THE TWO-POINT FANO AND IDEAL BINARY CLUTTERS

AHMAD ABDI AND BERTRAND GUENIN

ABSTRACT. Let $\mathbb{F}$ be a binary clutter. We prove that if $\mathbb{F}$ is non-ideal, then either $\mathbb{F}$ or its blocker $b(\mathbb{F})$ has one of $\mathbb{L}_7, \mathbb{O}_5, \mathbb{LC}_7$ as a minor. $\mathbb{L}_7$ is the non-ideal clutter of the lines of the Fano plane, $\mathbb{O}_5$ is the non-ideal clutter of odd circuits of the complete graph K_5, and the *two-point Fano* $\mathbb{LC}_7$ is the ideal clutter whose sets are the lines, and their complements, of the Fano plane that contain exactly one of two fixed points. In fact, we prove the following stronger statement: if $\mathbb{F}$ is a minimally non-ideal binary clutter different from $\mathbb{L}_7, \mathbb{O}_5, b(\mathbb{O}_5)$, then through every element, either $\mathbb{F}$ or $b(\mathbb{F})$ has a two-point Fano minor.

1. INTRODUCTION

Let E be a finite set. A *clutter* $\mathbb{F}$ over *ground set* $E(\mathbb{F}) := E$ is a family of subsets of E, where no subset is contained in another. We say that $\mathbb{F}$ is *binary* if the symmetric difference of any odd number of sets in $\mathbb{F}$ contains a set of $\mathbb{F}$. We say that $\mathbb{F}$ is *ideal* if the polyhedron

$$Q(\mathbb{F}) := \left\{x \in \mathbb{R}^E_+ : \sum(x_e : e \in C) \geq 1 \quad C \in \mathbb{F}\right\}$$

has only integral extreme points; otherwise it is *non-ideal*. When is a binary clutter ideal? We will be studying this question.

Let us describe some examples of ideal and non-ideal binary clutters. Given a graph G and distinct vertices s, t, the clutter of st-paths of G over the edge-set is binary. An immediate consequence of Menger's theorem [12], as well as Ford and Fulkerson's theorem [6], is that this binary clutter is ideal [3]. The clutter of *lines of the Fano plane*

$$\mathbb{L}_7 := \big\{\{1,2,6\},\{1,4,7\},\{1,3,5\},\{2,5,7\},\{2,3,4\},\{3,6,7\},\{4,5,6\}\big\}$$

is binary, and it is non-ideal as $\left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)$ is an extreme point of $Q(\mathbb{L}_7)$. (See Figure 1.) The clutter of odd circuits of K_5 over its ten edges, denoted $\mathbb{O}_5$, is also binary, and it is non-ideal as $\left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)$ is an extreme point of $Q(\mathbb{O}_5)$.

We say that two clutters are *isomorphic* if relabeling the ground set of one yields the other. There are two fundamental clutter operations that preserve being binary and ideal, let us describe them. The *blocker of* $\mathbb{F}$, denoted $b(\mathbb{F})$, is another clutter over the same ground set whose sets are the (inclusionwise) minimal sets in $\{B \subseteq E : B \cap C \neq \emptyset \; \forall C \in \mathbb{F}\}$. It is well-known that $b(b(\mathbb{F})) = \mathbb{F}$ [5]. We may therefore call $\mathbb{F}, b(\mathbb{F})$ a *blocking* pair. A clutter $\mathbb{F}$ is binary if, and only if, $|B \cap C|$ is odd for all $B \in b(\mathbb{F})$ and $C \in \mathbb{F}$ [9]. Hence, if $\mathbb{F}$ is binary,

---

Date: August 14, 2018.

This work is supported by NSERC CGS and Discovery grants and by U.S. Office of Naval Research grants under award numbers N00014-15-1-2171 and N00014-18-1-2078.

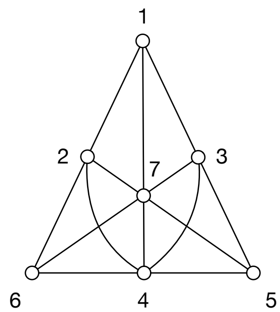


Figure 1. The Fano plane

then so is $b(\mathbb{F})$. Lehman's Width-Length Inequality shows that if $\mathbb{F}$ is ideal, then so is $b(\mathbb{F})$ [10]. In particular, since $\mathbb{L}_7$ and $\mathbb{O}_5$ are non-ideal, then so are $b(\mathbb{L}_7) = \mathbb{L}_7$ and $b(\mathbb{O}_5)$. Let I, J be disjoint subsets of E. Denote by $\mathbb{F} \setminus I/J$ the clutter over $E - (I \cup J)$ of minimal sets of $\{C - J : C \in \mathbb{F}, C \cap I = \emptyset\}$.[1] We say that $\mathbb{F} \setminus I/J$, and any clutter isomorphic to it, is a *minor of* $\mathbb{F}$ obtained after *deleting* I and *contracting* J. If $I \cup J \neq \emptyset$, then $\mathbb{F} \setminus I/J$ is a *proper* minor of $\mathbb{F}$. It is well-known that $b(\mathbb{F} \setminus I/J) = b(\mathbb{F})/I \setminus J$ [16]. If a clutter is binary, then so is every minor of it, and if a clutter is ideal, then so is every minor of it [17].

Let $\mathbb{F}$ be a binary clutter. Regrouping what we discussed, if $\mathbb{F}$ or $b(\mathbb{F})$ has one of $\mathbb{L}_7, \mathbb{O}_5$ as a minor, then it is non-ideal. Seymour [17] (page 200) conjectures the converse is also true:

The flowing conjecture. Let $\mathbb{F}$ be a non-ideal binary clutter. Then $\mathbb{F}$ or $b(\mathbb{F})$ has one of $\mathbb{L}_7, \mathbb{O}_5$ as a minor.

The *two-point Fano clutter*, denoted by $\mathbb{LC}_7$, is the clutter over ground set $\{1, \ldots, 7\}$ whose sets are the lines, and their complements, of the Fano plane that intersect $\{1, 4\}$ exactly once, i.e.

$$\mathbb{LC}_7 = \big\{\{1,2,6\}, \{3,4,5,7\}, \{1,3,5\}, \{2,4,6,7\}, \{2,3,4\}, \{1,5,6,7\}, \{4,5,6\}, \{1,2,3,7\}\big\}.$$

(Notice that changing the two points $1, 4$ yields an isomorphic clutter.) It can be readily checked that $\mathbb{LC}_7$ is binary *and* ideal. In this paper, we prove the following weakening of the flowing conjecture:

Theorem 1. *Let $\mathbb{F}$ be a non-ideal binary clutter. Then $\mathbb{F}$ or $b(\mathbb{F})$ has one of $\mathbb{L}_7, \mathbb{O}_5, \mathbb{LC}_7$ as a minor.*

What makes this result attractive is its relatively simple proof. The techniques used in the proof give hope of – and pave the way for – resolving the flowing conjecture. An interesting feature of the proof is the interplay between the clutter $\mathbb{F}$ and its blocker $b(\mathbb{F})$; if we fail to find one of the desired minors in the clutter, we switch to the blocker and find a desired minor there. Theorem 1 is a consequence of a stronger result stated in the next section.

2. Preliminaries and the main theorem

2.1. **Minimally non-ideal binary clutters.** A clutter is *minimally non-ideal (mni)* if it is non-ideal and every proper minor of it is ideal. Notice that every non-ideal clutter has an mni minor, and if a clutter is mni, then so

[1] Given sets A, B we denote by $A - B$ the set $\{a \in A : a \notin B\}$ and, for element a, we write $A - a$ instead of $A - \{a\}$.

is its blocker. Justified by this observation, instead of working with non-ideal binary clutters, we will work with mni binary clutters. The three clutters $\mathbb{L}_7, \mathbb{O}_5, b(\mathbb{O}_5)$ are mni, and the flowing conjecture predicts that these are the *only* mni binary clutters. We will need the following result of the authors:

Theorem 2 ([1])**.** $\mathbb{L}_7, \mathbb{O}_5$ *are the only mni binary clutters with a set of size* 3*.*

We will also need the following intermediate result of Alfred Lehman on mni clutters, stated only for binary clutters. Let $\mathbb{F}$ be a clutter over ground set E. Denote by $\bar{\mathbb{F}}$ the clutter of minimum size sets of $\mathbb{F}$. Denote by $M(\mathbb{F})$ the $0-1$ matrix whose columns are labeled by E and whose rows are the incidence vectors of the sets of $\mathbb{F}$. For an integer $r \geq 1$, a square $0-1$ matrix is *r-regular* if every row and every column has precisely r ones.

Theorem 3 ([11, 15, 2])**.** *Let $\mathbb{F}$ be an mni binary clutter where $n := |E(\mathbb{F})|$, and let $\mathbb{K} := b(\mathbb{F})$. Then*

(1) $M(\bar{\mathbb{F}})$ and $M(\bar{\mathbb{K}})$ are square and non-singular matrices,

(2) $M(\bar{\mathbb{F}})$ is r-regular and $M(\bar{\mathbb{K}})$ is s-regular, for some integers $r \geq 3$ and $s \geq 3$ such that $rs - n$ is even and $rs - n \geq 2$,

(3) after possibly permuting the rows of $M(\bar{\mathbb{K}})$, we have that

$$M(\bar{\mathbb{F}})M(\bar{\mathbb{K}})^\top = J + (rs-n)I = M(\bar{\mathbb{K}})^\top M(\bar{\mathbb{F}}).$$

Here, J denotes the all-ones matrix, and I the identity matrix. Given a ground set E and a set $C \subseteq E$, denote by $\chi_C \subseteq \{0,1\}^E$ the incidence vector of C. We will make use of the following corollary:

Corollary 4. *Let $\mathbb{F}$ be an mni binary clutter. Then the following statements hold:*

(1) For $C_1, C_2 \in \bar{\mathbb{F}}$, the only sets of $\mathbb{F}$ contained in $C_1 \cup C_2$ are C_1, C_2 ([7, 8]*).*

(2) Choose $C_1, C_2, C_3 \in \bar{\mathbb{F}}$ and $e \in E(\mathbb{F})$ such that $C_1 \cap C_2 = C_2 \cap C_3 = C_3 \cap C_1 = \{e\}$. If C, C' are sets of $\mathbb{F}$ such that $C \cup C' \subseteq C_1 \cup C_2 \cup C_3$ and $C \cap C' \subseteq \{e\}$, then $\{C, C'\} = \{C_i, C_j\}$ for some distinct $i, j \in \{1,2,3\}$.

Proof. Denote by r the minimum size of a set in $\mathbb{F}$. **(1)** Take a set $C \in \mathbb{F}$ such that $C \subseteq C_1 \cup C_2$. Since $\mathbb{F}$ is binary, $C_1 \triangle C_2 \triangle C$ contains another set C' of $\mathbb{F}$. Then

$$2r = |C_1| + |C_2| = |C_1 \cap C_2| + |C_1 \cup C_2| \geq |C \cap C'| + |C \cup C'| = |C| + |C'| \geq 2r,$$

so equality must hold throughout. In particular, $C, C' \in \bar{\mathbb{F}}$ and $\chi_{C_1} + \chi_{C_2} = \chi_C + \chi_{C'}$. Since $M(\mathbb{F})$ is non-singular by Theorem 3 (1), we get that $\{C, C'\} = \{C_1, C_2\}$. **(2)** As $\mathbb{F}$ is binary, $C_1 \triangle C_2 \triangle C_3 \triangle C \triangle C'$ contains another set C'' of $\mathbb{F}$. Notice that $C'' \cap C \subseteq \{e\}$ and $C'' \cap C' \subseteq \{e\}$. If k many of C, C', C'' contain e, then

$$3r - 3 = |(C_1 \cup C_2 \cup C_3) - e| \geq |(C \cup C' \cup C'') - e| = |C| + |C'| + |C''| - k \geq 3r - k,$$

implying in turn that $k = 3$ and equality must hold throughout. In particular, $C, C', C'' \in \bar{\mathbb{F}}$ and $\chi_{C_1} + \chi_{C_2} + \chi_{C_3} = \chi_C + \chi_{C'} + \chi_{C''}$, so as $M(\bar{\mathbb{F}})$ is non-singular, we get that $\{C_1, C_2, C_3\} = \{C, C', C''\}$. $\square$

2.2. **Signed matroids.** All matroids considered in this paper are binary; we follow the notation used in Oxley [14]. Let M be a matroid over ground set E. Recall that a circuit is a minimal dependent set of M and a cocircuit is a minimal dependent set of the dual $M^\star$. A *cycle* is the symmetric difference of circuits, and a *cocycle* is the symmetric difference of cocircuits. It is well-known that a nonempty cycle is a disjoint union of circuits ([14], Theorem 9.1.2). Let $\Sigma \subseteq E$. The pair (M, Σ) is called a *signed matroid* over ground set E. An *odd circuit of* (M, Σ) is a circuit C of M such that $|C \cap \Sigma|$ is odd.

Proposition 5 ([9, 13], also see [4])**.** *The clutter of odd circuits of a signed matroid is binary. Conversely, a binary clutter is the clutter of odd circuits of a signed matroid.*

A *representation* of a binary clutter $\mathbb{F}$ is a signed matroid whose clutter of odd circuits is $\mathbb{F}$. By the preceding proposition, every binary clutter has a representation. For instance, $\mathbb{L}_7$ is represented as $\big(F_7, E(F_7)\big)$, where F_7 is the Fano matroid. A *signature of* (M, Σ) is any subset of the form $\Sigma \triangle D$, where D is a cocycle of M; to *resign* is to replace (M, Σ) by $(M, \Sigma \triangle D)$. Notice that resigning does not change the family of odd cycles. We say that two signed matroids are *isomorphic* if one can be obtained from the other after a relabeling of the ground set and a resigning.

Remark 6. *Take an arbitrary element ω of F_7. Then $(F_7, E(F_7) - \omega)$ represents $\mathbb{LC}_7$.*

Proof. Suppose that $E(F_7) = \{1, \ldots, 7\}$. By the symmetry between the elements of $E(F_7)$, we may assume that $\omega = 7$. Consider the following representation of F_7,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

where the columns are labeled $1, \ldots, 7$ from left to right. Since $\{2, 3, 5, 6\}$ is a cocycle of F_7, $(F_7, \{1, \ldots, 6\})$ is isomorphic to $(F_7, \{1, \ldots, 6\} \triangle \{2, 3, 5, 6\}) = (F_7, \{1, 4\})$. It can be readily checked that the odd circuits of $(F_7, \{1, 4\})$ are precisely the sets of $\mathbb{LC}_7$, thereby proving the remark. $\square$

Proposition 7 ([9, 13], also see [8])**.** *In a signed matroid, the clutter of minimal signatures is the blocker of the clutter of odd circuits.*

Let I, J be disjoint subsets of E. The *minor* $(M, \Sigma) \setminus I/J$ obtained after *deleting* I and *contracting* J is the signed matroid defined as follows: if J contains an odd circuit, then $(M, \Sigma) \setminus I/J := (M \setminus I/J, \emptyset)$, and if J does not contain an odd circuit, then there is a signature Σ' of (M, Σ) disjoint from J by the preceding proposition, and we let $(M, \Sigma) \setminus I/J := (M \setminus I/J, \Sigma' - I)$. Observe that minors are defined up to resigning.

Proposition 8 ([13], also see [4])**.** *Let $\mathbb{F}$ be a binary clutter represented as (M, Σ), and take disjoint $I, J \subseteq E(\mathbb{F})$. Then $\mathbb{F} \setminus I/J$ is represented as $(M, \Sigma) \setminus I/J$.*

2.3. **Hubs and the main theorem.** Let (M, Σ) be a signed matroid, and take $e \in E(M)$. An *e-hub of* (M, Σ) is a triple (C_1, C_2, C_3) satisfying the following conditions:

(h1) C_1, C_2, C_3 are odd circuits such that, for distinct $i, j \in \{1,2,3\}$, $C_i \cap C_j = \{e\}$,

(h2) for distinct $i, j \in \{1,2,3\}$, the only nonempty cycles contained in $C_i \cup C_j$ are $C_i, C_j, C_i \triangle C_j$,

(h3) a cycle contained in $C_1 \cup C_2 \cup C_3$ is odd if and only if it contains e.

A *strict* e-hub is an e-hub (C_1, C_2, C_3) such that the following holds:

(h4) if C, C' are odd cycles contained in $C_1 \cup C_2 \cup C_3$ such that $C \cap C' = \{e\}$, then for some distinct $i, j \in \{1,2,3\}$, $\{C, C'\} = \{C_i, C_j\}$.

Given $I \subseteq E$, denote by $M|I$ the minor $M \setminus (E - I)$, and by $(M, \Sigma)|I$ the minor $(M, \Sigma) \setminus (E - I)$. The following is the main result of the paper:

Theorem 9. *Let $\mathbb{F}, \mathbb{K}$ be a blocking pair of mni binary clutters over ground set E, neither of which has a set of size 3. Let (M, Σ) represent $\mathbb{F}$ and let (N, Γ) represent $\mathbb{K}$. Then, for a given $e \in E$, the following statements hold:*

(1) (M, Σ) has a strict e-hub (C_1, C_2, C_3) and (N, Γ) has a strict e-hub (B_1, B_2, B_3) where for $i, j \in \{1,2,3\}$,

$$|C_i \cap B_j| \begin{cases} \geq 3 & \text{if } i = j \\ = 1 & \text{if } i \neq j, \end{cases}$$

(2) either $M|(C_1 \cup C_2 \cup C_3)$ or $N|(B_1 \cup B_2 \cup B_3)$ is non-graphic,

(3) if $M|(C_1 \cup C_2 \cup C_3)$ is non-graphic, then $(M, \Sigma) \setminus I / J \cong (F_7, E(F_7) - \omega)$ for some disjoint $I, J \subseteq E - e$, and similarly,

if $N|(B_1 \cup B_2 \cup B_3)$ is non-graphic, then $(N, \Gamma) \setminus I / J \cong (F_7, E(F_7) - \omega)$ for some disjoint $I, J \subseteq E - e$.

In particular, for each $e \in E$, either $\mathbb{F}$ or $\mathbb{K}$ has a two-point Fano minor going through e.

Given this result, let us prove Theorem 1:

Proof of Theorem 1. Let $\mathbb{F}$ be a non-ideal binary clutter, let $\mathbb{F}'$ be an mni minor of $\mathbb{F}$, and let $\mathbb{K}' := b(\mathbb{F}')$. If $\mathbb{F}'$ has a set of size 3, then by Theorem 2, $\mathbb{F}' \cong \mathbb{L}_7$ or $\mathbb{O}_5$. If $\mathbb{K}'$ has a set of size 3, then by Theorem 2, $\mathbb{K}' \cong \mathbb{L}_7$ or $\mathbb{O}_5$. Thus, if one of $\mathbb{F}', \mathbb{K}'$ has a set of size 3, then either $\mathbb{F}$ or $b(\mathbb{F})$ has one of $\mathbb{L}_7, \mathbb{O}_5$ as a minor. We may therefore assume that neither $\mathbb{F}'$ nor $\mathbb{K}'$ has a set of size 3. Let (M, Σ) represent $\mathbb{F}'$ and let (N, Γ) represent $\mathbb{K}'$, whose existence are guaranteed by Proposition 5. It then follows from Theorem 9 (2)-(3) that either (M, Σ) or (N, Γ) has an $(F_7, E(F_7) - \omega)$ minor. By Remark 6 and Proposition 8, we see that either $\mathbb{F}'$ or $\mathbb{K}'$ has an $\mathbb{LC}_7$ minor, implying in turn that either $\mathbb{F}$ or $b(\mathbb{F})$ has an $\mathbb{LC}_7$ minor, as required. □

Before proving Theorem 9, let us say a few words about our proof in particular and our strategy for tackling the flowing conjecture in general. Starting with an mni binary clutter $\mathbb{F}$ and its representation (M, Σ), we pick an arbitrary element e and identify a local structure around it, namely a strict e-hub. Our idea is to build an excluded minor on top of the local structure, by carefully adding elements from outside the strict e-hub. This idea is certainly not a new one; the second author used it to prove the flowing conjecture when M is a graphic matroid [7]. However, this approach does not always lead to structure resembling any of the excluded minors. To overcome this shortcoming, we introduce a new idea. We switch to the blocker $b(\mathbb{F})$ and its representation (N, Γ). There we identify a strict e-hub, one that is tied to the original e-hub in (M, Σ). We then repeat our attempt, this time in (N,Γ). Theorem 9 shows an implementation of this new idea, and how it can be incorporated with the old one. Clearly our approach has its limitations, but we firmly believe that a final resolution (or refutation) of the flowing conjecture would build on these two ideas.

In the remainder of this paper, we prove Theorem 9.

3. Proof of Theorem 9 part (1)

Let $\mathbb{F},\mathbb{K}$ be blocking mni binary clutters over ground set E, neither of which has a set of size 3. By Theorem 3, there are integers $r\geq 4$ and $s\geq 4$ such that $M(\bar{\mathbb{F}})$ is r-regular, $M(\bar{\mathbb{K}})$ is s-regular, and after possibly permuting the rows of $M(\bar{\mathbb{K}})$, $M(\bar{\mathbb{F}})M(\bar{\mathbb{K}})^\top=J+(rs-n)I=M(\bar{\mathbb{K}})^\top M(\bar{\mathbb{F}})$. Thus, there is a labeling $\bar{\mathbb{F}}=\{C_1,\ldots,C_n\}$ and $\bar{\mathbb{K}}=\{B_1,\ldots,B_n\}$ so that, for all $i,j\in\{1,\ldots,n\}$,

$$(\star)\qquad |C_i\cap B_j|=\begin{cases} rs-n+1 & \text{if } i=j\\ 1 & \text{if } i\neq j\end{cases}$$

and for all $g,h\in E$,

$$(\diamond)\qquad |\{i\in\{1,\ldots,n\}:g\in C_i,h\in B_i\}|=\begin{cases} rs-n+1 & \text{if } g=h\\ 1 & \text{if } g\neq h.\end{cases}$$

Take an element $e\in E$. Since $rs-n\geq 2$, we may assume by $(\diamond)$ that $e\in C_i\cap B_i$ for $i\in\{1,2,3\}$. Recall that (M,Σ) represents $\mathbb{F}$ and that (N,Γ) represents $\mathbb{K}$. We will show that (C_1,C_2,C_3) is a strict e-hub of (M,Σ).

1. *C_1,C_2,C_3 are odd circuits of (M,Σ) such that, for distinct $i,j\in\{1,2,3\}$, $C_i\cap C_j=\{e\}$, i.e. (h1) holds.*

Subproof. By definition, C_1,C_2,C_3 are odd circuits of (M,Σ). To see $C_1\cap C_2=\{e\}$, notice that if $f\in(C_1\cap C_2)-e$, then $\{1,2\}\subseteq\{i\in\{1,\ldots,n\}:f\in C_i,e\in B_i\}$, which cannot be the case as the latter set has size 1 by $(\diamond)$. Similarly, $C_2\cap C_3=C_3\cap C_1=\{e\}$. ♢

2. *For distinct $i,j\in\{1,2,3\}$, the only nonempty cycles of M contained in $C_i\cup C_j$ are $C_i,C_j,C_i\triangle C_j$, so (h2) holds.*

Subproof. By symmetry, we may only analyze the cycles of M contained in $C_1\cup C_2$. By Corollary 4 (1), the only odd circuits of (M,Σ) contained in $C_1\cup C_2$ are C_1,C_2.

We first show that C_1,C_2 are the only odd cycles of (M,Σ) in $C_1\cup C_2$. Suppose otherwise. Let A be an odd cycle different from C_1,C_2. Write C as the disjoint union of circuits $A_1,\ldots,A_k$ for some $k\geq 2$. Since $|\Sigma\cap A|=\sum_{i=1}^k|\Sigma\cap A_i|$ and $|\Sigma\cap A|$ is odd, we may assume that $|\Sigma\cap A_1|$ is odd, so $A_1\in\{C_1,C_2\}$, and we may assume that $A_1=C_1$. But then $A_2\subseteq C_2-e$, a contradiction as both A_2,C_2 are circuits of M.

Let C be a nonempty cycle of M contained in $C_1\cup C_2$. If C is an odd cycle of (M,Σ), then as we just showed, $C\in\{C_1,C_2\}$. Otherwise, C is an even cycle, so $C\triangle C_1$ is an odd cycle, so $C\triangle C_1\in\{C_1,C_2\}$, implying in turn that $C=C_1\triangle C_2$, as required. ♢

3. *A cycle of (M,Σ) contained in $C_1\cup C_2\cup C_3$ uses e if and only if it contains e, so (h3) holds.*

Subproof. Since $s \geq 4$ and $M(\bar{\mathbb{K}})$ is s-regular, there is a $B \in \bar{\mathbb{K}} - \{B_1, B_2, B_3\}$ such that $e \in B$. Then, for each $i \in \{1, 2, 3\}$, $|B \cap C_i| = 1$ by $(\star)$, so $B \cap (C_1 \cup C_2 \cup C_3) = \{e\}$. It follows from Proposition 7 that B is a signature of (M, Σ). Thus, if C is an odd cycle of (M, Σ) contained in $C_1 \cup C_2 \cup C_3$, then $|C \cap B|$ is odd and therefore nonzero, so $e \in C$. So every odd cycle contained in $C_1 \cup C_2 \cup C_3$ contains e. Thus, if C is an even cycle contained in $C_1 \cup C_2 \cup C_3$, then $C \triangle C_1$ is an odd cycle contained in $C_1 \cup C_2 \cup C_3$, so $e \in C \triangle C_1$, implying in turn that $e \notin C$. So every even cycle contained in $C_1 \cup C_2 \cup C_3$ excludes e, thereby proving the claim. ♢

4. *If C, C' are odd cycles of (M, Σ) contained in $C_1 \cup C_2 \cup C_3$ such that $C \cap C' = \{e\}$, then for some distinct $i, j \in \{1, 2, 3\}$, $\{C, C'\} = \{C_i, C_j\}$, so (h4) holds.*

Subproof. Let D, D' be odd circuits contained in C, C', respectively. It follows from Corollary 4 (2) that, for some distinct $i, j \in \{1, 2, 3\}$, $\{D, D'\} = \{C_i, C_j\}$. Since there is no even cycle contained in $(C_1 \cup C_2 \cup C_3) - (C_i \triangle C_j)$, it follows that $D = C$ and $D' = C'$, and the claim follows. ♢

Hence, (C_1, C_2, C_3) is a strict e-hub of (M, Σ). Similarly, (B_1, B_2, B_3) is a strict e-hub of (N, Γ). This finishes the proof of Theorem 9 part (1). □

4. HYPERGRAPHS, THE TRIFOLD, AND GRAPHIC HUBS

Let M be a binary matroid over ground set E. By definition, the cycles of M form a linear space modulo 2, so there is a $0{-}1$ matrix A such that the incidence vectors of the cycles in M are $\{x \in \{0,1\}^E : Ax \equiv \mathbf{0} \pmod 2\}$. The matrix A is referred to as a *representation of* M. Notice that elementary row operations modulo 2 applied to A yield another representation, and if $a \in \{0,1\}^E$ belongs to the row space of A modulo 2, then $\begin{pmatrix} A \\ a^\top \end{pmatrix}$ is also a representation. A *hypergraphic representation of* M is a representation where every column has an even number of ones. If $a^\top$ is the sum of the rows of A modulo 2, then $\begin{pmatrix} A \\ a^\top \end{pmatrix}$ is a hypergraphic representation. In particular, a binary matroid always has a hypergraphic representation.

A *hypergraph* is a pair $G = (V, E)$, where V is a finite set of *vertices* and E is a family of even subsets of V, called *edges*. Note that if A is a hypergraphic representation of M, then A may be thought of as a hypergraph whose vertices are labeled by the rows and whose edges are labeled by the columns. For instance, the Fano matroid F_7 may be represented as a hypergraph on vertices $\{1, \ldots, 4\}$ and edges $\{T \subseteq \{1, \ldots, 4\} : |T| \in \{2, 4\}\}$. Denote by S_8 the binary matroid represented as the hypergraph displayed in Figure 2, which has vertices $\{1, \ldots, 5\}$ and edges $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,3,4,5\}$. Label $\gamma := \{2,3,4,5\} \in E(S_8)$. A *trifold* is any signed matroid isomorphic to $\big(S_8, E(S_8) - \gamma\big)$.

Remark 10. *A trifold has an $(F_7, E(F_7))$ minor.*

Proof. Observe that $S_8/\gamma \cong F_7$, implying in turn that $(S_8, E(S_8) - \gamma)/\gamma \cong (F_7, E(F_7))$. □

Given a hypergraph $G = (V, E)$ and $F \subseteq E$, let $\text{odd}_G(F) := \triangle(e : e \in F) \subseteq V$. Observe that $\text{odd}_G(F)$ is an even subset of V. We will make use of the following remark throughout the paper:

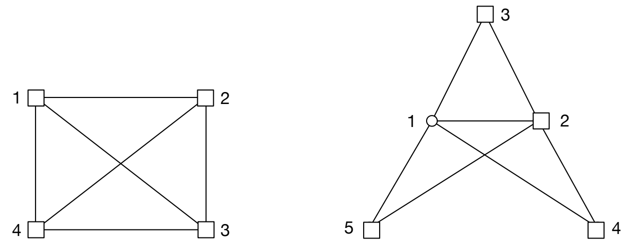


Figure 2. The hypergraph on the left represents F_7, and the one on the right represents S_8. Line segments represent edges of size 2, and square vertices form the edges of size 4.

Remark 11. *Let M be a binary matroid over ground set $E\cup\{e\}$, where $M\setminus e$ is represented by the hypergraph $G=(V,E)$. If for some $F\subseteq E$, $F\cup\{e\}$ is a cycle of M, then the hypergraph on vertices V and edges $E\cup\{odd_G(F)\}$ represents M.*

Recall that a binary matroid is graphic if it can be represented by a graph. We will also need the following:

Proposition 12. *Take a signed matroid (M,Σ), $e\in E(M)$ and an e-hub (C_1,C_2,C_3). Then there is a signature Σ' such that $\Sigma'\cap(C_1\cup C_2\cup C_3)=\{e\}$. Moreover, the following statements are equivalent:*

(i) $M|(C_1\cup C_2\cup C_3)$ *is graphic,*
(ii) $C_1,C_2,C_3,C_1\triangle C_2\triangle C_3$ *are the only odd cycles contained in* $C_1\cup C_2\cup C_3$.

Proof. By (h3), $(C_1\cup C_2\cup C_3)-e$ does not have an odd circuit, so its complement intersects every odd circuit. Thus, by Proposition 7, there is a minimal signature Σ' contained in the complement of $(C_1\cup C_2\cup C_3)-e$, and as $\Sigma'\cap C_1\neq\emptyset$, it follows that $\Sigma'\cap(C_1\cup C_2\cup C_3)=\{e\}$, as required.

(i) $\Rightarrow$ **(ii)**: Let $G=(V,C_1\cup C_2\cup C_3)$ be a graph representing $M|(C_1\cup C_2\cup C_3)$. Then by (h1), C_1,C_2,C_3 are circuits of G that are pairwise edge-disjoint except at e. In fact, it follows from (h2) that C_1,C_2,C_3 are pairwise vertex-disjoint except at the ends of e. Clearly, $C_1,C_2,C_3,C_1\triangle C_2\triangle C_3$ are the only cycles contained in $C_1\cup C_2\cup C_3$ that use e. Therefore, since $\Sigma'\cap(C_1\cup C_2\cup C_3)=\{e\}$, it follows immediately that $C_1,C_2,C_3,C_1\triangle C_2\triangle C_3$ are the only odd cycles contained in $C_1\cup C_2\cup C_3$.

(ii) $\Rightarrow$ **(i)**: We claim that $\emptyset,C_1\triangle C_2,C_2\triangle C_3,C_3\triangle C_1$ are the only even cycles contained in $C_1\cup C_2\cup C_3$. To this end, let $C\subseteq C_1\cup C_2\cup C_3$ be an even cycle. Then $C\triangle C_1$ is an odd cycle, so it is one of $C_1,C_2,C_3,C_1\triangle C_2\triangle C_3$, implying in turn that C is one of $\emptyset,C_1\triangle C_2,C_1\triangle C_3,C_2\triangle C_3$, as claimed. Thus,

$$\{\emptyset,C_1,C_2,C_3,C_1\triangle C_2,C_2\triangle C_3,C_3\triangle C_1,C_1\triangle C_2\triangle C_3\}$$

are the only cycles contained in $C_1\cup C_2\cup C_3$. It is now clear that $M|(C_1\cup C_2\cup C_3)$ may be represented by a graph $G=(V,C_1\cup C_2\cup C_3)$, where C_1,C_2,C_3 are circuits of G that are pairwise vertex-disjoint except at e. In particular, $M|(C_1\cup C_2\cup C_3)$ is graphic. $\square$

5. Proof of Theorem 9 part (2)

Let $\mathbb{F}, \mathbb{K}$ be blocking mni binary clutters over ground set E, neither of which has a set of size 3. Recall that (M, Σ) represents $\mathbb{F}$ and that (N, Γ) represents $\mathbb{K}$. Take an element $e \in E$. By Theorem 9 part (1), (M, Σ) has a (strict) e-hub (C_1, C_2, C_3) and (N, Γ) has a (strict) e-hub (B_1, B_2, B_3), where for $i, j \in \{1, 2, 3\}$,

$$|C_i \cap B_j| \begin{cases} \geq 3 & \text{if } i = j \\ = 1 & \text{if } i \neq j. \end{cases}$$

We need to show that either $M|(C_1 \cup C_2 \cup C_3)$ or $N|(B_1 \cup B_2 \cup B_3)$ is non-graphic. By Proposition 12, after a possible resigning of (M, Σ), we may assume that $\Sigma \cap (C_1 \cup C_2 \cup C_3) = \{e\}$. Notice further that by Proposition 7, the odd circuits of (N, Γ) are (minimal) signatures of (M, Σ).

Suppose for a contradiction that both $M|(C_1 \cup C_2 \cup C_3)$ and $N|(B_1 \cup B_2 \cup B_3)$ are graphic. Since the latter is graphic, it follows from Proposition 12 that B_1, B_2, B_3 are the only odd circuits of (N, Γ) contained in $B_1 \cup B_2 \cup B_3$. In other words, the only sets of $\mathbb{K}$ contained in $B_1 \cup B_2 \cup B_3$ are B_1, B_2, B_3.

1. *There is an odd circuit C of (M, Σ) such that $e \notin C$ and, for each $i \in \{1, 2, 3\}$, $C \cap B_i \subseteq C_i$.*

Subproof. Let B be the union of $(B_1 \cup B_2 \cup B_3) - (C_1 \cup C_2 \cup C_3)$ and $\{e\}$. Since $B_1 \cap C_1 \neq \{e\}$, it follows that $B_1 \not\subseteq B$. Similarly, $B_2 \not\subseteq B$ and $B_3 \not\subseteq B$. Thus, since the only sets of $\mathbb{K}$ contained in $B_1 \cup B_2 \cup B_3$ are B_1, B_2, B_3, we get that B does not contain a set of $\mathbb{K} = b(\mathbb{F})$. In other words, there is a set $C \in \mathbb{F}$ such that $C \cap B = \emptyset$. By definition, C is an odd circuit of (M, Σ). Clearly, $e \notin C$. Consider the intersection $C \cap B_1$. Since $C \cap B = \emptyset$, it follows that $C \cap B_1 \subseteq C_1 \cup C_2 \cup C_3$. Moreover, as $B_1 \cap C_2 = B_1 \cap C_3 = \{e\}$, we see that $C \cap B_1 \subseteq C_1$. Similarly, $C \cap B_2 \subseteq C_2$ and $C \cap B_3 \subseteq C_3$. ♢

Since $e \notin C$, we get that $C \cap \Sigma \subseteq C - (C_1 \cup C_2 \cup C_3)$, and as C is odd, it follows that $C \not\subseteq C_1 \cup C_2 \cup C_3$.

2. $(M, \Sigma)|(C_1 \cup C_2 \cup C_3 \cup C)$ *has a trifold minor.*

Subproof. Let S be a minimal subset of $C - (C_1 \cup C_2 \cup C_3)$ such that

(m1) $M|(C_1 \cup C_2 \cup C_3 \cup S)$ has a cycle containing S,

(m2) $|S \cap \Sigma|$ is odd.

Note that S is well-defined, since $C - (C_1 \cup C_2 \cup C_3)$ satisfies both (m1)-(m2). Let

$$(M', \Sigma') := (M, \Sigma)|(C_1 \cup C_2 \cup C_3 \cup S).$$

We claim that the elements of S are in series in M'. Suppose otherwise. Then there is a cycle D of M' such that $S \cap D$ is a nonempty and proper subset of S. Notice that (m1) is satisfied by both $S \cap D$ (because of cycle D) and $S - D$ (because of cycle $D \triangle C'$, where C' is a cycle of $M|(C_1 \cup C_2 \cup C_3 \cup S)$ containing S). However, one of $S \cap D, S - D$ also satisfies (m2), thereby contradicting the minimality of S. Thus, the elements of S are in series in M'. In particular, after a possible resigning, we may assume that $\Sigma' \cap (C_1 \cup C_2 \cup C_3 \cup S) = \{e, f\}$ for some element $f \in S$. Let

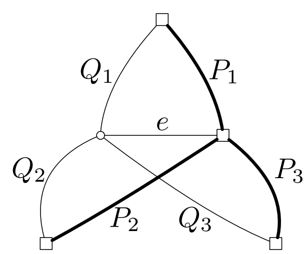
$$(M'', \{e, f\}) := (M', \Sigma')/(S - f).$$

Notice that

($\star$) $B_1 \cap C_1$ is a signature for $(M'', \{e,f\})$.

To see this, note that B_1 is a signature for (M,Σ), and by our choice of C, we have $B_1 \cap (C_1 \cup C_2 \cup C_3 \cup C) = B_1 \cap C_1$. This means that $B_1 \cap C_1$ is a signature for (M', Σ'). Since $(B_1 \cap C_1) \cap S = \emptyset$, it follows that $B_1 \cap C_1$ is also a signature for $(M'', \{e,f\})$.

We have $M'' \setminus f = M'/(S-f) \setminus f = M' \setminus S = M|(C_1 \cup C_2 \cup C_3)$, where the second equality follows from the fact that the elements of M' in S are in series. Since $M|(C_1 \cup C_2 \cup C_3)$ is graphic, $M'' \setminus f$ can be represented as a graph $G = (V, C_1 \cup C_2 \cup C_3)$. It follows from (h2) that the circuits C_1, C_2, C_3 are pairwise vertex-disjoint except at the ends of $e = \{x,y\} \subseteq V$. By (m1), $M|(C_1 \cup C_2 \cup C_3 \cup S)$ has a cycle containing S, so M'' has a cycle $P \cup \{f\}$, for some $P \subseteq C_1 \cup C_2 \cup C_3$. By replacing P by $P \triangle C_1$, if necessary, we may assume that $e \notin P$. For each $i \in \{1,2,3\}$, let $P_i := P \cap C_i$ and $Q_i := C_i - (P_i \cup \{e\})$. After possibly rearranging the edges of G within each series class $C_i - e$, we may assume that each P_i is a path that starts from x. It follows from Remark 11 that M'' is represented as the hypergraph on vertices V and edges $C_1 \cup C_2 \cup C_3 \cup \{\mathrm{odd}_G(P)\}$. We may therefore label $f = \mathrm{odd}_G(P)$, and represent M'' with the following hypergraph



where f consists of the square vertices. We claim that $P_1 \neq \emptyset$ and $Q_1 \neq \emptyset$. Since $(P \cup \{f\}) \cap \{e,f\} = \{f\}$, it follows that $P \cup \{f\}$ is an odd cycle of $(M'', \{e,f\})$. Thus, since $B_1 \cap C_1$ is a signature for $(M'', \{e,f\})$ by ($\star$), we get that $(P \cup \{f\}) \cap (B_1 \cap C_1)$ has odd size. However,

$$(P \cup \{f\}) \cap (B_1 \cap C_1) = P_1 \cap B_1,$$

so P_1 contains an odd number of edges in B_1, thus $P_1 \neq \emptyset$. Since B_1 has an even number of edges in $C_1 - e$, this means that Q_1 also picks an odd number of edges in B_1, so $Q_1 \neq \emptyset$. Similarly, for each $i \in \{1,2,3\}$, $P_i \neq \emptyset$ and $Q_i \neq \emptyset$, so there are $p_i \in P_i$ and $q_i \in Q_i$. Since $\{e, p_1, p_2, p_3, q_1, q_2, q_3\}$ is a signature for $(M'', \{e,f\})$, we see that

$$(M'', \{e,f\}) \cong (M'', \{e, p_1, p_2, p_3, q_1, q_2, q_3\}).$$

Observe however that the right signed matroid has a trifold minor, obtained after contracting each $C_i - \{e, p_i, q_i\}$. As a result, $(M,\Sigma)|(C_1 \cup C_2 \cup C_3 \cup C)$ has a trifold minor. $\Diamond$

However, by Remark 10, a trifold has an $(F_7, E(F_7))$ minor, so (M,Σ) has an $(F_7, E(F_7))$ minor. As a consequence, Proposition 8 implies that $\mathbb{F}$ has an $\mathbb{L}_7$ minor. Since $\mathbb{F}$ is mni, we must have that $\mathbb{F} \cong \mathbb{L}_7$, but $\mathbb{F}$ has no set of size 3, a contradiction. This finishes the proof of Theorem 9 part (2). $\square$

6. NON-GRAPHIC STRICT HUBS

In this section, we prove two results needed for the proof of Theorem 9 part (3).

Proposition 13. *Take a signed matroid (M,Σ), $e \in E(M)$ and a strict e-hub (C_1,C_2,C_3) such that $M|(C_1 \cup C_2 \cup C_3)$ is non-graphic. Then there exist $I \subseteq C_3 - e$ and distinct $g_1, g_2 \in (C_3 - I) - e$ where*

(1) $(C_1, C_2, C_3 - I)$ is an e-hub of $(M,\Sigma)/I$,
(2) $(M/I)|\big(C_1 \cup C_2 \cup \{g_i\}\big)$ has a circuit containing g_i, for each $i \in \{1,2\}$,
(3) $(M/I)|\big(C_1 \cup C_2 \cup \{g_1, g_2\}\big)$ is non-graphic.

Proof. By Proposition 12, after a possible resigning, we may assume that $\Sigma \cap (C_1 \cup C_2 \cup C_3) = \{e\}$.

1. *Let I be a maximal subset of $C_3 - e$ such that every cycle of $M|(C_1 \cup C_2 \cup I)$ is disjoint from I. Then the following statements hold:*

(i) $(C_1, C_2, C_3 - I)$ is an e-hub of $(M,\Sigma)/I$,
(ii) for each $g \in (C_3 - I) - e$, $(M/I)|\big(C_1 \cup C_2 \cup \{g\}\big)$ has a cycle containing g,
(iii) $(M/I)|(C_1 \cup C_2 \cup (C_3 - I))$ is non-graphic.

Subproof. Let $(M', \Sigma) := (M,\Sigma)/I$ and $C_3' := C_3 - I$ (note, $\Sigma \cap I = \emptyset$). **(i)** Since $I \subseteq C_3 - e$ and $e \in C_3'$, we get that C_3' is an odd circuit of (M',Σ), and by our choice of I, we see that C_1, C_2 are still odd circuits of (M',Σ). Moreover, $C_1 \cap C_2 = C_2 \cap C_3' = C_3' \cap C_1 = \{e\}$, so (h1) holds. Our choice of I also implies that the only nonempty cycles contained in $C_1 \cup C_2$ are $C_1, C_2, C_1 \triangle C_2$, and since $I \subseteq C_3 - e$, we get that the only nonempty cycles in $C_i \cup C_3'$ are $C_i, C_3', C_i \triangle C_3'$, for $i \in \{1,2\}$, so (h2) holds. Lastly, as $\Sigma \cap (C_1 \cup C_2 \cup C_3') = \{e\}$, every odd cycle of (M',Σ) contained in $C_1 \cup C_2 \cup C_3'$ uses e, so (h3) holds. Thus, (C_1, C_2, C_3') is an e-hub of (M',Σ). **(ii)** follows immediately from our maximal choice of I. **(iii)** By Proposition 12, it is sufficient to provide an odd cycle of $(M',\Sigma)|(C_1 \cup C_2 \cup C_3')$ distinct from $C_1, C_2, C_3', C_1 \triangle C_2 \triangle C_3'$. Since $M|(C_1 \cup C_2 \cup C_3)$ is non-graphic, the same proposition implies the existence of an odd cycle C of $(M,\Sigma)|(C_1 \cup C_2 \cup C_3)$ different from $C_1, C_2, C_3, C_1 \triangle C_2 \triangle C_3$. Then $e \in C$. Notice that by (h2), $\{e\} \subsetneq C \cap C_1 \subsetneq C_1$; for if $\{e\} = C \cap C_1$ then $C \in \{C_2, C_3\}$, and if $C \cap C_1 = C_1$, then $C \triangle C_1 \in \{\emptyset, C_2 \triangle C_3\}$, neither of which are the case. Let $C' := C - I$. Observe that C' is an odd cycle of $(M',\Sigma)|(C_1 \cup C_2 \cup C_3')$ with $C' \cap C_1 = C \cap C_1$, so $\{e\} \subsetneq C' \cap C_1 \subsetneq C_1$, implying in turn that C' is different from $C_1, C_2, C_3', C_1 \triangle C_2 \triangle C_3'$. Thus, Proposition 12 implies that $M'|(C_1 \cup C_2 \cup C_3')$ is non-graphic, as required. $\Diamond$

2. *Let I be a maximal subset of $C_3 - e$ such that every cycle of $M|(C_1 \cup C_2 \cup I)$ is disjoint from I. Then there exist $h_1, h_2 \in (C_3 - I) - e$ such that one of the following statements holds:*

- *$(M/I)|(C_1 \cup C_2 \cup \{h_1, h_2\})$ is non-graphic, or*
- *$(M/I)|(C_1 \cup C_2 \cup \{h_1, h_2\})$ has cycles D_1, D_2 where $D_1 \cap \{h_1, h_2\} = \{h_1\}$, $D_2 \cap \{h_1, h_2\} = \{h_2\}$ and $D_1 \cap D_2 = \{e\}$.*

Subproof. Let $(M', \{e\}) := (M,\Sigma)|(C_1 \cup C_2 \cup C_3)/I$ and $C_3' := C_3 - I$. By Claim 1, M' is non-graphic and (C_1, C_2, C_3') is an e-hub of $(M', \{e\})$. Take an element $g \in C_3' - e$. Claim 1 also tells us that there is a cycle D_g of $M'|(C_1 \cup C_2 \cup \{g\})$ using g, where after possibly replacing D_g by $D_g \triangle C_1$, we may assume that $e \notin D_g$. Notice that $D_g \triangle C_1 \triangle C_2$ is another cycle of $M'|(C_1 \cup C_2 \cup \{g\})$ using g that excludes e. We will refer to $D_g - g$ and $(D_g \triangle C_1 \triangle C_2) - g$ as the *outer joins of* g. Note that each outer join intersects both C_1 and C_2, as $(M', \{e\})$ is an e-hub and therefore satisfies (h2). We have defined two outer joins for each $g \in C_3' - e$. Pick $h_1 \in C_3' - e$ so that one of its outer joins, call it J_{h_1}, is minimal among all the defined outer joins.

For each $g \in C_3' - \{e, h_1\}$, pick an arbitrary outer join J_g of g. We claim that

$$(\star) \qquad \triangle\big(J_g : g \in C_3' - e\big) = C_1 - e \quad \text{or} \quad C_2 - e.$$

Each D_g is a cycle so $\triangle\big(D_g : g \in C_3' - e\big)$ is a cycle of M', implying in turn that

$$C_3' \triangle \big[\triangle\big(D_g : g \in C_3' - e\big)\big] = \{e\} \cup \big[\triangle\big(D_g - g : g \in C_3' - e\big)\big]$$

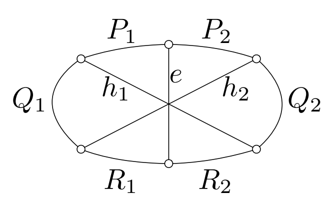
is a cycle of M' contained in $C_1 \cup C_2$, so by (h2), $\triangle\big(D_g - g : g \in C_3' - e\big)$ is either $C_1 - e$ or $C_2 - e$. Since $\triangle\big(J_g : g \in C_3' - e\big)$ is either $\triangle\big(D_g - g : g \in C_3' - e\big)$ or $(C_1 \triangle C_2) \triangle \big[\triangle\big(D_g - g : g \in C_3' - e\big)\big]$, $(\star)$ follows.

Let $a \in J_{h_1} \cap C_1$ and $b \in J_{h_1} \cap C_2$. Since $(\star)$ holds, there is an $h_2 \in C_3' - e$ such that $|J_{h_2} \cap \{a, b\}| = 1$. We claim that h_1, h_2 are the desired elements. If $M'|(C_1 \cup C_2 \cup \{h_1, h_2\})$ is non-graphic, then we are done. Otherwise, there is a graph $G = (V, C_1 \cup C_2 \cup \{h_1, h_2\})$ representing $M'|(C_1 \cup C_2 \cup \{h_1, h_2\})$. By (h2), the circuits C_1, C_2 are pairwise vertex-disjoint except at the ends of e. Notice that J_{h_1}, J_{h_2} are paths contained in $C_1 \triangle C_2$. The minimality of J_{h_1} implies that $J_{h_2} \not\subseteq J_{h_1}$ and $J_{h_2} \triangle C_1 \triangle C_2 \not\subseteq J_{h_1}$, and since $\{a, b\} \subseteq J_{h_1}$ while $|J_{h_2} \cap \{a, b\}| = 1$, we see that J_{h_1}, J_{h_2} are crossing paths:

$$J_{h_1} \cap J_{h_2} \neq \emptyset, \qquad J_{h_1} - J_{h_2} \neq \emptyset, \qquad J_{h_2} - J_{h_1} \neq \emptyset, \quad \text{and} \quad J_{h_1} \cup J_{h_2} \neq C_1 \triangle C_2.$$

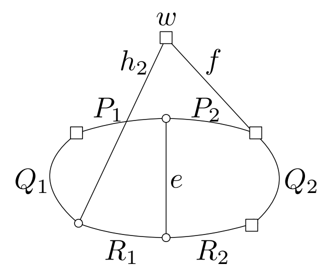
If $J_{h_2} \cap \{a, b\} = \{b\}$, let $D_1 := D_{h_1} \triangle C_1$ and $D_2 := D_{h_2} \triangle C_2$, and otherwise, let $D_1 := D_{h_1} \triangle C_1$ and $D_2 := D_{h_2} \triangle C_1$. Notice that $D_1 \cap \{h_1, h_2\} = \{h_1\}$, $D_2 \cap \{h_1, h_2\} = \{h_2\}$, and as the two paths J_{h_1}, J_{h_2} are crossing, it follows that $D_1 \cap D_2 = \{e\}$, as required. ◊

Now let I be a maximal subset of $C_3 - e$ such that every cycle of $M|(C_1 \cup C_2 \cup I)$ is disjoint from I. If there are elements $g_1, g_2 \in C_3 - I$ such that $(M/I)|(C_1 \cup C_2 \cup \{g_1, g_2\})$ is non-graphic, then (1)-(3) hold by Claim 1 (i)-(ii), and we are done. Otherwise, by Claim 2, there are elements $h_1, h_2 \in (C_3 - I) - e$ such that $(M/I)|(C_1 \cup C_2 \cup \{h_1, h_2\})$ has cycles D_1, D_2 where $D_1 \cap \{h_1, h_2\} = \{h_1\}$, $D_2 \cap \{h_1, h_2\} = \{h_2\}$ and $D_1 \cap D_2 = \{e\}$. In particular, $(M/I)|(C_1 \cup C_2 \cup \{h_1, h_2\})$ is graphic, and can be represented as the graph $H = (V, C_1 \cup C_2 \cup \{h_1, h_2\})$ displayed below where $C_1 = \{e\} \cup P_1 \cup Q_1 \cup R_1$ and $C_2 = \{e\} \cup P_2 \cup Q_2 \cup R_2$



are vertex-disjoint except at the ends of e, $D_1 = \{e, h_1\} \cup P_1 \cup R_2$ and $D_2 = \{e, h_2\} \cup \{P_2, R_1\}$. Since (C_1, C_2, C_3') is an e-hub of $(M, \Sigma)/I$, we get from (h2) that, for $i \in \{1, 2\}$, $P_i, Q_i, R_i \neq \emptyset$.

For $i \in \{1,2\}$, let D_i' be a cycle of M such that $D_i \subseteq D_i' \subseteq D_i \cup I$; as $D_i' \cap \Sigma = \{e\}$, D_i' is an odd cycle of (M,Σ). Note further, for $i \in \{1,2\}$, that D_i' is different from C_1, C_2, C_3. Thus, since (C_1, C_2, C_3) is a strict e-hub of (M,Σ) and therefore satisfies (h4), we must have that $\{e\} \subsetneq D_1' \cap D_2'$. Because $D_1 \cap D_2 = \{e\}$, there is an element $f \in I$ such that $\{e,f\} \subseteq D_1' \cap D_2'$. Consider now the minor $(M,\Sigma)/(I-f)$; note that $D_1 \cup \{f\}$ and $D_2 \cup \{f\}$ are odd cycles of this signed matroid. We may represent $M/(I-f)$ as a hypergraph $G = (V \cup \{w\}, C_1 \cup C_2 \cup \{h_1, h_2, f\})$ obtained from H by adding a vertex w, displayed below



where the square vertices form the edge h_1, as $D_1 \cup \{f\} = \{e, h_1, f\} \cup P_1 \cup R_2$ is a cycle of $M/(I-f)$. Now let $J := I \triangle \{f, h_2\}$. Observe that $(M/J)|(C_1 \cup C_2 \cup \{f, h_1\})$ is non-graphic, as it has an F_7 minor obtained after contracting $P_1 \cup R_2$ and contracting each of Q_1, R_1, P_2, Q_2 to a single edge. Observe further that

($\diamond$) J is a maximal subset of $C_3 - e$ such that every cycle of $M|(C_1 \cup C_2 \cup J)$ is disjoint from J.

Let us first show that every cycle of $M|(C_1 \cup C_2 \cup J)$ is disjoint from J. Suppose otherwise. Then there is a cycle C of $M|(C_1 \cup C_2 \cup J)$ such that $C \cap J \neq \emptyset$. By our choice of I, $h_2 \in C$. But then $C \triangle D_2'$ is a cycle of $M|(C_1 \cup C_2 \cup I)$ containing $f \in I$, a contradiction. Thus, every cycle of $M|(C_1 \cup C_2 \cup J)$ is disjoint from J. To see maximality, take $g \in (C_3 - J) - e$. If $g = f$, then D_2' is a cycle of $M|(C_1 \cup C_2 \cup J \cup \{g\})$ containing g. Otherwise, $g \neq f$, so there is a cycle C of $M|(C_1 \cup C_2 \cup I \cup \{g\})$ containing g. If $f \notin C$, then C is also a cycle of $M|(C_1 \cup C_2 \cup J \cup \{g\})$ containing g, and if $f \in C$, then $C \triangle D_2'$ is a cycle of $M|(C_1 \cup C_2 \cup J \cup \{g\})$ containing g. Thus, ($\diamond$) holds. Hence, by Claim 1 (i)-(ii), we get that J and $\{f, h_1\}$ satisfy (1)-(3). This finishes the proof of Proposition 13. □

Let us define two signed matroids, displayed in Figure 3. Take a hypergraph $G = (V, E)$ where $V = \{x, y, s_1, s_2, t\}$. If

$$E = \big\{\{x,y\}, \{y,s_1\}, \{s_1,t\}, \{t,x\}, \{x,s_2\}, \{s_2,y\}, \{s_1,s_2\}, \{t,s_2\}, \{t,x,s_1,s_2\}\big\},$$

we say that G is a *Type I hypergraph*, and if

$$E = \big\{\{x,y\}, \{y,s_1\}, \{s_1,t\}, \{t,x\}, \{x,s_2\}, \{s_2,y\}, \{s_1,s_2\}, \{t,y\}, \{t,x,y,s_1\}\big\},$$

we say that G is a *Type II hypergraph*. Let $\Sigma := \big\{\{x,y\}, \{s_1,s_2\}\big\} \subseteq E$. Given that M is the matroid represented by the Type I hypergraph (respectively, Type II hypergraph), we refer to (M,Σ) as the *Type I signed matroid* (respectively, *Type II signed matroid*). (The careful reader will notice that the Type I and Type II signed matroids are isomorphic.)

Remark 14. *The Type I and Type II signed matroids have $(F_7, E(F_7))$ as a minor.*

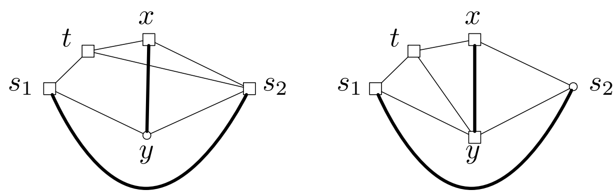


Figure 3. The left signed hypergraph represents the Type I signed matroid, while the right one represents the Type II signed matroid. In each signed hypergraph, the square vertices form the edge of size 4, and the two bold edges form the signature.

Proof. For the Type I signed matroid, note that the minor obtained after deleting edge $\{x, s_2\}$ and contracting edge $\{x, y\}$ is isomorphic to $(F_7, E(F_7))$. As for the Type II signed matroid, the minor obtained after deleting edge $\{y, s_1\}$ and contracting edge $\{s_1, s_2\}$ is isomorphic to $(F_7, E(F_7))$. $\square$

7. Proof of Theorem 9 part (3)

Let $\mathbb{F}, \mathbb{K}$ be blocking mni clutters over ground set E, neither of which has a set of size 3, where (M, Σ) represents $\mathbb{F}$ and (N, Γ) represents $\mathbb{K}$. By Theorem 9 part (1), (M, Σ) has a strict e-hub (C_1, C_2, C_3) and (N, Γ) has a strict e-hub (B_1, B_2, B_3) such that, for $i, j \in \{1, 2, 3\}$,

$$|C_i \cap B_j| \begin{cases} \geq 3 & \text{if } i = j \\ = 1 & \text{if } i \neq j. \end{cases}$$

Assume further that $M|(C_1 \cup C_2 \cup C_3)$ is non-graphic. We need to show that (M, Σ) has an $(F_7, E(F_7) - \omega)$ minor going through e. By Proposition 12, after a possible resigning, we may assume that $\Sigma \cap (C_1 \cup C_2 \cup C_3) = \{e\}$. By Proposition 13, there exist $I \subseteq C_3 - e$ and distinct $g_1, g_2 \in (C_3 - I) - e$ where

(1) $(C_1, C_2, C_3 - I)$ is an e-hub of $(M, \Sigma)/I$,
(2) $(M/I)|\big(C_1 \cup C_2 \cup \{g_i\}\big)$ has a circuit D_i containing g_i, for each $i \in \{1, 2\}$,
(3) $(M/I)|\big(C_1 \cup C_2 \cup \{g_1, g_2\}\big)$ is non-graphic.

For each $i \in \{1, 2\}$, after possibly replacing D_i by $D_i \triangle C_1$, we may assume that $e \notin D_i$; as $(C_1, C_2, C_3 - I)$ is an e-hub of $(M, \Sigma)/I$, it follows from (h2) that $D_i \cap C_1 \neq \emptyset$ and $D_i \cap C_2 \neq \emptyset$. Notice that, for each $i \in \{1, 2\}$, $B_i \cap I = \emptyset$, so B_i is a signature of $(M, \Sigma)/I$.

1. *There exists an odd circuit C of $(M, \Sigma)/I$ such that $e \notin C$ and, for each $i \in \{1, 2\}$, $C \cap B_i \subseteq C_i$.*

Subproof. Let B be the union of $\{e\}$ and $(B_1 \cup B_2) - (C_1 \cup C_2)$. Since $B_1 \cap C_1 \neq \{e\}$ and $B_2 \cap C_2 \neq \{e\}$, it follows that $B_1 \not\subseteq B$ and $B_2 \not\subseteq B$. Since (B_1, B_2, B_3) is an e-hub of (N, Γ), it follows from (h2) that B_1, B_2 are the only odd circuits of (N, Γ) contained in $B_1 \cup B_2$, implying in turn that B_1, B_2 are the only sets of $\mathbb{K}$ contained in $B_1 \cup B_2$. As a result, B does not contain any set of $\mathbb{K} = b(\mathbb{F})$, so there is a set $C' \in \mathbb{F}$ such that $C' \cap B = \emptyset$. Observe that C' is an odd circuit of (M, Σ) such that $e \notin C'$ and, for each $i \in \{1, 2\}$, $C' \cap B_i \subseteq C_i$. Note that $C' - I$ is an odd cycle of $(M, \Sigma)/I$, so it contains an odd circuit C, which is the desired set. $\Diamond$

Let $(M', \Sigma) := (M, \Sigma)/I$. Let S be a minimal subset of $C - (C_1 \cup C_2)$ such that (m1) $M'|(C_1 \cup C_2 \cup S)$ has a cycle containing S,

(m2) $|S \cap \Sigma|$ is odd.

Note that S is well-defined as $C - (C_1 \cup C_2)$ satisfies (m1)-(m2).

2. *$S \cap \{g_1, g_2\} = \emptyset$, and the elements of S are in series in $M'|(C_1 \cup C_2 \cup \{g_1, g_2\} \cup S)$.*

Subproof. If $g_1 \in S$, then $S - g_1$ satisfies (m1) (given that D is a cycle of $M'|(C_1 \cup C_2 \cup S)$ containing S, $D \triangle D_1$ is a cycle of $M'|(C_1 \cup C_2 \cup (S - g_1))$ containing $S - g_1$) and (m2) (as $g_1 \notin \Sigma$), which is not the case by the minimality of S. Thus, $g_1 \notin S$ and similarly, $g_2 \notin S$. Suppose for a contradiction that the elements of S are not in series in $M'|(C_1 \cup C_2 \cup \{g_1, g_2\} \cup S)$. Then there is a cycle D of $M'|(C_1 \cup C_2 \cup \{g_1, g_2\} \cup S)$ such that $S \cap D$ is a nonempty and proper subset of S. After possibly replacing D by $D \triangle D_1$, we may assume that $g_1 \notin D$, and after possibly replacing D by $D \triangle D_2$, we may assume that $g_2 \notin D$. Notice now that both $S \cap D$ and $S - D$ satisfy (m1), and one of them satisfies (m2), thereby contradicting the minimality of S. This finishes the proof of the claim. $\Diamond$

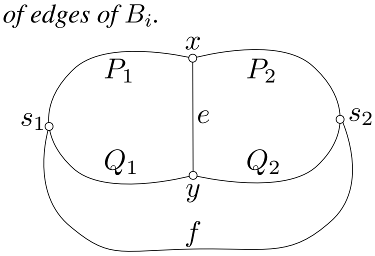
Thus, there exists a signature Σ' of (M', Σ) such that $\Sigma' \cap (C_1 \cup C_2 \cup \{g_1, g_2\} \cup S) = \{e, f\}$, for some $f \in S$. Consider the minor

$$(M'', \{e, f\}) := (M', \Sigma')|(C_1 \cup C_2 \cup \{g_1, g_2\} \cup S)/(S - f).$$

For each $i \in \{1, 2\}$, our choice of C implies that $B_i \cap S = \emptyset$, so $B_i \cap (C_1 \cup C_2 \cup \{g_1, g_2\}) = B_i \cap C_i$ is a signature of $(M'', \{e, f\})$.

3. *Up to rearranging edges within series classes, there is a unique graph $G = (V, C_1 \cup C_2 \cup \{f\})$ representing $M'' \setminus \{g_1, g_2\}$, where*

- *$e = \{x, y\}$ and $f = \{s_1, s_2\}$ for distinct vertices x, y, s_1, s_2,*
- *$V(C_1) \cup V(C_2) = V$, $V(C_1) \cap V(C_2) = \{x, y\}$, $s_1 \in V(C_1)$ and $s_2 \in V(C_2)$,*
- *if P_i (respectively, Q_i) is the path in $C_i - e$ with ends s_i, x (respectively, s_i, y), then P_i (respectively, Q_i) contains an odd number of edges of B_i.*



Subproof. Since $(C_1, C_2, C_3 - I)$ is an e-hub of (M', Σ) and $M'|(C_1 \cup C_2) = M''|(C_1 \cup C_2)$, it follows from (h2) that $\emptyset, C_1, C_2, C_1 \triangle C_2$ are the only cycles of M'' contained in $C_1 \cup C_2$. Thus, $M''|(C_1 \cup C_2)$ can be represented as a graph $H = (V, C_1 \cup C_2)$ where C_1, C_2 are circuits vertex-disjoint except at the ends of e. Write $e = \{x, y\}$, and note that $V(C_1) \cup V(C_2) = V$ and $V(C_1) \cap V(C_2) = \{x, y\}$. By (m1), there is a cycle D of $M''|(C_1 \cup C_2 \cup \{f\})$ containing f. After replacing D by $D \triangle C_1$, if necessary, we may assume that $e \notin D$. In particular, D is an odd cycle of $(M'', \{e, f\})$. Thus, D contains an odd number of edges of signature $B_i \cap C_i$, for each $i \in \{1,2\}$. After rearranging the edges of H in series classes $C_1 - e, C_2 - e$, we may assume that $D - f$ is a path in $C_1 \cup C_2$ using vertex y. Let $\{s_1, s_2\} := \mathrm{odd}_H(D - f)$ for $s_1 \in V(C_1)$ and $s_2 \in V(C_2)$. By Remark 11, $M''|(C_1 \cup C_2 \cup \{f\})$ is represented as the graph $G = (V, C_1 \cup C_2 \cup \{f\})$, where $f = \{s_1, s_2\}$. Write $D = Q_1 \cup Q_2 \cup \{f\}$, $C_1 - e = P_1 \cup Q_1$ and $C_2 - e = P_2 \cup Q_2$. Then, for $i \in \{1,2\}$, P_i and Q_i each contain an odd number edges of B_i, so in particular, $P_i \neq \emptyset$ and $Q_i \neq \emptyset$. Thus, x, y, s_1, s_2 are distinct vertices of G. It is easily seen that, up to rearranging the edges within series classes, G is the unique graphic representation of $M''|(C_1 \cup C_2 \cup \{f\})$. $\Diamond$

4. *For each $g \in \{g_1, g_2, f\}$, if $M'' \setminus g$ is non-graphic, then it has an F_7 minor obtained after contracting some elements of $C_1 \triangle C_2$.*

Subproof. We leave this as an easy exercise for the reader. $\Diamond$

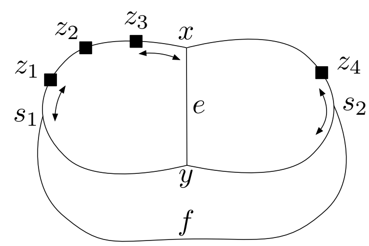
5. *If $M'' \setminus g_i$ is graphic for each $i \in \{1,2\}$, then $(M'', \{e, f\})$ has an $(F_7, E(F_7))$ minor.*

Subproof. Suppose $M'' \setminus g_i$ is graphic for each $i \in \{1,2\}$. Then there is a graph $G_1 = (V, C_1 \cup C_2 \cup \{f, g_1\})$ representing $M'' \setminus g_2$. By the uniqueness of G from Claim 3, we may assume that $G_1 \setminus g_1 = G$. Recall that D_1 is a circuit of G containing g_1 such that $D_1 \cap \{e, f\} = \emptyset$. Thus, D_1 is an even cycle of $(M'', \{e, f\})$, implying in turn that D_1 contains an even number of edges of $B_i \cap C_i$, for each $i \in \{1,2\}$. Thus, assuming that $g_1 = \{t_1, t_2\}$ for $t_1 \in V(C_1)$ and $t_2 \in V(C_2)$, then $t_1 \neq s_1$ and $t_2 \neq s_2$. Moreover, as $D_1 \cap C_1 \neq \emptyset$ and $D_1 \cap C_2 \neq \emptyset$, it follows that $t_1 \notin \{x, y\}$ and $t_2 \notin \{x, y\}$. As a result, the distinct vertices x, y, s_1, s_2, t_1, t_2 break the circuit $C_1 \triangle C_2$ into 6 nonempty paths, which are series classes of $M''|(C_1 \cup C_2 \cup \{f, g_1\})$. Recall that D_2 is a circuit of $M''|(C_1 \cup C_2 \cup \{g_2\})$ containing g_2 such that $e \notin D_2$. Let $J := D_2 - g_2 \subseteq C_1 \triangle C_2$. After rearranging the edges of G_1 within the mentioned 6 series classes, if necessary, we may assume that J is the union of at most 3 paths. Thus, $|\mathrm{odd}_{G_1}(J)| \leq 6$. By Remark 11, M'' is represented as the hypergraph $H_2 = (V, C_1 \cup C_2 \cup \{f, g_1, g_2\})$, where $g_2 = \mathrm{odd}_{G_1}(J)$. Since $M'' \setminus f = M'|(C_1 \cup C_2 \cup \{g_1, g_2\})$ is non-graphic by (3), it follows that $|g_2| \in \{4, 6\}$. In the case analysis below, in various ways we will take advantage of the fact that $M'' \setminus g_1$ is graphic and therefore has no F_7 minor. (For instance, if g_2 picks an odd number of vertices in each of $V(P_1) - \{x, s_1\}, V(P_2) - \{x, s_2\}, V(Q_1) - \{y, s_1\}, V(Q_2) - \{y, s_2\}$, then $M'' \setminus g_2$ has an F_7 minor.)

Case 1: $|g_2| = 4$. Write $g_2 = \{z_1, z_2, z_3, z_4\}$. As $M'' \setminus g_1$ has no F_7 minor, two vertices of g_2 belong to one of $V(P_1) - x, V(P_2) - x, V(Q_1) - y, V(Q_2) - y$. By symmetry, we may assume that $z_1, z_2 \in V(P_1) - x$. Once again, as $M'' \setminus g_1$ has no F_7 minor, either

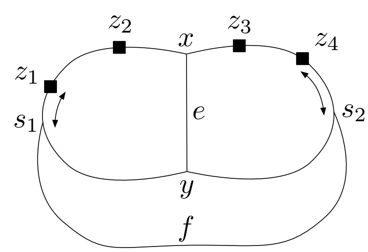
(i) $z_3 \in V(P_1)$ and $z_4 \in V(C_2) - \{x, y\}$, or
(ii) $z_3, z_4 \in V(P_2) - x$.

Suppose in the first case that (i) holds. We may assume that z_2 lies strictly between z_1, z_3. Then $H_2 \setminus g_1$ is displayed as the figure below,



where z_1, z_3, z_4 can move as indicated by the arrows. As $M'' \setminus f$ is non-graphic by (3), it follows that t_1 lies strictly between z_1, z_3. Observe now that $(M'', \{e, f\})$ has the Type I signed matroid as a minor, so by Remark 14, $(M'', \{e, f\})$ has an $(F_7, E(F_7))$ minor.

Suppose in the remaining case that (ii) holds. We may assume that z_2, z_3 lie between z_1 and z_4. Then $H_2 \setminus g_1$ is displayed as

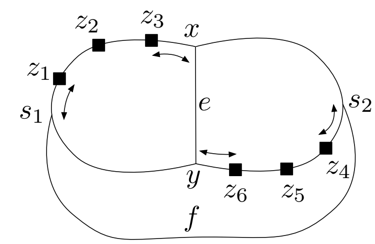


where z_1, z_4 can move as indicated by the arrows. Denote by $P'_1 \subseteq C_1 - e$ the path between z_2, x, and by $P'_2 \subseteq C_2 - e$ the path between z_3, x. Since $M'' \setminus f$ is non-graphic, it follows from Claim 4 that either $M'' \setminus f / P'_1$ or $M'' \setminus f / P'_2$ has an F_7 minor. By symmetry, we may assume that $M'' \setminus f / P'_2$ has an F_7 minor, implying in particular that $M'' \setminus f / P'_2$ is non-graphic. A similar argument as in (i) now tells us that $(M'', \{e, f\})/P'_2$ has an $(F_7, E(F_7))$ minor.

Case 2: $|g_2| = 6$. Write $g_2 = \{z_1, z_2, z_3, z_4, z_5, z_6\}$. Since $M'' \setminus g_1$ has no F_7 minor, one of the following statements holds, up to relabeling P_1, Q_1, P_2, Q_2 and relabeling $z_1, \ldots, z_6$:

(i) $z_1, z_2, z_3, z_4, z_5 \in V(P_1)$ and $z_6 \in V(C_2) - \{x, y\}$,
(ii) $z_1, z_2, z_3, z_4, z_5, z_6 \in V(P_1) \cup V(P_2)$,
(iii) $z_1, z_2, z_3 \in V(P_1)$ and $z_4, z_5, z_6 \in V(Q_2)$.

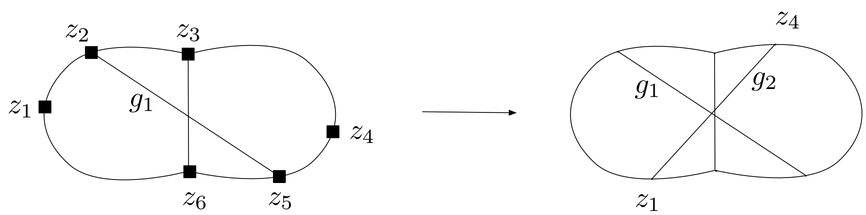
If (i) or (ii) hold, then the edges of G_1 can be rearranged within its 6 series classes in $C_1 \triangle C_2$ so as to bring the size of $g_2 = \mathrm{odd}_{G_1}(J)$ down to 4, so by Case 1, $(M'', \{e, f\})$ has an $(F_7, E(F_7))$ minor. Otherwise, (iii) holds. We may assume that $s_1, z_1, z_2, z_3, x, s_2, z_4, z_5, z_6, y$ appear in this cyclic order on $C_1 \triangle C_2$. We may therefore display $H_2 \setminus g_1$ as



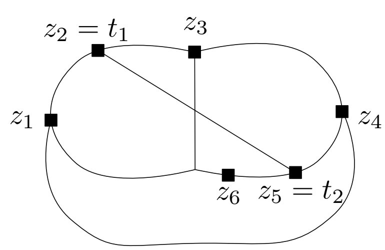
where z_1, z_3, z_4, z_6 can move as indicated by the arrows. Let us analyze where the ends t_1, t_2 of g_1 lie on $C_1 \triangle C_2$. If the edges of G_1 can be rearranged within its 6 series classes in $C_1 \triangle C_2$ so as to bring the size of $g_2 = \text{odd}_{G_1}(J)$ down to 4, then by Case 1, $(M'', \{e, f\})$ has an $(F_7, E(F_7))$ minor. Otherwise, t_1 lies strictly between z_1, z_3 and t_2 lies strictly between z_4, z_6. Since $M'' \setminus f$ is non-graphic, we have

$$(z_2, z_3, z_5, z_6) \neq (t_1, x, t_2, y).$$

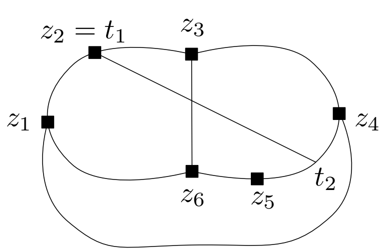
(For if not, the current representation of $M'' \setminus f$, displayed in the left figure below,

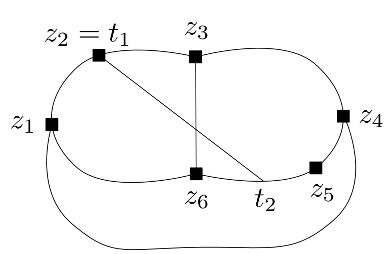


can be replaced by the graphic representation, displayed in the right figure above.) By symmetry, we may assume that $(z_5, z_6) \neq (t_2, y)$. If $z_6 \neq y$, then



$(M'', \{e, f\})$ has the Type I signed matroid as a minor (obtained after contracting the $z_4 z_6$-path in $C_2 - e$ and contracting all but one element from each series class), so by Remark 14, $(M'', \{e, f\})$ has an $(F_7, E(F_7))$ minor. Otherwise, $z_5 \neq t_2$. In this case, $(M'', \{e, f\})$ has one of the following as a minor:





The signed matroid displayed on the left has the Type I signed matroid as a minor (obtained after contracting the z_4t_2- and z_5z_6-paths in $C_2 - e$ and contracting all but one element from each series class), so by Remark 14, it has an $(F_7, E(F_7))$ minor. The signed matroid displayed on the right, on the other hand, has the Type II signed matroid as a minor (obtained after contracting the z_4z_5- and t_2z_6-paths in $C_2 - e$ and contracting all but one element from each series class), so by Remark 14, it has an $(F_7, E(F_7))$ minor. We have shown that in both cases, $(M'', \{e, f\})$ has an $(F_7, E(F_7))$ minor, thereby finishing the proof of the claim. $\Diamond$

Assume that $M'' \setminus g_i$ is graphic for each $i \in \{1, 2\}$. Then by the preceding claim, $(M'', \{e, f\})$ has an $(F_7, E(F_7))$ minor, implying in turn that (M, Σ) has an $(F_7, E(F_7))$ minor. So by Proposition 8, $\mathbb{F}$ has an $\mathbb{L}_7$ minor, and since $\mathbb{F}$ is mni, this means $\mathbb{F} \cong \mathbb{L}_7$, which cannot be as $\mathbb{F}$ has no set of size 3. Hence, by symmetry, we may assume that $M'' \setminus g_2$ is non-graphic. Thus, by Claims 3 and 4, we see that there exists $I \subseteq C_1 \triangle C_2$ such that $M'' \setminus g_2/I \cong F_7$. Then $(M'', \{e, f\}) \setminus g_2/I \cong (F_7, E(F_7) - \omega)$, and so (M, Σ) has an $(F_7, E(F_7) - \omega)$ minor going through e, as required. This finishes the proof of Theorem 9 part (3). $\square$

ACKNOWLEDGMENTS

We would like to thank the anonymous referees for carefully reading the manuscript; their suggestions helped improved the presentation of the paper.

REFERENCES

[1] Abdi, A. and Guenin, B.: The minimally non-ideal binary clutters with a triangle. Submitted.
[2] Bridges, W.G. and Ryser H.J.: Combinatorial designs and related systems. J. Algebra **13**, 432–446 (1969)
[3] Cornuéjols, G.: Combinatorial optimization, packing and covering. SIAM, Philadelphia (2001)
[4] Cornuéjols, G. and Guenin, B.: Ideal binary clutters, connectivity, and a conjecture of Seymour. SIAM J. Discrete Math. **15**(3), 329–352 (2002)
[5] Edmonds, J. and Fulkerson, D.R.: Bottleneck extrema. J. Combin. Theory Ser. B **8**, 299–306 (1970)
[6] Ford, L.R. and Fulkerson, D.R.: Maximal flow through a network. Canadian J. Math. **8**, 399–404 (1956)
[7] Guenin, B.: A characterization of weakly bipartite graphs. J. Combin. Theory Ser. B **83**, 112–168 (2001)
[8] Guenin, B.: Integral polyhedra related to even-cycle and even-cut matroids. Math. Oper. Res. **27**(4), 693–710 (2002)
[9] Lehman, A.: A solution of the Shannon switching game. Society for Industrial Appl. Math. **12**(4), 687–725 (1964)
[10] Lehman, A.: On the width-length inequality. Math. Program. **17**(1), 403–417 (1979)
[11] Lehman, A.: The width-length inequality and degenerate projective planes. DIMACS Vol. **1**, 101–105 (1990)
[12] Menger, K.: Zur allgemeinen Kurventheorie. Fundamenta Mathematicae **10**, 96–115 (1927)
[13] Novick, B. and Sebő, A.: On combinatorial properties of binary spaces. IPCO Vol. **4**, 212–227 (1995)
[14] Oxley, J.: Matroid theory, second edition. Oxford University Press, New York (2011)
[15] Seymour, P.D.: On Lehman's width-length characterization. DIMACS Vol. **1**, 107–117 (1990)
[16] Seymour, P.D.: The forbidden minors of binary matrices. J. London Math. Society **2**(12), 356–360 (1976)
[17] Seymour, P.D.: The matroids with the max-flow min-cut property. J. Combin. Theory Ser. B **23**, 189–222 (1977)