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AN ALGORITHM FOR THE SOLUTION OF LINEAR PROGRAMMING PROBLEMS

H. P. WILLIAMS 1970

Abstract

This paper describes an algorithm for solving linear programming problems. It is quite different from the simplex and is based on a decision procedure used in logic. A small problem is fully worked out. Advantages of the method over the simplex are discussed.

This algorithm is based on a decision procedure for the formal theory of dense linear order. The procedure is due to Langford [1] . It is described in Mendelson [2] . A linear programming problem consists of a number of equalities and inequalities. In the simplex method all inequalities are converted to equalities by the introduction of slack variables. As a result the simplex carries out calculations with equations but has to deal with the possibility of some of the variables being negative (infeasible). The algorithm described here does not introduce slacks but carries out calculations with equations and inequalities making use of the algebra of those relations. The algorithm is most easily justified by formal logic. The notation is defined and the logical equivalences should be intuitive but further explanations can be found in [2] . Although the description uses logical formalism this is not necessary once the validity of the transformations is realised.

Logical Notation

The following symbols are taken from formal logic. Their meanings and the truth of the equivalences in which they are used should be intuitive.

$\sim$	means	"not"
$\vee$	means	"or"
$\cdot$	means	"and"
$\rightarrow$	means	"implies"
$\forall$	means	"for all"
$\exists$	means	"there exists"

Logical and Arithmetical Equivalences

In the following equivalences A and B represent statements. P(x) is a statement involving the arithmetic variable x. The logical symbols defined above combine these statements to form other statements. The symbols s and t represent arithmetic expressions. When such expressions are related by the relations $\leq$ and = statements are produced. T represents a statement that is always true (a tautology). The following equivalences are used in the algorithm.

(i) $A \rightarrow B \equiv \sim A \vee B$

(ii) $\sim\sim A \equiv A$

(iii) $\sim(A \vee B) \equiv \sim A . \sim B$

(iv) $\sim(A.B) \equiv \sim A \vee \sim B$

(v) $\forall x\, P(x) \equiv \sim \exists x \sim P(x)$

(vi) $\sim(s \leq t) \equiv s > t$

In the following equivalences the symbol $\prod$ is used to denote a conjunction of a number of statements using " . " . Hence :

$$\prod_{i=1}^{n} A_i \text{ is } A_1 . A_2 \cdots A_n$$

(vii) $$\exists x \left[s = x . \prod_{i=1}^{n} (x = t_i) . \prod_{i=1}^{m} (x \leq u_i) . \prod_{i=1}^{p} (v_i \leq x) \right]$$
$$\equiv \prod_{i=1}^{n} (s = t_i) . \prod_{i=1}^{m} (s \leq u_i) . \prod_{i=1}^{p} (v_i \leq s)$$

(viii) $$\exists x \left[\prod_{i=1}^{m} (s_i \leq x) . \prod_{j=1}^{n} (x \leq t_j) \right] \equiv \prod_{\substack{i=1 \\ j=1}}^{m,n} (s_i \leq t_j)$$

(ix) $$\exists x \left[\prod_{i=1}^{n} (t_i \leq x) \right] \equiv T \text{ (always true)}$$

In case (viii) the conjunction $\prod_{\substack{i=1 \\ j=1}}^{m,n} (s_i \leq t_j)$ is taken over all possible combinations of i and j .

Logical Formulation and Solution of an L.P. Problem

All L.P. problems will be considered as maximisation problems. This can be achieved by reversing the signs of the coefficients in the objective function for a minimisation problem. All L.P. problems can therefore be stated in the following form.

Maximise $U = \sum_{j=1}^{n} c_j x_j$

subject to the following restraints

Restraints R_i, $1 \leq i \leq m$

$$\left\{ \begin{array}{ll} \sum_{j=1}^{n} a_{ij} x_j - b_i = 0 & 1 \leq i \leq m_1 \\ \sum_{j=1}^{n} a_{ij} x_j - b_i \leq 0 & m_1 + 1 \leq i \leq m_2 \\ \sum_{j=1}^{n} a_{ij} x_j - b_i \geq 0 & m_2 + 1 \leq i \leq m \end{array} \right.$$

Restraints R_{m+j} $\qquad x_j \geq 0 \qquad 1 \leq j \leq n$

Suppose a value of U is chosen greater than or equal to the maximum. Then all sets of values of the variables x_j which satisfy the restraints $R_i (1 \leq i \leq m+n)$ will give a value to the objective function less than or equal to U . Logically this can be expressed by

(A) $$\forall x_n \forall x_{n-1} \cdots \forall x_1 \left(R_1 . R_2 \cdots R_{m+n} \rightarrow \sum c_j x_j \leq U \right)$$

The problem is therefore to find the least value of U for which the above statement is true.

Using the logical equivalences this statement is equivalent to :

$$\forall x_n \forall x_{n-1} \cdots \forall x_1 (\sim(R_1 . R_2 \cdots R_{m+n}) \vee \sum c_j x_j \leq U) \quad \text{by (i)}$$

$$\equiv \forall x_n \forall x_{n-1} \cdots \forall x_2 \sim \exists x_1 \sim (\sim(R_1 . R_2 \cdots R_{m+n}) \vee \sum c_j x_j \leq U) \quad \text{by (v)}$$

$$\equiv \forall x_n \forall x_{n-1} \cdots \forall x_2 \sim \exists x_1 (R_1 . R_2 \cdots R_{m+n} . \sim (\sum c_j x_j \leq U)) \quad \text{by (iii)}$$

$$\equiv \forall x_n \forall x_{n-1} \cdots \forall x_2 \sim \exists x_1 (R_1 . R_2 \cdots R_{m+n} . \sum c_j x_j > U) \quad \text{by (vi)}$$

$$\equiv \sim \exists x_n \exists x_{n-1} \cdots \exists x_1 (R_1 . R_2 \cdots R_{m+n} . \sum c_j x_j > U)$$

by successive applications of (v)

For simplicity the relations "$>$" and "$<$" will be treated as the relations "$\geq$" and "$\leq$". This is permissible since it is the continuum of rational numbers which are being considered. The statement :

$$\sum c_j x_j - U \geq 0 \quad \text{will be represented by } R_{m+n+1}$$

The problem has therefore been reduced to the following form :

Find the least U such that :

$$\sim \exists x_n \exists x_{n-1} \cdots \exists x_1 (R_1 . R_2 \cdots R_{m+n} . R_{m+n+1})$$

This is a well known form for logical statements. The symbol $\exists$ is a quantifier. A method of reduction known as the "elimination of quantifiers" will be used to successively eliminate each variable x_j and its corresponding quantifier $\exists x_j$. When all these variables have been eliminated the problem will be in a form where the least value of U is immediate. To eliminate a variable x_j the following procedures are carried out :

1. Examine only those statements R_i which involve the variable x_j. The other statements remain unchanged during the consideration of this variable.

2. In the selected statements R_i divide through by the coefficient of x_j. If this coefficient is negative in an inequality statement change a less-than inequality to a greater-than inequality or vice-versa.

   The statements R_i are now in the following possible forms :

   (E) $\quad x_j + s_i = 0$

   (L) $\quad x_j + t_i \leq 0$

   (G) $\quad x_j + v_i \geq 0$

   where s_i, t_i, v_i represent expressions involving the variables from among $x_1, x_2, \cdots, x_n, U$ which have not been eliminated, together with the constants.

3. The following three possibilities must be distinguished :

(i) There exists at least one R_i of the form (E).

(ii) All R_i are of the form (L) or all R_i are of the form (G)

(iii) There do not exist any R_i of the form (E) and not all R_i are of the same form.

In case (i) a substitution is made of $-s_i$ for the variable x_j in all other statements. This can be effected by subtracting the statement from all the others. This transformation is a result of the equivalence (vii).

In case (ii) the problem is unbounded. This is a result of equivalence (ix). Since the statement (A) is true regardless of the value of U , the least U will be infinite.

In case (iii) the equivalence (viii) is applied. This can be effected by subtracting each statement of the form (G) from each statement of the form (L) in turn and making the result a less-than inequality. For example if :

$$R_{i1} \equiv x_j + t_{i1} \leq 0$$
$$R_{i2} \equiv x_j + v_{i2} \geq 0$$

combining R_{i1} and R_{i2} produces the statement :

$$t_{i1} - v_{i2} \leq 0$$

All such combinations of R_{i1} and R_{i2} are produced where R_{i1} ranges over all statements of the form (L) and R_{i2} ranges over all statements of the form (G).

After following the above procedures for the variable x_j it is eliminated. The procedures are repeated for each of the remaining variables in turn. If at any stage a statement of the form :

$$b \leq 0$$

results where b is a constant there are two possibilities :

(i) b is positive
(ii) b is negative or zero

In case (i) the statement is clearly false. This means that a number of the initial restraints must have been self contradictory i.e. the problem is infeasible.
In case (ii) the statement is clearly true and does not further effect the problem. The statement need not therefore be considered in the rest of the calculation.

When the above procedures have been performed for all the variables x_j statements remain involving only the variable U .
Procedure 2. is carried out for the variable U . The resulting statements P_i are of the following possible forms :

(a) $U - k_i = 0$

(b) $U - k_i \leq 0$

(c) $U - k_i \geq 0$

where the k_i are constants.
The problem has therefore been reduced to the following form :

Find the least U such that

(B) $\quad \sim (P_1 . P_2 \ldots P_r)$

Using the equivalence (iv) this can be stated as :

Find the least U such that

(C) $\quad \sim P_1 \vee \sim P_2 \vee \ldots \vee \sim P_r$

The statements $\sim P_i$ are of the following possible forms corresponding to the forms of P_i above.

(a) $\quad U - k_i \neq 0$

(b) $\quad U - k_i > 0$

(c) $\quad U - k_i < 0$

Statements of the form (a) will not arise in practice since the variable U initially only occurs in R_{m+n+1} which is an inequality relation. In statements of the forms (b) and (c) the relations " $>$ " and " $<$ " are really " $\geqslant$ " and " $\leqslant$ ". The reason for the appearance of the other symbols was the earlier simplification regarding the use of only one type of inequality sign.

It is necessary to distinguish two possibilities in the final statements above.

(i) There exist statements of the form (c)

(ii) There only exist statements of the form (b)

In case (i) there is clearly no least U such that

$$U < k_i$$

The problem is therefore unbounded.
In case (ii) the least value of U satisfying the statements is the least value of k_i appearing. This value is therefore the maximum possible value which can be attained by the objective function.

It remains to find the values of the variables x_j which give this maximum value to the objective function. The statement :

$$U - k > 0$$

which gives this maximum value will have resulted from successive combinations of other statements using the procedures above. These other statements will have been equalities and inequalities. Since the Optimum value of U satisfies the equation :

$$U - k = 0$$

any inequalities which were used would have to be converted to equalities. This argument can be applied backwards successively to show that the above equation results from a certain set of the original equalities and inequalities when they are taken as equalities. This set of equations can be solved to provide the values of the variables x_i.

The answer to the problem is independent of the order in which the variables x_j are eliminated. This order, however, greatly affects the amount of computation. When all the rows R_i are inequalities the variable to be eliminated should be chosen so that procedure 3 (iii) results in the smallest possible number of new rows. This will be apparent in the example.

General Considerations

i) The above description uses logical symbolism to provide a rigorous justification for the algorithm. In any calculation, of course, this is not necessary and the execution of the above procedures appears much simpler as is demonstrated by the example.

ii) Although logical notation makes the proof concise it is possible to justify each step without it. Each elimination of a variable x_j transforms the L.P. problem to another problem having the same answer but with a lesser number of variables. This process is repeated until a problem is produced with only one variable u. The solution to such a problem is immediate.

iii) To obtain the values of the variables x_j giving the optimum solution it is necessary to solve a set of equations. These equations will be a subset of the original rows R_i of the problem. They will give a unique (optimal) value of u. There will sometimes be alternative values for the variables x_j as is well known from the simplex.

iv) It is possible to consider the answer to the problem as a basis in the sense of the simplex. In general certain of the equations to be solved will have resulted from restraints R_{m+j} and will therefore be :

$$x_j = 0$$

Such variables can be considered as out of the basis. The other variables will provide structurals for the basis. Corresponding to the original rows R_i $(1 \leq i \leq m)$ which do not appear in the final set of equations there will be slacks in the basis. Apart from the case of alternative solutions mentioned before the basis will be a square matrix which can be inverted. If there are alternate solutions the matrix will have more columns than rows and a solution can be obtained by selecting any non-singular square matrix from the set of columns.

v) Although this algorithm works by transformations on rows there is a dual where the transformations are on columns. In this case an elimination is performed for each row in the problem. Since most problems have more columns than rows this would probably be more efficient.

An example is now given of a small L.P. problem solved using the algorithm. Possible advantages of the algorithm over the simplex are then discussed.

Example of the Solution of an L.P. Problem

Maximise $\quad U = 4X_1 + 7X_2 + X_3 \quad$ subject to the following constraints:

$$
\begin{array}{l}
4X_1 - 3X_2 - X_3 = 0 \\
2X_1 + X_2 + 2X_3 \geqslant -20 \\
3X_1 - X_2 - X_3 \geqslant -1 \\
2X_1 - 3X_2 \leqslant -1 \\
3X_2 + X_3 \leqslant 24
\end{array}
$$

Writing the statements R_i enclosed in the brackets in (A) as separate rows together with R_9 and converting to the form described in Procedure 2 yields the following tableau.

$$
\begin{array}{ll}
R_1 & X_1 - \frac{3}{4}X_2 - \frac{1}{4}X_3 = 0 \\
R_2 & X_1 + \frac{1}{2}X_2 + X_3 + 10 \geqslant 0 \\
R_3 & X_1 - \frac{1}{3}X_2 - \frac{1}{3}X_3 + \frac{1}{3} \geqslant 0 \\
R_4 & X_1 - \frac{3}{2}X_2 + \frac{1}{2} \leqslant 0 \\
R_5 & 3X_2 + X_3 - 24 \leqslant 0 \\
R_6 & X_1 \geqslant 0 \\
R_7 & X_2 \geqslant 0 \\
R_8 & X_3 \geqslant 0 \\
R_9 & X_1 + \frac{7}{4}X_2 + \frac{1}{4}X_3 - \frac{1}{4}U \geqslant 0
\end{array}
$$

Since there is an equality (R_1) involving X_1 a substitution may be made for X_1 in the other rows involving X_1. The figures on the left indicate from which original rows the new rows result.

$$
\begin{array}{ll}
1,2 & \frac{5}{4}X_2 + \frac{5}{4}X_3 + 10 \geqslant 0 \\
1,3 & \frac{5}{12}X_2 - \frac{1}{12}X_3 + \frac{1}{3} \geqslant 0 \\
1,4 & -\frac{3}{4}X_2 + \frac{1}{2} \leqslant 0 \\
5 & 3X_2 + X_3 - 24 \leqslant 0 \\
1,6 & \frac{3}{4}X_2 + \frac{1}{4}X_3 \geqslant 0 \\
7 & X_2 \geqslant 0 \\
8 & X_3 \geqslant 0 \\
1,9 & \frac{5}{2}X_2 + \frac{1}{2}X_3 - \frac{1}{4}U \geqslant 0
\end{array}
$$

Either X_2 or X_3 can now be eliminated. Examination of the signs of the coefficients of these variables together with the direction of the inequalities indicates that elimination of X_2 will result in the fewer number of rows. Procedure 2 is applied to yield the following tableau

1, 2	$x_2 + x_3$	$+ 8$	≥ 0
1, 3	$x_2 - \frac{1}{5}x_3$	$+ \frac{4}{5}$	≥ 0
1, 4	x_2	$- \frac{2}{3}$	≥ 0
5	$x_2 + \frac{1}{3}x_3$	$- 8$	≤ 0
1, 6	$x_2 + \frac{1}{3}x_3$		≥ 0
7	x_2		≥ 0
8	x_3		≥ 0
1, 9	$x_2 + \frac{1}{5}x_3 - \frac{1}{10}U$		≥ 0

Procedure 3(iii) is applied to eliminate x_2 and results in the following tableau.

5, 1, 2	$-\frac{2}{3}x_3$	-16	≤ 0
5, 1, 3	$\frac{8}{15}x_3$	$-\frac{44}{5}$	≤ 0
5, 1, 4	$\frac{1}{3}x_3$	$-\frac{22}{3}$	≤ 0
5, 1, 6		$- 8$	≤ 0
5, 7	$\frac{1}{3}x_3$	$- 8$	≤ 0
8	x_3		≥ 0
5, 1, 9	$\frac{2}{15}x_3 + \frac{1}{10}U$	$- 8$	≤ 0

Procedure 2 is applied and x_3 eliminated to give:

5, 1, 3, 5, 1, 2		$-\frac{81}{2}$	≤ 0
5, 1, 3, 8		$-\frac{33}{2}$	≤ 0
5, 1, 4, 5, 1, 2		$- 46$	≤ 0
5, 1, 4, 8		$- 22$	≤ 0
5, 7, 5, 1, 2		$- 48$	≤ 0
5, 7, 8		$- 24$	≤ 0
5, 1, 9, 5, 1, 2	$\frac{3}{4}U$	$- 84$	≤ 0
5, 1, 9, 8	$\frac{3}{4}U$	$- 60$	≤ 0

In those statements which do not involve the variable U the constants are negative. Therefore the problem is not infeasible.

Procedure 2 is carried out on the other statements to yield the following tableau.

$5, 1, 9, 5, 1, 2 \qquad U - 112 \leq 0$

$5, 1, 9, 8 \qquad U - 80 \leq 0$

These statements occur when the problem is in form (B). After converting the problem to form (C) all the statements are greater-than inequalities showing that the problem is not unbounded. The least value of U is 80 arising from the statement

$$U - 80 > 0$$

This statement arises from the following rows of the original tableau :

1, 5, 8, 9

Here the optimal solution to the problem is the solution to the following set of simultaneous equations

$$\begin{aligned} 4x_1 - 3x_2 - x_3 &= 0 \\ 3x_2 + x_3 &= 24 \\ x_3 &= 0 \\ 4x_1 + 7x_2 + x_3 &= U \end{aligned}$$

This gives $\qquad x_1 = 6, \; x_2 = 8, \; x_3 = 0, \; U = 80$

This solution can be regarded in terms of a basis in the following way. Taking the formulation on page 2 with $m = 5$ there is one restraint R_{m+j} $(1 \leq j \leq 3)$, namely R_8, which becomes an equality. Therefore the variable x_3 is out of the basis. From among the restraints R_i $(1 \leq i \leq 5)$ the following do not appear R_2, R_3, R_4. These rows are therefore in the basis. Hence the optimal basis consists of variables x_1 and x_2 and rows R_2, R_3, R_4.

Comparison of the Algorithm With the Simplex

The original simplex method has undergone many refinements. Analogous refinements could be made to this algorithm. In particular the following two modifications could be made :

i) Instead of actually subtracting rows represent the subtractions by multiplication by elementary matrices. This would save redundant computation.

ii) Instead of explicitly writing each row $x_j \geq 0$ $(1 \leq j \leq n)$ assume the existence of such a row when eliminating x_j. Any restraints representing bounds could be represented by giving each variable a lower and upper bound.

The Simplex method is made up of iterations. Each iteration consists of selecting a variable, taking a row in which it occurs and using this row to eliminate the variable from all the other rows. Some of these variables will be slacks and all the rows will be equality rows.

The algorithm described here also consists of eliminating variables between rows. Assuming the problem to be feasible and bounded two cases must be distinguished.

i) The variable to be eliminated occurs in at least one equality row.

ii) The variable to be eliminated occurs (with like signs) in m_1, less-than rows, m_2 greater-than rows.

These variables will consist only of the original variables (structurals) in the problem.

In the case of (i) one of the equality rows in which the variable occurs is used to eliminate the variable from all the other rows. The amount of computation involved will be comparable with an elimination in the Simplex. In case (ii) eliminations will be made between all possible pairs of dissimilar inequalities in which the variable occurs. Hence there will be $m_1 m_2$ subtractions. The amount of computation will be at least as great as a Simplex elimination. It should be remembered, however, that

(i) Most matrices are sparse and a particular variable does not occur in many rows.

(ii) The variable to be eliminated is chosen to make $m_1 m_2$ as small as possible.

The average amount of computation for eliminating a variable will probably, therefore, be greater with this algorithm than with the simplex. With the Simplex, however, the total number of eliminations (iterations) is indeterminate. For this algorithm each variable has to be eliminated once only.

In most problems solved with the simplex there are two phases in the calculation.

(i) Obtaining a feasible solution

(ii) Obtaining an optimal solution

With the algorithm described here there is no such distinction.

Many problems solved with the Simplex exhibit " degeneracy ". A lot of iterations have no beneficial effect. Such a difficulty will not arise with this algorithm.

Redundant iterations often take place with the Simplex in which a variable enters and leaves the basis a number of times. There is no corresponding phenomenon here.

With the Simplex on an infeasible problem it is often difficult to tell which are the contradictory restraints. This algorithm shows a problem to be infeasible when a row of the form

$$b \leq 0$$

where b is positive, is obtained. The original rows of the problem from which this row arises give the contradictory restraints.

All these factors contribute to making the amount of computation with the Simplex unpredictable. With this algorithm the amount of computation is much more determinate. It seems likely that either the algorithm or its dual could be an improvement on the Simplex. This, however, requires testing by computer on realistic problems.

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References

1. Langford, Some theorems on deducibility.
   Ann. of Math I   Vol. 28 pp 16-40
   II   Vol. 28 pp 459-471

2. Mendelson. Introduction to Mathematical Logic
   Von Nostrand 1963