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Exact Simulation of Generalised Vervaat Perpetuities

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Abstract

We consider a generalised Vervaat perpetuity of the form

\[ X = Y_1 W_1 + Y_2 W_1 W_2 + \ldots, \]

where \( W_i \sim U^{1/t} \) and \((Y_i)_{i\geq 0}\) is an i.i.d sequence of random variables independent from \((W_i)_{i\geq 0}\). Based on a distributional decomposition technique, we propose a novel method for exactly simulating the generalised Vervaat perpetuity. The general framework relies on the exact simulation of the truncated Gamma process, which we develop using a marked renewal representation for its paths. Furthermore, a special case arises when \( Y_i = 1 \), and \( X \) has the generalised Dickman distribution, for which we present an exact simulation algorithm using the marked renewal approach. In particular, this new algorithm is much faster than existing algorithms illustrated in Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012), and Cloud and Huber (2017) as well as being applicable to the general payments case. Examples and numerical analysis are provided to demonstrate the accuracy and effectiveness of our method.

Keywords: Vervaat perpetuities; Dickman distribution; Truncated Gamma process; Exact simulation; Marked Renewal Process.

Mathematics Subject Classification (2010): Primary: 60G51 ; Secondary: 60K15 - 65C05

1 Introduction

A stochastic perpetuity is a random variable which takes the form

\[ X = Y_1 W_1 + Y_2 W_1 W_2 + Y_3 W_1 W_2 W_3 + \ldots + Y_n W_1 W_2 \ldots W_n + \ldots, \]  (1.1)
where the $W_i$ and $Y_i$ are i.i.d random variables independent of each other. Perpetuities arise in a diverse range of fields such as economics (Embrechts et al., 1996), insurance mathematics (Nyrhinen, 2001), astrophysics (Chandrasekhar and Münch, 1950), analytic number theory (De Bruijn, 1951), and in the analysis of Hoare’s selection algorithm (Mahmoud et al., 1995). We refer to Vervaat (1979) and Embrechts and Goldie (1994) for a huge variety of references and examples of applications, ranging from the brightness of the Milky Way to ARCH processes of financial modelling. In the context of economics, the quantity $Y_i$ represents a random cashflow at time $i$ and $W_i$ represents the discount rate from time $i$ to time $i-1$. The random variable $X$ then represents the net present value of the stochastic cashflows.

When $W_i \overset{D}{=} U^{1/1}$, for $U \sim \text{Uniform}(0,1)$, the random variable $X$ is known as a generalised Vervaat perpetuity. It was shown in Vervaat (1979) that the series converges if and only if $\mathbb{E} \left( \log^+ |Y_i| \right) < \infty$, and it satisfies the distributional identity

$$X \overset{D}{=} W(X + Y), \quad (1.2)$$

where $W$ and $Y$ represent random variables distributed like $W_i$ and $Y_i$ respectively. Vervaat perpetuity has been much studied. Vervaat (1979) showed that the random variable $X$ is infinitely divisible, and obtained its Lévy Khintchine representation. This is the basis of our simulation framework, and we further show that under the mild assumption that the Laplace transform of $Y$ exists, $X$ turns out to be a Lévy process with Lévy measure proportional to $\frac{1}{y} \mathbb{P}(Y > y)$. There is also no loss of generality to consider only the case when $Y > 0$, as it can be shown that a two-sided perpetuity can be split into the difference of two one-sided perpetuities. Goldie and Grübel (1996) and Grübel and Rösler (1996) studied the tails of these perpetuities. Random variables of this type also appear as the limit of shot noise processes (Takács, 1954, 1955).

In this paper, we are interested in exactly simulating a generalised Vervaat perpetuity. Much attention has been given in the literature to the special case when $Y_i = 1$ almost surely, in which case $X$ has the generalised Dickman distribution (Dickman, 1930; Arratia, 1998). Even so, explicit expressions for the distribution of $X$ are difficult to obtain. The first simulation method is based on a density approximation given in Devroye (2001). Later, Fill and Huber (2010) and Devroye and Fawzi (2010) using a dominated coupling from the past procedure for Markov chains. Cloud and Huber (2017) proposed
an improvement by running both an upper and lower bound on the Markov chain. Chi (2012) suggested an alternative approach based on a rejection sampling procedure. However, these algorithms are efficient only for the Dickman distribution, where the parameter $t = 1$, and their speed slows down considerably when $t$ gets larger than 1.

We develop a faster and more efficient simulation algorithm for the generalised Dickman distribution with parameter $t$. Since it is viewed here as a Lévy process over the range of parameters $t$, with Lévy measure $\nu(dy) = \frac{1}{y} \mathbb{1}_{\{y<1\}} dy$, we thus call it the Dickman process. The paths of the Dickman process can be characterised by a marked renewal process where the renewal intervals are the hitting times of level 1, and the jumps are the overshoots at this time. By simulating its path along each hitting time of level 1, we obtain a simulation algorithm for the Dickman process. Similar to our method, the approach in Chi (2012) also utilises the truncated density of the perpetuity. However, the advantage of our algorithm is that it is based on a marked renewal process, which not only provides a pathwise probabilistic interpretation of the corresponding Lévy process, it also means that the incremental random variables we need to simulate are i.i.d. pairs, allowing us to employ vectorisation to reduce total computation time. This is not the case for the algorithm suggested in Chi (2012), where the incremental variables depend on the previous ones. We provide comparisons of the simulation speed of our algorithm with the algorithms suggested by Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012), and Cloud and Huber (2017). We found that our method outperforms the other algorithms and even more substantially when $t$ is large.

Blanchet and Sigman (2011) proposed a modification of the dominated coupling from the past algorithm to sample from generalised Vervaat perpetuities. The algorithm imposes similar regularity conditions as ours on the density of $Y_i$. However, it also requires $Y_i$ to have a heavy tail. In this paper, we develop a framework for simulating from generalised Vervaat perpetuities. In particular, we prove a distributional decomposition result, which states that a generalised Vervaat perpetuity $X$ can be represented as the sum of a Gamma or truncated Gamma process and independent compound Poisson processes. Thus, it can be sampled by generating each of these components separately and adding them up. The condition we need is

$$\lim_{y \downarrow 0} \frac{\mathbb{P}(Y < y)}{y} < \infty,$$  \hspace{1cm} (1.3)
which holds for a wide range of distributions, such as the Gamma (shape parameter larger than 1), Pareto, Weibull (light tailed), Beta, and Normal distributions as well as all discrete distributions (to avoid technicalities 0 is not a possible value). Whenever a density exists, it should possess a finite derivative at 0.

Using the marked renewal representation, we develop an exact simulation algorithm for the truncated Gamma process, which we define to be the Lévy process with Lévy measure $\nu(dy) = \frac{e^{-\mu y}}{y} \mathbb{1}_{(y<b)} dy$ for some truncation level $b$. The truncated Gamma process can be obtained from the Gamma process by restricting the size of its jumps to be bounded from above by $b$. It is also a generalised Vervaat perpetuity with $Y_i \sim V \wedge 1$, where $V \sim \text{Exp}(\mu)$. This algorithm is important, as it forms the basis of other algorithms for more general distributions of $Y_i$. We provide specific examples to demonstrate the methodology, and numerical results and graphical illustrations are given for the cases when $Y_i$ has a Gamma distribution, and when $Y_i$ has a Normal distribution.

This paper will be organised as follows. Section 2 gives a representation of the generalised Vervaat perpetuities as Gamma related processes. We also prove that a two sided perpetuity is the difference of two positive perpetuities. In Section 3, we study the specific cases of a Dickman process and a truncated Gamma process. The marked renewal path representation is described and the exact simulation algorithms developed. Numerical analysis and comparison with existing algorithms are provided. In Section 4, we prove the distributional decomposition result, which enables us to exactly simulate a generalised Vervaat perpetuity. Numerical examples are also given to demonstrate the accuracy and efficiency of our algorithms.

2 Generalised Vervaat Perpetuities

We obtain the following distributional decomposition result for a positive perpetuity $X > 0$.

**Theorem 2.1** Consider the generalised Vervaat perpetuity $X$ with $Y_i > 0$. We define the Laplace transforms $\hat{f}(\beta) = \mathbb{E}(\exp(-\beta X))$ and $\hat{g}(\beta) = \mathbb{E}(\exp(-\beta Y_i))$. Then, we have

$$\hat{f}(\beta) = \exp\left(-t \int_0^\infty \frac{1 - e^{-\beta y}}{y} \mathbb{P}(Y > y) dy\right).$$

(2.1)
Hence, \( \{ X \} \overset{D}{=} \{ Z_t \} \), where \( Z_t \) is a Lévy process with Lévy measure

\[
v(dy) = \frac{1}{y} \mathbb{P}(Y > y) dy.
\]  

(2.2)

\textbf{Proof.} For \( W = U^{1/t} \), the density of \( W \) is \( tw^{t-1} \). Then from (1.2), we have

\[
\hat{f}(\beta) = \int_0^1 \hat{f}(\beta w) \hat{g}(\beta w) tw^{t-1} dw,
\]  

(2.3)

and therefore,

\[
\beta^t \hat{f}(\beta) = \int_0^\beta \hat{f}(w) \hat{g}(w) tw^{t-1} dw.
\]

Differentiating with respect to \( \beta \) on both sides, we have

\[
\frac{d}{d\beta} \hat{f}(\beta) = -t \frac{1 - \hat{g}(\beta)}{\beta} \hat{f}(\beta).
\]

Solving the first order differential equation with respect to \( \beta \), we obtain

\[
\hat{f}(\beta) = \exp \left( -t \int_0^\beta \frac{1 - \hat{g}(u)}{u} du \right) = \exp \left( -t \int_0^\infty \frac{1 - e^{-\beta y}}{y} \mathbb{P}(Y > y) dy \right).
\]  

(2.4)

and the result (2.1) follows. Thus, \( \{ X \} \overset{D}{=} \{ Z_t \} \), where \( Z_t \) has Lévy measure defined in (2.2). \( \blacksquare \)

We can see that \( Z_t \) is a Gamma related Lévy process with Lévy measure density proportional to \( \mathbb{P}(Y > y)y^{-1} \). For a two-sided perpetuity where \( Y \) takes values in the whole real line, the following result shows that it is the difference of two positive perpetuities and thus it suffices to study only positive perpetuities.

\textbf{Corollary 2.2} Suppose now that \( \mathbb{P}(Y < 0) > 0 \). Then,

\[
\hat{f}(\beta) = \exp \left( -t \left( p \int_0^\infty \frac{1 - e^{-\beta y}}{y} \mathbb{P}(Y > y|Y > 0) dy + (1 - p) \int_0^\infty \frac{1 - e^{\beta y}}{y} \mathbb{P}(Y < -y|Y < 0) dy \right) \right),
\]

(2.5)

where \( p = \mathbb{P}(Y > 0) \).
Proof. For two sided $Y$, the Laplace transform of the perpetuity $X$ satisfies the following,

\[
\exp \left( -t \int_0^\infty \frac{1 - e^{-uy}}{u} \, du \right) = \exp \left( -t \left( \int_0^\infty \int_0^\infty \frac{1 - e^{-uy}}{u} \, dy \, du + \int_0^\infty \int_0^\infty e^{uy} \, P(Y < -y) \, dy \, du \right) \right) \\
= \exp \left( -t \left( \int_0^\infty \int_0^\infty \frac{1 - e^{-uy}}{u} \, dy \, du + \int_0^\infty \int_0^\infty e^{uy} \, P(Y < -y) \, dy \, du \right) \right) \\
= \exp \left( -t \left( \int_0^\infty \frac{1 - e^{-\beta y}}{y} \, P(Y > y) \, dy \, dy + \int_0^\infty \frac{1 - e^{\beta y}}{y} \, P(Y < -y) \, dy \, dy \right) \right) \\
= \exp \left( -t \left( \int_0^\infty \frac{1 - e^{-\beta y}}{y} \, P(Y > y|Y > 0) \, dy + (1 - p) \int_0^\infty \frac{1 - e^{\beta y}}{y} \, P(Y < -y|Y < 0) \, dy \right) \right),
\]

which is easily seen as the difference of two positive perpetuities, one with $W_i \overset{D}{=} U^{1/pt}$ and $Y_i \overset{D}{=} Y|Y > 0$, and the other with $W_i \overset{D}{=} U^{1/(1-p)t}$ and $Y_i \overset{D}{=} -Y|Y < 0$. ■

Our aim is to simulate the generalised Vervaat perpetuity $X$ for a given distribution of $Y_i$. In the simplest case when $Y_i = 1$ almost surely, $X$ is known as the Vervaat perpetuity and it reduces to the Dickman process. When $Y_i \sim V$, where $V \sim Exp(\mu)$, we obtain the Gamma process with parameter $\mu$, and when $Y_i \sim V \land 1$, we obtain the truncated Gamma process. Making use of a marked renewal representation for the paths, we develop fast and exact simulation algorithms for these perpetuities. These cases are important as they form the basis of the other algorithms. We will then give other examples to demonstrate how our methodology can be adapted to general random variables $Y_i$. Our method works for any $Y_i$ satisfying (1.3).

3 Generalised Dickman distribution and the Truncated Gamma process

When $Y \sim Exp(\mu)$, we have $P(Y > y) = e^{-\mu y}$, and thus

\[
\hat{f}(\beta) = \exp \left( -t \int_0^\infty (1 - e^{-\beta y}) \frac{1}{y} e^{-\mu y} \, dy \right),
\]

which is the Laplace transform of a Gamma process with parameter $\mu$. This can be simulated easily by sampling from the Gamma distribution.
When $Y = 1$ almost surely, we have $\mathbb{P}(Y > y) = 1_{\{y < 1\}}$, and thus
\[
\hat{f}(\beta) = \exp \left( -t \int_0^1 \frac{1 - e^{-\beta y}}{y} \, dy \right),
\]
which is the Laplace transform of a generalised Dickman distribution with parameter $t$. This is also a Lévy process $Z_t$ with Lévy measure $\frac{1}{y} 1_{\{y < 1\}}$, and hence we coin it the Dickman process.

When $Y = V \land 1$, where $V \sim \text{Exp}(\mu)$, we have $\mathbb{P}(Y > y) = e^{-\mu y} 1_{\{y < 1\}}$, and thus
\[
\hat{f}(\beta) = \exp \left( -t \int_0^1 \left( 1 - e^{-\beta y} \right) \frac{e^{-\mu y}}{y} \, dy \right),
\]
which is the Laplace transform of a truncated Gamma process with parameter $\mu$. The truncated Gamma process refers to the Gamma process with truncated Lévy measure $\frac{e^{-\mu y}}{y} 1_{\{y < 1\}}$.

To simulate the Dickman process and the truncated Gamma process, we utilise a marked renewal process representation of its sample paths. We now describe the sample paths of truncated subordinators using a marked renewal process.

### 3.1 The marked renewal process

The paths of the truncated subordinator $Z_t$ can be characterised by a marked renewal process. First, we define a sequence of hitting times $T_1, T_2, T_3, \ldots$, and denoting $S_i = \sum_{j=1}^i T_j$, let
\[
T_i := \inf \{ t > S_{i-1} | Z_t > Z_{S_{i-1}} + 1 \}, \quad i = 2, 3, \ldots
\]
These $T_1, T_2, \ldots$ are treated as the holding times for the renewal process. We further define $M_1, M_2, \ldots$ to be the overshoots at time $S_1, S_2, \ldots$, i.e.
\[
M_i := Z_{S_i} - Z_{S_{i-1}} - 1.
\]

Hence, at time $S_i$ the process will increase by $(1 + M_i)$ units for all $i$. Since the process $Z_t$ has independent and stationary increments, each pair of $(T_i, M_i)$ are independent and identically distributed with joint density given in Theorem 3.1. In addition, $M_i$ will be bounded by 0 and 1 for all $i$ as the jump sizes of the process are restricted by 1. The value of the process at time $S_n$ will be $Z_{S_n} = \sum_{i=1}^n (1 + M_i)$. 
The position of the process $Z_t$ at time $t$ can therefore be expressed as the sum of a marked renewal process with a random variable,

$$Z_t = \sum_{i=1}^{N_t} (1 + M_i) + (Z_t - Z_{S_{N_t}}).$$

(3.6)

where $N_t = \sum_{i=1}^{\infty} 1_{\{S_i \leq t\}}$ is the renewal process determined via (3.4) such that $S_{N_t} \leq t < S_{N_t+1}$, and we have

$$Z_t - Z_{S_{N_t}} \overset{D}{=} Z_{t-S_{N_t}}|S_{N_t+1} > t \overset{D}{=} Z_t | T > t,$$

(3.7)

where the last equality is due to the strong Markov property of $Z_t$.

We illustrate the marked renewal framework graphically in Figure 1. The sum in the first part of (3.6) represents the position of the truncated process at time $S_{N_t}$ just before it reaches $t$. The second term in (3.6) represents the final movement of the process within time $t - N_t$. Thus, $Z_t$ can be simulated by generating pairs of hitting times and overshoots $(T_i, M_i)$, stopping when the sum of $T_i$ that have
been generated, say $S_{N_i + 1}$, becomes larger than the input $t$. We then generate the final part $\{Z_t | T > t\}$ and return the entire sum (3.6).

3.2 Hitting times and overshoots

Now, we obtain analytically the first hitting times and overshoots of the truncated Gamma process and the Dickman process. This is given in the following Theorem 3.1.

**Theorem 3.1** Let $T$ be the first hitting time of level 1 of a truncated Gamma process $Z_t$ with $\mu \geq 0$, and $M$ the overshoot at this time. Then the joint distribution of $(T, M)$ is given by

$$f_{T,M}(t, m) = \frac{1}{m} \int_{\mu}^{\infty} \frac{e^{\mu(1+m)} - e^{-\mu(t)}}{\Gamma(t)} \frac{y^{m-1} e^{-y}}{1 + m - y} dy, \quad \text{for } \mu > 0,$$

(3.8)

where $t \in (0, \infty)$, $m \in (0, 1)$.

For the Dickman process, the joint density of $(T, M)$ is the limit of (3.8) as $\mu \to 0$,

$$f_{T,M}(t, m) = \frac{1}{m} \int_{\gamma}^{\infty} \frac{y^{m-1} e^{-y}}{\Gamma(t)} dy,$$

(3.9)

where $\gamma$ is the Euler–Mascheroni constant, $t \in (0, \infty)$, and $m \in (0, 1)$.

**Proof.** Let $f(x; t)$ denote the density of $Z_t$. For $\mu > 0$, the density of $Z_t$ restricted to $(0, 1)$ can be derived though its Laplace transform. We have

$$f(x) = \mathcal{L}^{-1}\left\{\mathbb{E}\left[e^{-\beta Z_t}\right]\right\}_{1 < x < 1}$$

$$= \mathcal{L}^{-1}\left\{\exp\left(-t \int_{0}^{\infty} (1 - e^{-\beta s}) s^{-1} e^{-\mu s} ds\right) \exp\left(t \int_{1}^{\infty} (1 - e^{-\beta s}) s^{-1} e^{-\mu s} ds\right)\right\}_{1 < x < 1}$$

$$= \mathcal{L}^{-1}\left\{\left(1 + \frac{\beta}{\mu}\right)^{-t} \exp\left(t \int_{1}^{\infty} s^{-1} e^{-\mu s} ds\right) \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \left(\int_{1}^{\infty} e^{-\beta s} s^{-1} e^{-\mu s} ds\right)\right\}_{1 < x < 1}$$

$$= \mathcal{L}^{-1}\left\{e^{\Gamma(0, \mu)} \left(1 + \frac{\beta}{\mu}\right)^{-t} + e^{\Gamma(0, \mu)} \left(1 + \frac{\beta}{\mu}\right)^{-t} \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} \left(\int_{1}^{\infty} e^{-\beta s} s^{-1} e^{-\mu s} ds\right)\right\}_{1 < x < 1}$$

$$= \mathcal{L}^{-1}\left\{e^{\Gamma(0, \mu)} \left(1 + \frac{\beta}{\mu}\right)^{-t} \right\}_{1 < x < 1}$$

$$= \frac{e^{\Gamma(0, \mu)} \mu^{t}}{\Gamma(t)} x^{m-1} e^{-\mu x} \mathbb{1}_{0 < x < 1}.$$

(3.10)
Using the strong Markov property of Lévy processes, we have

\[
P(T \in dt, M > m) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} P(Z_{t-\epsilon} \leq 1, Z_t > 1 + m)
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^1 P(Z_{t-\epsilon} \in dy) P(Z_\epsilon > 1 + m - y)
\]
\[
= \int_0^1 \frac{e^{\Gamma(0,\mu)t} \mu^t}{\Gamma(t)} y^{t-1} e^{-\mu y} \int_{1+m-y}^{\infty} z^{-1} e^{-\mu^2 z} dz dy,
\]

(3.11)

Differentiating (3.11) with respect to \(m\), the joint density of \((T, M)\) in (3.8) directly follows. For the Dickman process, we take the limit as \(\mu \to 0\). The density of \(Z_t\) within \((0,1)\) is obtained by taking limit of (3.10), i.e.

\[
f(x, t) = \lim_{\mu \to 0} \frac{e^{[\Gamma(0,\mu)+\log(\mu)]t}}{\Gamma(t)} x^{t-1} e^{-\mu x} = \frac{e^{-\gamma t}}{\Gamma(t)} x^{t-1}, \quad 0 < x < 1,
\]

(3.12)

with \(\gamma\) being the Euler–Mascheroni constant. Hence, associated distribution of \((T, M)\) directly follows (3.9).

Given that the first passage time of \(Z_t\) hitting level 1 is greater than \(t\), the distribution of the truncated process \(Z_t\) is characterised via its density within \((0,1)\). The details are in Theorem 3.2.

**Theorem 3.2** The density of \(Z_t|T > t\) is given by

\[
f_{Z_t|T > t}(x; t) = \begin{cases} 
\frac{\mu^t}{\Gamma(t,\mu)} x^{t-1} e^{-\mu x}, & \mu > 0, \\
rx^{t-1}, & \mu = 0,
\end{cases}
\]

(3.13)

where \(0 < x < 1\).

**Proof.** We know that

\[
Z_t|T > t \overset{D}{=} Z_t|Z_t < 1,
\]

(3.14)

since \(\{T > t\}\) is equivalent to \(\{Z_t < 1\}\). The density therefore satisfies

\[
f_{Z_t|T > t}(x; t) = f_{Z_t|Z_t < 1}(x; t) = \frac{f(x; t)}{\int_0^1 f(x; t)dx}.
\]

Thus, the density immediately follows (3.13). \(\blacksquare\)
3.3 Exact simulation of the generalised Dickman distribution

Based on the marked renewal framework and choosing suitable acceptance-rejection (A/R) envelopes for \( f_{T,M}(t,m) \) and \( f_{Z|T>\tau}(x;\tau) \), we have the following simulation algorithm for the Dickman process, or equivalently, the generalised Dickman distribution.

**Algorithm 3.3 Simulation of Dickman process (Generalised Dickman distribution)**

1. Set \( S = 0, \gamma = \text{digamma}(1) \).

2. Generate \((T, M)\) via the following steps:
   (a) Generate \( U_1 \sim U(0,1) \), and set \( T = -\frac{1}{0.8} \log U_1 \).
   (b) Generate \( Y \sim \text{Beta}(T,0.5) \), \( U_2 \sim U(0,1) \), and set
   \[
   M = Y - 1 + (1 - Y) \exp (-U_2 \log (1 - Y)).
   \]
   (c) Generate \( V \sim U(0,1) \), and if
   \[
   V \leq \frac{1}{2.35} \frac{\Gamma(0.5) e^{-0.8 \gamma} (\log(1 - Y))}{0.8 \Gamma(T + 0.5) (1 - Y)^{-0.5}}.
   \] then accept \((T, M)\). Otherwise, reject this candidate and go back to Step 2(a).

3. If \( T > t \), continue to Step 4. Otherwise, set \( S = S + 1 + M, t = t - T, \) and go back to Step 2.

4. Generate \( U \sim U(0,1) \) and set \( X = U^\dagger \).

5. Return \( S + X \).

**Proof.** For the joint density \( f_{T,M}(t,m) \), we simulate \((T, M, Y)\) jointly from the integral in (3.9),
\[
 f(t,m,y) = \frac{e^{-\gamma y} y^{\gamma - 1}}{\Gamma(t)} \frac{1}{1 + m - y},
\]
for \( t \in (0, \infty), 0 < m < y < 1 \). The A/R envelope chosen is
\[
 g(t,m,y) = \sigma e^{-\sigma t} \frac{\Gamma(t + \eta)}{\Gamma(t)} \frac{1}{\Gamma(\eta)} \frac{1}{y^{\gamma - 1} (1 - y)^{\eta - 1}} \frac{(-\log(1 - y))}{(1 + m - y)},
\]
where \( \sigma = 0.8 \) and \( \eta = 0.5 \) are chosen numerically via two-dimensional optimisation. Furthermore, \( f_{Z|T>\tau}(x;\tau) \) can be simulated directly via inverse transformation resulting in Step 4. \( \blacksquare \)
3.4 Exact simulation of the truncated Gamma process

For the truncated Gamma process, one could generate a Dickman process and apply a final A/R scheme. However, the algorithm is not very efficient, especially for large $\mu$ as the acceptance rate is exponentially decreasing in $\mu$. Thereby, we propose the following algorithm based on the marked renewal framework to sample truncated Gamma process.

**Algorithm 3.4 Simulation of truncated Gamma process (Marked renewal approach)**

1. Set $S = 0$.

2. Generate $(T, M)$ via the following steps:

   (a) Numerically minimise

   $C(\theta, \delta) = \frac{\Gamma(\delta)e^{-\mu}e^{\Gamma(0, \mu)\exp(\Gamma(0, \mu) + \theta + \log(\mu))} - \mu\exp(\Gamma(0, \mu) + \theta + \log(\mu))}{\theta(1 - \delta)e^{\Gamma(\exp(\Gamma(0, \mu) + \theta + \log(\mu)) + \delta)}}$, (3.18)

   record the optimal value $\theta^*$ and $\delta^*$ and set $C = C(\theta^*, \delta^*)$.

   (b) Generate $U_1 \sim U(0, 1)$ and set $T = -\frac{1}{\delta^*}\log(U_1)$.

   (c) Generate $Y \sim \text{Beta}(T, \delta^*)$, $U_2 \sim U(0, 1)$ and set

   $M = Y - 1 + (1 - Y)\exp(-U_2\log(1 - Y))$.

   (d) Generate $V_1 \sim U(0, 1)$. If

   $V_1 \leq \frac{1}{C}\Gamma(0, \mu)\exp(\Gamma(0, \mu) \cdot \mu\theta^* e^{\theta^* T} - \log(1 - Y) - (1 - Y)^{\delta^* - 1} - \mu(1 + M)}$, (3.19)

   then accept $(T, M)$. Otherwise, reject this candidate and go back to Step 2(b).

3. If $T > t$, continue to Step 4. Otherwise, set $S = S + 1 + M$, $t = t - T$ and go back to Step 2.

4. Generate $X$ via the following steps:

   (a) Generate $U_3 \sim U(0, 1)$ and set $X = U_3^{\frac{1}{2}}$.

   (b) Generate $V_2 \sim U(0, 1)$. If

   $V_2 \leq \exp(-\mu X)$, (3.20)

   then accept $X$. Otherwise, reject this candidate and go back to Step 4(a).
5. Return $S + X$.

**Proof.** For the joint density $f_{T,M}(t,m)$, we simulate $(T, M, Y)$ jointly from the integral in (3.8),

$$f(t,m,y) = e^{\Gamma(0,\mu)t \mu^t y^t \Gamma(1) - \mu(1+m)} \frac{1}{1 + m - y},$$

where $t \in (0, \infty), 0 < m < y < 1$. We choose the envelope

$$g(t,m,y) = \theta e^{-b_1 \Gamma(1 + \delta) y^t \Gamma(1 - y) \delta^{-1} \frac{1}{1 + m - y}}.$$

The ratio of the densities is bounded by

$$\frac{f(t,m,y)}{g(t,m,y)} \leq \frac{\Gamma(\delta \mu \Gamma(0,\mu)^t \mu^t e^{\alpha t}}{\alpha(1 - \delta) e^{\Gamma(t^* + \delta)}} \leq C(\theta, \delta),$$

where we obtained by maximising the ratio with respect to $t$,

$$t^* = \exp \left( \Gamma(0,\mu) + \alpha + \log(\mu) + \mathcal{O} \left( \frac{1}{\Gamma(t^*)} \right) \right) \approx \exp \left( \Gamma(0,\mu) + \alpha + \log(\mu) \right).$$

We then minimise $C(\theta, \delta)$ with respect to $\theta$ and $\delta$ via two dimensional numerical optimisation. Finally, \{\{Z_t|T > t\}\} can be simulated directly via A/R scheme by choosing an envelop with density $g(x; t) = \frac{x^{t-1} \mathbb{1}_{[0 < x < 1]}}{t^t}$.

### 3.5 Numerical Results

In this section, we present some numerical results of the exact simulation methods developed in Algorithm 3.3, 3.4. Numerical validation and tests for our simulation algorithms are all based on the true mean. The associated errors are reported by the difference, the percentage errors, and root mean square error (RMSE). The detailed numerical results are reported in Table 1. We can see that each algorithm can achieve a very high level of accuracy, which is reflected by the difference in the theoretical mean and associated percentage errors. The simulation time is measured by the elapsed CPU time in seconds. We can see that both algorithms perform well. In particular, for Algorithm 3.4, the simulation time to generate the truncated Gamma process is not highly influenced by $\mu$.

---

1RMSE is calculated by $RMSE = \sqrt{SE^2 + Bias^2}$
Table 1: Comparison between the true means and the associated simulation results for Algorithm 3.3, 3.4 based on the parameter setting $t = 0.5, 1, 2.5, 3$, $\mu = 0, 0.5, 1$, respectively.

<table>
<thead>
<tr>
<th>Paths</th>
<th>True</th>
<th>Simulation</th>
<th>Difference</th>
<th>Error%</th>
<th>RMSE</th>
<th>Time</th>
<th>True</th>
<th>Simulation</th>
<th>Difference</th>
<th>Error%</th>
<th>RMSE</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,000</td>
<td>0.5000</td>
<td>0.4879</td>
<td>-0.0121</td>
<td>-2.4%</td>
<td>0.0155</td>
<td>0.11</td>
<td>2.5000</td>
<td>2.4421</td>
<td>-0.0579</td>
<td>-2.37%</td>
<td>0.0343</td>
<td>0.08</td>
</tr>
<tr>
<td>4,000</td>
<td>0.5000</td>
<td>0.5057</td>
<td>0.0057</td>
<td>1.12%</td>
<td>0.0079</td>
<td>0.13</td>
<td>2.5000</td>
<td>2.5148</td>
<td>0.0148</td>
<td>0.59%</td>
<td>0.0178</td>
<td>0.11</td>
</tr>
<tr>
<td>16,000</td>
<td>0.5000</td>
<td>0.5011</td>
<td>0.0011</td>
<td>0.22%</td>
<td>0.0040</td>
<td>0.38</td>
<td>2.5000</td>
<td>2.5073</td>
<td>0.0073</td>
<td>0.29%</td>
<td>0.0089</td>
<td>0.36</td>
</tr>
<tr>
<td>64,000</td>
<td>0.5000</td>
<td>0.5008</td>
<td>0.0008</td>
<td>0.16%</td>
<td>0.0020</td>
<td>1.09</td>
<td>2.5000</td>
<td>2.4975</td>
<td>-0.0025</td>
<td>-0.10%</td>
<td>0.0044</td>
<td>1.17</td>
</tr>
<tr>
<td>256,000</td>
<td>0.5000</td>
<td>0.5002</td>
<td>0.0002</td>
<td>0.05%</td>
<td>0.0010</td>
<td>4.35</td>
<td>2.5000</td>
<td>2.5058</td>
<td>0.0058</td>
<td>0.23%</td>
<td>0.0022</td>
<td>4.43</td>
</tr>
<tr>
<td>1,024,000</td>
<td>0.5000</td>
<td>0.5009</td>
<td>0.0009</td>
<td>0.19%</td>
<td>0.0005</td>
<td>16.63</td>
<td>2.5000</td>
<td>2.5003</td>
<td>0.0003</td>
<td>0.01%</td>
<td>0.0011</td>
<td>17.19</td>
</tr>
<tr>
<td>$t = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,000</td>
<td>0.7869</td>
<td>0.7890</td>
<td>0.0020</td>
<td>0.25%</td>
<td>0.0183</td>
<td>0.14</td>
<td>0.6321</td>
<td>0.6426</td>
<td>0.0105</td>
<td>1.63%</td>
<td>0.0168</td>
<td>0.19</td>
</tr>
<tr>
<td>4,000</td>
<td>0.7869</td>
<td>0.7909</td>
<td>0.0039</td>
<td>0.49%</td>
<td>0.0095</td>
<td>0.28</td>
<td>0.6321</td>
<td>0.6318</td>
<td>-0.0003</td>
<td>-0.04%</td>
<td>0.0081</td>
<td>0.28</td>
</tr>
<tr>
<td>16,000</td>
<td>0.7869</td>
<td>0.7809</td>
<td>-0.0060</td>
<td>-0.76%</td>
<td>0.0047</td>
<td>0.66</td>
<td>0.6321</td>
<td>0.6388</td>
<td>0.0017</td>
<td>0.26%</td>
<td>0.0041</td>
<td>0.83</td>
</tr>
<tr>
<td>64,000</td>
<td>0.7869</td>
<td>0.7891</td>
<td>0.0021</td>
<td>0.26%</td>
<td>0.0024</td>
<td>2.12</td>
<td>0.6321</td>
<td>0.6304</td>
<td>-0.0017</td>
<td>-0.27%</td>
<td>0.0020</td>
<td>2.18</td>
</tr>
<tr>
<td>256,000</td>
<td>0.7869</td>
<td>0.7855</td>
<td>-0.0014</td>
<td>-0.18%</td>
<td>0.0012</td>
<td>8.38</td>
<td>0.6321</td>
<td>0.6324</td>
<td>0.0002</td>
<td>0.03%</td>
<td>0.0010</td>
<td>8.55</td>
</tr>
<tr>
<td>1,024,000</td>
<td>0.7869</td>
<td>0.7865</td>
<td>-0.0004</td>
<td>-0.05%</td>
<td>0.0006</td>
<td>33.09</td>
<td>0.6321</td>
<td>0.6327</td>
<td>0.0005</td>
<td>0.08%</td>
<td>0.0005</td>
<td>31.48</td>
</tr>
<tr>
<td>$t = 3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,000</td>
<td>2.3608</td>
<td>2.3699</td>
<td>0.0091</td>
<td>0.38%</td>
<td>0.0326</td>
<td>0.17</td>
<td>1.8964</td>
<td>1.9056</td>
<td>0.0092</td>
<td>0.48%</td>
<td>0.0289</td>
<td>0.16</td>
</tr>
<tr>
<td>4,000</td>
<td>2.3608</td>
<td>2.3431</td>
<td>-0.0177</td>
<td>-0.75%</td>
<td>0.0163</td>
<td>0.30</td>
<td>1.8964</td>
<td>1.8995</td>
<td>0.0051</td>
<td>0.16%</td>
<td>0.0139</td>
<td>0.17</td>
</tr>
<tr>
<td>16,000</td>
<td>2.3608</td>
<td>2.3731</td>
<td>0.0123</td>
<td>0.51%</td>
<td>0.0083</td>
<td>0.73</td>
<td>1.8964</td>
<td>1.8964</td>
<td>0.0000</td>
<td>0.00%</td>
<td>0.0070</td>
<td>0.50</td>
</tr>
<tr>
<td>64,000</td>
<td>2.3608</td>
<td>2.3629</td>
<td>0.0020</td>
<td>0.08%</td>
<td>0.0041</td>
<td>2.12</td>
<td>1.8964</td>
<td>1.8965</td>
<td>0.0002</td>
<td>0.01%</td>
<td>0.0035</td>
<td>1.94</td>
</tr>
<tr>
<td>256,000</td>
<td>2.3608</td>
<td>2.3608</td>
<td>0.0000</td>
<td>0.00%</td>
<td>0.0021</td>
<td>8.76</td>
<td>1.8964</td>
<td>1.8977</td>
<td>0.0013</td>
<td>0.06%</td>
<td>0.0018</td>
<td>6.98</td>
</tr>
<tr>
<td>1,024,000</td>
<td>2.3608</td>
<td>2.3579</td>
<td>-0.0029</td>
<td>-0.12%</td>
<td>0.0010</td>
<td>32.33</td>
<td>1.8964</td>
<td>1.8960</td>
<td>-0.0003</td>
<td>-0.01%</td>
<td>0.0009</td>
<td>31.60</td>
</tr>
</tbody>
</table>

For the generalised Dickman distribution, we implement a comparison between Algorithm 3.3 against the algorithms suggested by Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012), and Cloud and Huber (2017). The detailed numerical results based on different parameter settings against $t$ are reported in Table 2 and the associated log-log plots for the RMSE against the CPU time are presented in Figure 2. In addition, we also compared the computation time to generate 100,000 samples using these five algorithms for different values of $t$, and details are presented in Table 3. We can see that sampling based on Algorithm 3.3 is much faster than algorithms in Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012), and Cloud and Huber (2017). For instance, when $t = 3$, Algorithm 3.3 is 4 times faster than Devroye and Fawzi (2010), 50 times faster than Fill and Huber (2010), 25 times faster than Chi (2012), and 7 times faster than Cloud and Huber (2017). Compared with the other algorithms, the main advantage of Algorithm 3.3 is that the incremental random variables $(T_i, M_i)_{i \geq 0}$ are i.i.d pairs, and this means that we can employ vectorisation to reduce total computation time, especially for large values of $t$. Although the expected simulation time is unbounded for Algorithm 3.3, the simulation idea based on the marked renewal approach still outperforms the dominated coupling approach and rejection sampling approach, and even more so when $t$ is large.
4 More general Vervaat perpetuities

In this section, we present a general method for simulating Vervaat perpetuities, for more general random variables $Y$. This method is applicable to a wide range of distributions, and we give several examples to demonstrate this. We have the following result which decomposes the Vervaat perpetuity $X$ into the sum of a Gamma process (or truncated Gamma process), and compound Poisson processes. For simplicity, we will let $Y$ have a continuous density $g$.

**Lemma 4.1** Consider a perpetuity $X$ defined in (1.1), with $Y \in \mathbb{R}^+$. If the density of $Y$ satisfies (1.3), then there exists $k \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$ such that $\Pr(Y > y) \geq e^{-ky}$ for all $y < b$. We then have the following two cases:
Figure 2: Convergence analysis of Algorithm 3.3 and Algorithms suggested in Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012), and Cloud and Huber (2017) based on parameter setting $t = 1, 3$, respectively, with associated detailed numerical results reported in Table 2.

Table 3: Comparison CPU time (sec) for Algorithms suggested in Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012) and Cloud and Huber (2017) with Algorithm 3.3 for 100,000 replications.

<table>
<thead>
<tr>
<th>Input</th>
<th>0.40</th>
<th>0.60</th>
<th>0.80</th>
<th>1.00</th>
<th>1.20</th>
<th>1.40</th>
<th>1.60</th>
<th>1.80</th>
<th>2.00</th>
<th>2.20</th>
<th>2.40</th>
<th>2.60</th>
<th>2.80</th>
<th>3.00</th>
<th>3.20</th>
<th>3.40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algorithm 3.3</td>
<td>1.85</td>
<td>1.81</td>
<td>1.68</td>
<td>1.60</td>
<td>1.71</td>
<td>1.79</td>
<td>1.82</td>
<td>1.83</td>
<td>1.78</td>
<td>2.03</td>
<td>1.72</td>
<td>1.77</td>
<td>1.83</td>
<td>2.02</td>
<td>1.83</td>
<td>1.95</td>
</tr>
<tr>
<td>Devroye and Fawzi (2010)</td>
<td>3.02</td>
<td>2.23</td>
<td>3.19</td>
<td>3.94</td>
<td>5.66</td>
<td>5.47</td>
<td>5.23</td>
<td>5.31</td>
<td>5.93</td>
<td>8.48</td>
<td>7.22</td>
<td>7.51</td>
<td>7.97</td>
<td>8.47</td>
<td>10.88</td>
<td>11.50</td>
</tr>
<tr>
<td>Fill and Huber (2010)</td>
<td>3.97</td>
<td>2.53</td>
<td>3.14</td>
<td>4.12</td>
<td>5.08</td>
<td>7.24</td>
<td>10.53</td>
<td>13.64</td>
<td>18.25</td>
<td>20.76</td>
<td>26.78</td>
<td>31.22</td>
<td>37.12</td>
<td>41.31</td>
<td>43.09</td>
<td>45.31</td>
</tr>
<tr>
<td>Chi (2012)</td>
<td>3.92</td>
<td>6.28</td>
<td>9.62</td>
<td>18.52</td>
<td>21.25</td>
<td>23.53</td>
<td>27.45</td>
<td>31.80</td>
<td>34.78</td>
<td>36.92</td>
<td>38.41</td>
<td>41.57</td>
<td>55.31</td>
<td>50.54</td>
<td>55.83</td>
<td>59.23</td>
</tr>
</tbody>
</table>

**Case 1:** There exists $k \in \mathbb{R}^+$ such that $P(Y > y) \geq e^{-ky}$ for all $y \in \mathbb{R}^+$. Then we have

$$B = \int_0^\infty \frac{P(Y > y) - e^{-ky}}{y} dy < \infty,$$

(4.1)

and thus

$$X \overset{D}{=} \Gamma_t + \sum_{i=1}^{N_t} J_i,$$  

(4.2)

where

- $\Gamma_t$ is a Gamma process such that $\Gamma_t \sim \Gamma(t, k)$;
- $\sum_{i=1}^{N_t} J_i$ is a compound Poisson process such that
  - $N_t$ is a Poisson process with rate $tB$,
  - $\{J_i\}_{i=1,2,...}$ are i.i.d jumps with density
    $$g_{J_i}(y) = \frac{P(Y \geq y) - e^{-ky}}{By}, \quad y \in \mathbb{R}^+.$$
**Case 2:** Otherwise, for the case that there only exists $k \in \mathbb{R}^+$ such that $\mathbb{P}(Y > y) \geq e^{-ky}$ for $y < b$. We have

$$D = \int_0^b \frac{\mathbb{P}(Y > y) - e^{-ky}}{y} \, dy < \infty,$$  

(4.3)

and thus

$$X \overset{D}{=} Z_t + \sum_{i=1}^{N_1^1} J_i^{(1)} + \sum_{k=1}^{N_2^2} J_k^{(2)},$$  

(4.4)

where

- $Z_t$ is a truncated Gamma process with Lévy measure

$$\nu(dz) = z^{-1}e^{-kz}1_{\{0 < z < b\}} \, dz;$$

- $\sum_{i=1}^{N_1^1} J_i^{(1)}$ is a compound Poisson process such that
  - $N_1^1$ is a poisson process with rate $tD$,
  - $\{J_i^{(1)}\}_{i=1,2,...}$ are i.i.d jumps with density

$$g_{J_i^{(1)}}(y) = \frac{\mathbb{P}(Y \geq y) - e^{-ky}}{Dy}, \quad y \in (0, b).$$

- $\sum_{k=1}^{N_2^2} J_k^{(2)}$ is a compound Poisson process such that
  - $N_2^2$ is a poisson process with rate $tE$, where

$$E = \int_b^\infty \frac{\mathbb{P}(Y \geq y)}{y} \, dy$$  

(4.5)

  - $\{J_k^{(2)}\}_{k=1,2,...}$ are i.i.d jumps with density

$$g_{J_k^{(2)}}(y) = \frac{\mathbb{P}(Y \geq y)}{Ey}, \quad y \in (b, \infty).$$

**Proof.** If the density of $Y$ satisfies (1.3), then $g(0) < \infty$. Hence, the survival function of $Y$, $\mathbb{P}(Y > y)$, decays at an exponential rate or slower at 0. This implies that there exists a $k \in \mathbb{R}^+$ such that $\mathbb{P}(Y > y) \geq e^{-ky}$ for all $y < b$, for some $b \in \mathbb{R}^+$.

**Case 1:** If, furthermore, there exists such $k \in \mathbb{R}^+$ so that $\mathbb{P}(Y > y) \geq e^{-ky}$ holds for all $y \in \mathbb{R}^+$, then
for $B$ defined as in (4.1), we have that the integral $B$ is finite since

$$
\int_0^{\infty} \frac{P(Y > y) - e^{-ky}}{y} dy = \int_0^{1} \frac{P(Y > y) - e^{-ky}}{y} dy + \int_1^{\infty} \frac{P(Y > y) - e^{-ky}}{y} dy.
$$

The second term is finite since we have assumed that the mean of $Y$ exists, and we also have that

$$
\int_0^{1} \frac{P(Y > y) - e^{-ky}}{y} dy = \left[ \log y \left( P(Y > y) - e^{-ky} \right) \right]_0^{1} + \int_0^{1} \log y \left( \frac{g(y) - e^{-ky}}{k} \right) dy
$$

since $g(0) < \infty$. Hence, the Laplace transform of $X$ can be expressed as

$$
E \left[ e^{-\beta X} \right] = \exp \left( -t \int_0^{\infty} \frac{1 - e^{-\beta y} e^{-ky}}{y} dy \right) \exp \left( -t B \int_0^{\infty} \frac{P(Y > y) - e^{-ky}}{By} dy \right)
$$

and we have the distributional decomposition result given in (4.2).

**Case 2:** In the case where we only have $P(Y > y) \geq e^{-ky}$ for $y < b$ for some $b \in \mathbb{R}^+$, we define the integral $D$ as in (4.3) and $E$ as in (4.5). A similar argument as above would show that they are finite, and we thus have

$$
E \left[ e^{-\beta X} \right] = \exp \left( -t \int_0^{b} \left( 1 - e^{-\beta y} \right) \frac{P(Y > y)}{y} dy \right) \exp \left( -t \int_0^{\infty} \frac{1 - e^{-\beta y} P(Y > y)}{y} dy \right)
$$

$$
\times \exp \left( -t E \int_b^{\infty} \left( 1 - e^{-\beta y} \right) \frac{P(Y > y)}{Ey} dy \right)
$$

$$
= E \left[ e^{-\beta Z_1} \right] E \left[ e^{-\beta \sum_{i=1}^{N_1} I_i^{(1)}} \right] E \left[ e^{-\beta \sum_{k=1}^{N_2} I_k^{(2)}} \right].
$$

The distributional decomposition result (4.4) thus follows. ■

We now have a general method for exactly simulating generalised Vervaat perpetuities for a wide range of distributions for $Y$. In particular, Case 2 is useful when the distribution of $Y$ has finite support, or when $Y$ has a tail which decays faster than exponential, in which case there does not exist $k \in \mathbb{R}^+$ such that $P(Y > y) \geq e^{-ky}$ for all $y$. In the rest of this section, we give some examples to demonstrate
Example 1: $Y \sim \text{Pareto}(\alpha, \sigma)$. From Theorem 2.1, the Laplace transform of $X$ is

$$
E[e^{-\beta X}] = \exp \left( -t \int_0^\sigma \frac{1 - e^{-\beta y}}{y} dy \right) \exp \left( -t \int_\sigma^\infty \frac{1 - e^{-\beta y}}{y} \left( \frac{\sigma}{y} \right)^\alpha dy \right). \tag{4.8}
$$

Hence, $X \overset{D}{=} Z_t + CP$, where $Z_t \sim \text{Dickman}$ process, and $CP$ is a Compound Poisson process, independent of each other. The Dickman process can be generated using Algorithm 3.3, and the Compound Poisson process can be generated easily via an A/R scheme.

Example 2: $Y \sim \text{Gamma}(\alpha, \beta)$. Since $g(0) < \infty$ for all $\alpha \geq 1$, we can simulate all Gamma perpetuities with $\alpha \geq 1$ and all values of $\beta$. Furthermore, for all $\alpha \geq 1$, there exists a $k$ such that $P(Y > y) \geq e^{-ky}$ for all $y$, thus it falls under Case 1 of Lemma 4.1. Thus the Gamma perpetuity $X$ can be split into $X \overset{D}{=} Z_t + CP$, where $Z_t$ is a Gamma process, and $CP$ is a Compound Poisson process independent of $Z_t$ by Lemma 4.1. Figure 3 shows the histograms of the Vervaat perpetuity $X$ for $Y \sim \Gamma(\alpha, \beta)$ with different shape parameters $\alpha$.

Example 3: $Y \sim \text{Weibull}(\kappa, \lambda)$. Since $g(0) < \infty$ for all $\kappa \geq 1$, we can simulate all Weibull perpetuities with $\kappa \geq 1$ and all values of $\lambda$. Furthermore, there only exists a $k$ such that $P(Y > y) \geq e^{-ky}$ for $y \in (0, b)$ for some $b \in \mathbb{R}^+$. Hence, the perpetuity falls into Case 2 of Lemma 4.1, and can be split into $X \overset{D}{=} Z_t + CP_1 + CP_2$, where $Z_t$ is a truncated Gamma process, $CP_1$ and $CP_2$ are Compound Poisson processes, independent of each other and $Z_t$. The truncated Gamma process can be generated using Algorithm 3.4 and the two compound Poisson processes can be easily generated via A/R schemes.

Example 4: $Y \sim \text{Beta}(\alpha, \beta)$. Since $g(0) < \infty$ for all $\alpha \geq 1$, we can simulate all Beta perpetuities with $\alpha \geq 1$ and all values of $\beta$. Furthermore, since the distribution has a finite support, there only exists a $k$ such that $P(Y > y) \geq e^{-ky}$ for $y \in (0, b)$ for some $b \in \mathbb{R}^+$. Hence, the perpetuity falls into Case 2 of Lemma 4.1, and the Lévy measure needs to be truncated. The Beta perpetuity can be split into $X \overset{D}{=} Z_t + CP_1 + CP_2$ according to Lemma 4.1, where $Z_t$ is a truncated Gamma process, and $CP_1$ and $CP_2$ are two compound Poisson processes, independent of each other and $Z_t$.

Example 5: $Y \sim \text{Normal}(\mu, \sigma^2)$. The Normal perpetuity $X$ is two-sided. According to Corollary 2.2, one could sample independent $X_1$ and $X_2$, each of which are half-normal perpetuities, and the
difference $X_1 - X_2$ is the realisation of the two-sided perpetuity $X$.

For a half-normal perpetuity $X_i$, where $Y_i \sim \text{Half-Normal}(\mu, \sigma)$, there only exists a $k$ such that $\Pr(Y > y) \geq e^{-ky}$ for $y \in (0, b)$ for some $b \in \mathbb{R}^+$. Hence, the perpetuity $X_i$ falls into Case 2 and can be split into a truncated Gamma process and two compound Poisson processes according to Lemma 4.1. The truncated Gamma process can be generated via Algorithm 3.4 and the two compound Poisson processes can be easily generated via A/R schemes. Figure 4 illustrates a sample density plot for the perpetuity $X_i$ with $Y_i \sim \text{Half-Normal}(0, 1)$. Figure 5 compares the distribution of $X_i$ under different parameter settings.

Now, we can simulate the Normal perpetuity $X$ by generating two independent half-normal perpetuities $X_1$ and $X_2$ and taking the difference $X \overset{D}{=} X_1 - X_2$. Figure 6 and 7 demonstrate the distribution behaviour of the two sided perpetuity $X$ via its histogram and density plot. We can also obtain other variations of the Normal perpetuities by introducing a new variable $B \sim \text{Bernoulli}(p)$ and set-
Figure 4: Probability density function of Perpetuity $X_i$ with $Y_i \sim \text{Half-Normal}(0,1)$.

Figure 5: Histograms of Perpetuity $X_i$ with $Y_i \sim \text{Half-Normal}(0,1)$ and $Y_i \sim \text{Half-Normal}(1,1)$ with $t = 1, 2$, respectively.

Letting $\hat{X} \overset{D}{=} pX_1 - (1 - p)X_2$. When $p = \frac{1}{2}$ and $X_i \sim \text{Normal}(0,1)$, this is identical to the $\text{Normal}(0,1)$ perpetuity. Figure 8 illustrates the differences of the perpetuity $\hat{X}$ and $X$ in terms of their histograms.

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Figure 6: Histogram and Density Plot of Perpetuity $X$ with $Y \sim \text{Normal}(0, 1)$.

Figure 7: Histogram and Density Plot of Perpetuity $X$ with $Y \sim \text{Normal}(1, 1)$.

References


Figure 8: Histogram of Perpetuity $\hat{X} = BX_1 - (1 - B)X_2$ with $B \sim \text{Bernoulli}(p)$ under parameter setting $p = 0.3, 0.5, 0.9$.


