial f_i}\left(2 + \frac{\partial \mathbb{E}p_{max}}{\partial p_U}\right)^{-1}.$$

When $f_i > f_1$, $\frac{\partial p_0}{\partial f_i} = 0$, $\frac{\partial p_0}{\partial p_U} < 0$ and $\frac{dp_U^*}{df_i} > 0$. Hence $\frac{dp_0^*}{df_i} > 0$. On the other hand, $\frac{\partial p_0}{\partial f_1} < 0$ and $\frac{dp_U^*}{df_i} > 0$, so $\frac{dp_0^*}{df_1} < 0$.

Similarly, when $1 < i \neq j$, $\frac{\partial \mathbb{E}p_j}{\partial f_i} \geq 0$ and $\frac{\partial \mathbb{E}p_j}{\partial p_U} < 0$, so we have $\frac{d\mathbb{E}p_j^*}{df_i} > 0$. On the other hand, if $j > 1$, $\frac{\partial \mathbb{E}p_j}{\partial f_1} \leq 0$, so $\frac{d\mathbb{E}p_j^*}{df_1} < 0$.

Since *ISP*$_j$'s profit is the stochastic quantity $I_j + \mathbb{E}_p \pi_j$ where $I_j = f_j \mathbb{E}_\beta \mathbb{E}_p p_j^*$ and $\mathbb{E}_p \pi_j = p_0^*(k - f_j)$, the result follows immediately. ■

VII. Conclusions

This article started with the observation that a dynamic pricing system for the Internet would ensure a more efficient allocation of resources. However, without forward contracting, providers would be exposed to substantial price risk due to the uncertainty in market demand. Could forward contracting remove this price risk? In the absence of any strategic interaction, e.g. in a communication network operated by a single provider, the answer is yes. When strategic interaction is considered in a multi-provider network, the situation is more complex. Forward contracting weakens a provider's strategic incentive to charge high prices. Thus, in the presence of an upstream monopoly, the optimal forward contracting strategy is a trade-off between reducing price risk and seeking to ensure high prices in the future. When the contracting provider is part of an oligopoly, the optimal contracting strategy will also be dependent on its competitors' strategies.

In this paper, we have analyzed the incentives for forward contracting by ISPs competing to supply bandwidth on a downstream network segment, when a single ISP with significant market power supplies bandwidth on a complementary upstream network segment. In order to determine the incentives for contracting, we have first studied the subsequent pricing equilibrium which arises in different contracting scenarios. Depending on the level of market potential compared with the available bandwidth capacity, the pricing outcome can be characterized as an equilibrium in pure or mixed strategies.

We can draw some conclusions on the choice of forward contracts over two stages assuming the market's price-sensitivity is random and the downstream firms are risk-averse. Note that in addition to the benefits, there are also risks associated with forward contracting. Provided that market potential is not so low that downstream prices are competed down to zero, we prove that any pure-strategy Nash equilibrium of positive contracting volumes must be asymmetric and have a unique lowest contracting volume. This gives rise to a version of the game of "Chicken": as the provider who chooses this lowest contracting volume is exposed to the risk of more price uncertainty than the other competitors, no selfish risk-averse provider would want to be the one choosing the lowest equilibrium contracting volume. In practice, this instability may discourage investment into bandwidth. The reason is that forward contracts have a negative impact on a provider's strategic incentives during the pricing stage. A natural low-contracting provider may arise in the presence of asymmetries, for example, in risk aversion or timing.

We further prove that the choice of contracting volumes causes externalities, both negative and positive. An increase in the lowest contracting volume has a negative marginal externality on other downstream ISPs. An increase in any other contracting volume has no externality for high market potential, but a positive marginal externality for intermediate market potential. In this sense, we can think of the downstream ISP with the least forward contracting as providing a public good to the oligopoly. A consequence is an incentive for providers to collude on contracting choice, as discussed below.

In summary, for risk averse ISPs operating under this market structure employing forward contracts, this paper provides some initial practical guidelines. First, if an ISP believes that every competitor will choose a high volume of forward contracting, then he would be well-advised to choose a low contracting volume. Second, a provider with a high contracting volume might want to act in such a way that a low-contracting provider would choose a lower contracting volume than would be privately optimal. It could achieve this through side-payments or other strategic behavior. Third, given that forward contracts have a negative impact on a provider's strategic incentives during the pricing stage, network providers might want to vertically integrate with the upstream provider in order to eliminate this effect. Of course, this paper is an initial investigation into this topic, and our model is somewhat restrictive. One interesting direction for future research would be to consider interactions between ISPs linked by other

[8] A *public good* is a good that is non-excludable and non-rivalrous, i.e., it is not possible to exclude someone from using the good, and one individual's usage does not prevent another from using it.

network shapes.

Finally, our framework could have other networking applications; for example, similar risk-return trade-offs might exist in last-hop wireless spectrum markets, see [23].

APPENDIX

Proof of Lemma 1:

(i) Assume market potential is (f_1, p_U)-high. At the given prices, assumptions (5)–(7) imply that capacity is exhausted, so no ISP_i has an incentive to lower his price. From (21) and (24), it follows that $\beta p_i \geq k - f_1 \geq k - f_i$, and it was shown in (20) that, together with the fact that capacity is exhausted, this implies that ISP_i has no incentive to raise his price. Therefore this point is indeed a pure-strategy equilibrium.
To establish uniqueness, consider any pure-strategy equilibrium. Note that every ISP_i must have a positive profit and, in particular, a positive market share $D_i > 0$ in equilibrium, since ISP_i can achieve a positive profit by choosing the price given in (24) regardless of its competitors' strategies. It follows that any two ISP_i and ISP_j must choose the same price $p_i = p_j$, since otherwise the ISP with the lower price would have an incentive to raise its price.
Next, assumptions (7)–(9) imply that, unless the price p_i equals the value given in (24), either each downstream ISP's market share is less than its capacity k or the total demand cannot be served by the downstream ISPs. In both cases, a downstream ISP would have an incentive to change its price, which shows that (24) must hold in equilibrium. This establishes uniqueness.

(ii) Assume market potential is (f_1, p_U)-low. If every downstream network chooses a price of zero, then from (5), (22) the total demand satisfies $d_{\text{market}} \leq k(n-1) + f_1$. Assumptions (7)–(8) imply that ISP_i's second-stage profit when choosing $p_i > 0$ is negative. Therefore, this point is a pure-strategy equilibrium.
For uniqueness, consider any pure-strategy equilibrium. We will show that if $p_i > 0$ then $\pi_i < 0$ so ISP_i has an incentive to set $p_i = 0$. Let ISP_i be the network choosing the highest price. First, if $D_i = 0$ then $\pi_i < 0$ is trivial.
Second, in the case where $D_i > 0$, suppose there are m downstream ISPs choosing price p_i. From assumptions (7)–(9), the market share obtained by each is $D_i = (\alpha - \beta p_U - \beta p_i - k(n-m))/m < k$, where the inequality follows from (22) and the fact that $f_1 < k$. If $m > 1$ then every provider choosing price p_i has an incentive to just undercut the other providers choosing price p_i, contradicting the equilibrium assumption. On the other hand, if $m = 1$, then assumptions (7)–(9), (22) imply that $\pi_i < 0$. We have therefore shown uniqueness.

(iii) Assume market potential is (f_1, p_U)-intermediate. The following results are direct consequences of the definitions stated in the lemma.

- We have
$$0 < p_0 < p_1^1 < \frac{k - f_1}{\beta}. \tag{42}$$
- For $p_0 \leq p \leq p_1^1$, we have
$$k(n-1) < d_{\text{market}}(p + p_U) < kn. \tag{43}$$
- The functions $h(p)$ and $H_1(p)$ are continuous and strictly increasing on $[p_0, p_1^1]$, with $H_1(p_0) = 0$, $H_1(p_1^1) = 1$.
- We have
$$p_0 = p_1^{n+1} \leq p_1^n \leq \cdots \leq p_1^2 = p_1^1, \tag{44}$$
where, for $1 < i < n$, $p_1^i = p_1^{i+1}$ if and only if $f_i = f_{i+1}$. Also $p_1^i > p_0$ since $f_i < k$.
- For any j, G_j is a continuous and strictly increasing function on $[p_0, p_1^j]$. For $j > 1$, $G_j(p_1^j) = 1$. $G_1(p_1^1) = \frac{k-f_2}{k-f_1}$.
- Finally, for $p_0 \leq p \leq p_1^i$, the cumulative density function of $\max_{j \neq i}\{p_j\}$ satisfies
$$G_{-i}(p) \equiv \prod_{j \neq i,\, p_1^j > p} G_j(p) = H_i(p). \tag{45}$$

Given this, we can show that the strategies defined in the lemma form a Nash equilibrium. Note that by inequality (43) and assumptions (7)–(8), if $\max_{j \neq i}\{p_j\} > p_i$, then ISP_i's market share $D_i = k$, whereas if $\max_{j \neq i}\{p_j\} < p_i$, then ISP_i's market share is the residual demand after the other $(n-1)$ downstream networks' capacities are exhausted, $D_i = d_{\text{market}}(p_i + p_U) - k(n-1)$. Thus ISP_i's market share depends only on p_i and $\max_{j \neq i}\{p_j\}$.
Since the probability distributions have no point mass at any $p_0 < p < p_1^1$, and at least one ISP_j with $j \neq i$ has $p_j > p_0$ almost surely, the event that $\max_{j \neq i}\{p_j\} = p$ has zero probability for any $p < p_1^1$.
Thus ISP_i's profit, when choosing some $p_0 \leq p < p_1^i$, is $\mathbb{E}\pi_i(p) = (1 - G_{-i}(p))p(k - f_i) + G_{-i}(p)p(d_{\text{market}}(p + p_U) - k(n-1) - f_i) = p_0(k - f_i) = \mathbb{E}\pi_i(p_0)$.
Moreover, for ISP_1, $\mathbb{E}\pi_1(p_1^1) = p_0(k - f_1) = \mathbb{E}\pi_1(p_0)$.
To establish the equilibrium, we just need to prove that, conditional on the other ISPs' strategies, no ISP_i can increase his profit by choosing a price p outside the support of G_i, $[p_0, p_1^i]$.
First, since each ISP_i can set price p_0 for a market share of k, setting a lower price $p < p_0$ leads to lower profits: $\mathbb{E}\pi_i(p) = p(k - f_i) < p_0(k - f_i) = \pi_i(p_0)$.
Second, if ISP_i sets price $p_1^i < p \leq p_1^1$, then by (44), $p_1^{j+1} \leq p \leq p_1^j$ for some j. Let $p_{max}^{(-j)} \equiv \max_{j \neq i}\{p_j\}$ and let $G_{max}^{(-j)}$ be the cumulative density function of $p_{max}^{(-j)}$. Observe that, under the equilibrium strategies, for $p_1^{j+1} \leq p \leq p_1^j$, we have

$$G_{max}^{(-j)}(p) = \prod_{l=1}^{j} G_l(p) = \left(h(p) \prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} h(p)$$
$$\geq \left(h(p_1^{j+1}) \prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} h(p) \geq H_i(p), \tag{46}$$

where the first inequality follows from the monotonicity of h and the second inequality follows from $i \leq j + 1$.

Thus ISP_i's profit is $\mathbb{E}\pi_i(p) = (1-G_{max}^{(-j)}(p))p(k-f_i) + G_{max}^{(-j)}(p)p(d_{\text{market}}(p+p_U) - k(n-1) - f_i) \le \mathbb{E}\pi_i(p_0)$, where the inequality follows from (43) and (46). This shows that ISP_i has no incentive to set a price $p_1^i < p \le p_1^1$.

It now only remains to consider a third case where ISP_i sets price $p > p_1^1$. His profit function takes the form $\mathbb{E}\pi_i(p) = p(d_{\text{market}}(p+p_U) - k(n-1) - f_i)$, which is a quadratic function attaining its maximum in $(0, p_1^1]$. However, from (26)–(27), (42) and $f_i \ge f_1$, we have $\mathbb{E}\pi_i(p_1^1) = (k-f_1)p_0 - (f_i - f_1)p_1^1 \le \mathbb{E}\pi_i(p_0)$, showing that ISP_i has no incentive to deviate by setting $p > p_1^1$. This establishes that ISP_i has no incentive to deviate from his equilibrium strategy, and therefore the given mixed strategies form a Nash equilibrium.

Conversely, to prove uniqueness, consider any mixed-strategy Nash equilibrium given by cumulative density functions $G_j(p) = \mathbb{P}\{p_j < p\}$. Consider the well-defined low- and high-price thresholds for each ISP_j

$$\overline{p_0^j} = \sup\{p : G_j(p) = 0\},$$
$$\overline{p_1^j} = \inf\{p : G_j(p) = 1\}.$$

Note that

- Every ISP_j obtains a positive expected profit $\mathbb{E}\pi_j$ in equilibrium. Indeed, we have already shown that ISP_1's profit is positive when choosing p_1^1 regardless of his competitors' strategies. But ISP_1 must then have a positive low-price threshold $\overline{p_0^1}$ and any competitor ISP_j can obtain a positive profit by slightly undercutting this price.
- In equilibrium, there is sufficient capacity for the total demand at each low-price threshold:

$$\alpha - \beta(\overline{p_0^j} + p_U) \le kn;$$

  and each ISP_j's market share is positive even at his high-price threshold:

$$\alpha - \beta(\overline{p_1^j} + p_U) > k(n-1).$$

  It is easy to check that the first inequality is required for ISP_j to have no incentive to play a mixed strategy with a higher $\overline{p_0^j}$, and that the second inequality is required for the ISP_j with the highest price $\overline{p_1^j}$ to have no incentive to play a mixed strategy with a lower $\overline{p_1^j}$.

It follows that every ISP_j has the low-price threshold p_0 defined in (27), i.e. $\overline{p_0^j} = p_0\,\forall j$; every ISP_j has the expected profit given in (34), i.e. $\mathbb{E}\pi_j = p_0(k-f_j)$; and ISP_1 has the high-price threshold p_1^1 defined in (26), and no high-price threshold exceeds it: $\overline{p_1^1} = p_1^1 \ge \overline{p_1^j}\,\forall j$.

Define cumulative density functions for $\max_{i\neq j}\{p_i\}$ as before:

$$G_{-j}(p) = \prod_{i\neq j} G_i(p).$$

From the equilibrium requirement that ISP_j should have no incentive to change his mixed strategy, it is straightforward to verify the following. There exists an open interval $U \supset [p_0, p_1^1]$ such that whenever $p \in U$, we have

$$G_{-j}(p) \ge H_j(p); \qquad (47)$$

and, whenever $G_{-j}(p) > H_j(p)$, we have

$$\exists\, \epsilon > 0 :\ G_j(p-\epsilon) = G_j(p+\epsilon). \qquad (48)$$

By the definition of $\overline{p_1^j}$, G_j cannot be locally constant at $\overline{p_1^j}$, so

$$G_{-j}(\overline{p_1^j}) = H_j(\overline{p_1^j}). \qquad (49)$$

Further, the following is easily shown:

- Each G_j is continuous on $(p_0,\ p_1^1]$. (So the mixed strategies have no point mass, except possibly at p_0 and p_1^1.) Moreover, for $j \neq 1$, G_j has no point mass at p_1^1, so

$$G_j(\overline{p_1^j}) = 1. \qquad (50)$$

- If $f_i < f_j$, then $\overline{p_1^j} \le \overline{p_1^i}$. Whenever $f_i = f_{i+1}$, we can re-order ISP_i, ISP_{i+1}, so that $\overline{p_1^{i+1}} \le \overline{p_1^i}$. Letting $\overline{p_1^{n+1}} \equiv \overline{p_0}$, without loss of generality

$$\overline{p_1^{n+1}} \equiv \overline{p_0} \le \overline{p_1^n} \le \overline{p_1^{n-1}} \le \cdots \le \overline{p_1^1} = p_1^1. \qquad (51)$$

  Then (49)–(51) imply, for $2 \le i \le n$:

$$G_{-i}(\overline{p_1^i}) = \prod_{j=1}^{i-1} G_j(\overline{p_1^i}) = H_i(\overline{p_1^i}). \qquad (52)$$

- We have $\overline{p_1^2} = p_1^1 = p_1^2$.

We are ready to prove that the mixed strategies employed are indeed those of our constructed equilibrium.

We now prove by induction that, for each $2 \le i \le n$:

(a) $\overline{p_1^i} = p_1^i$,
(b) $G_j(p) = \widetilde{G_j^i}(p)$ piecewise for $p \in [\overline{p_1^{i+1}}, p_1^i]$, $i \ge j$,

$$\widetilde{G_j^i}(p) \equiv \begin{cases} \dfrac{\left(\prod_{l\le i,\, l\neq j} H_l(p)\right)^{\frac{1}{i-1}}}{(H_j(p))^{\frac{i-2}{i-1}}} & \text{if } p > p_0, \\ 0 & \text{if } p = p_0. \end{cases}$$

For the case $i = 2$, we already know $\overline{p_1^2} = p_1^2 = p_1^1$, so (a) holds. For part (b), we have already shown that $G_j(p_1^1) = 1 = \widetilde{G_j^2}(p_1^1)$ for $j \neq 1$. For the case $j = 1$, equation (52) implies that $G_{-2}(\overline{p_1^2}) = G_1(p_1^1) = H_2(p_1^1) = \frac{k-f_2}{k-f_1} = \widetilde{G_1^1}(p_1^1)$, so (b) holds.

Now assume the inductive hypothesis holds for some $i - 1 < n$. We first show (a). Using part (b) of the inductive hypothesis for $i-1$ allows us to rearrange (52) as

$$h(\overline{p_1^i}) = \frac{(k-f_i)^{i-2}}{\prod_{j=1}^{i-1}(k-f_j)}.$$

The unique solution of this equation is $\overline{p_1^i} = p_1^i$, by the definition (30) of p_1^i, so (a) holds.

We now show (b). In the case $\overline{p_1^{i+1}} = p_1^i$, we have $G_j(p) = G_j(p_1^i) = \widetilde{G_j^{i-1}}(p_1^i) = \widetilde{G_j^i}(p_1^i)$ by the inductive hypothesis and the definition (30) of p_1^i, so (b) holds.

Consider the case $\overline{p_1^{i+1}} < p_1^i$. For every $\overline{p_1^{i+1}} \leq p \leq p_1^i$, if $p > p_0$, then

$$G_j(p) = \frac{\left(\prod_{l\leq i,\, l\neq j} G_{-l}(p)\right)^{\frac{1}{i-1}}}{(G_{-j}(p))^{\frac{i-2}{i-1}}}. \quad (53)$$

To establish (b), it is sufficient to show that $G_{-l}(p) = H_l(p)$ for every $l \leq i$, $p \in (\overline{p_1^{i+1}}, \overline{p_1^i})$: then $G_j(p) = \widetilde{G_j^i}(p)$ for $p \in [\overline{p_1^{i+1}}, \overline{p_1^i}]$ by (53) (using continuity at the interval bounds).

Suppose, for a contradiction, that there exists some $l \leq i$, $p \in (\overline{p_1^{i+1}}, \overline{p_1^i})$ such that $G_{-l}(p) \neq H_l(p)$. Then $G_{-l}(p) > H_l(p)$ by property (47). We start by showing that, for this value p, we have $G_j(p) > \widetilde{G_j^i}(p)$ for every $j \leq i$. We show this separately for j such that $G_{-j}(p) = H_j(p)$ and j such that $G_{-j}(p) > H_j(p)$. First, for every j such that $G_{-j}(p) = H_j(p)$, we have $G_j(p) > \widetilde{G_j^i}(p)$ by (53). Second, for every l satisfying $G_{-l}(p) > H_l(p)$, define

$$\overline{p_l} = \sup\{q : G_{-l}(q') > H_l(q')\ \forall p \leq q' \leq q\}. \quad (54)$$

By the inductive hypothesis for $i-1$, property (48) does not hold at p_1^i, so the supremum exists and $\overline{p_l} \leq p_1^i$.

$$G_{-l}(\overline{p_l}) = H_l(\overline{p_l}) \quad (55)$$

follows by continuity if $\overline{p_l} < p_1^i$, and by the inductive hypothesis for $i-1$ if $\overline{p_l} = p_1^i$. Using expression (53) for $G_l(\overline{p_l})$, equation (55), and inequality (47) for G_{-k}, $k \neq l$, gives

$$G_l(\overline{p_l}) \geq \widetilde{G_l^i}(\overline{p_l}). \quad (56)$$

Note that by the choice of l and (55), we must have $p < \overline{p_l}$. From (54), for $p \leq q' < \overline{p_l}$, we have $G_{-l}(q') > H_l(q')$, so property (48) implies that G_l is constant on $(p, \overline{p_l})$. Continuity at p and left-continuity at $\overline{p_l}$ imply $G_l(p) = G_l(\overline{p_l})$. From (56) and the fact that $\widetilde{G_l^i}$ is strictly increasing: $G_l(p) = G_l(\overline{p_l}) \geq \widetilde{G_l^i}(\overline{p_l}) > \widetilde{G_l^i}(p)$. Thus we have shown that $G_j(p) > \widetilde{G_j^i}(p)$ for every $j \leq i$.

It follows directly that, for every $j \leq i$,

$$G_{-j}(p) = \prod_{l\leq i,\, l\neq j} G_l(p) > H_j(p).$$

Next, note that the set $S \equiv \{p' \in [\overline{p_1^{i+1}}, p] : G_l(p'') > \widetilde{G_l^i}(p'')\ \forall p' \leq p'' \leq p,\ l \leq i\}$ is open in $[\overline{p_1^{i+1}}, p]$, since each G_l is locally constant at every point inside it, and each $\widetilde{G_l^i}$ is increasing. Again, using the monotonicity of $\widetilde{G_l^i}$, it is easy to check that $S = \{p' \in [\overline{p_1^{i+1}}, p] : G_l(p') = G_l(p)\ \forall l \leq i\}$, which is closed by continuity of G_l. But since S is non-empty, open and closed, it must be the entire interval $[\overline{p_1^{i+1}}, p]$.

To obtain the desired contradiction, we consider the cases $i = n$ and $i \leq n$ separately. First, in the case $i = n$, we have $\overline{p_0} = \overline{p_1^{i+1}} \in S$, so $G_l(\overline{p_0}) > \widetilde{G_l^i}(\overline{p_0})$ for every $l \leq i$. This implies that property (48) holds at $\overline{p_0}$, which contradicts the definition of $\overline{p_0}$. Second, in the case $i < n$, each G_l, $l \leq i$, is constant on S, so $G_{-(i+1)}$ is constant on S. Thus $G_{-(i+1)}(\overline{p_1^{i+1}}) = G_{-(i+1)}(p) \geq H_{i+1}(p) > H_{i+1}(\overline{p_1^{i+1}})$, where the inequalities follow from (47) and the fact that H_{i+1} is strictly increasing. This contradicts (52). In both cases, we have a contradiction, so we have shown part (b) of the inductive hypothesis. This completes the inductive argument.

Since $G_j(p) = 0$ for $p \leq p_0$, $1 \leq j \leq n$, we have proved that the cumulative density functions specifying the mixed strategies employed by the ISP_j in any equilibrium coincide with those in the equilibrium we have explicitly constructed. Hence the mixed-strategy equilibrium of our game is unique.

Continuous differentiability of $\mathbb{E}p_{max}$ as a function of p_U is trivial inside the regions of $(f_1,\ p_U)$-high and -low market potential. For $(f_1,\ p_U)$-intermediate market potential, it is obvious that p_1^1 and p_0 are continuously differentiable functions of p_U. The existence of a continuous derivative of p_1^i follows from $p_1^i = p_1^1$ when $f_i = f_1$, and from $h'(p_1^i) > 0$ by the implicit function theorem when $f_i \neq f_1$. We can write

$$\mathbb{E}p_{max} = \int_0^\infty (1 - \mathbb{P}\{p_{max} < p\})\, dp$$
$$= p_0 + \sum_{i=2}^{n} \int_{p_1^{i+1}}^{p_1^i} \left(1 - \left(\prod_{l=1}^{i} H_l(p)\right)^{\frac{1}{i-1}}\right) dp. \quad (57)$$

We now check that $\mathbb{E}p_{max}$ is continuously differentiable with respect to p_U. The limits of each integral are continuously differentiable with respect to p_U. Moreover, each integrand is continuously differentiable with respect to p_U and with respect to p, where the derivative with respect to p_U can be bounded above by an integrable function independently of p_U, for values of p_U in some sufficiently small interval. These conditions are sufficient for continuous differentiability of each integral with respect to p_U. Therefore $\mathbb{E}p_{max}$ is a continuously differentiable function of p_U for $(f_1,\ p_U)$-intermediate market potential. Continuity and lack of differentiability are easy to verify at the boundary points, completing the proof of the lemma. ■

Proof of Lemma 2: As we have seen in the proof of Lemma 1, the function $\mathbb{E}p_{max}$ satisfies the assumptions required for the existence of a continuous derivative which can be found by differentiating the expression in (57) after substituting the definition of H_l given in (29):

$$\frac{\partial \mathbb{E}p_{max}}{\partial p_U} = -\frac{1}{2}\left(1 - \frac{k-f_2}{k-f_1}\right) - Q, \quad (58)$$

where

$$Q \equiv \sum_{j=2}^{n} \int_{p_1^{j+1}}^{p_1^j} \frac{j}{j-1} \left(h(p) \prod_{l=1}^{j} (k-f_l)\right)^{\frac{1}{j-1}} \frac{\partial h(p)}{\partial p_U}\, dp. \quad (59)$$

The proof that $\frac{\partial \mathbb{E}p_{max}}{\partial p_U}$ is always greater than -2 is done in two parts: for $\delta \equiv \alpha - 2(n-1)k - 2f_1$ smaller than $\frac{12}{5}(k-f_1)$, and for δ greater than $\frac{20}{9}(k-f_1)$. (Note that these regions overlap.)

Consider the first case, $\delta < \frac{12}{5}(k-f_1)$. The function h is increasing, and $p \leq p_1^j$ in each integral in (59), so using (30):

$$\left(h(p)\prod_{l=1}^{j}(k-f_l)\right)^{\frac{1}{j-1}} \leq k-f_j \leq k-f_1, \qquad (60)$$

Using $j \geq 2$ and (60) in (59):

$$Q \leq 2(k-f_1)\sum_{j=2}^{n}\int_{p_1^{j+1}}^{p_1^j}\frac{\partial h(p)}{\partial p_U}\,dp$$

$$= 2(k-f_1)\int_{p_0}^{p_1^1}\frac{\beta\left(1-\frac{p_0}{p_1^1}\right)(p_1^1-p)}{p\left(\frac{p_1^1}{p_0}\beta(p_1^1-p_0)-\beta(p_1^1-p)\right)^2}\,dp \equiv \overline{Q}.$$

Letting $\gamma \equiv \frac{p_0}{p_1^1}$, with the change of variable $t \equiv \frac{p-p_0}{p_1^1-p_0}$, yields

$$\overline{Q} = 2\gamma\frac{(\gamma-1)(\log\gamma - \log(1-\gamma)) + (2\gamma-1)}{(2\gamma-1)^2}. \qquad (61)$$

Although the evaluated integral is undefined for $\gamma = \frac{1}{2}$, an application of L'Hôpital's Rule shows that it can be extended to this point, giving a continuous function of γ on $(0,\ 1)$. Note that $\overline{Q}$ is an increasing function of γ.

By the definitions in Lemma 1, $\mathbb{E}p_{max} \leq p_1^1$. From (35), $p_U = \frac{\alpha-\beta\mathbb{E}p_{max}}{2\beta} \geq \frac{\alpha-\beta p_1^1}{2\beta}$. Substituting this inequality into the definition of p_1^1 gives $p_1^1 \leq \frac{\delta}{3\beta}$. This together with (27) gives $\gamma = \frac{p_0}{p_1^1} = \frac{\beta p_1^1}{k-f_1} \leq \frac{\delta}{3(k-f_1)} \leq \frac{4}{5}$. Since $\overline{Q}$ is increasing everywhere on $0<\gamma<1$ and $\overline{Q}(\frac{4}{5})<\frac{3}{2}$, it follows that $\overline{Q}(\gamma)<\frac{3}{2}$ for any $0<\gamma\leq\frac{4}{5}$. Substituting $f_1 \leq f_2$ and $Q \leq \overline{Q} < \frac{3}{2}$ into (58) establishes the lemma for $\delta < \frac{12}{5}(k-f_1)$.

Consider now the second case, $\delta > \frac{20}{9}(k-f_1)$. The following bound is straightforward to verify:

$$\frac{\partial h(p)}{\partial p_U} \leq \frac{\partial h(p)}{\partial p} + \frac{h(p)}{p}. \qquad (62)$$

Substituting this inequality into (59) gives

$$Q \leq Q_1 + Q_2, \qquad (63)$$

where

$$Q_1 \equiv \sum_{j=2}^{n}\int_{p_1^{j+1}}^{p_1^j}\frac{j}{j-1}\left(h(p)\prod_{l=1}^{j}(k-f_l)\right)^{\frac{1}{j-1}}\frac{\partial h(p)}{\partial p}\,dp,$$

$$Q_2 \equiv \sum_{j=2}^{n}\int_{p_1^{j+1}}^{p_1^j}\frac{j}{j-1}\left(h(p)\prod_{l=1}^{j}(k-f_l)\right)^{\frac{1}{j-1}}\frac{h(p)}{p}\,dp.$$

Integrating:

$$Q_1 = \sum_{j=2}^{n}\frac{(k-f_j)^j-(k-f_{j+1})^j}{\prod_{l=1}^{j}(k-f_l)} = \frac{k-f_2}{k-f_1}. \qquad (64)$$

Using $j \geq 2$ with $p \geq p_0$ and $\mathbb{E}p_{max} \geq p_0$:

$$Q_2 \leq \frac{2}{p_0}(p_1^1 - \mathbb{E}p_{max}) \leq 2\left(\frac{k-f_1}{\beta p_1^1}-1\right). \qquad (65)$$

From $\mathbb{E}p_{max} \geq p_0$ and (35) we have $p_U \leq \frac{\alpha-\beta p_0}{2\beta}$, whence

$$p_1^1 \geq \frac{\alpha+\beta p_0 - 2k(n-1) - 2f_1}{4\beta}.$$

Using the definition of p_0, we can re-state this as

$$(p_1^1)^2 - \frac{4(k-f_1)}{\beta}p_1^1 + \frac{\delta(k-f_1)}{\beta^2} \leq 0.$$

Thus p_1^1 is at least as large as the smaller root of the quadratic:

$$p_1^1 \geq \frac{2(k-f_1)}{\beta}\left(1-\sqrt{1-\frac{\delta(\alpha)}{4(k-f_1)}}\right) > \frac{2(k-f_1)}{3\beta}, \qquad (66)$$

where the second inequality follows from the assumption that $\delta(\alpha) > \frac{20}{9}(k-f_1)$.

We now substitute (66) into (65), obtaining $Q_2 < 1$. Combining (58), (63), (64) and $Q_2 < 1$, we get

$$\frac{\partial\mathbb{E}p_{max}}{\partial p_U} > -\frac{1}{2}\left(1-\frac{k-f_2}{k-f_1}\right) - \frac{k-f_2}{k-f_1} - 1 \geq -2,$$

which establishes the lemma for $\delta > \frac{20}{9}(k-f_1)$. ■

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