Strategic trading and welfare in a dynamic market

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Abstract

This paper studies a dynamic model of a financial market with N strategic agents. Agents receive random stock endowments at each period and trade to share dividend risk. Endowments are the only private information in the model. We find that agents trade slowly even when the time between trades goes to 0. In fact, welfare loss due to strategic behavior increases as the time between trades decreases. In the limit when the time between trades goes to 0, welfare loss is of order $1/N$, and not $1/N^2$ as in the static models of the double auctions literature. The model is very tractable and closed-form solutions are obtained in a special case.

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1 Introduction

Large traders, such as dealers, mutual funds, and pension funds, play an increasingly important role in financial markets. These agents' trades exceed the average daily volume of many securities and, according to a number of empirical studies, have a significant price impact.[1] Recent studies have also shown that large agents strategically reduce the price impact of their trades by spreading them over several days.[2] One explanation for the price impact of large trades is that they reveal inside information. However, since large traders do not outperform the market, the majority of their trades cannot be attributed to such information.[3]

In this paper we study a dynamic model of a financial market with large agents who trade to share risk. We address the following questions. First, what trading strategies do large agents employ in order to minimize their price impact? Second, what influences market liquidity? Third, how well does the market perform its basic function of matching supply and demand, i.e. how small is the fraction of the gains from trade lost because of strategic behavior?

We consider a discrete-time, infinite-horizon economy with a consumption good and two investment opportunities. The first is a riskless technology and the second is a risky stock that pays a random dividend at each period. The only agents in the economy are N infinitely-lived, risk-averse large traders. For simplicity, no trades are motivated by inside information, i.e. dividend information is public. Instead, agents receive random stock endowments at each period, and trade to share dividend risk. Trades have a price impact because there is a finite number of risk-averse agents. An agent's endowment is private information. Trade in the stock takes place at each period and is organized as a Walrasian auction where agents submit demand functions.

Our assumptions fit particularly well inter-dealer markets, i.e. markets where dealers trade to share the risk of the inventories they accumulate from their customers. Inter-dealer markets are very active for government bonds, foreign exchange, and London Stock Exchange and Nasdaq stocks.[4] Dealers are large and generally are the only participants in these markets. Their endowments are the trades they receive from their customers, and are private information. Finally, in many inter-dealer markets trade is centralized and conducted through a limit-order book

This paper is related to three different literatures: the market microstructure literature, the literature on durable goods monopoly, and the literature on double auctions. The relation to the market microstructure literature is through the asset trading process. Starting with Glosten and Milgrom (1985) and Kyle (1985), the market microstructure literature has mainly focused on how inside information is revealed through the trading process. The literature carefully models insiders' strategic behavior but takes as exogenous the behavior of agents who trade for reasons other than inside information.[5] This paper focuses instead on the strategic behavior of these agents. Our large traders are similar to insiders to some extent, since they are trying to minimize price impact. However, the "noise" traders, who are essential for trade in an insider model, are not present in our model.

This paper is also closely related to the literature on durable goods monopoly.[6] Large traders as well as durable goods monopolists delay trades in order to practice price discrimination. In both cases there is a welfare loss due to delaying trades: in this paper dividend risk is not optimally shared while in the durable goods monopoly literature future gains are discounted. The main difference from most of this literature, where the monopolist's cost is public information, is that agents' endowments are private information.[7] Moreover, by assuming that private information is continuously generated over time (since agents receive endowments at each period), this paper differs from previous models with private information.[8]

Finally, this paper is related to the double auctions literature since it studies how strategic behavior and welfare loss depend on the size of the market. Most of this literature considers static models with risk-neutral agents who have 0-1 demands.[9] Instead, we consider a more realistic (in a financial market framework) dynamic model where agents are risk-averse and have multi-unit demands.

Our results are the following. First, agents trade slowly even when the time between trades, h, goes to 0, i.e. even when there are many trading opportunities. The increase in trading opportunities has a direct and an indirect effect on agents' behavior. To determine the direct effect, we assume that price impact stays constant. With more trading opportunities, agents have more flexibility. They break their trades into many small trades and, at the same time, complete their trades very quickly. Price impact increases, however, since a trade signals many more trades in the same direction. The increase in price impact induces agents to trade more slowly, which further increases price impact, and so on. Agents trade slowly because of this indirect effect. Our result is contrary to the Coase conjecture studied in the durable goods monopoly literature. To compare our result to that literature, we study the case where agents' endowments are public information. Consistent with the Coase conjecture, agents trade very quickly as h goes to 0.

We next study the welfare loss due to strategic behavior. Our second result is that welfare loss increases as the time between trades, h, decreases. Therefore, welfare loss is maximum in the limit when h goes to 0. Our result implies that in the presence of private information, dynamic competitive and non-competitive models differ more than their static counterparts.

We finally study how quickly the market becomes competitive, i.e. how quickly welfare loss goes to 0, as the number of agents, N, grows. Our third result is that welfare loss is of order $1/N^2$ for a fixed h, but of order $1/N$ in the limit when h goes to 0. Therefore, in the presence of private information, dynamic non-competitive models become competitive more slowly than their static counterparts. The $1/N^2$ result was also obtained in the static models of the double auctions literature.

A practical implication of our results is that a switch from a discrete call market to a continuous market may reduce liquidity and not substantially increase welfare.[10] Our results also suggest that if dealers' trades in the customer market are disclosed immediately, trading in the inter-dealer market will be more efficient.

The rest of the paper is structured as follows. In section 2 we present the model. In section 3 we study the benchmark case where agents behave competitively and take prices as given. In sections 4 and 5 we assume that agents behave strategically. In section 4 we study the case where agents' endowments are private information, and in section 5 we study the case where endowments are public information. Section 6 presents the welfare analysis and section 7 contains some concluding remarks. All proofs are in the appendix.

2 The Model

Time is continuous and goes from 0 to ∞. Activity takes place at times ℓh, where $\ell = 0, 1, 2, ..$ and $h > 0$. We refer to time ℓh as period ℓ. There is a consumption good and two investment opportunities. The first investment opportunity is a riskless technology with a continuously compounded rate of return r. One unit of the consumption good invested in this technology at period $\ell - 1$ returns e^{rh} units at period ℓ. The second investment opportunity is a risky stock that pays a dividend $d_\ell h$ at period ℓ. We set $d_0 = d$ and

$$d_\ell = d_{\ell-1} + \delta_\ell. \tag{2.1}$$

The dividend shock δ_ℓ is independent of $\delta_{\ell'}$ for $\ell \neq \ell'$, i.e. dividends follow a random walk, and is normal with mean 0 and variance $\sigma^2 h$. All agents learn δ_ℓ at period ℓ, i.e. dividend information is public.

There are N infinitely-lived agents. Agent i consumes $c_{i,\ell} h$ at period ℓ. His utility over consumption is exponential with coefficient of absolute risk-aversion α and discount rate β, i.e.

$$-h \sum_{\ell=0}^{\infty} exp(-\alpha c_{i,\ell} - \beta \ell h). \tag{2.2}$$

Agent i is endowed with M units of the consumption good and e shares of the stock at period 0. At period ℓ, $\ell \geq 1$, he is endowed with $\epsilon_{i,\ell}$ shares of the stock and

$$-d_\ell \frac{h}{1 - e^{-rh}} \epsilon_{i,\ell}$$

units of the consumption good. The consumption good endowment is the negative of the present value of expected dividends, $d_\ell h/(1 - e^{-rh})$, times the stock endowment, $\epsilon_{i,\ell}$.[11] The stock endowment, $\epsilon_{i,\ell}$, is independent of $\epsilon_{i',\ell'}$ for $i \neq i'$ or $\ell \neq \ell'$, independent of $\delta_{\ell'}$, and normal with mean 0 and variance $\sigma_e^2 h$. We generally assume that $\epsilon_{i,\ell}$ is private information to agent i and is revealed to him at period ℓ. In section 5 we study the public information case where all agents learn $\epsilon_{i,\ell}$ at period ℓ.

Trade in the stock takes place at each period $\ell \geq 1$. The trading mechanism is a Walrasian auction as in Kyle (1989). Agents submit demands that are continuous functions of the price p_ℓ. The market-clearing price is then found and all trades take place at this price. If there are many market-clearing prices, the price with minimum absolute value is selected (if there are ties, the positive price is selected). If there is no market-clearing price, there is either positive excess demand at all prices or negative excess demand at all prices, since demands are continuous. In the former case the price is ∞ and all buyers receive negatively infinite utility, while in the latter case the price is $-\infty$ and all sellers receive negatively infinite utility.

The sequence of events at period ℓ is as follows. First, agents receive their endowments and learn the dividend shock, δ_ℓ. Next, trade takes place. Then, the stock pays the dividend, $d_\ell h$, and finally agents consume $c_{i,\ell}h$. We denote by $M_{i,\ell}$ and $e_{i,\ell}$ the units of the consumption good and shares of the stock that agent i holds at period ℓ, after trade takes place and before the dividend is paid. We allow $M_{i,\ell}$ and $e_{i,\ell}$ to negative, interpreting them as short positions. We denote by $x_{i,\ell}(p_\ell)$ the demand of agent i at period ℓ. We only make explicit the dependence of $x_{i,\ell}(p_\ell)$ on the period ℓ price, p_ℓ. However, $x_{i,\ell}(p_\ell)$ can depend on all other information available to agent i at period ℓ, i.e. $\delta_{\ell'}$ for $\ell' \leq \ell$, $p_{\ell'}$ for $\ell' < \ell$, $\epsilon_{i,\ell'}$ for $\ell' \leq \ell$, and, in the public information case, $\epsilon_{j,\ell'}$ for $j \neq i$ and $\ell' \leq \ell$.

3 The Competitive Case

In this section we study the benchmark case where agents behave competitively and take prices as given. We first define candidate demands and deduce market-clearing prices. We then provide conditions for demands to be optimal given the prices.

3.1 Candidate Demands and Prices

The demand of agent i at period ℓ is

$$x_{i,\ell}(p_\ell) = Ad_\ell - Bp_\ell - a(e_{i,\ell-1} + \epsilon_{i,\ell}). \tag{3.1}$$

Demand, $x_{i,\ell}(p_\ell)$, is a linear function of the dividend, d_ℓ, the price, p_ℓ, and agent i's stock holdings before trade at period ℓ. Stock holdings are the sum of stock holdings after trade at period $\ell - 1$, $e_{i,\ell-1}$, and of the stock endowment at period ℓ, $\epsilon_{i,\ell}$. The parameters A, B, and a are determined in section 3.2. The market-clearing condition,

$$\sum_{i=1}^{N} x_{i,\ell}(p_\ell) = 0, \tag{3.2}$$

and the definition of $x_{i,\ell}(p_\ell)$ imply that the price at period ℓ is

$$p_\ell = \frac{A}{B}d_\ell - \frac{a}{B}\frac{\sum_{i=1}^{N}(e_{i,\ell-1} + \epsilon_{i,\ell})}{N}. \tag{3.3}$$

The price, p_ℓ, is a linear function of the dividend, d_ℓ, and of average stock holdings, $\sum_{i=1}^{N}(e_{i,\ell-1} + \epsilon_{i,\ell})/N$.

The stock holdings of agent i after trade at period ℓ are

$$e_{i,\ell} = (e_{i,\ell-1} + \epsilon_{i,\ell}) + x_{i,\ell}(p_\ell), \tag{3.4}$$

i.e. are the sum of stock holdings before trade, $e_{i,\ell-1} + \epsilon_{i,\ell}$, and of the trade, $x_{i,\ell}(p_\ell)$. (p_ℓ now denotes the market-clearing price, 3.3.) To determine $x_{i,\ell}(p_\ell)$, we plug p_ℓ back in the demand 3.1 and get

$$x_{i,\ell}(p_\ell) = a\left(\frac{\sum_{j=1}^{N}(e_{j,\ell-1} + \epsilon_{j,\ell})}{N} - (e_{i,\ell-1} + \epsilon_{i,\ell})\right). \tag{3.5}$$

Equations 3.4 and 3.5 imply that

$$e_{i,\ell} = (1-a)(e_{i,\ell-1} + \epsilon_{i,\ell}) + a\frac{\sum_{j=1}^{N}(e_{j,\ell-1} + \epsilon_{j,\ell})}{N}. \tag{3.6}$$

Stock holdings after trade are a weighted average of stock holdings before trade and average stock holdings. Trade thus reduces the dispersion in stock holdings. The parameter a measures the speed at which disperse stock holdings become identical, and dividend risk is optimally shared. In section 3.2 we show that a is equal to 1. In the competitive case stock holdings become identical after one trading round.

3.2 Demands are Optimal

Our candidate demands and prices were defined as to satisfy the market-clearing condition. To show that they constitute a competitive equilibrium, we only need to show that demands are optimal given the prices. In this section we study agents' optimization problem and provide conditions for demands to be optimal. The conditions are on the parameters A, B, and a.

We formulate agent i's optimization problem as a dynamic programming problem. The "state" at period ℓ is evaluated after trade takes place and before the stock pays the dividend. There are four state variables: the agent's consumption good holdings, $M_{i,\ell}$, the dividend, d_ℓ, the agent's stock holdings, $e_{i,\ell}$, and the average stock holdings, $\sum_{j=1}^N e_{j,\ell}/N$. There are two control variables chosen between the state at period $\ell - 1$ and the state at period ℓ: the consumption, $c_{i,\ell-1}$, and the demand, $x_{i,\ell}(p_\ell)$. The dynamics of $M_{i,\ell}$ are given by the budget constraint

$$M_{i,\ell} = e^{rh}(M_{i,\ell-1} + d_{\ell-1}e_{i,\ell-1}h - c_{i,\ell-1}h) - d_\ell \frac{h}{1-e^{-rh}}\epsilon_{i,\ell} - p_\ell x_{i,\ell}(p_\ell). \tag{3.7}$$

The agent behaves competitively and takes prices as given. Therefore the price, p_ℓ, in the budget constraint is independent of the agent's demand and given by equation 3.3. The dynamics of d_ℓ and $e_{i,\ell}$ are given by equations 2.1 and 3.4, respectively. (Equation 3.6 gives the dynamics of $e_{i,\ell}$ only when the agent submits his equilibrium demand.) Finally, the dynamics of $\sum_{j=1}^N e_{j,\ell}/N$ are given by

$$\frac{\sum_{j=1}^N e_{j,\ell}}{N} = \frac{\sum_{j=1}^N (e_{j,\ell-1} + \epsilon_{j,\ell})}{N}, \tag{3.8}$$

since average stock holdings are equal before and after trade. Note that the agent does not observe other agents' stock holdings directly. However he can infer average stock holdings from the price, p_ℓ, using equation 3.3.

Summarizing, agent i's optimization problem, (P_c), is

$$\sup_{c_{i,\ell}, x_{i,\ell}(p_\ell)} -E_0(h\sum_{\ell=0}^{\infty} exp(-\alpha c_{i,\ell} - \beta \ell h))$$

subject to

$$p_\ell = \frac{A}{B} d_\ell - \frac{a}{B} \frac{\sum_{i=1}^N (e_{i,\ell-1} + \epsilon_{i,\ell})}{N},$$

$$M_{i,\ell} = e^{rh}(M_{i,\ell-1} + d_{\ell-1} e_{i,\ell-1} h - c_{i,\ell-1} h) - d_\ell \frac{h}{1-e^{-rh}} \epsilon_{i,\ell} - p_\ell x_{i,\ell}(p_\ell),$$

$$d_\ell = d_{\ell-1} + \delta_\ell,$$

$$e_{i,\ell} = (e_{i,\ell-1} + \epsilon_{i,\ell}) + x_{i,\ell}(p_\ell),$$

$$\frac{\sum_{j=1}^N e_{j,\ell}}{N} = \frac{\sum_{j=1}^N (e_{j,\ell-1} + \epsilon_{j,\ell})}{N},$$

and the transversality condition

$$\lim_{\ell \to \infty} E_0 V_c \left(M_{i,\ell}, d_\ell, e_{i,\ell}, \frac{\sum_{j=1}^N e_{j,\ell}}{N} \right) exp(-\beta \ell h) = 0, \tag{3.9}$$

where V_c is the value function.[12] Our candidate value function is

$$V_c \left(M_{i,\ell}, d_\ell, e_{i,\ell}, \frac{\sum_{j=1}^N e_{j,\ell}}{N} \right) = -exp(-\alpha(\frac{1-e^{-rh}}{h} M_{i,\ell} + d_\ell e_{i,\ell} + F(Q, \begin{pmatrix} e_{i,\ell} \\ \frac{\sum_{j=1}^N e_{j,\ell}}{N} \end{pmatrix}) + q)). \tag{3.10}$$

In expression 3.10, $F(Q, \mathbf{v}) = (1/2)\mathbf{v}^t Q \mathbf{v}$ for a matrix Q and a vector $\mathbf{v}$ ($\mathbf{v}^t$ is the transpose of $\mathbf{v}$), Q is a symmetric 2×2 matrix, and q is a constant.

In proposition A.1, proven in appendix A, we provide sufficient conditions for the demand 3.1 to solve (P_c), and for the function 3.10 to be the value function. The conditions are on the parameters A, B, a, Q, and q, and are derived from the Bellman equation

$$V_c \left(M_{i,\ell-1}, d_{\ell-1}, e_{i,\ell-1}, \frac{\sum_{j=1}^N e_{j,\ell-1}}{N} \right) =$$

$$\sup_{c_{i,\ell-1}, x_{i,\ell}(p_\ell)} \left\{ -exp(-\alpha c_{i,\ell-1})h + E_{\ell-1} V_c \left(M_{i,\ell}, d_\ell, e_{i,\ell}, \frac{\sum_{j=1}^N e_{j,\ell}}{N} \right) exp(-\beta h) \right\}. \tag{3.11}$$

There are two sets of conditions, the "optimality conditions" and the "Bellman conditions". To derive the optimality conditions, we write that the demand 3.1 maximizes the RHS of the Bellman equation. To derive the Bellman conditions, we write that the value function 3.10 solves the Bellman equation.

We now derive heuristically the optimality conditions and later use them to compare the competitive case to the private and public information cases. There are 3 optimality conditions. The first optimality condition concerns A/B, the sensitivity of price to the dividend. To derive this condition, we set $e_{i,\ell-1} + \epsilon_{i,\ell}$ and $\sum_{j=1}^{N}(e_{j,\ell-1} + \epsilon_{j,\ell})/N$ to 0, and d_ℓ to 1. If agent i submits the demand 3.1, his trade is 0 by equation 3.5. Suppose that he modifies his demand and buys Δx shares. His holdings of the consumption good decrease by $p_\ell \Delta x$, which is $(A/B)\Delta x$ by equation 3.3. His stock holdings become Δx, while average stock holdings remain equal to 0. Equation 3.10 implies that the value function does not change in the first-order in Δx if

$$-\frac{1-e^{-rh}}{h}\frac{A}{B} + 1 = 0. \tag{3.12}$$

Therefore, A/B, the sensitivity of price to the dividend, is $h/(1-e^{-rh})$, the present value of expected dividends.

The second optimality condition concerns a/B, the sensitivity of price to average stock holdings. To derive this condition, we set d_ℓ to 0, and $e_{i,\ell-1} + \epsilon_{i,\ell}$ and $\sum_{j=1}^{N}(e_{j,\ell-1} + \epsilon_{j,\ell})/N$ to 1. If agent i submits the demand 3.1, his trade is 0 by equation 3.5. Suppose that he modifies his demand and buys Δx shares. His holdings of the consumption good decrease by $p_\ell \Delta x$, which is $-(a/B)\Delta x$ by equation 3.3. His stock holdings become Δx, while average stock holdings remain equal to 1. The value function does not change in the first-order if

$$\frac{1-e^{-rh}}{h}\frac{a}{B} + Q_{1,1} + Q_{1,2} = 0. \tag{3.13}$$

($Q_{i,j}$ denotes the i,j'th element of the matrix Q.)

The third optimality condition concerns the speed of trade, a. To derive this condition, we set d_ℓ and $\sum_{j=1}^{N}(e_{j,\ell-1} + \epsilon_{j,\ell})/N$ to 0, and $e_{i,\ell-1} + \epsilon_{i,\ell}$ to 1. Agent i thus needs to sell one share to the other agents. If he submits the demand 3.1, he sells a shares. Suppose that he modifies his demand and sells Δx fewer shares. His holdings of the consumption good do not change since $p_\ell = 0$. His stock holdings become $1 - a + \Delta x$, while average stock holdings remain equal to 0. The value function does not change in the first-order if

$$(1-a)Q_{1,1} = 0. \tag{3.14}$$

Equation 3.14 implies that $a = 1$. In the competitive case the agent equates his marginal valuation to the price, and sells the one share immediately. His stock holdings become equal to average stock holdings, and dividend risk is optimally shared.

To derive the Bellman conditions we need to compute the expectation of the value function. The value function is the exponential of a quadratic function of a normal vector, and its expectation is complicated. Therefore, the Bellman conditions are complicated as well. In appendix A we show that the optimality conditions and the Bellman conditions can be reduced to a system of 3 non-linear equations in the 3 elements of the symmetric 2×2 matrix Q.

The Bellman conditions simplify dramatically in the limit when endowment risk, σ_e^2, goes to 0. In proposition A.2, proven in appendix A, we solve the non-linear system in closed-form for $\sigma_e^2 = 0$ and use the implicit function theorem to extend the solution for small σ_e^2. The Bellman conditions have a very intuitive interpretation for $\sigma_e^2 = 0$. To illustrate this interpretation, we determine $Q_{1,1} + Q_{1,2}$ and $Q_{1,1}$ using the Bellman conditions. The expressions for $Q_{1,1} + Q_{1,2}$ and $Q_{1,1}$ will also be valid in the private and public information cases. In the next two sections we will combine these expressions with the optimality conditions, to study market liquidity and the speed of trade.

Adding up the Bellman conditions for $Q_{1,1}$ and $Q_{1,2}$, i.e. equations A.23 and A.25, we get

$$Q_{1,1} + Q_{1,2} = (-\alpha\sigma^2 h + (Q_{1,1} + Q_{1,2}))e^{-rh}. \tag{3.15}$$

The LHS, $Q_{1,1} + Q_{1,2}$, represents agent i's marginal benefit of holding Δx shares at period ℓ, when $d_\ell = 0$ and $e_{i,\ell} = \sum_{j=1}^N e_{j,\ell} = 1$. This marginal benefit is the sum of two terms. The first term, $-\alpha\sigma^2 he^{-rh}$, represents the marginal benefit of holding Δx shares between periods ℓ and $\ell + 1$. This term is in fact a marginal cost, due to dividend risk. The marginal cost is increasing in the coefficient of absolute risk-aversion, α, and in the dividend risk, σ^2. The second term, $(Q_{1,1} + Q_{1,2})e^{-rh}$, represents the marginal benefit of holding Δx shares at period $\ell + 1$. It is the same as the marginal benefit at period ℓ (except for discounting) because $d_{\ell+1}$, $e_{i,\ell+1}$, and $\sum_{j=1}^N e_{j,\ell+1}$ are equal in expectation to their period ℓ values. The Bellman conditions simplify for $\sigma_e^2 = 0$ because the marginal benefit between periods ℓ and $\ell + 1$ is only due to dividend risk. For $\sigma_e^2 > 0$, the marginal benefit is also due to endowment risk and is a complicated function of Q. Equation 3.15 implies that

$$Q_{1,1} + Q_{1,2} = -\frac{\alpha\sigma^2 h e^{-rh}}{1 - e^{-rh}}. \tag{3.16}$$

Therefore, $Q_{1,1} + Q_{1,2}$ is simply a present value of marginal costs due to dividend risk.

The Bellman equation for $Q_{1,1}$ is equation A.23, i.e.

$$Q_{1,1} = (-\alpha\sigma^2 h + (1-a)^2 Q_{1,1})e^{-rh}. \tag{3.17}$$

The LHS, $Q_{1,1}$, represents the marginal benefit of holding Δx shares at period ℓ, when $d_\ell = \sum_{j=1}^N e_{j,\ell} = 0$ and $e_{i,\ell} = 1$. This marginal benefit is the sum of two terms. The first term represents the marginal benefit of holding Δx shares between periods ℓ and $\ell + 1$. The second term represents the marginal benefit of holding $(1-a)\Delta x$ shares at period $\ell + 1$.[13] This marginal benefit is obtained from the marginal benefit at period ℓ by discounting and multiplying by $(1-a)^2$. The "first" $1-a$ comes because $e_{i,\ell+1}$ is $1-a$ instead of 1 (in expectation) and the "second" because we consider $(1-a)\Delta x$ shares instead of Δx. Equation 3.17 implies that

$$Q_{1,1} = -\frac{\alpha\sigma^2 h e^{-rh}}{1 - (1-a)^2 e^{-rh}}. \tag{3.18}$$

Therefore, $Q_{1,1}$ is a present value of marginal costs due to dividend risk, exactly as $Q_{1,1} + Q_{1,1}$. However, the discount rate is higher than r, and incorporates the parameter a that measures the speed of trade. This is because for $Q_{1,1} + Q_{1,2}$ the agent expects to hold one share forever, while for $Q_{1,1}$ the agent expects to sell the share over time.

4 The Private Information Case

In this section we study the case where agents behave strategically. We refer to this case as the private information case in order to contrast it to the case studied in the next section, where agents behave strategically and where endowments are public information. We first construct candidate demands, and provide conditions for these demands to constitute a Nash equilibrium. We then study market liquidity and the speed of trade.

4.1 Demands

The demand of agent i at period ℓ is

$$x_{i,\ell}(p_\ell) = Ad_\ell - Bp_\ell - a(e_{i,\ell-1} + \epsilon_{i,\ell}). \tag{4.1}$$

Demand is given by the same expression as in the competitive case. Therefore the price, the trade, and stock holdings after trade are also given by the same expressions. The parameters A, B, and a will however be different. In particular, the parameter a, that measures the speed of trade, will be smaller than 1. Note that agent i's demand does not depend on his expectation of other agents' stock holdings, which is $\sum_{j\neq i} e_{j,\ell-1}$ since stock holdings are $\sum_{j\neq i}(e_{j,\ell-1}+\epsilon_{j,\ell})$.[14] In fact, if we introduce a term in $\sum_{j\neq i} e_{j,\ell-1}$ in the demand 4.1, we will find that its coefficient is 0. This is surprising: if $\sum_{j\neq i} e_{j,\ell-1}$ increases, agent i expects larger future sales from the other agents. His demand at period ℓ should thus decrease, holding price constant. Demand does not change because "holding price constant" means that the period ℓ stock endowments, $\sum_{j\neq i} \epsilon_{j,\ell}$, are such that stock holdings, $\sum_{j\neq i}(e_{j,\ell-1} + \epsilon_{j,\ell})$, remain constant.

Our candidate demands constitute a Nash equilibrium if it is optimal for agent i to submit his candidate demand when all other agents submit their candidate demands.[15] We now study agent i's optimization problem and provide conditions for the demand 4.1 to be optimal. The only difference between this optimization problem and the optimization problem in the competitive case, concerns the price, p_ℓ. In the competitive case p_ℓ is independent of the agent's demand and given by equation 3.3. In the private information case p_ℓ is given by the market-clearing condition

$$\sum_{j\neq i}(Ad_\ell - Bp_\ell - a(e_{j,\ell-1}+\epsilon_{j,\ell})) + x_{i,\ell}(p_\ell) = 0. \tag{4.2}$$

The price in the private information case coincides with the price in the competitive case only when agent i submits his equilibrium demand.

Agent i's optimization problem, (P_{pr}), is

$$\sup_{c_{i,\ell}, x_{i,\ell}(p_\ell)} -E_0(h\sum_{\ell=0}^{\infty} exp(-\alpha c_{i,\ell} - \beta \ell h))$$

subject to

$$\sum_{j\neq i}(Ad_\ell - Bp_\ell - a(e_{j,\ell-1}+\epsilon_{j,\ell})) + x_{i,\ell}(p_\ell) = 0,$$

$$M_{i,\ell} = e^{rh}(M_{i,\ell-1} + d_{\ell-1}e_{i,\ell-1}h - c_{i,\ell-1}h) - d_\ell \frac{h}{1-e^{-rh}}\epsilon_{i,\ell} - p_\ell x_{i,\ell}(p_\ell),$$

$$d_\ell = d_{\ell-1} + \delta_\ell,$$

$$e_{i,\ell} = (e_{i,\ell-1}+\epsilon_{i,\ell}) + x_{i,\ell}(p_\ell),$$
$$\frac{\sum_{j=1}^N e_{j,\ell}}{N} = \frac{\sum_{j=1}^N (e_{j,\ell-1}+\epsilon_{j,\ell})}{N},$$

and the transversality condition

$$\lim_{\ell\to\infty} E_0 V_{pr}\left(M_{i,\ell}, d_\ell, e_{i,\ell}, \frac{\sum_{j=1}^N e_{j,\ell}}{N}\right) exp(-\beta \ell h) = 0,$$

where V_{pr} is the value function. Our candidate value function is

$$V_{pr}\left(M_{i,\ell}, d_\ell, e_{i,\ell}, \frac{\sum_{j=1}^N e_{j,\ell}}{N}\right) = -exp(-\alpha(\frac{1-e^{-rh}}{h}M_{i,\ell} + d_\ell e_{i,\ell} + F(Q, \begin{pmatrix} e_{i,\ell} \\ \frac{\sum_{j=1}^N e_{j,\ell}}{N}\end{pmatrix}) + q)). \tag{4.3}$$

The value function is given by the same expression as in the competitive case. The parameters Q and q will however be different. In proposition B.1, proven in appendix B, we provide sufficient conditions for the demand 4.1 to solve (P_{pr}), and for the function 4.3 to be the value function. These conditions are the optimality conditions and the Bellman conditions.

The first two optimality conditions are the same as in the competitive case, namely

$$-\frac{1-e^{-rh}}{h}\frac{A}{B} + 1 = 0 \tag{4.4}$$

and

$$\frac{1-e^{-rh}}{h}\frac{a}{B}+Q_{1,1}+Q_{1,2}=0. \tag{4.5}$$

The conditions are the same as in the competitive case because they are derived for $e_{i,\ell-1}+\epsilon_{i,\ell}$ equal to $\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})/N$, and equal to 0 or 1. In both cases agent i does not trade if he submits the demand 4.1. Therefore, if he modifies his demand, the first-order change in the value function will be independent of whether the price changes or not. To derive the third optimality condition, we set d_ℓ and $\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})/N$ to 0, and $e_{i,\ell-1}+\epsilon_{i,\ell}$ to 1. Agent i thus needs to sell one share to the other agents. If he submits the demand 4.1, he sells a shares. Suppose that he modifies his demand and sells Δx fewer shares. The market-clearing condition 4.2 implies that p_ℓ increases by $\Delta x/(N-1)B$, and thus becomes $\Delta x/(N-1)B$ instead of 0. Therefore the agent's holdings of the consumption good increase by $(\Delta x/(N-1)B)(a-\Delta x)$. His stock holdings become $1-a+\Delta x$, while average stock holdings remain equal to 0. The value function does not change in the first-order if

$$\frac{1-e^{-rh}}{h}\frac{a}{(N-1)B}+(1-a)Q_{1,1}=0. \tag{4.6}$$

Condition 4.6 differs from condition 3.14 in the competitive case, because of the first term. This term represents the price improvement from trading more slowly. Because of this term, a will be smaller than 1.

The Bellman conditions are the same as in the competitive case. In appendix B we show that the optimality conditions and the Bellman conditions can be reduced to a system of 4 non-linear equations in a and in the 3 elements of the symmetric 2×2 matrix Q. In proposition B.2, proven in appendix B, we solve the non-linear system in closed-form for $\sigma_e^2=0$ and use the implicit function theorem to extend the solution for small σ_e^2.

4.2 Market Liquidity and the Speed of Trade

4.2.1 Market Liquidity

Before defining market liquidity, we study a/B, the sensitivity of price to average stock holdings. Combining the optimality condition 4.5 with the Bellman condition

3.16, we get

$$\frac{1-e^{-rh}}{h}\frac{a}{B} = \frac{\alpha\sigma^2 he^{-rh}}{1-e^{-rh}}. \tag{4.7}$$

To provide some intuition for equation 4.7, we set d_ℓ to 0, and $e_{i,\ell-1} + \epsilon_{i,\ell}$ and $\sum_{j=1}^{N}(e_{j,\ell-1} + \epsilon_{j,\ell})/N$ to 1. Agent i thus does not trade, and expects to hold one share forever. Since he does not trade, the price is equal to his marginal valuation. The LHS of equation 4.7 corresponds to the price, which is $-a/B$. The RHS corresponds to the marginal valuation, which is $-\alpha\sigma^2 he^{-rh}/(1-e^{-rh})$, i.e. is a present value of marginal costs due to dividend risk.

To define market liquidity, we assume that an agent deviates from his equilibrium strategy and sells one more share. Market liquidity is the inverse of the price impact, i.e. of the change in price. The market-clearing condition 4.2 implies that if agent i sells one more share, the price decreases by $1/(N-1)B$. Therefore, price impact is $1/(N-1)B$. Note that price impact is equal to the sensitivity of price to average stock holdings, a/B, divided by $a(N-1)$. This is because when agent i deviates from his equilibrium strategy and sells one more share, the other agents incorrectly infer that he did not deviate but received an endowment shock higher by $N/(a(N-1))$. Indeed, they do not observe agent i's endowment and buy $1/(N-1)$ shares in both cases. In the second case, price impact is the product of the increase in average stock holdings, $1/(a(N-1))$, times the price sensitivity, a/B. Equation 4.7 implies that price impact is

$$\frac{1}{(N-1)B} = \frac{1}{a(N-1)}\frac{\alpha\sigma^2 h^2 e^{-rh}}{(1-e^{-rh})^2}. \tag{4.8}$$

Note that price impact is large when a is small, since a sale of one share signals that many more shares will follow. As the time between trades, h, goes to 0, a will go to 0. Since price sensitivity goes to a strictly positive limit, price impact will go to ∞ and market liquidity to 0.

4.2.2 Speed of Trade

We now study the parameter a that measures the speed of trade. Combining the optimality condition 4.6 with the Bellman condition 3.18, we get

$$\frac{1-e^{-rh}}{h}\frac{a}{(N-1)B} = (1-a)\frac{\alpha\sigma^2 he^{-rh}}{1-(1-a)^2 e^{-rh}}. \tag{4.9}$$

To provide some intuition for equation 4.9, we set d_ℓ and $\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})/N$ to 0 and $e_{i,\ell-1}+\epsilon_{i,\ell}$ to 1. Agent i thus needs to sell one share to the other agents. At period ℓ he sells a shares. If he sells fewer shares, he trades off an increase in the price at period ℓ with an increase in dividend risk at future periods. The LHS of equation 4.9 represents the price improvement. It is proportional to the price impact, $1/(N-1)B$, and the trade, a. The RHS represents the increase in dividend risk and is a present value of marginal costs. Using equation 4.7 to eliminate B, we get

$$a^2 e^{-rh} - a(2e^{-rh} + (N-1)(1-e^{-rh})) + (N-2)(1-e^{-rh}) = 0. \tag{4.10}$$

Equation 4.10 gives a as a function of the number of agents, N, and the time between trades, h, for $\sigma_e^2 = 0$. To study how a depends on N and h, we use equation 4.10 for $\sigma_e^2 = 0$ and extend our results by continuity for small σ_e^2. In proposition 4.1, proven in appendix B, we study how a depends on N and h.

Proposition 4.1 *For small σ_e^2, a increases in N and h.*

The intuition for proposition 4.1 is the following. If N is large, the price improvement obtained by delaying trades is small. If h is large, there are few trading opportunities. Therefore, the increase in dividend risk from delaying trades is large. We finally study a for two limit values of h. The first value is ∞ and corresponds to the benchmark static case. In the static case a is $(N-2)/(N-1)$.[16] The second value is 0 and corresponds to the continuous-time case. In proposition 4.2 we show that a is of order h and thus goes to 0 as h goes to 0. Proposition 4.2 is proven in appendix B.

Proposition 4.2 *For small σ_e^2, a/h goes to $\bar{a} > 0$ as h goes to 0.*

To understand the implications of proposition 4.2 for the speed of trade, assume that at calendar time t agent i holds one share and average stock holdings are 0. The agent thus needs to sell one share, and sells a shares at time t. At time $t+h$ he needs to sell $1-a$ shares, and sells $a(1-a)$ shares. At time $t+2h$ he needs to sell $(1-a)^2$ shares, and so on. Therefore, at calendar time t' such that $t'-t$ is a multiple of h, the agent needs to sell $(1-a)^{\frac{t'-t}{h}}$ shares. Since a/h goes to $\bar{a} > 0$ as h goes to 0, $(1-a)^{\frac{t'-t}{h}}$ goes to $e^{-\bar{a}(t'-t)} \in (0,1)$. The agent thus sells slowly.

The result of proposition 4.2 is somewhat surprising since it implies that agents trade slowly even when there are many trading opportunities. To provide some intuition for this result, we assume that h goes to 0, and proceed in two steps. First, we assume that price impact, $1/(N-1)B$, stays constant, and determine the direct effect of h on a. Second, we take into account the change in price impact, and determine the indirect effect.

Equation 4.9 implies that if price impact stays constant, a is of order $\sqrt{h}$. Therefore, both a and $(1-a)^{\frac{t'-t}{h}}$ go to 0 as h goes to 0. Since a goes to 0, agents break their trades into many small trades. At the same time, since $(1-a)^{\frac{t'-t}{h}}$ goes to 0, agents complete their trades very quickly. The intuition is that with constant price impact, agents' problem is similar to trading against an exogenous, downward-sloping demand curve. As h goes to 0, agents can "go down" the demand curve very quickly, achieving perfect price-discrimination and bearing little dividend risk. Note that both the LHS and the RHS of equation 4.9 are of order $\sqrt{h}$, and thus go to 0 as h goes to 0. The LHS represents the price improvement obtained by delaying. Since agents break their trades into many small trades, the price improvement concerns a small trade and goes to 0. The RHS represents the increase in dividend risk from delaying. It goes to 0 since agents complete their trades very quickly and bear little dividend risk.

Equation 4.8 implies that as h and a go to 0, price impact does not stay constant, but goes to ∞. The intuition is that a trade signals many more trades in the same direction. The increase in price impact implies a decrease in a, which implies a further increase in price impact, and so on. To determine this indirect effect of h on a, we note that the LHS of equation 4.9 does not go to 0 as h goes to 0. Indeed, the price improvement obtained by delaying is proportional to the trade, a, which is small, and the price impact, $1/(N-1)B$, which is large. The product of the trade and the price impact is in turn proportional to the price sensitivity, a/B, which goes to a strictly positive limit. Since the LHS does not go to 0, the RHS does not go to 0. The increase in dividend risk from delaying does not go to 0 only when agents trade slowly and a is of order h.

The result of proposition 4.2 is contrary to the Coase conjecture studied in the durable goods monopoly literature. According to the Coase conjecture, a durable goods monopolist sells very quickly as the time between trades goes to 0. Our result is contrary to the Coase conjecture because endowments are private information. With private information, price impact is larger than with public information, and agents have an incentive to trade slowly. Price impact is larger because if an agent sells more shares, the other agents incorrectly infer that he received a higher endowment shock. Therefore, they expect larger sales in the future. In next section we study the case where endowments are public information. We show that, as h goes to 0, price impact goes to 0 and agents trade very quickly.

5 The Public Information Case

In this section we study the case where agents behave strategically and where endowments are public information. This case serves as a useful benchmark, allowing us to compare our results to the durable goods monopoly literature. In the inter-dealer market interpretation of the model, the public information case corresponds to mandatory disclosure of trades that the dealers receive from their customers. We first construct candidate demands and provide conditions for these demands to constitute a Nash equilibrium. We show that there exists a continuum of Nash equilibria and select one using a trembling-hand type refinement. We then study market liquidity and the speed of trade.

5.1 Demands

The demand of agent i at period ℓ is

$$x_{i,\ell}(p_\ell) = Ad_\ell - Bp_\ell - A_e \frac{\sum_{j=1}^{N}(e_{j,\ell-1} + \epsilon_{j,\ell})}{N} - a(e_{i,\ell-1} + \epsilon_{i,\ell}). \tag{5.1}$$

The only difference with the private information case is that agent i's demand depends on other agents' stock holdings. We will give some intuition for this result later. The price at period ℓ is

$$p_\ell = \frac{A}{B} d_\ell - \frac{A_e + a}{B} \frac{\sum_{i=1}^{N}(e_{i,\ell-1} + \epsilon_{i,\ell})}{N}. \tag{5.2}$$

The only difference with the private information case is that the sensitivity of price to average stock holdings is $(A_e + a)/B$ and not a/B. Agent i's trade and stock holdings after trade are given by the same expressions as in the private information case.

Our candidate demands constitute a Nash equilibrium if it is optimal for agent i to submit his candidate demand when all other agents submit their candidate demands. We now study agent i's optimization problem and provide conditions for the demand 5.1 to be optimal. There are two differences between this optimization problem and the optimization problem in the private information case. First, agent i's demand, $x_{i,\ell}(p_\ell)$, can depend on other agents' stock holdings, $e_{j,\ell-1} + \epsilon_{j,\ell}$ for $j \neq i$, since these are public information. Second, the market-clearing condition is

$$\sum_{j \neq i} \left(Ad_\ell - Bp_\ell - A_e \frac{\sum_{k=1}^{N}(e_{k,\ell-1} + \epsilon_{k,\ell})}{N} - a(e_{j,\ell-1} + \epsilon_{j,\ell})\right) + x_{i,\ell}(p_\ell) = 0, \tag{5.3}$$

instead of 4.2, since demands depend on average stock holdings.

Agent i's optimization problem, (P_p), is

$$\sup_{c_{i,\ell}, x_{i,\ell}(p_\ell)} -E_0(h\sum_{\ell=0}^{\infty} exp(-\alpha c_{i,\ell} - \beta \ell h))$$

subject to

$$\sum_{j\neq i}(Ad_\ell - Bp_\ell - A_e\frac{\sum_{k=1}^N(e_{k,\ell-1}+\epsilon_{k,\ell})}{N} - a(e_{j,\ell-1}+\epsilon_{j,\ell})) + x_{i,\ell}(p_\ell) = 0,$$

$$M_{i,\ell} = e^{rh}(M_{i,\ell-1} + d_{\ell-1}e_{i,\ell-1}h - c_{i,\ell-1}h) - d_\ell\frac{h}{1-e^{-rh}}\epsilon_{i,\ell} - p_\ell x_{i,\ell}(p_\ell),$$

$$d_\ell = d_{\ell-1} + \delta_\ell,$$

$$e_{i,\ell} = (e_{i,\ell-1} + \epsilon_{i,\ell}) + x_{i,\ell}(p_\ell),$$

$$\frac{\sum_{j=1}^N e_{j,\ell}}{N} = \frac{\sum_{j=1}^N(e_{j,\ell-1}+\epsilon_{j,\ell})}{N},$$

and the transversality condition

$$\lim_{\ell\to\infty} E_0V_p\left(M_{i,\ell}, d_\ell, e_{i,\ell}, \frac{\sum_{j=1}^N e_{j,\ell}}{N}\right) exp(-\beta\ell h) = 0,$$

where V_p is the value function. Our candidate value function is

$$V_p\left(M_{i,\ell}, d_\ell, e_{i,\ell}, \frac{\sum_{j=1}^N e_{j,\ell}}{N}\right) = -exp(-\alpha(\frac{1-e^{-rh}}{h}M_{i,\ell} + d_\ell e_{i,\ell} + F(Q, \begin{pmatrix} e_{i,\ell} \\ \frac{\sum_{j=1}^N e_{j,\ell}}{N}\end{pmatrix}) + q)). \tag{5.4}$$

The value function is given by the same expression as in the private information case. In proposition C.1, proven in appendix C, we provide sufficient conditions for the demand 5.1 to solve (P_p), and for the function 5.4 to be the value function. These conditions are the optimality conditions and the Bellman conditions. The optimality conditions are

$$-\frac{1-e^{-rh}}{h}\frac{A}{B} + 1 = 0, \tag{5.5}$$

$$\frac{1-e^{-rh}}{h}\frac{A_e + a}{B} + Q_{1,1} + Q_{1,2} = 0, \tag{5.6}$$

and

$$\frac{1-e^{-rh}}{h}\frac{a}{(N-1)B} + (1-a)Q_{1,1} = 0. \tag{5.7}$$

Conditions 5.5 and 5.7 are the same as in the private information case. Condition 5.6 is different, since the sensitivity of price to average stock holdings is $(A_e + a)/B$ and not a/B. The Bellman conditions are the same as in the private information case.

Since there are as many conditions as in the private information case, but there is the additional parameter A_e, there exists a continuum of Nash equilibria. The intuition for the indeterminacy is that in the public information case agents know the market-clearing price, and are indifferent as to the demand they submit for all other prices. Therefore the slope of the demand function, B, is indeterminate.[17] This intuition is the same as in Wilson (1979), Klemperer and Meyer (1989), and Back and Zender (1993).

To select one Nash equilibrium, we use a refinement that has the flavor of trembling-hand perfection in the agent-strategic form.[18] We consider the following "perturbed" game. Each consumption and demand choice of agent i is made by a different "incarnation" of this agent. All incarnations maximize the same utility. They take the choices of the other incarnations, and of the incarnations of the other agents, as given. Consumption is optimally chosen, but demand is $x_{i,\ell}(p_\ell) + u_{i,\ell}$, where $x_{i,\ell}(p_\ell)$ is the optimal demand and $u_{i,\ell}$ is a "tremble". For tractability, $u_{i,\ell}$ is independent of $u_{i',\ell'}$ for $i \neq i'$ or $\ell \neq \ell'$. Trembles are thus independent over agents. Trembles are also independent over time, i.e. over incarnations of an agent, and this is why we need to consider trembling-hand perfection in the agent-strategic form (that involves incarnations) rather than trembling-hand perfection. For tractability, $u_{i,\ell}$ is normal with mean 0 and variance σ_u^2. Since the normal is a continuous distribution, agent i trembles with probability 1. However, behavior has a rational flavor since expected demand is $x_{i,\ell}(p_\ell)$, the optimal demand. A Nash equilibrium of the original game satisfies our refinement if and only if it is the limit of Nash equilibria of the perturbed games as σ_u^2 goes to 0.

In appendix C we show that our refinement selects one Nash equilibrium. The intuition is that an agent does not know the market-clearing price, because of the trembles of the other agents. Therefore, he is not indifferent as to the demand he submits for each price, and the slope of the demand function can be determined. More precisely, our refinement implies an additional optimality condition, that concerns the slope of the demand function. To derive heuristically this condition, we set d_ℓ, $e_{i,\ell-1} + \epsilon_{i,\ell}$, and $\sum_{j=1}^N (e_{j,\ell-1} + \epsilon_{j,\ell})/N$ to 0. We also assume that $u_{i,\ell} = 0$, i.e. agent i does not tremble, and that $\sum_{j=1}^N u_{j,\ell}/N = 1$. If agent i submits his equilibrium demand 5.1, he sells one share to the other agents at price $1/B$. This follows from the market-clearing condition

$$\sum_{j \neq i} (Ad_\ell - Bp_\ell - A_e \frac{\sum_{k=1}^N (e_{k,\ell-1} + \epsilon_{k,\ell})}{N} - a(e_{j,\ell-1} + \epsilon_{j,\ell}) + u_{j,\ell}) + x_{i,\ell}(p_\ell) = 0, \quad (5.8)$$

i.e. condition 5.3 adjusted for the trembles. If agent i modifies his demand and sells Δx fewer shares, the price becomes $1/B + \Delta x/(N-1)B$. The agent's holdings of the consumption good increase by $(1/B + \Delta x/(N-1)B)(1 - \Delta x) - 1/B$. His stock holdings become $-(1 - \Delta x)$, while average stock holdings remain equal to 0. The value function does not change in the first-order if

$$\frac{1 - e^{-rh}}{h} \frac{N-2}{(N-1)B} + Q_{1,1} = 0. \quad (5.9)$$

In appendix C we show that the optimality conditions and the Bellman conditions can be reduced to a system of 3 non-linear equations in the 3 elements of the symmetric 2×2 matrix Q. In proposition C.3, proven in appendix C, we solve the non-linear system in closed-form for $\sigma_e^2 = 0$ and use the implicit function theorem to extend the solution for small σ_e^2.

In the public information case, agent i's demand depends on other agents' stock holdings, $\sum_{j \neq i}(e_{j,\ell-1} + \epsilon_{j,\ell})$. By contrast, in the private information case, agent i's demand does not depend on his expectation of other agents' stock holdings, $\sum_{j \neq i} e_{j,\ell-1}$. The intuition for the difference is the following. Suppose that in the private information case $\sum_{j \neq i} e_{j,\ell-1}$ increases, holding price constant. Holding price constant means that the period ℓ stock endowments, $\sum_{j \neq i} \epsilon_{j,\ell}$, are such that stock holdings, $\sum_{j \neq i}(e_{j,\ell-1} + \epsilon_{j,\ell})$, remain constant. Therefore, agent i does not expect larger future sales from the other agents, and his demand remains constant. By contrast, suppose that in the public information case $\sum_{j \neq i}(e_{j,\ell-1} + \epsilon_{j,\ell})$ increases, holding price constant. Holding price constant now means that the other agents "trembled" and by mistake sold less at period ℓ. Therefore, agent i still expects larger future sales, and his demand decreases.

5.2 Market Liquidity and the Speed of Trade

5.2.1 Market Liquidity

The market-clearing condition 5.8 implies that if agent i deviates from his equilibrium strategy and sells one more share, the price decreases by $1/(N-1)B$. Therefore, price impact is $1/(N-1)B$. Combining the optimality condition 5.9 with the Bellman condition 3.18, we get

$$\frac{1}{(N-1)B} = \frac{\alpha\sigma^2 h^2 e^{-rh}}{(N-2)(1-e^{-rh})(1-(1-a)^2 e^{-rh})}. \tag{5.10}$$

Price impact is thus different than in the private information case. In fact, it is easy to check that for any $a \in [0, (N-2)/(N-1)]$, price impact is smaller than in the private information case. The intuition is that if in the private information case agent i deviates and sells more shares, the other agents incorrectly infer that he received a higher endowment shock. Therefore, they expect larger sales in the future. By contrast, if in the public information case agent i deviates and sells more shares, the other agents correctly infer that he deviated (or "trembled"). Therefore, they expect smaller sales in the future, and the price does not decrease by as much.

Note that price impact is small when a is large, i.e. when dividend risk is shared quickly. To provide some intuition for this result, we suppose that agent i deviates and sells more shares. The other agents correctly infer that i deviated and that total stock holdings remain constant. The deviation increases their stock holdings only for a short period, since dividend risk is shared quickly. Therefore, the price decrease is small. As the time between trades, h, goes to 0, a will go to a strictly positive limit, i.e. agents will trade very quickly. Equation 5.10 implies that price impact will go to 0, and market liquidity to ∞.

5.2.2 Speed of Trade

Combining the optimality condition 5.7 with the Bellman condition 3.18, we get

$$\frac{1-e^{-rh}}{h}\frac{a}{(N-1)B} = (1-a)\frac{\alpha\sigma^2 h e^{-rh}}{1-(1-a)^2 e^{-rh}}. \tag{5.11}$$

Equation 5.11 determines a, and is the same as in the private information case. The LHS represents the price improvement obtained by delaying. The RHS represents the increase in dividend risk. Using equation 5.10 to eliminate B, we get

$$a = \frac{N-2}{N-1}. \tag{5.12}$$

Equation 5.12 is valid not only for $\sigma_e^2 = 0$ but for any σ_e^2, since it can be derived directly from equations 5.7 and 5.9.

The main implication of equation 5.12 is that a is independent of the time between trades, h. Therefore, agents trade very quickly as h goes to 0. Indeed, assume that at calendar time t agent i holds one share and average stock holdings are 0. At calendar time t' such that $t'-t$ is a multiple of h, the agent holds $(1-a)^{\frac{t'-t}{h}}$ shares. Since a is independent of h, $(1-a)^{\frac{t'-t}{h}}$ goes to 0 as h goes to 0. The agent thus sells very quickly.

To provide an intuition for why agents trade very quickly, we assume that h goes to 0, and proceed as in the private information case. First, we assume that price impact stays constant, and determine the direct effect of h on a. Second, we take into account the change in price impact, and determine the indirect effect. Equation 5.11 implies that if price impact stays constant, a is of order $\sqrt{h}$, and thus agents trade very quickly. This direct effect is the same as in the private information case. Price impact does not stay constant, however. Equation 5.10 implies that if a is of order $\sqrt{h}$, price impact goes to 0 as h goes to 0. The decrease in price impact implies an increase in a, which implies a further decrease in price impact, and so on. This indirect effect reinforces the result that agents trade very quickly. Note that the indirect effect works in the opposite direction relative to the private information case.

6 Welfare Analysis

In the private and public information cases, stock holdings do not become identical after one trading round. Therefore, dividend risk is not optimally shared. This entails a welfare loss. In this section we define welfare loss and study two questions. First, how does welfare loss depend on the time between trades, h? In particular, how does welfare loss change when we move from a static to a dynamic model? Second, how quickly does the market become competitive as the number of agents, N, grows, i.e. how quickly does welfare loss go to 0 as N grows?

6.1 Definition of Welfare Loss

To define welfare loss, we introduce some notation. We denote the time 0 certainty equivalents of an agent in the competitive, private information, and public information cases by CEQ_c, CEQ_{pr}, and CEQ_p, respectively. (Since agents are symmetric, certainty equivalents do not depend on i.) We also denote the time 0 certainty equivalent of an agent in the case where trade is not allowed by CEQ_n. The "no-trade" case is studied in appendix D. Welfare loss, L, is

$$L = \frac{CEQ_c - CEQ_{pr}}{CEQ_c - CEQ_n} \tag{6.1}$$

in the private information case and

$$L = \frac{CEQ_c - CEQ_p}{CEQ_c - CEQ_n} \tag{6.2}$$

in the public information case. The numerators of expressions 6.1 and 6.2 represent the welfare loss in the private and public information cases, respectively, relative to the competitive case. We normalize this welfare loss, dividing by the maximum welfare loss, i.e. the welfare loss in the no-trade case relative to the competitive case. This definition of welfare loss is the same as in the double auctions literature. Proposition 6.1 shows that in the limit when σ_e^2 goes to 0, L is given by a very simple expression. The proposition is proven in appendix E.

Proposition 6.1 *In both the private and public information cases*

$$\lim_{\sigma_e^2 \to 0} L = \frac{(1-a)^2(1-e^{-rh})}{1-(1-a)^2 e^{-rh}}. \tag{6.3}$$

To provide some intuition for expression 6.3, we rewrite it as

$$((1-a)^2 + (1-a)^4 e^{-rh} + .. + (1-a)^{2(\ell+1)} e^{-r\ell h} + ..)(1-e^{-rh}).$$

We also assume that at time t agent i holds one share and average stock holdings are 0. The agent thus needs to sell one share, but only sells a shares at time t. The welfare loss of not selling $(1-a)$ shares corresponds to the term $(1-a)^2$. At time $t+h$ the agent needs to sell $(1-a)$ shares but only sells $a(1-a)$ shares. The welfare loss of not selling $(1-a)^2$ shares, discounted at time t, is $(1-a)^4 e^{-rh}$, and so on. Note that expression 6.3 goes from 0 to 1 as a goes from 1 to 0.

6.2 Welfare Loss and h

We now study how welfare loss, L, depends on the time between trades, h. In particular, we study how welfare loss changes when we move from a static model ($h = \infty$) to a dynamic model. This exercise allows us to compare our results to the double auctions literature that mainly considers static models. Corollary 6.1 follows easily from proposition 6.1 and is proven in appendix E.

Corollary 6.1 *Suppose that σ_e^2 is small. In the public information case L increases in h, but in the private information case L decreases in h.*

In the public information case, welfare loss decreases as the time between trades, h, decreases. This result is consistent with our results on the speed of trade. Indeed, in the public information case, agents trade very quickly as h goes to 0. Therefore, welfare loss goes to 0 as h goes to 0. In the private information case, welfare loss increases as h decreases. This result is not a simple consequence of our earlier results. Indeed, proposition 4.2 implies that agents trade slowly as h goes to 0. This means only that welfare loss goes to a strictly positive limit as h goes to 0.

To provide some intuition for why welfare loss increases as h decreases, we assume that at time t agent i holds one share and average stock holdings are 0. Equation 4.9, that we reproduce below, decribes the agent's trade-off.

$$\frac{1-e^{-rh}}{h} \frac{a}{(N-1)B} = (1-a) \frac{\alpha \sigma^2 h e^{-rh}}{1-(1-a)^2 e^{-rh}}. \tag{6.4}$$

The LHS represents the price improvement obtained by selling fewer shares at time t, while the RHS represents the increase in dividend risk. The increase in dividend risk corresponds to the marginal welfare loss. Total welfare loss is proportional to marginal welfare loss and to $1-a$, the number of shares that agent i does not sell at time t. As h decreases, a decreases. If marginal welfare loss does not change, total welfare loss increases, since agent i sells fewer shares at time t. The change in marginal welfare loss is equal to the change in price improvement. Price improvement changes only slightly, since the increase in price impact, $1/(N-1)B$, offsets the decrease in the trade, a.

Corollary 6.1 does not imply that welfare in the private information case, CEQ_{pr}, decreases as h decreases. Indeed, as h decreases, there are more trading opportunities, and welfare in the competitive case, CEQ_c, increases. Therefore, CEQ_{pr} may increase.[19] Corollary 6.1 rather implies that in the presence of private information, dynamic competitive and non-competitive models differ more than their static counteparts.

6.3 Convergence to a Competitive Market

Finally, we study how quickly the market becomes competitive, i.e. how quickly welfare loss goes to 0, as the number of agents, N, grows. Corollary 6.2 answers this question in the limit when σ_e^2 goes to 0. The corollary follows easily from proposition 6.1 and is proven in appendix E.

Corollary 6.2 *In the public information case,*

$$\lim_{\sigma_e\to 0} L = \frac{1}{(N-1)^2}\frac{1-e^{-rh}}{1-\frac{1}{(N-1)^2}e^{-rh}}. \tag{6.5}$$

In the private information case

$$\lim_{\sigma_e\to 0} L = \frac{1}{(N-1)^2(1-e^{-rh})} + o(\frac{1}{N^2}) \tag{6.6}$$

and

$$\lim_{h\to 0}\lim_{\sigma_e\to 0} L = \frac{1}{N-1}. \tag{6.7}$$

In the public information case, welfare loss is of order $1/N^2$. In the private information case, welfare loss is of order $1/N^2$, for a fixed h. However, in the limit when h goes to 0, welfare loss is of order $1/N$. The two results are consistent: for a fixed small h, welfare loss is "of order" $1/N$, except when N is very large in which case welfare loss is of order $1/N^2$. The $1/N^2$ result was also obtained in the static models of the double auctions literature.

Corollary 6.2 implies that in the presence of private information, dynamic non-competitive models become competitive more slowly than their static counterparts. To provide some intuition for this result, we assume that at time t agent i holds one share and average stock holdings are 0. Equation 6.4 implies that price improvement is of order $1/N$. Therefore, marginal welfare loss is also of order $1/N$. Total welfare loss is proportional to marginal welfare loss and to $1-a$, the number of shares that agent i does not sell at time t. In the static case, agent i keeps the $1-a$ shares forever. Therefore, marginal welfare loss is proportional to $1-a$. Since marginal welfare loss is of order $1/N$, $1-a$ is of order $1/N$, and total welfare loss is of order $1/N^2$. (In fact, $1-a = 1/(N-1)$ since in the static case $a = (N-2)/(N-1)$.) As h goes to 0, a goes to 0 and $1-a$ goes to 1. Therefore, total welfare loss is of order $1/N$. For a fixed small h, $1-a$ is close to 1, except when N is very large. When N is very large, the benefit of delaying is small and $1-a$ is of order $1/N$ as in the static case.

7 Concluding Remarks

This paper studies a dynamic model of a financial market with N strategic agents. Agents receive random stock endowments at each period and trade to share dividend risk. Endowments are the only private information in the model. We find that agents trade slowly even when the time between trades goes to 0. In fact, welfare loss due to strategic behavior increases as the time between trades decreases. In the limit when the time between trades goes to 0, welfare loss is of order $1/N$, and not $1/N^2$ as in the static models of the double auctions literature. The model is very tractable and closed-form solutions are obtained in a special case.

We made two important simplifying assumptions. The first assumption is that the only agents in the market are large and strategic. This assumption implies that the equilibrium price fully reveals their endowments. If small "noise" traders were present in the market, price would not be fully revealing and large agents would not be identified immediately. However, if we introduce noise traders in this model, we run into the infinite regress problem, i.e. into an infinite number of state variables. The intuition is that since price is not fully revealing, an agent uses his expectation of other agents' stock holdings when forming his demand. To form this expectation, he uses past prices and his own past stock holdings. Since other agents are forming their expectations in the same way, the agent needs to form expectations of other agents' past stock holdings, and so on. Vayanos (1998) studies a model with one large trader who receives random stock endowments, a competitive risk-averse market-maker, and noise traders. Since the large trader has better information than the market-maker, there is no infinite regress problem. When the time between trades goes to 0, the trading process consists of two phases. During the first phase, the large trader sells very quickly a fraction of his endowment and is identified by the market-maker. He completes slowly his trades during the second phase. If there are many noise traders, the large trader may "manipulate" the market, overselling during the first phase and buying during the second phase.

The second assumption is that dividend information is public. This assumption rules out trades motivated by inside information. Chau (1998) studies a model with one large trader who receives both inside information and random stock endowments, a competitive market-maker, and noise traders. He also finds that the large trader may manipulate the market.

Appendix

A The Competitive Case

We first provide conditions for the demand 3.1 to solve (P_c), in proposition A.1. We then show that the conditions can be reduced to a system of 3 non-linear equations. Finally, we solve the system in closed-form for $\sigma_e^2 = 0$ and extend the solution for small σ_e^2, in proposition A.2.

A.1 The Conditions

To state proposition A.1, we define the following symmetric 2×2 matrices. First, we define Q' by

$$Q'_{1,1} = (1-a)^2 Q_{1,1}, \tag{A.1}$$

$$Q'_{1,2} = -\frac{1-e^{-rh}}{h}\frac{a^2}{B} + a(1-a)Q_{1,1} + (1-a)Q_{1,2}, \tag{A.2}$$

and

$$Q'_{2,2} = \frac{1-e^{-rh}}{h}\frac{2a^2}{B} + a^2 Q_{1,1} + 2aQ_{1,2} + Q_{2,2}. \tag{A.3}$$

Next, we define Σ^2, the variance-covariance matrix of the vector $(\epsilon_{i,\ell}, \sum_{j=1}^N \epsilon_{j,\ell}/N)$. This matrix is

$$\Sigma^2 = \sigma_e^2 h \begin{pmatrix} 1 & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N} \end{pmatrix} \tag{A.4}$$

Finally, we define R, R', and P by

$$R = I + \alpha Q' \Sigma^2, \tag{A.5}$$

$$R' = \alpha Q' \Sigma^2 R^{-1} Q', \tag{A.6}$$

and

$$P = (Q' - R' - \alpha\sigma^2 h \Gamma)e^{-rh}. \tag{A.7}$$

In equation A.7, Γ is a 2×2 matrix whose elements are 1 for $(i,j) = (1,1)$ and 0 otherwise.

Proposition A.1 *The demand 3.1 solves* (P_c) *and the function 3.10 is the value function, if the following conditions hold. First, the optimality conditions 3.12, 3.13, 3.14, and the inequality* $Q_{1,1} < 0$. *Second, the Bellman conditions*

$$Q = P \tag{A.8}$$

and

$$q = \frac{log(|R|)e^{-rh}}{2\alpha(1-e^{-rh})} + \frac{(\beta e^{-rh} - r)h}{\alpha(1-e^{-rh})} - \frac{1}{\alpha}log(\frac{h}{e^{rh}-1}). \tag{A.9}$$

In equation A.9, $|R|$ *denotes the determinant of the matrix* R.

Proof: The demand 3.1 solves (P_c) and the function 3.10 is the value function, if the following are true. First, the function 3.10 solves the Bellman equation 3.11, for the demand 3.1 and the optimal consumption. Second, the demand 3.1 and the optimal consumption satisfy the transversality condition 3.9. We will show that conditions 3.12, 3.13, 3.14, $Q_{1,1} < 0$, A.8, and A.9 are sufficient for the Bellman equation and the transversality condition.

Bellman Equation

We proceed in 3 steps. First, we show that the optimality conditions 3.12, 3.13, 3.14, and $Q_{1,1} < 0$ are sufficient for the demand 3.1 to maximize the RHS of the Bellman equation. Second, we compute the expectation of the RHS w.r.t. period $\ell - 1$ information. Finally, we show that the Bellman conditions A.8 and A.9 are sufficient for the function 3.10 to satisfy the Bellman equation.

Step 1: Optimal Demand

Before trade at period ℓ, agent i knows δ_ℓ and $\epsilon_{i,\ell}$, but not $\epsilon_{j,\ell}$ for $j \neq i$. He thus chooses his demand, $x_{i,\ell}(p_\ell)$, to maximize

$$-E_{\epsilon_{j,\ell}, j\neq i} exp(-\alpha(\frac{1-e^{-rh}}{h}\left(-d_\ell \frac{h}{1-e^{-rh}}\epsilon_{i,\ell} - p_\ell x_{i,\ell}(p_\ell)\right)$$

$$+ d_\ell(e_{i,\ell-1} + \epsilon_{i,\ell} + x_{i,\ell}(p_\ell)) + F(Q, \begin{pmatrix} e_{i,\ell-1} + \epsilon_{i,\ell} + x_{i,\ell}(p_\ell) \\ \frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N} \end{pmatrix}))),$$

where the price, p_ℓ, is given by equation 3.3. The agent can infer $\sum_{j\neq i}\epsilon_{j,\ell}$ from the price. Therefore his problem is the same as knowing $\sum_{j\neq i}\epsilon_{j,\ell}$ and choosing a trade,

x, to maximize

$$-\frac{1-e^{-rh}}{h}\left(\frac{A}{B}d_\ell - \frac{a}{B}\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}\right)x - d_\ell\epsilon_{i,\ell}$$

$$+d_\ell(e_{i,\ell-1}+\epsilon_{i,\ell}+x) + F(Q,\begin{pmatrix} e_{i,\ell-1}+\epsilon_{i,\ell}+x \\ \frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}\end{pmatrix}).$$

If the trade 3.5 is optimal, then the demand 3.1 is optimal, since it produces this trade. Setting

$$x = a\left(\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N} - (e_{i,\ell-1}+\epsilon_{i,\ell})\right) + \Delta x,$$

the demand 3.1 is optimal if $\Delta x = 0$ maximizes

$$-\frac{1-e^{-rh}}{h}\left(\frac{A}{B}d_\ell - \frac{a}{B}\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}\right)(a\left(\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N} - (e_{i,\ell-1}+\epsilon_{i,\ell})\right)+\Delta x)$$

$$-d_\ell\epsilon_{i,\ell} + d_\ell\left((1-a)(e_{i,\ell-1}+\epsilon_{i,\ell}) + a\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N} + \Delta x\right)$$

$$+F(Q,\begin{pmatrix} (1-a)(e_{i,\ell-1}+\epsilon_{i,\ell}) + a\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N} + \Delta x \\ \frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}\end{pmatrix}). \tag{A.10}$$

Conditions 3.12, 3.13, and 3.14 ensure that the first-order condition is satisfied for $\Delta x = 0$. Condition $Q_{1,1} < 0$ ensures that expression A.10 is concave.

Using the definition of the matrix Q', we can write expression A.10 for $\Delta x = 0$ as

$$d_\ell e_{i,\ell-1} + F(Q',\begin{pmatrix} e_{i,\ell-1}+\epsilon_{i,\ell} \\ \frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}\end{pmatrix}). \tag{A.11}$$

Step 2: Computing the Expectation

We have to compute the expectation of the period ℓ value function w.r.t. period $\ell-1$ information, i.e. w.r.t. δ_ℓ and $\epsilon_{j,\ell}$. Using the budget constraint 3.7 and expression A.11, we have to compute

$$E_{\ell-1}exp(-\alpha(\frac{1-e^{-rh}}{h}e^{rh}(M_{i,\ell-1}+d_{\ell-1}e_{i,\ell-1}h - c_{i,\ell-1}h) + (d_{\ell-1}+\delta_\ell)e_{i,\ell-1}$$

$$+F(Q',\begin{pmatrix} e_{i,\ell-1}+\epsilon_{i,\ell} \\ \frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}\end{pmatrix}) + q) - \beta h). \tag{A.12}$$

Computing expectations w.r.t δ_ℓ is straightforward. We get

$$E_{\epsilon_{j,\ell}} exp(-\alpha(\frac{1-e^{-rh}}{h} e^{rh}(M_{i,\ell-1} - c_{i,\ell-1}h) + e^{rh} d_{\ell-1} e_{i,\ell-1} - \frac{1}{2}\alpha\sigma^2 h e_{i,\ell-1}^2$$

$$+F(Q', \begin{pmatrix} e_{i,\ell-1} + \epsilon_{i,\ell} \\ \frac{\sum_{j=1}^N (e_{j,\ell-1}+\epsilon_{j,\ell})}{N} \end{pmatrix}) + q) - \beta h). \tag{A.13}$$

To compute expectations w.r.t. $\epsilon_{j,\ell}$, we use the formula

$$E(exp(-\alpha(a+b^t x+\frac{1}{2}x^t cx))) = exp(-\alpha(a-\frac{1}{2}\alpha b^t \Sigma^2 (I+\alpha c\Sigma^2)^{-1} b + \frac{1}{2\alpha} log|I+\alpha c\Sigma^2|)), \tag{A.14}$$

where x is a $n \times 1$ normal vector with mean 0 and variance-covariance matrix Σ^2, I the $n \times n$ identity matrix, a a number, b an $n \times 1$ vector, and c an $n \times n$ symmetric matrix. (Formula A.14 gives simply the moment generating function of the normal distribution for $c = 0$. We can always assume $c = 0$ by also assuming that x is a normal vector with mean 0 and variance-covariance matrix $\Sigma^2(I + \alpha c\Sigma^2)^{-1}$.)

We set

$$x = \begin{pmatrix} \epsilon_{i,\ell} \\ \frac{\sum_{j=1}^N \epsilon_{j,\ell}}{N} \end{pmatrix},$$

Σ^2 the matrix defined by equation A.4,

$$a = \frac{1-e^{-rh}}{h} e^{rh}(M_{i,\ell-1} - c_{i,\ell-1}h) + e^{rh} d_{\ell-1} e_{i,\ell-1} - \frac{1}{2}\alpha\sigma^2 h e_{i,\ell-1}^2$$

$$+F(Q', \begin{pmatrix} e_{i,\ell-1} \\ \frac{\sum_{j=1}^N e_{j,\ell-1}}{N} \end{pmatrix}) + q + \frac{\beta h}{\alpha},$$

$$b = Q' \begin{pmatrix} e_{i,\ell-1} \\ \frac{\sum_{j=1}^N e_{j,\ell-1}}{N} \end{pmatrix},$$

and $c = Q'$. Using the definitions of R and R', we can write expression A.12 as

$$exp(-\alpha(\frac{1-e^{-rh}}{h} e^{rh}(M_{i,\ell-1} - c_{i,\ell-1}h) + e^{rh} d_{\ell-1} e_{i,\ell-1} - \frac{1}{2}\alpha\sigma^2 h e_{i,\ell-1}^2$$

$$+F(Q' - R', \begin{pmatrix} e_{i,\ell-1} \\ \frac{\sum_{j=1}^N e_{j,\ell-1}}{N} \end{pmatrix}) + \frac{1}{2\alpha} log|R| + q) - \beta h).$$

Finally, using the definition of P, we can rewrite this expression as

$$exp(-\alpha(\frac{1-e^{-rh}}{h} e^{rh}(M_{i,\ell-1} - c_{i,\ell-1}h) + e^{rh} d_{\ell-1} e_{i,\ell-1}$$

$$+e^{rh}F(P,\begin{pmatrix}\frac{e_{i,\ell-1}}{\frac{\sum_{j=1}^{N}e_{j,\ell-1}}{N}}\end{pmatrix})+\frac{1}{2\alpha}log|R|+q)-\beta h). \quad (A.15)$$

Step 3: Bellman Equation

To compute the RHS of the Bellman equation, we have to maximize w.r.t. $c_{i,\ell-1}$

$$-exp(-\alpha c_{i,\ell-1})h-exp(-\alpha(\frac{1-e^{-rh}}{h}e^{rh}(M_{i,\ell-1}-c_{i,\ell-1}h)+e^{rh}d_{\ell-1}e_{i,\ell-1}$$

$$+e^{rh}F(P,\begin{pmatrix}\frac{e_{i,\ell-1}}{\frac{\sum_{j=1}^{N}e_{j,\ell-1}}{N}}\end{pmatrix})+\frac{1}{2\alpha}log|R|+q)-\beta h). \quad (A.16)$$

Simple calculations show that the maximum is

$$-exp(-\alpha(\frac{1-e^{-rh}}{h}M_{i,\ell-1}+d_{\ell-1}e_{i,\ell-1}+F(P,\begin{pmatrix}\frac{e_{i,\ell-1}}{\frac{\sum_{j=1}^{N}e_{j,\ell-1}}{N}}\end{pmatrix})$$

$$+\frac{1}{2\alpha}log|R|e^{-rh}+qe^{-rh}+\frac{(\beta e^{-rh}-r)h}{\alpha}-\frac{1}{\alpha}(1-e^{-rh})log(\frac{h}{e^{rh}-1}))). \quad (A.17)$$

This is equal to the period $\ell-1$ value function if conditions A.8 and A.9 hold.

Transversality condition

It is easy to check that, by substituting the optimal $c_{i,\ell-1}$ in the second term of expression A.16, we find expression A.17 times e^{-rh}. Therefore the expectation of the period ℓ value function at period $\ell-1$, is the period $\ell-1$ value function times e^{-rh}. Recursive use of this equation implies equation 3.9. Q.E.D.

A.2 The System

The system will follow from the Bellman condition A.8. In this condition, the matrix P is indirectly a function of Q', which is in turn a function of B, a, and Q. To have Q as the only unknown, we eliminate first B and then a in the definition of Q'. Using condition 3.13, we can write Q' as a function of a and Q, as follows

$$Q'_{1,1}=(1-a)^2Q_{1,1}, \quad (A.18)$$

$$Q'_{1,2}=a(2-a)Q_{1,1}+Q_{1,2} \quad (A.19)$$

and

$$Q'_{2,2} = -a(2-a)Q_{1,1} + Q_{2,2}. \tag{A.20}$$

Since $a = 1$ from condition 3.14, Q' becomes a function of Q only. Therefore, the system is the Bellman condition A.8, where P, R', and R are given by equations A.7, A.6, and A.5, respectively, Q' by equations A.18, A.19, and A.20, and $a = 1$. Having solved this system, we can solve for A, B, and q, using equations 3.12, 3.13, and A.9, respectively. We also have to check that inequality $Q_{1,1} < 0$ is satisfied.

A.3 The Solution

Proposition A.2 *The system has a solution for small σ_e^2. For $\sigma_e^2 = 0$ the solution is*

$$Q_{1,1} = -\alpha\sigma^2 h e^{-rh} < 0, \quad Q_{1,2} = -Q_{2,2} = -\frac{\alpha\sigma^2 h e^{-2rh}}{1-e^{-rh}}. \tag{A.21}$$

Proof: We first solve the system for $\sigma_e^2 = 0$. We then use the implicit function theorem to extend the solution for small σ_e^2.

The Solution for σ_e^2=0

For $\sigma_e^2 = 0$, $R' = 0$ and

$$Q = P = (Q' - \alpha\sigma^2 h\Gamma)e^{-rh}. \tag{A.22}$$

To solve the system, we use equations A.18 to A.20 and A.22. We treat a as a parameter in equations A.18 to A.20, obtain Q as a function of a, and then set $a = 1$ to obtain the Q of the proposition. We treat a as a parameter because the expression of Q as a function of a will also be valid in the private and public information cases.

The equation for $Q_{1,1}$ is

$$Q_{1,1} = ((1-a)^2 Q_{1,1} - \alpha\sigma^2 h)e^{-rh}, \tag{A.23}$$

and is satisfied for

$$Q_{1,1} = -\frac{\alpha\sigma^2 h e^{-rh}}{1-(1-a)^2 e^{-rh}}. \tag{A.24}$$

The equation for $Q_{1,2}$ is

$$Q_{1,2} = (a(2-a)Q_{1,1} + Q_{1,2})e^{-rh}, \tag{A.25}$$

and is satisfied for

$$Q_{1,2} = \frac{a(2-a)Q_{1,1}e^{-rh}}{1-e^{-rh}}. \tag{A.26}$$

Finally, the equation for $Q_{2,2}$ is

$$Q_{2,2} = (-a(2-a)Q_{1,1} + Q_{2,2})e^{-rh}, \tag{A.27}$$

and is satisfied for

$$Q_{2,2} = -\frac{a(2-a)Q_{1,1}e^{-rh}}{1-e^{-rh}}. \tag{A.28}$$

Setting $a = 1$, we obtain the Q of the proposition.

Small σ_e^2

To extend the solution for small σ_e^2, we consider the function

$$G(\overline{Q}_{1,1}, Q_{1,2}, Q_{2,2}, \sigma^2, \sigma_e^2, h, N) = \frac{1}{h}\begin{pmatrix} \overline{Q}_{1,1}h - P_{1,1} \\ Q_{1,2} - P_{1,2} \\ Q_{2,2} - P_{2,2} \end{pmatrix}.$$

The matrices P, R', R, and Q' are defined by equations A.7, A.6, A.5, A.18, A.19, and A.20, for $a = 1$ and $Q_{1,1} = \overline{Q}_{1,1}h$. We use $\overline{Q}_{1,1}$ instead of $Q_{1,1}$ to deal with the case $h = 0$. We apply the implicit function theorem to G at the point $\mathcal{A}$, where $\sigma_e^2 = 0$, $\overline{Q}_{1,1} = -\alpha\sigma^2 e^{-rh}$, and $Q_{1,2}$ and $Q_{2,2}$ are given by expressions A.21 for $Q_{1,1} = \overline{Q}_{1,1}h$. h can be 0, in which case we extend G, $Q_{1,2}$, and $Q_{2,2}$ by continuity. The function G is equal to 0 at $\mathcal{A}$, since Q solves the system. To apply the implicit function theorem, we only need to show that the Jacobian matrix of G w.r.t. $\overline{Q}_{1,1}$, $Q_{1,2}$, and $Q_{2,2}$ is invertible. Using equation A.22 (which is valid since we compute partial derivatives for $\sigma_e^2 = 0$) it is easy to check the following. First, the partial derivatives of $G_{1,1}$ w.r.t. $\overline{Q}_{1,1}$, $G_{1,2}$ w.r.t. $Q_{1,2}$, and $G_{2,2}$ w.r.t. $Q_{2,2}$, are strictly positive. Second, the partial derivatives of $G_{1,1}$ w.r.t. $Q_{1,2}$, $Q_{2,2}$, and $G_{1,2}$ w.r.t. $Q_{2,2}$ are 0. Therefore, the Jacobian matrix is invertible. Since $\overline{Q}_{1,1} < 0$ at $\mathcal{A}$, $\overline{Q}_{1,1} < 0$ and $Q_{1,1} < 0$ for small σ_e^2. Q.E.D.

B The Private Information Case

We first provide conditions for the demand 4.1 to solve (P_{pr}), in proposition B.1. We then show that the conditions can be reduced to a system of 4 non-linear equations. We solve the system in closed-form for $\sigma_e^2 = 0$ and extend the solution for small σ_e^2, in proposition B.2. Finally, we prove propositions 4.1 and 4.2.

B.1 The Conditions

To state proposition B.1, we use the matrices Q', Σ^2, R, R', and P, defined in appendix A.

Proposition B.1 *The demand 4.1 solves (P_{pr}) and the function 4.3 is the value function, if the following conditions hold. First, the optimality conditions 4.4, 4.5, 4.6, and the inequality*

$$-\frac{1-e^{-rh}}{h}\frac{2}{(N-1)B} + Q_{1,1} < 0. \tag{B.1}$$

Second, the Bellman conditions

$$Q = P \tag{B.2}$$

and

$$q = \frac{log(|R|)e^{-rh}}{2\alpha(1-e^{-rh})} + \frac{(\beta e^{-rh} - r)h}{\alpha(1-e^{-rh})} - \frac{1}{\alpha}log(\frac{h}{e^{rh}-1}). \tag{B.3}$$

Proof: We will show that the optimality conditions 4.4, 4.5, 4.6, and B.1 are sufficient for the demand 4.1 to maximize the RHS of the Bellman equation. The rest of the proof is as in the competitive case.

Before trade at period ℓ, agent i knows δ_ℓ and $\epsilon_{i,\ell}$, but not $\epsilon_{j,\ell}$ for $j \neq i$. He thus chooses his demand, $x_{i,\ell}(p_\ell)$, to maximize

$$-E_{\epsilon_{j,\ell}, j\neq i} exp(-\alpha(\frac{1-e^{-rh}}{h}\left(-d_\ell \frac{h}{1-e^{-rh}}\epsilon_{i,\ell} - p_\ell x_{i,\ell}(p_\ell)\right)$$

$$+d_\ell(e_{i,\ell-1} + \epsilon_{i,\ell} + x_{i,\ell}(p_\ell)) + F(Q, \begin{pmatrix} e_{i,\ell-1} + \epsilon_{i,\ell} + x_{i,\ell}(p_\ell) \\ \frac{\sum_{j=1}^N (e_{j,\ell-1}+\epsilon_{j,\ell})}{N} \end{pmatrix}))),$$

where the price, p_ℓ, is given by equation 4.2 and depends on $x_{i,\ell}(p_\ell)$. Instead of solving this problem, we proceed as in Kyle (1989) and solve a simpler problem.

We assume that the agent knows the residual demand of the other agents (i.e. knows $\sum_{j \neq i} \epsilon_{j,\ell}$) and simply chooses a trade, x, on this residual demand. The market-clearing condition 4.2 implies that the price, p_ℓ, is

$$\frac{A}{B} d_\ell - \frac{a}{B} \frac{\sum_{j \neq i}(e_{j,\ell-1} + \epsilon_{j,\ell})}{N-1} + \frac{x}{(N-1)B}.$$

Therefore, the agent's problem is to maximize w.r.t. x

$$-\frac{1-e^{-rh}}{h} \left(\frac{A}{B} d_\ell - \frac{a}{B} \frac{\sum_{j \neq i}(e_{j,\ell-1} + \epsilon_{j,\ell})}{N-1} + \frac{x}{(N-1)B} \right) x - d_\ell \epsilon_{i,\ell}$$

$$+d_\ell(e_{i,\ell-1} + \epsilon_{i,\ell} + x) + F(Q, \begin{pmatrix} e_{i,\ell-1} + \epsilon_{i,\ell} + x \\ \frac{\sum_{j=1}^N (e_{j,\ell-1} + \epsilon_{j,\ell})}{N} \end{pmatrix}).$$

If the trade 3.5 solves this problem, then the demand 4.1 is optimal since it produces this trade for all values of $\sum_{j \neq i} \epsilon_{j,\ell}$. Defining Δx as in the competitive case, the demand 4.1 is optimal if $\Delta x = 0$ maximizes

$$-\frac{1-e^{-rh}}{h} \left(\frac{A}{B} d_\ell - \frac{a}{B} \frac{\sum_{j=1}^N (e_{j,\ell-1} + \epsilon_{j,\ell})}{N} + \frac{\Delta x}{(N-1)B} \right)$$

$$(a \left(\frac{\sum_{j=1}^N (e_{j,\ell-1} + \epsilon_{j,\ell})}{N} - (e_{i,\ell-1} + \epsilon_{i,\ell}) \right) + \Delta x)$$

$$-d_\ell \epsilon_{i,\ell} + d_\ell \left((1-a)(e_{i,\ell-1} + \epsilon_{i,\ell}) + a \frac{\sum_{j=1}^N (e_{j,\ell-1} + \epsilon_{j,\ell})}{N} + \Delta x \right)$$

$$+F(Q, \begin{pmatrix} (1-a)(e_{i,\ell-1} + \epsilon_{i,\ell}) + a \frac{\sum_{j=1}^N (e_{j,\ell-1} + \epsilon_{j,\ell})}{N} + \Delta x \\ \frac{\sum_{j=1}^N (e_{j,\ell-1} + \epsilon_{j,\ell})}{N} \end{pmatrix}). \tag{B.4}$$

Conditions 4.4, 4.5, and 4.6 ensure that the first-order condition is satisfied for $\Delta x = 0$. Condition B.1 ensures that expression B.4 is concave. Proceeding as in the competitive case, we can rewrite expression B.4 for $\Delta x = 0$ as

$$d_\ell e_{i,\ell-1} + F(Q', \begin{pmatrix} e_{i,\ell-1} + \epsilon_{i,\ell} \\ \frac{\sum_{j=1}^N (e_{j,\ell-1} + \epsilon_{j,\ell})}{N} \end{pmatrix}).$$

Q.E.D.

B.2 The System and the Solution

The first 3 equations follow from the Bellman condition B.2, where P, R', and R are given by equations A.7, A.6, and A.5, respectively, and Q' by equations A.18, A.19, and A.20. The last equation is

$$\frac{1}{N-1}(Q_{1,1}+Q_{1,2}) = (1-a)Q_{1,1}. \tag{B.5}$$

This equation follows from conditions 4.5 and 4.6 by eliminating B.

Proposition B.2 *The system has a solution for small σ_e^2. For $\sigma_e^2 = 0$ the matrix Q is given by*

$$Q_{1,1} = -\frac{\alpha\sigma^2 h e^{-rh}}{1-(1-a)^2 e^{-rh}}, \quad Q_{1,2} = -Q_{2,2} = \frac{a(2-a)Q_{1,1}e^{-rh}}{1-e^{-rh}}, \tag{B.6}$$

and a is the unique solution in $[0, (N-2)/(N-1)]$ of

$$a^2 e^{-rh} - a(2e^{-rh} + (N-1)(1-e^{-rh})) + (N-2)(1-e^{-rh}) = 0. \tag{B.7}$$

Proof: We first solve the system for $\sigma_e^2 = 0$. To obtain Q as a function of a, we proceed exactly as in the competitive case. To obtain equation B.7, we eliminate $Q_{1,1}$ and $Q_{1,2}$ in equation B.5. Finally, we check that inequality B.1 is satisfied. Using equation 4.5, we can write inequality B.1 as

$$\frac{2}{N-1}\frac{Q_{1,1}+Q_{1,2}}{a} + Q_{1,1} < 0.$$

Since $0 < a < 1$, $Q_{1,1} < 0$. Since in addition $Q_{1,1}+Q_{1,2} = -\alpha\sigma^2 h e^{-rh}/(1-e^{-rh}) < 0$, the inequality is satisfied.

To extend the solution for small σ_e^2, we consider the function

$$G(Q_{1,1}, Q_{1,2}, Q_{2,2}, \bar{a}, \sigma^2, \sigma_e^2, h, N) = \frac{1}{h}\begin{pmatrix} Q_{1,1} - P_{1,1} \\ Q_{1,2} - P_{1,2} \\ Q_{2,2} - P_{2,2} \\ h\left(\frac{1}{N-1}(Q_{1,1}+Q_{1,2}) - (1-\bar{a}h)Q_{1,1}\right) \end{pmatrix}.$$

The matrices P, R', R, and Q' are defined by equations A.7, A.6, A.5, A.18, A.19, and A.20, for $a = \bar{a}h$. We use $\bar{a}$ instead of a to deal with the case $h = 0$. We apply the implicit function theorem to G at the point $\mathcal{A}$, where $\sigma_e^2 = 0$, Q is given by expressions B.6 of the proposition for $a = \bar{a}h$, and $\bar{a}$ is the unique solution in $(0, (N-2)/(N-1)h)$ of

$$(\bar{a}h)^2 e^{-rh} - \bar{a}h(2e^{-rh} + (N-1)(1-e^{-rh})) + (N-2)(1-e^{-rh}) = 0, \tag{B.8}$$

i.e. of equation B.7 for $a = \bar{a}h$. h can be 0, in which case we extend G, Q, and $\bar{a}$ by continuity. The function G is equal to 0 at $\mathcal{A}$, since Q and $a = \bar{a}h$ solve the system. To apply the implicit function theorem, we only need to show that the Jacobian matrix of G w.r.t. the $Q_{i,j}$'s and $\bar{a}$ is invertible. It is easy to check that the partial derivative of a component of G, other than $G_{2,2}$, w.r.t. $Q_{2,2}$ is 0, and that the partial derivative of $G_{2,2}$ w.r.t. $Q_{2,2}$ is strictly positive. Therefore, the Jacobian matrix of G is invertible if and only if the Jacobian matrix of $G_{1,1}$, $G_{1,2}$, and $G_{\bar{a}}$ w.r.t. $Q_{1,1}$, $Q_{1,2}$, and $\bar{a}$ is invertible. This matrix is

$$\begin{pmatrix} \frac{1-(1-\bar{a}h)^2 e^{-rh}}{h} & 0 & 2Q_{1,1}(1-\bar{a}h)e^{-rh} \\ -\bar{a}(2-\bar{a}h)e^{-rh} & \frac{1-e^{-rh}}{h} & -2Q_{1,1}(1-\bar{a}h)e^{-rh} \\ \bar{a}h - \frac{N-2}{N-1} & \frac{1}{N-1} & Q_{1,1}h \end{pmatrix}.$$

To compute the determinant, we add the first row to the second and factor out $Q_{1,1}(1-e^{-rh})/h$. We get

$$Q_{1,1}\frac{1-e^{-rh}}{h} \begin{vmatrix} \frac{1-(1-\bar{a}h)^2 e^{-rh}}{h} & 0 & 2(1-\bar{a}h)e^{-rh} \\ 1 & 1 & 0 \\ \bar{a}h - \frac{N-2}{N-1} & \frac{1}{N-1} & h \end{vmatrix}.$$

Subtracting the second column from the first and expanding the determinant we get

$$Q_{1,1}\frac{1-e^{-rh}}{h}\left(\frac{1-(1-\bar{a}h)^2 e^{-rh}}{h} h + 2(1-\bar{a}h)^2 e^{-rh}\right),$$

which is strictly negative. Q.E.D.

We now prove proposition 4.1.

Proof: Equation B.8 implies that $a = \bar{a}h$ is increasing in N and decreasing in e^{-rh}, i.e. increasing in h. By continuity it is increasing in N and h for σ_e^2 small. Q.E.D.

Finally, we prove proposition 4.2.

Proof: Proposition B.2 implies that $\bar{a} = a/h$ is a continuous function of h and σ_e^2. Equation B.8 implies that $\bar{a} = (N-2)r/2$ for $h = 0$ and $\sigma_e^2 = 0$. By continuity $\bar{a} > 0$ for $h = 0$ and σ_e^2 small. Q.E.D.

C The Public Information Case

We first provide conditions for the demand 5.1 to solve (P_p), in proposition C.1. We then study the perturbed game, and show that our refinement implies an additional optimality condition. We next show that the optimality conditions and the Bellman conditions can be reduced to a system of 3 non-linear equations. Finally, we solve the system in closed-form for $\sigma_e^2 = 0$ and extend the solution for small σ_e^2, in proposition C.3.

C.1 The Conditions for (P_p)

To state proposition C.1, we use the matrices Q', Σ^2, R, R', and P, defined in appendix A. However, we change the definitions of $Q'_{1,2}$ and $Q'_{2,2}$ to

$$Q'_{1,2} = -\frac{1-e^{-rh}}{h}\frac{a(A_e+a)}{B} + a(1-a)Q_{1,1} + (1-a)Q_{1,2}, \tag{C.1}$$

and

$$Q'_{2,2} = \frac{1-e^{-rh}}{h}\frac{2a(A_e+a)}{B} + a^2 Q_{1,1} + 2aQ_{1,2} + Q_{2,2}. \tag{C.2}$$

Proposition C.1 *The demand 5.1 solves (P_p) and the function 5.4 is the value function, if the following conditions hold. First, the optimality conditions 5.5, 5.6, 5.7, and the inequality*

$$-\frac{1-e^{-rh}}{h}\frac{2}{(N-1)B} + Q_{1,1} < 0. \tag{C.3}$$

Second, the Bellman conditions

$$Q = P \tag{C.4}$$

and

$$q = \frac{log(|R|)e^{-rh}}{2\alpha(1-e^{-rh})} + \frac{(\beta e^{-rh} - r)h}{\alpha(1-e^{-rh})} - \frac{1}{\alpha}log(\frac{h}{e^{rh}-1}). \tag{C.5}$$

Proof: We will show that the optimality conditions 5.5, 5.6, 5.7, and C.3 are sufficient for the demand 5.1 to maximize the RHS of the Bellman equation. The rest of the proof is as in the competitive case.

Before trade at period ℓ, agent i knows δ_ℓ and $\epsilon_{j,\ell}$, $\forall j$. He thus chooses his demand, $x_{i,\ell}(p_\ell)$, to maximize

$$-exp(-\alpha(\frac{1-e^{-rh}}{h}\left(-d_\ell\frac{h}{1-e^{-rh}}\epsilon_{i,\ell} - p_\ell x_{i,\ell}(p_\ell)\right)$$

$$+d_\ell(e_{i,\ell-1}+\epsilon_{i,\ell}+x_{i,\ell}(p_\ell))+F(Q,\left(\frac{e_{i,\ell-1}+\epsilon_{i,\ell}+x_{i,\ell}(p_\ell)}{\frac{\sum_{j=1}^N(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}}\right))),$$

where the price, p_ℓ, is given by equation 5.3 and depends on $x_{i,\ell}(p_\ell)$. The agent knows the residual demand of the other agents. Therefore his problem is the same as choosing a trade, x, on this residual demand. The market-clearing condition 5.3 implies that the agent's problem is to maximize w.r.t. x

$$-\frac{1-e^{-rh}}{h}\left(\frac{A}{B}d_\ell-\frac{A_e}{B}\frac{\sum_{k=1}^N(e_{k,\ell-1}+\epsilon_{k,\ell})}{N}-\frac{a}{B}\frac{\sum_{j\neq i}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N-1}+\frac{x}{(N-1)B}\right)x$$

$$-d_\ell\epsilon_{i,\ell}+d_\ell(e_{i,\ell-1}+\epsilon_{i,\ell}+x)+F(Q,\left(\frac{e_{i,\ell-1}+\epsilon_{i,\ell}+x}{\frac{\sum_{j=1}^N(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}}\right)).$$

If the trade 3.5 solves this problem, then the demand 5.1 is optimal since it produces this trade. Defining Δx as in the competitive case, the demand 4.1 is optimal if $\Delta x=0$ maximizes

$$-\frac{1-e^{-rh}}{h}\left(\frac{A}{B}d_\ell-\frac{A_e+a}{B}\frac{\sum_{j=1}^N(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}+\frac{\Delta x}{(N-1)B}\right)$$

$$(a\left(\frac{\sum_{j=1}^N(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}-(e_{i,\ell-1}+\epsilon_{i,\ell})\right)+\Delta x)$$

$$-d_\ell\epsilon_{i,\ell}+d_\ell\left((1-a)(e_{i,\ell-1}+\epsilon_{i,\ell})+a\frac{\sum_{j=1}^N(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}+\Delta x\right)$$

$$+F(Q,\left(\frac{(1-a)(e_{i,\ell-1}+\epsilon_{i,\ell})+a\frac{\sum_{j=1}^N(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}+\Delta x}{\frac{\sum_{j=1}^N(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}}\right)). \tag{C.6}$$

Conditions 5.5, 5.6, and 5.7 ensure that the first-order condition is satisfied for $\Delta x=0$. Condition C.3 ensures that expression C.6 is concave. Proceeding as in the competitive case, we can rewrite expression C.6 for $\Delta x=0$ as

$$d_\ell e_{i,\ell-1}+F(Q',\left(\frac{e_{i,\ell-1}+\epsilon_{i,\ell}}{\frac{\sum_{j=1}^N(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}}\right)).$$

Q.E.D.

C.2 The Perturbed Game

We first provide conditions for the demands 5.1 to constitute a Nash equilibrium of the perturbed game. We then study the limit of these conditions as σ_u^2 goes to 0, and show that our refinement implies an additional optimality condition.

The demands 5.1 constitute a Nash equilibrium of the perturbed game if there exist consumption choices, $c_{i,\ell}$, such that the following are true. First, the demand 5.1 maximizes the utility of the incarnation of agent i that chooses demand at period ℓ (the "(i,ℓ) demand" incarnation). Second, the consumption $c_{i,\ell}$ maximizes the utility of the incarnation of agent i that chooses consumption at period ℓ (the "(i,ℓ) consumption" incarnation). Both incarnations take the choices of the other incarnations as given, i.e. assume that the (j,ℓ') demand incarnation chooses the demand 5.1 plus the noise $u_{j,\ell'}$, and that the (j,ℓ') consumption incarnation chooses $c_{j,\ell'}$.

In proposition C.2 we provide sufficient conditions for the demands 5.1 to constitute a Nash equilibrium of the perturbed game. These conditions are on the parameters A, B, A_e, a, Q, and q. The parameters Q and q correspond to the value function, which is still given by expression 5.4. To state proposition C.2, we define the following matrices. First, the symmetric 4×4 matrix $\hat{Q}'$. The top left 2×2 submatrix of $\hat{Q}'$ is the matrix Q' defined at the beginning of appendix C. The remaining elements of $\hat{Q}'$ are

$$\hat{Q}'_{1,3} = (1-a)Q_{1,1}, \qquad \hat{Q}'_{1,4} = \frac{1-e^{-rh}}{h}\frac{a}{B} - (1-a)Q_{1,1},$$

$$\hat{Q}'_{2,3} = \frac{1-e^{-rh}}{h}\frac{A_e + a}{B} + aQ_{1,1} + Q_{1,2}, \qquad \hat{Q}'_{2,4} = -\frac{1-e^{-rh}}{h}\frac{A_e + 2a}{B} - aQ_{1,1} - Q_{1,2},$$

$$\hat{Q}'_{3,3} = Q_{1,1}, \qquad \hat{Q}'_{3,4} = -\frac{1-e^{-rh}}{h}\frac{1}{B} - Q_{1,1}, \qquad \hat{Q}'_{4,4} = \frac{1-e^{-rh}}{h}\frac{2}{B} + Q_{1,1}.$$

We next define $\hat{\Sigma}^2$, the variance-covariance matrix of the vector $(\epsilon_{i,\ell}, \sum_{j=1}^{N} \epsilon_{j,\ell}/N, u_{i,\ell}, \sum_{j=1}^{N} u_{j,\ell})$, and $\hat{R}$ and $\hat{R}'$ by

$$\hat{R} = I + \alpha \hat{Q}' \hat{\Sigma}^2$$

and

$$\hat{R}' = \alpha \hat{Q}' \hat{\Sigma}^2 \hat{R}^{-1} \hat{Q}'.$$

Finally, we define P by equation A.7, where R' now denotes the top left 2×2 submatrix of $\hat{R}'$.

Proposition C.2 *The demands 5.1 constitute a Nash equilibrium of the perturbed game, if the following conditions hold. First, the optimality conditions 5.5, 5.6, 5.7, 5.9, and the inequality C.3. Second, the Bellman conditions*

$$Q = P \tag{C.7}$$

and

$$q = \frac{log(|\hat{R}|)e^{-rh}}{2\alpha(1-e^{-rh})} + \frac{(\beta e^{-rh} - r)h}{\alpha(1-e^{-rh})} - \frac{1}{\alpha} log(\frac{h}{e^{rh}-1}). \tag{C.8}$$

Proof: We assume that the value function after trade at period ℓ is given by expression 5.4. We first show that the demand 5.1 maximizes the utility of the (i,ℓ) demand incarnation. We then study the optimization problem of the $(i,\ell-1)$ consumption incarnation and determine $c_{i,\ell-1}$. We also show that the function 5.4 satisfies the Bellman equation, i.e. that the value function after trade at period $\ell-1$ is given by expression 5.4. The transversality condition follows as in the competitive case. Therefore, the value function is indeed given by expression 5.4.

Step 1: Optimal Demand

The optimal demand maximizes

$$-E_{u_{j,\ell},j\neq i} exp(-\alpha(\frac{1-e^{-rh}}{h}\left(-d_\ell \frac{h}{1-e^{-rh}}\epsilon_{i,\ell} - p_\ell x_{i,\ell}(p_\ell)\right)$$

$$+d_\ell(e_{i,\ell-1}+\epsilon_{i,\ell}+x_{i,\ell}(p_\ell)) + F(Q, \begin{pmatrix} e_{i,\ell-1}+\epsilon_{i,\ell}+x_{i,\ell}(p_\ell) \\ \frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N} \end{pmatrix}))),$$

where the price, p_ℓ, is given by equation 5.8 and depends on $x_{i,\ell}(p_\ell)$. Instead of solving this problem, we proceed as in the private information case, and solve a simpler problem. We assume that the residual demand of the incarnations of the other agents is known (i.e. $\sum_{j\neq i} u_{j,\ell}$ is known) and choose a trade, x, on this residual demand. The market-clearing condition 5.3 implies that x maximizes

$$-\frac{1-e^{-rh}}{h}\left(\frac{A}{B}d_\ell - \frac{A_e}{B}\frac{\sum_{k=1}^{N}(e_{k,\ell-1}+\epsilon_{k,\ell})}{N} - \frac{a}{B}\frac{\sum_{j\neq i}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N-1} + \frac{1}{B}\frac{\sum_{j\neq i} u_{j,\ell}}{N-1}\right.$$

$$\left. + \frac{x}{(N-1)B}\right)x - d_\ell\epsilon_{i,\ell} + d_\ell(e_{i,\ell-1}+\epsilon_{i,\ell}+x) + F(Q, \begin{pmatrix} e_{i,\ell-1}+\epsilon_{i,\ell}+x \\ \frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N} \end{pmatrix}).$$

If the trade

$$a\left(\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}-(e_{i,\ell-1}+\epsilon_{i,\ell})\right)-\frac{\sum_{j\neq i}u_{j,\ell}}{N}$$

solves this problem, then the demand 5.1 is optimal since it produces this trade for all values of $\sum_{j\neq i}u_{j,\ell}$. Setting

$$x=a\left(\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}-(e_{i,\ell-1}+\epsilon_{i,\ell})\right)-\frac{\sum_{j\neq i}u_{j,\ell}}{N}+\Delta x,$$

the demand 5.1 is optimal if $\Delta x=0$ maximizes

$$\begin{aligned}
&-\frac{1-e^{-rh}}{h}\left(\frac{A}{B}d_\ell-\frac{A_e+a}{B}\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}+\frac{1}{B}\frac{\sum_{j\neq i}u_{j,\ell}}{N}+\frac{\Delta x}{(N-1)B}\right)\\
&\quad(a\left(\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}-(e_{i,\ell-1}+\epsilon_{i,\ell})\right)-\frac{\sum_{j\neq i}u_{j,\ell}}{N}+\Delta x)\\
&-d_\ell\epsilon_{i,\ell}+d_\ell\left((1-a)(e_{i,\ell-1}+\epsilon_{i,\ell})+a\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}-\frac{\sum_{j\neq i}u_{j,\ell}}{N}+\Delta x\right)\\
&+F(Q,\left(\frac{(1-a)(e_{i,\ell-1}+\epsilon_{i,\ell})+a\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}-\frac{\sum_{j\neq i}u_{j,\ell}}{N}+\Delta x}{\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}}\right)).
\end{aligned}\tag{C.9}$$

Conditions 5.5, 5.6, 5.7, and 5.9 ensure that the first-order condition is satisfied for $\Delta x=0$. Condition C.3 ensures that expression C.9 is concave.

Expression C.9, evaluated for $\Delta x=0$, corresponds to the case where the (i,ℓ) demand incarnation submits the demand 5.1. Since it submits the demand 5.1 plus the noise $u_{i,\ell}$, expression C.9 becomes

$$\begin{aligned}
&-\frac{1-e^{-rh}}{h}\left(\frac{A}{B}d_\ell-\frac{A_e+a}{B}\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}+\frac{1}{B}\frac{\sum_{j=1}^{N}u_{j,\ell}}{N}\right)\\
&\quad(a\left(\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}-(e_{i,\ell-1}+\epsilon_{i,\ell})\right)+u_{i,\ell}-\frac{\sum_{j=1}^{N}u_{j,\ell}}{N})\\
&-d_\ell\epsilon_{i,\ell}+d_\ell\left((1-a)(e_{i,\ell-1}+\epsilon_{i,\ell})+a\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}+u_{i,\ell}-\frac{\sum_{j=1}^{N}u_{j,\ell}}{N}\right)\\
&+F(Q,\left(\frac{(1-a)(e_{i,\ell-1}+\epsilon_{i,\ell})+a\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}+u_{i,\ell}-\frac{\sum_{j=1}^{N}u_{j,\ell}}{N}}{\frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N}}\right)).
\end{aligned}$$

Proceeding as in the competitive case, we can rewrite this expression as

$$d_\ell e_{i,\ell-1} + F(\hat{Q}', \begin{pmatrix} (e_{i,\ell-1} + \epsilon_{i,\ell}) \\ \frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N} \\ u_{i,\ell} \\ \frac{\sum_{j=1}^{N} u_{j,\ell}}{N} \end{pmatrix}). \tag{C.10}$$

Step 2: Bellman Equation

We first compute the expectation of the period ℓ value function w.r.t. period $\ell - 1$ information, i.e. w.r.t. δ_ℓ, $\epsilon_{j,\ell}$, and $u_{j,\ell}$. Using the budget constraint 3.7 and expression C.10, we have to compute

$$E_{\ell-1} exp(-\alpha(\frac{1-e^{-rh}}{h} e^{rh}(M_{i,\ell-1} + d_{\ell-1}e_{i,\ell-1}h - c_{i,\ell-1}h) + (d_{\ell-1} + \delta_\ell)e_{i,\ell-1}$$

$$+F(\hat{Q}', \begin{pmatrix} (e_{i,\ell-1} + \epsilon_{i,\ell}) \\ \frac{\sum_{j=1}^{N}(e_{j,\ell-1}+\epsilon_{j,\ell})}{N} \\ u_{i,\ell} \\ \frac{\sum_{j=1}^{N} u_{j,\ell}}{N} \end{pmatrix}) + q) - \beta h).$$

Proceeding as in the competitive case, we get

$$exp(-\alpha(\frac{1-e^{-rh}}{h} e^{rh}(M_{i,\ell-1} - c_{i,\ell-1}h) + e^{rh}d_{\ell-1}e_{i,\ell-1} - \frac{1}{2}\alpha\sigma^2 h e_{i,\ell-1}^2$$

$$+F(\hat{Q}' - \hat{R}', \begin{pmatrix} e_{i,\ell-1} \\ \frac{\sum_{j=1}^{N} e_{j,\ell-1}}{N} \\ 0 \\ 0 \end{pmatrix}) + \frac{1}{2\alpha} log|\hat{R}| + q) - \beta h).$$

Finally, using the definition of P, we can rewrite this expression as

$$exp(-\alpha(\frac{1-e^{-rh}}{h} e^{rh}(M_{i,\ell-1} - c_{i,\ell-1}h) + e^{rh}d_{\ell-1}e_{i,\ell-1}$$

$$e^{rh}F(P, \begin{pmatrix} e_{i,\ell-1} \\ \frac{\sum_{j=1}^{N} e_{j,\ell-1}}{N} \end{pmatrix}) + \frac{1}{2\alpha} log|\hat{R}| + q) - \beta h). \tag{C.11}$$

The $(i, \ell - 1)$ consumption incarnation chooses consumption, $c_{i,\ell-1}$, to maximize

$$-exp(-\alpha c_{i,\ell-1})h - exp(-\alpha(\frac{1-e^{-rh}}{h} e^{rh}(M_{i,\ell-1} - c_{i,\ell-1}h) + e^{rh}d_{\ell-1}e_{i,\ell-1}$$

$$+e^{rh}F(P,\begin{pmatrix} e_{i,\ell-1} \\ \frac{\sum_{j=1}^{N} e_{j,\ell-1}}{N} \end{pmatrix}) + \frac{1}{2\alpha}log|\hat{R}| + q) - \beta h).$$

As in the competitive case, the maximum is

$$-exp(-\alpha(\frac{1-e^{-rh}}{h}M_{i,\ell-1} + d_{\ell-1}e_{i,\ell-1} + F(P,\begin{pmatrix} e_{i,\ell-1} \\ \frac{\sum_{j=1}^{N} e_{j,\ell-1}}{N} \end{pmatrix})$$

$$+\frac{1}{2\alpha}log|\hat{R}|e^{-rh} + qe^{-rh} + \frac{(\beta e^{-rh} - r)h}{\alpha} - \frac{1}{\alpha}(1-e^{-rh})log(\frac{h}{e^{rh}-1}))).$$

This is equal to the period $\ell-1$ value function if conditions C.7 and C.8 hold. Q.E.D.

The optimality conditions of proposition C.2 are those of proposition C.1, plus condition 5.9. The Bellman conditions are different than in proposition C.1, since the matrices R' and P are different, and $|\hat{R}| \neq |R|$. However, it is easy to check that for $\sigma_u^2 = 0$ the Bellman conditions are the same. Therefore, our refinement implies the additional optimality condition 5.9.

C.3 The System and the Solution

The system is the Bellman condition C.4, where P, R', and R are given by equations A.7, A.6, and A.5, respectively, Q' by equations A.18, A.19, and A.20, and $a = (N-2)/(N-1)$. In the public information case, equations A.19 and A.20 follow from equations C.1, C.2, and 5.6. The fact that $a = (N-2)/(N-1)$ follows from equations 5.7 and 5.9.

Proposition C.3 *The system has a solution for small σ_e^2. For $\sigma_e^2 = 0$ the matrix Q is given by*

$$Q_{1,1} = -\frac{\alpha\sigma^2 he^{-rh}}{1-\frac{1}{(N-1)^2}e^{-rh}}, \quad Q_{1,2} = -Q_{2,2} = \frac{\frac{N(N-2)}{(N-1)^2}Q_{1,1}e^{-rh}}{1-e^{-rh}}. \tag{C.12}$$

Proof: We proceed exactly as in the competitive case, but with $a = (N-2)/(N-1)$ instead of $a = 1$. Q.E.D.

D The No-Trade Case

We first study agents' optimization problem and, in proposition D.1, provide a set of sufficient conditions. The conditions can be reduced to a non-linear equation. We solve the equation in closed-form for $\sigma_e^2 = 0$ and extend the solution for small σ_e^2, in proposition D.2.

D.1 The Optimization Problem

We formulate agent i's optimization problem as a dynamic programming problem. The state variables are the consumption good holdings, $M_{i,\ell}$, the dividend, d_ℓ, the stock holdings, $e_{i,\ell}$, and the average stock holdings, $\sum_{j=1}^{N} e_{j,\ell}/N$. Average stock holdings will, of course, be irrelevant. We include them to facilitate the comparison of the no-trade case to the competitive, private information, and public information cases. The only control variable is the consumption, $c_{i,\ell-1}$. The dynamics of $M_{i,\ell}$, d_ℓ, and $\sum_{j=1}^{N} e_{j,\ell}/N$ are given by equations 3.7, 2.1, and 3.8, respectively. The dynamics of $e_{i,\ell}$ are simply

$$e_{i,\ell} = e_{i,\ell-1} + \epsilon_{i,\ell}.$$

Agent i's optimization problem, (P_n), is

$$\sup_{c_{i,\ell}} -E_0(h \sum_{\ell=0}^{\infty} exp(-\alpha c_{i,\ell} - \beta \ell h))$$

subject to

$$M_{i,\ell} = e^{rh}(M_{i,\ell-1} + d_{\ell-1} e_{i,\ell-1} h - c_{i,\ell-1} h) - d_\ell \frac{h}{1 - e^{-rh}} \epsilon_{i,\ell},$$

$$d_\ell = d_{\ell-1} + \delta_\ell,$$

$$e_{i,\ell} = (e_{i,\ell-1} + \epsilon_{i,\ell}),$$

$$\frac{\sum_{j=1}^{N} e_{j,\ell}}{N} = \frac{\sum_{j=1}^{N} (e_{j,\ell-1} + \epsilon_{j,\ell})}{N},$$

and the transversality condition

$$\lim_{\ell \to \infty} E_0 V_n \left(M_{i,\ell}, d_\ell, e_{i,\ell}, \frac{\sum_{j=1}^{N} e_{j,\ell}}{N} \right) exp(-\beta \ell h) = 0,$$

where V_n is the value function. Our candidate value function is

$$V_n\left(M_{i,\ell}, d_\ell, e_{i,\ell}, \frac{\sum_{j=1}^N e_{j,\ell}}{N}\right) = -exp(-\alpha(\frac{1-e^{-rh}}{h}M_{i,\ell}+d_\ell e_{i,\ell}+F(Q,\left(\begin{array}{c} e_{i,\ell} \\ \frac{\sum_{j=1}^N e_{j,\ell}}{N} \end{array}\right))+q)). \tag{D.1}$$

In proposition D.1 we provide sufficient conditions for the function D.1 to be the value function. To state the proposition, we set $Q' = Q$, and use the matrices Σ^2, R, R', and P, defined in appendix A.

Proposition D.1 *The function D.1 is the value function, if the Bellman conditions*

$$Q = P \tag{D.2}$$

and

$$q = \frac{log(|R|)e^{-rh}}{2\alpha(1-e^{-rh})} + \frac{(\beta e^{-rh} - r)h}{\alpha(1-e^{-rh})} - \frac{1}{\alpha}log(\frac{h}{e^{rh}-1}) \tag{D.3}$$

hold.

Proof: To show the Bellman equation, we proceed in two steps. First, we compute the expectation of the RHS w.r.t. period $\ell - 1$ information. Second, we show that the Bellman conditions D.2 and D.3 are sufficient for the function D.1 to satisfy the Bellman equation. The proof, and the proof of the transversality condition, are as in the competitive case. Q.E.D.

D.2 The Equation and the Solution

The equation follows from the Bellman condition D.2, where P, R', and R are given by equations A.7, A.6, and A.5, respectively, and $Q' = Q$. Setting $Q_{1,2} = Q_{2,2} = 0$, it is easy to check that the equations corresponding to $Q_{1,2}$ and $Q_{2,2}$ are satisfied. We are left with the equation corresponding to $Q_{1,1}$.

Proposition D.2 *The equation has a solution for small σ_e^2. For $\sigma_e^2 = 0$ the solution is*

$$Q_{1,1} = -\frac{\alpha\sigma^2 h e^{-rh}}{1-e^{-rh}}. \tag{D.4}$$

Proof: To solve the equation for $\sigma_e^2 = 0$ we proceed as in the competitive case. To extend the solution for small σ_e^2, we apply the implicit function theorem to the function

$$G(Q_{1,1}, \sigma^2, \sigma_e^2, h, N) = \frac{Q_{1,1} - P_{1,1}}{h},$$

at the point $\mathcal{A}$, where $\sigma_e^2 = 0$ and $Q_{1,1}$ is given by expression D.4 of the proposition. It is easy to check that the partial derivative of G w.r.t. $Q_{1,1}$ is strictly positive. Q.E.D.

E Welfare Analysis

We first prove proposition 6.1.

Proof: Noting that $d_0 = d$, $M_{i,0} = M$, and $e_{i,0} = e$, each CEQ is

$$\frac{1-e^{-rh}}{h}M + de + (Q_{1,1} + 2Q_{1,2} + Q_{2,2})e^2 + q.$$

The 4 CEQ's are equal for $\sigma_e^2 = 0$. Indeed, propositions A.2, B.2, C.3, and D.2 imply that

$$Q_{1,1} + Q_{1,2} = -\frac{\alpha\sigma^2 h e^{-rh}}{1-e^{-rh}}, \quad Q_{1,2} + Q_{2,2} = 0. \tag{E.1}$$

Therefore, $Q_{1,1} + 2Q_{1,2} + Q_{2,2}$ is the same in the competitive, private information, public information, and no-trade cases. To show that q is also the same, we need to show that R is the same. Indeed, in all 4 cases $R = I$. Since the 4 CEQ's are equal, we need to use l'Hospital's rule and replace them by their partial derivatives w.r.t. σ_e^2 at $\sigma_e^2 = 0$. We first show that the partial derivative of $Q_{1,1} + 2Q_{1,2} + Q_{2,2}$ is the same in all 4 cases. We then compute the partial derivative of q.

Partial Derivative of $Q_{1,1} + 2Q_{1,2} + Q_{2,2}$

We differentiate implicitly the Bellman condition $Q = P$ w.r.t. σ_e^2 at $\sigma_e^2 = 0$. Noting that $\Sigma^2 = 0$ and $R = I$, we get

$$dQ = dP = (dQ' - \alpha Q' d\Sigma^2 Q')e^{-rh}.$$

From this matrix equation, we "extract" one scalar equation (equation $\mathcal{S}$), multiplying from the left by $x^t = (1, 1)$ and from the right by x. To show that the partial derivative of $Q_{1,1} + 2Q_{1,2} + Q_{2,2}$ is the same in the competitive, private information, public information, and no-trade cases, we will show that (i) equation $\mathcal{S}$ is the same in all 4 cases, and (ii) the terms in $dQ_{i,j}$ in equation $\mathcal{S}$ are only in $d(Q_{1,1} + 2Q_{1,2} + Q_{2,2})$. Since the terms obtained from $\alpha Q' d\Sigma^2 Q'$ are in $Q'_{1,1} + Q'_{1,2}$ and $Q'_{1,2} + Q'_{2,2}$, and since, by equations A.18, A.19, A.20, and E.1, these are the same in all 4 cases, equation $\mathcal{S}$ is the same in all 4 cases. Moreover since, by equations A.18, A.19, and A.20,

$$d(Q'_{1,1} + 2Q'_{1,2} + Q'_{2,2}) = d(Q_{1,1} + 2Q_{1,2} + Q_{2,2}),$$

the terms in $dQ_{i,j}$ in equation $\mathcal{S}$ are only in $d(Q_{1,1} + 2Q_{1,2} + Q_{2,2})$.

Partial Derivative of q

We differentiate the Bellman condition

$$q = \frac{log(|R|)e^{-rh}}{2\alpha(1-e^{-rh})} + \frac{(\beta e^{-rh} - r)h}{\alpha(1-e^{-rh})} - \frac{1}{\alpha} log(\frac{h}{e^{rh}-1})$$

w.r.t. σ_e^2 at $\sigma_e^2 = 0$. Noting that $d|R| = dR_{1,1} + dR_{2,2}$ and $dR = \alpha Q' d\Sigma^2$, we get

$$\frac{\partial q}{\partial \sigma_e^2} = \frac{he^{-rh}}{2(1-e^{-rh})}(Q'_{1,1} + \frac{2Q'_{1,2} + Q'_{2,2}}{N}).$$

Since $Q'_{1,1} + Q'_{1,2}$ and $Q'_{1,2} + Q'_{2,2}$ are the same in all 4 cases, l'Hospital's rule implies that

$$L = \frac{(Q'_{1,1})_c - (Q'_{1,1})_{pr}}{(Q'_{1,1})_c - (Q'_{1,1})_n},$$

in the private information case and

$$L = \frac{(Q'_{1,1})_c - (Q'_{1,1})_p}{(Q'_{1,1})_c - (Q'_{1,1})_n},$$

in the public information case, where $(Q'_{1,1})_c$, $(Q'_{1,1})_{pr}$, $(Q'_{1,1})_p$, and $(Q'_{1,1})_n$ denote $Q'_{1,1}$ in the competitive, private information, public information, and no-trade cases, respectively. Using equation A.18 and propositions A.2, B.2, C.3, and D.2, we get equation 6.3. Q.E.D.

We now prove corollary 6.1.

Proof: We use equation 6.3 for $\sigma_e^2 = 0$. We then use continuity of $\partial L/\partial h$ to extend our results for small σ_e^2. In the public information case, we set $a = (N-2)/(N-1)$ in equation 6.3. In the private information case, we combine equations 4.9 and 6.3 into

$$L = (1-a)\frac{(1-e^{-rh})^2}{\alpha\sigma^2 h^2 e^{-rh}} \frac{a}{(N-1)B}.$$

Using equation 4.8, we can write this equation as

$$L = \frac{1-a}{N-1}. \tag{E.2}$$

Since a increases in h, L decreases in h. Q.E.D.

We finally prove corollary 6.2.

Proof: In the public information case, we set $a = (N-2)/(N-1)$ in equation 6.3. In the private information case, equation 6.6 follows from equation E.2 and

$$a = 1 - \frac{1}{(N-1)(1-e^{-rh})} + o(\frac{1}{N-1}).$$

To prove this fact, we set $x = 1/(N-1)$, write equation 4.10 as

$$xa^2e^{-rh} - a(2xe^{-rh} + (1-e^{-rh})) - (x-1)(1-e^{-rh}) = 0,$$

and differentiate implicitly w.r.t. x at $x = 0$, $a = 1$. Equation 6.7 follows from equation E.2 and the fact that a goes to 0 as h goes to 0. Q.E.D.

Notes

[1]See, for instance, Kraus and Stoll (1972), Holthausen, Leftwich and Mayers (1987, 1990), Hausman, Lo, and McKinlay (1992), Chan and Lakonishok (1993), and Keim and Madhavan (1996).

[2]See, for instance, Chan and Lakonishok (1995) and Keim and Madhavan (1995,1996,1998).

[3]For instance, the growth and growth and income funds in the 1994 Morningstar CD turn over 76.8% of their portfolios every year. However, they underperform the market by .5%. See Chevalier and Ellison (1998).

[4]In the foreign exchange market, inter-dealer trading is 80% of total volume (Lyons (1996)). In the London Stock Exchange and the Nasdaq the numbers are 35% and 15% (Reiss and Werner (1995) and Gould and Kleidon (1994)).

[5]For surveys of the market microstructure literature, see Admati (1991) and O'Hara (1995). Admati and Pfleiderer (1988) endogenize the behavior of uninformed agents in a limited way. Bertsimas and Lo (1998) study the behavior of a large agent who may or may not be informed. However, they do not endogenize prices.

[6]See, for instance, Stokey (1981), Bulow (1982), and Gul, Sonnenschein, and Wilson (1986).

[7]Another difference is that this paper introduces risk aversion and a different trading mechanism. The closest paper in that respect is DeMarzo and Bizer (1998) which allows for increasing marginal costs and a similar trading mechanism.

[8]Cramton (1984), Cho (1990), and Ausubel and Deneckere (1992) assume that the monopolist's cost is private information.

[9]See, for instance Gresik and Satterthwaite (1989), Satterthwaite and Williams (1989), and Rustichini, Satterthwaite, and Williams (1994). Wilson (1986) considers a dynamic model with risk-neutral agents who have 0-1 demands.

[10]Economides and Schwartz (1995) suggest this as one of the reasons why the NYSE

should incorprorate a call market into its continuous trading system.

[11]We introduce the consumption good endowment for tractability. With this endowment, the risk represented by the stock endowment is independent of the dividend level. The model without the consumption good endowment is somewhat more complicated but produces the same results. The consumption good endowment is consistent with the inter-dealer market interpretation of the model. If a dealer receives a positive endowment shock, i.e. buys stock from a customer, he pays the customer in return.

[12]The transversality condition 3.9 is standard for optimal consumption-investment problems. See, for instance, Merton (1969) and Wang (1994).

[13]We assume that the agent sells $a\Delta x$ shares at period $\ell+1$, while for $Q_{1,1}+Q_{1,2}$ we assumed that the agent keeps all Δx shares. By the envelope theorem, the two assumptions are equivalent. In the private and public information cases, it is easier to determine $Q_{1,1}$ making the first assumption. Indeed, if the agent sells $a\Delta x$ shares, $p_{\ell+1}$ is independent of Δx and equal to 0. If the agent keeps all Δx shares, we have to take into account the change in $p_{\ell+1}$. For $Q_{1,1}+Q_{1,2}$ the change does not matter because the agent does not trade in expectation at period $\ell+1$.

[14]Agent i can infer $\sum_{j\neq i} e_{j,\ell-1}$ from the period $\ell-1$ price, $p_{\ell-1}$.

[15]Note that there may exist other Nash equilibria in which demands are non-linear or depend on more variables than p_ℓ, d_ℓ, and $e_{i,\ell-1}+\epsilon_{i,\ell}$ (such as "trigger-strategy" equilibria). Studying these equilibria is beyond the scope of this paper.

[16]For $\sigma_e^2=0$, the result follows from equation 4.10. For $\sigma_e^2>0$ (and not necessarily small) the proof is available upon request.

[17]It might seem that we get indeterminacy only because we allow agent i's demand to depend on other agents' stock holdings, thus introducing the additional parameter A_e. This is incorrect. If in the private information case we allow agent i's demand to depend on his expectations of other agents' stock holdings, we will obtain an additional optimality condition. This condition will imply that $A_e=0$.

[18]See Selten (1975) and chapter 8 in Fudenberg and Tirole (1991).

[19]We do not study how CEQ_{pr} depends on h because when h changes, the dividend process and preferences change. This problem can be avoided by considering a "continuous" model where dividends follow a Brownian motion and preferences are over consumption flow. We do not present the continuous model because it is complicated. (The Bellman conditions are differential rather than algebraic equations.) We should note that the continuous model produces the same welfare loss, L, as the discrete model.

References

ADMATI, A. (1991), "The Informational Role of Prices: A Review Essay", *Journal of Monetary Economics*, **28**, 347-361.

ADMATI, A. and PFLEIDERER, P. (1988), "A Theory of Intraday Patterns: Volume and Price Variability", *The Review of Financial Studies*, **1**, 3-40.

AUSUBEL, L. and DENECKERE, R. (1992), "Durable Goods Monopoly with Incomplete Information", *Review of Economic Studies*, **59**, 795-812.

BACK, K. and ZENDER, J. (1993), "Auctions of Divisible Goods: On the Rationale for the Treasury Experiment", *Review of Financial Studies*, **6**, 733-764.

BERTSIMAS, D. and LO, A. (1998), "Optimal Control of Execution Costs", mimeo, MIT.

BULOW, J. (1982), "Durable Goods Monopolists", *Journal of Political Economy*, **90**, 314-322.

CHAN, L. and LAKONISHOK, J. (1993), "Institutional Trades and Intra-Day Stock Price Behavior", *Journal of Financial Economics*, **33**, 173-199.

CHAN, L. and LAKONISHOK, J. (1995), "The Behavior of Stock Prices Around Institutional Trades", *Journal of Finance*, **50**, 1147-1174.

CHAU, M. (1998), "Dynamic Trading and Market-Making with Inventory Costs and Asymmetric Information", mimeo MIT.

CHEVALIER, J. and ELLISON, G. (1998), "Do Mutual Fund Managers Matter? An Empirical Investigation of the Performance, Career Concerns, and Behavior of Fund Managers", mimeo MIT.

CHO, I-K. (1990), "Uncertainty and Delay in Bargaining", *Review of Economic Studies*, **57**, 575-596.

CRAMTON, P. (1984), "Bargaining with Incomplete Information: An Infinite-Horizon Model with Continuous Uncertainty", *Review of Economic Studies*, **51**, 579-593.

DEMARZO, P. and BIZER, D. (1998), "Sequential Trade and the Coase Conjecture: A Time-Consistent Model of Monopoly with Applications to Finance", mimeo, University of California Berkeley.

ECONOMIDES, N. and SCHWARTZ, R. (1995), "Electronic Call Market Trading", *The Journal of Portfolio Management*, **21**, 10-18.

FUDENBERG, D. and TIROLE, J. (1991), *Game Theory*, MIT press.

GLOSTEN, L. and MILGROM, P. (1985), "Bid, Ask and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders", *Journal of Financial Economics*, **13**, 71-100.

GOULD, J. and KLEIDON, A. (1994), "Market-Maker Activity on Nasdaq: Implications for Trading Volume", *Stanford Journal of Law, Business and Finance*, **1**, 1-17.

GRESIK, T. and SATTERTHWAITE, M. (1989), "The Rate at which a Simple Market Converges to Efficiency as the Number of Traders Increases. An Asymptotic Result for Optimal Trading Mechanisms", *Journal of Economic Theory*, **48**, 304-332.

GUL, F., SONNENSCHEIN, H. and WILSON, R. (1986), "Foundations of Dynamic Monopoly and the Coase Conjecture", *Journal of Economic Theory*, **39**, 155-190.

HAUSMAN, J., LO, A. and MCKINLAY, C. (1992), "An Ordered Probit Analysis of Transaction Stock Prices", *Journal of Financial Economics*, **31**, 319-379.

HOLTHAUSEN, R., LEFTWICH, R. and MAYERS, D. (1987), "The Effect of Large Block Transactions on Security Prices: A Cross-Sectional Analysis", *Journal of Financial Economics*, **19**, 237-267.

HOLTHAUSEN, R., LEFTWICH, R. and MAYERS, D. (1990), "Large Block Transactions, the Speed of Response and Temporary and Permanent Stock-Price Effects", *Journal of Financial Economics*, **26**, 71-95.

KEIM, D. and MADHAVAN, A. (1995), "The Anatomy of the Trading Process", *Journal of Financial Economics*, **37**, 371-398.

KEIM, D. and MADHAVAN, A. (1996), "The Upstairs Market for Large-Block Transactions: Analysis and Measurement of Price Effects", *Review of Financial Studies*, **9**, 1-36.

KEIM, D. and MADHAVAN, A. (1998), "Execution Costs and Investment Performance: An Empirical Analysis of Institutional Equity Trades", mimeo, USC.

KLEMPERER, P. and MEYER, M. (1989), "Supply Function Equilibria in Oligopoly under Uncertainty", *Econometrica*, **57**, 1243-1277.

KRAUS, A. and STOLL, H. (1972), "Price Impacts of Block Trading on the New York Stock Exchange", *Journal of Finance*, **27**, 569-588.

KYLE, A. (1985), "Continuous Auctions and Insider Trading", *Econometrica*, **53**, 1315-1336.

KYLE, A. (1989), "Informed Speculation and Imperfect Competition", *Review of Economic Studies*, **56**, 517-556.

LYONS, R. (1996), "Optimal Transparency in a Dealership Market with an Application to Foreign Exchange", *Journal of Financial Intermediation*, **5**, 225-254.

MERTON, R. (1969), "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case", *Review of Economics and Statistics*, **51**, 247-257.

O'HARA, M. (1995), *Market Microstructure Theory*, Blackwell Publishers.

REISS, P. and WERNER, I. (1995), "Transaction Costs in Dealer Markets: Evidence from the London Stock Exchange", *The Industrial Organization and Regulation of the Securities Industry*, A. Lo (ed.), University of Chicago Press.

RUSTICHINI, A., SATTERTHWAITE, M and WILLIAMS, S. (1994), "Convergence to Efficiency in a Simple Market with Incomplete Information", *Econometrica*, **62**, 1041-1063.

SATTERTHWAITE, M. and WILLIAMS, S. (1989), "The Rate of Convergence to Efficiency in the Buyer's Bid Double Auction as the Market Becomes Large", *Review of Economic Studies*, **56**, 477-498.

SELTEN, R. (1975), "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games", *International Journal of Game Theory*, **4**, 25-55.

STOKEY, N. (1981), "Rational Expectations and Durable Goods Pricing", *Bell Journal of Economics*, **12**, 112-128.

VAYANOS, D. (1998), "Strategic Trading in a Dynamic Noisy Market", mimeo MIT.

WANG, J. (1994), "A Model of Competitive Stock Trading Volume", *Journal of Political Economy*, **102**, 127-168.

WILSON, R. (1979), "Auctions of Shares", *Quarterly Journal of Economics*, **93**, 675-689.

WILSON, R. (1986), "Equilibria in Bid-Ask Markets", *Arrow and the Ascent of Economic Theory: Essays in the Honor of Kenneth J. Arrow*, G. Feiwel (ed.), The Macmillan Press Ltd.