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THE BASS AND TOPOLOGICAL STABLE RANKS FOR ALGEBRAS OF ALMOST PERIODIC FUNCTIONS ON THE REAL LINE, II

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This paper is dedicated to the memory of Ronald Douglas

Abstract. Let Λ be either a subgroup of the integers $\mathbb{Z}$, a semigroup in $\mathbb{N}$, or $\Lambda = \mathbb{Q}$, respectively $\mathbb{Q}^+$. We determine the Bass and topological stable ranks of the algebras $\mathrm{AP}_\Lambda = \{f \in \mathrm{AP} : \sigma(f) \subseteq \Lambda\}$ of almost periodic functions on the real line and with Bohr spectrum in Λ. This answers a question in the first part of this series of papers under the same heading, where it was shown that, in contrast to the present situation, these ranks were infinite for each semigroup Λ of real numbers for which the $\mathbb{Q}$-vector space generated by Λ had infinite dimension.

Introduction

According to our definitions, $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Let AP be the uniform closure in $C_b(\mathbb{R}, \mathbb{C})$ of the set of all functions of the form

$$Q(t) := \sum_{j=1}^{N} a_j e^{i\lambda_j t},$$

where $a_j \in \mathbb{C}$, $\lambda_j \in \mathbb{R}$ and $N \in \mathbb{N}^*$. We call Q a *generalized trigonometric polynomial*. Under the usual pointwise algebraic operations, AP is a point separating function algebra on $\mathbb{R}$ with the property that $f \in \mathrm{AP}$ implies that $\overline{f} \in \mathrm{AP}$. Harald Bohr developed the basic theory for this space in a series of papers [5, 6]. We also refer to the books by Corduneanu [10] and Besicovich [3] for an introduction into this important class of functions. Modern treatments and applications to operator theory can be found for example in [7].

In the following we provide a solution to a question concerning the Bass and topological stable ranks of a certain class of subalgebras of AP (see below for the definition) and that was left open in the first part [13] of this series of papers. Recall that D. Suárez [14] was the first who determined these ranks for the algebra AP itself. The present work is conceptually a mixture of a research paper and a survey paper.

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1. **Our working setting**

Let us recall from [13] the definitions of the fundamental notions in connection with almost periodic functions. Proofs are for instance in [10].

Definition 1.1. Let $f \in \mathrm{AP}$. If $\lambda \in \mathbb{R}$, the associated *Fourier-Bohr coefficient* $\widehat{f}(\lambda)$ is defined as

$$\widehat{f}(\lambda) = \lim_{|I|\to\infty} \frac{1}{|I|} \int_I f(t) e^{-i\lambda t}\, dt,$$

where I runs through the set of all compact intervals in $\mathbb{R}$.

Proposition A *If $f \in \mathrm{AP}$, then $\widehat{f}(\lambda)$ exists for every $\lambda \in \mathbb{R}$ and $\widehat{f}(\lambda) \neq 0$ for at most a countable number of λ.*

Definition 1.2. If $f \in \mathrm{AP}$, then the *Bohr spectrum*, $\sigma(f)$, of f is the set of all $\lambda \in \mathbb{R}$ for which the associated Fourier-Bohr coefficient $\widehat{f}(\lambda)$ is not zero. If $\sigma(f) = \{\lambda_n : n \in I\}$, $I \subseteq \mathbb{N}$, then the *Fourier-Bohr series* associated with f is the formal series

$$f \sim \sum_{n\in I} \widehat{f}(\lambda_n) e^{i\lambda_n t}.$$

One of our main tools will be the following approximation theorem (see [10] or [7]).

Theorem B *The following assertions hold:*

1. *The Fourier-Bohr series uniquely determines f whenever $f \in \mathrm{AP}$.*
2. *Let f be an almost periodic function on $\mathbb{R}$ with Bohr spectrum $\sigma(f)$. Then there exists a sequence (q_n) of generalized trigonometric polynomials with $\sigma(q_n) \subseteq \sigma(f)$ converging uniformly to f.*

In the present paper we are concerned with the algebras

$$\mathrm{AP}_\Lambda = \{f \in \mathrm{AP} : \sigma(f) \subseteq \Lambda\},$$

where Λ runs through a certain class of groups/semigroups of the additive group $(\mathbb{R}, +)$. Recall that a semigroup of $\mathbb{R}$ or $\mathbb{N}$ is just an additive subset of $\mathbb{R}$, respectively $\mathbb{N}$, containing the origin. In [13] it was shown that AP_Λ is a closed subalgebra of AP. Let us point out that AP has many other closed subalgebras that are not of this form. For example, take $f_n(t) = e^{i(2n-1)t} + e^{i2nt}$. Then the uniform closure A of the linear span of these functions f_j together with the constants is such a counterexample . To see this, just observe that the Fourier-Bohr coefficients of the f_j have the property that an even one coincides with the odd one just preceding it. This carries over to the linear hull of the f_j with 1 and to the closure.

One of the main results in [13] is that the stable ranks of AP_Λ are infinite provided the dimension of the $\mathbb{Q}$-vector space $[\Lambda]$ generated by Λ is infinite. In particular, $\mathrm{bsr}\,\mathrm{AP} = \mathrm{bsr}\,\mathrm{AP}^+ = \infty$ where $\mathrm{AP}^+ := \mathrm{AP}_{\mathbb{R}^+}$.

It remained an open problem whether there are any algebras of this type AP_Λ with infinite stable ranks, but for which $\dim[\Lambda] < \infty$. The algebra $\mathrm{AP}_\mathbb{Q}$ seemed to be a potential candidate. In the present paper we show that this is not the case. More precisely, we will prove the following Theorem:

Theorem 1.3.
(1) $\operatorname{tsr} \mathrm{AP}_{\mathbb{Q}} = 1$ *and* $\operatorname{tsr} \mathrm{AP}_{\mathbb{Q}^+} = 2$.
(2) $\operatorname{bsr} \mathrm{AP}_{\mathbb{Q}} = \operatorname{bsr} \mathrm{AP}_{\mathbb{Q}^+} = 1$.

We will also determine the stable ranks of the algebras AP_Λ, where Λ is a subgroup of $\mathbb{Z}$ or a sub-semigroup of $\mathbb{N}$. Let us recall from [13, Theorem 3.6] (also stated in the Epilog as Theorem F) that $\mathrm{AP}_{\mathbb{Z}}$ is isometrically isomorphic to $C(\mathbb{T}, \mathbb{C})$ and $\mathrm{AP}_{\mathbb{N}}$ isometrically isomorphic to the disk algebra $A(\mathbb{D})$. Here is our second set of new results:

Theorem 1.4. *Let Λ be a sub-semigroup of $\mathbb{N}$ such that $\gcd \Lambda = 1$. Then the following assertions hold:*
(1) *$S := \mathbb{N} \setminus \Lambda$ is finite.*
(2) *AP_Λ is isometrically isomorphic to*
$$\{f \in A(\mathbb{D}) : f^{(j)}(0) = 0 \text{ for } j \in S\}.$$
(3) *$\operatorname{bsr} \mathrm{AP}_\Lambda = 1$ and $\operatorname{tsr} \mathrm{AP}_\Lambda = 2$.*

For the reader's convenience, we also give the pertinent definitions of the algebraic concepts of stable ranks, introduced by Bass and Rieffel and first studied in the realm of function spaces by Vaserstein [16].

Definition 1.5. Let A be a commutative unital algebra (real or complex) with identity element denoted by $\mathbf{1}$.

[(1)] An n-tuple $(f_1, \dots, f_n) \in A^n$ is said to be *invertible* (or *unimodular*), if there exists $(x_1, \dots, x_n) \in A^n$ such that the Bézout equation $\sum_{j=1}^n x_j f_j = \mathbf{1}$ is satisfied. The set of invertible n-tuples is denoted by $U_n(A)$. Note that $U_1(A) = A^{-1}$, the group of invertible elements in A.

An $(n+1)$-tuple $(f_1, \dots, f_n, g) \in U_{n+1}(A)$ is called *reducible* (in A) if there exists $(a_1, \dots, a_n) \in A^n$ such that $(f_1 + a_1 g, \dots, f_n + a_n g) \in U_n(A)$.

[(2)] The *Bass stable rank* of A, denoted by $\operatorname{bsr} A$, is the smallest integer n such that every element in $U_{n+1}(A)$ is reducible. If no such n exists, then $\operatorname{bsr} A = \infty$.

[(3)] If, additionally, A is endowed with a topology $\mathcal{T}$, then the *topological stable rank*, $\operatorname{tsr} A := \operatorname{tsr}_{\mathcal{T}} A$, of $(A, \mathcal{T})$ is the least integer n for which $U_n(A)$ is dense in A^n, or infinite if no such n exists. Some people also call this the *Rieffel rank*.

It is well known that for Banach algebras one has $\operatorname{bsr} A \le \operatorname{tsr} A$. For the most basic function algebras we have the following facts:

Let X be a normal space. Then
$$\operatorname{tsr} C(X, \mathbb{R}) = \operatorname{bsr} C(X, \mathbb{R}) = \dim X + 1,$$
$$\operatorname{tsr} C(X, \mathbb{C}) = \operatorname{bsr} C(X, \mathbb{C}) = \left\lfloor \tfrac{\dim X}{2} \right\rfloor + 1,$$
where $\dim X$ is the covering dimension of X. If X is not compact, then one endows $C(X, \mathbb{K})$ with the topology of uniform convergence, a basis of which is given by the system
$$U_\varepsilon(f) = \{g \in C(X, \mathbb{K}) : \sup_{x \in X} |f(x) - g(x)| < \varepsilon\}, \ \varepsilon > 0.$$

A similar statement holds for the algebras $C_b(X,\mathbb{K})$ of bounded continuous functions. These results are due to Vaserstein [16]. In particular, since for a subgroup Λ of $\mathbb{R}$ the space $A := \mathrm{AP}_\Lambda$ is a selfadjoint, uniformly closed subalgebra of $C_b(\mathbb{R},\mathbb{C})$, hence (due to the Gelfand-Naimark Theorem) isomorphic isometric to $C(M(A),\mathbb{C})$, we have

$$\operatorname{bsr} A = \operatorname{tsr} A = \left\lfloor \frac{\dim M(A)}{2} \right\rfloor + 1,$$

where $M(A)$ is the spectrum (=maximal ideal space) of A.

If $A = A(\mathbb{D})$ or $A = H^\infty(\mathbb{D})$, the algebra of bounded holomorphic functions on $\mathbb{D}$, then $\operatorname{bsr} A = 1$ and $\operatorname{tsr} A = 2$ (Jones-Marshall-Wolff [11] and Corach-Suárez [8] for $A(\mathbb{D})$) and Treil [15] for $H^\infty(\mathbb{D})$. Actually, Treil [15] proved the following refined version with norm control on the solutions of the equation $1 = uf + Gg$, u invertible:

Theorem C **[Treil]** *Let $f, g \in H^\infty(\mathbb{D})$ satisfy $||f||_\infty \le 1, ||g||_\infty \le 1$ and*

$$\inf_{z\in\mathbb{D}} (|f(z)| + |g(z)|) =: \delta > 0.$$

Then there exist $G \in H^\infty(\mathbb{D})$ and $u \in H^\infty(\mathbb{D})$ such that u is invertible in $H^\infty(\mathbb{D})$,

$$1 = uf + Gg$$

and

$$||G||_\infty + ||u||_\infty + ||u^{-1}||_\infty \le c$$

for some constant $c > 0$ depending only on δ.

As a consequence, we obtain the following solution to the generalized Bézout equation for polynomials. That result will be the key for constructing solutions to this equation in the AP_Λ-setting. It is based on the use of the following function spaces:

Definition 1.6. The Wiener algebra

$$W^+(\mathbb{D}) = \left\{ \sum_{n=0}^\infty a_n z^n : ||f||_{W^+} = \sum_{n=0}^\infty |a_n| < \infty \right\},$$

and its associate Wiener-AP algebra

$$\mathrm{APW}_{\mathbb{Q}^+} := \left\{ \sum_{r_n \in \mathbb{Q}^+} a_n e^{ir_n t} \in \mathrm{AP}_{\mathbb{Q}^+} : \sum_{n=0}^\infty |a_n| < \infty \right\},$$

where (r_n) is an enumeration of $\mathbb{Q}^+ := \{q \in \mathbb{Q} : q \ge 0\}$.

To better distinguish the different norms appearing here, we additionally use the following notation: if $E \subseteq \mathbb{C}$, then for a function f defined at least on E, we put $||f||_E := \sup\{|f(z)| : z \in E\}$.

Corollary 1.7. *Let $p, q \in \mathbb{C}[z]$ satisfy $||p||_\mathbb{T} \le 1, ||q||_\mathbb{T} \le 1$ and*

$$\inf_{z\in\mathbb{D}} (|p(z)| + |q(z)|) =: \delta > 0.$$

Then there exists $\chi \in \mathbb{C}[z]$ *such that*

$$\varphi := p + \chi q \text{ is invertible in } W^+(\mathbb{D})$$

and

$$||\chi||_{\mathbb{D}} + ||\varphi||_{\mathbb{D}} + ||\varphi^{-1}||_{\mathbb{D}} \leq C$$

for some constant $C > 0$ *depending only on* δ.

Proof. By Treil's result there are $u, G \in H^\infty(\mathbb{D})$, u invertible in $H^\infty(\mathbb{D})$, such that $1 = up + Gq$ and

$$||G||_\infty + ||u||_\infty + ||u^{-1}||_\infty \leq c$$

for some constant $c = c(\delta)$. We may assume that $c \geq 1$.

Now consider the dilations F_r given by $F_r(z) := F(rz)$, $0 < r \leq 1$, of all four functions appearing here. Since $p, q \in A(\mathbb{D})$ we may choose r so close to 1 that

$$||p - p_r||_{\mathbb{D}} \leq \frac{1}{2c}, \ ||q - q_r||_{\mathbb{D}} \leq \frac{1}{2c}.$$

Due to $1 = u_r p_r + G_r q_r$, we conclude that

$$v := u_r p + G_r q \in H^\infty(\mathbb{D}) \cap H^\infty(\mathbb{D})^{-1},$$

because

$$|v| = |1 + u_r(p - p_r) + G_r(q - q_r)| \geq 1 - 1/2 = 1/2.$$

Dividing by u_r yields:

$$\psi := \frac{v}{u_r} = p + \frac{G_r}{u_r} q.$$

Since u_r has no zeros on $\overline{\mathbb{D}}$, $h := G_r/u_r \in A(\mathbb{D})$ and $\psi \in H^\infty(\mathbb{D}) \cap H^\infty(\mathbb{D})^{-1}$. Moreover, since $|p| \leq 1$ and $|q| \leq 1$,

$$\frac{1/2}{c} \leq |\psi| = \frac{|v|}{|u_r|} \leq \frac{|u_r| + |G_r|}{|u_r|} \leq \frac{c}{1/c} = c^2.$$

Now we approximate h by a polynomial χ such that $||h - \chi||_{\mathbb{D}} \leq 1/(4c)$. Then

$$\varphi := p + \chi q \in \mathbb{C}[z]$$

has no zeros on $\overline{\mathbb{D}}$, because (with $\psi = p + hq$)

$$|\varphi| \geq |\psi| - |\psi - \varphi| = |\psi| - |q|\,|\chi - h| \geq \frac{1}{2c} - \frac{1}{4c} = \frac{1}{4c} > 0.$$

Hence, φ is invertible in $W^+(\mathbb{D})$. Finally, by noticing that $|h| = |G_r|/|u_r| \leq c^2$,

$$|\chi| \leq 1 + ||h||_{\mathbb{D}} \leq 1 + c^2,$$

and so

$$||\chi||_{\mathbb{D}} + ||\varphi||_{\mathbb{D}} + ||\varphi^{-1}||_{\mathbb{D}} \leq (1 + c^2) + (2 + c^2) + 4c \leq 9c^2 =: C. \qquad \square$$

We also need the following general results on stable ranks. Theorem D, due to C. Badea [2], gives a nice characterization of the topological stable rank.

Theorem D [Badea] *Let* $A = (A, |\cdot|)$ *be a commutative unital Banach algebra over* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$. *For* $\boldsymbol{a} = (a_1, \ldots, a_n) \in A^n$, *let* $||\boldsymbol{a}|| = \sum_{j=1}^n |a_j|$ *be a fixed norm on the product space. Then the following assertions are equivalent:*

(1) $\operatorname{tsr} A \le n$;
(2) *For every* $(\boldsymbol{a}, g) \in U_{n+1}(A)$ *and given* $\varepsilon > 0$, *there is* $\boldsymbol{v} \in U_n(A)$ *and* $\boldsymbol{y} \in A^n$ *such that*
  i) $||\boldsymbol{v} - \boldsymbol{a}|| < \varepsilon$,
  ii) $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{y}\, g$.

Part (3) of the next result was already given in [12]. For the reader's convenience we reprove it here in order to have a better comparison possibility with the proofs of the other three items.

Theorem 1.8. *Let* A *be a commutative unital algebra over the field* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ *and let* I *be a proper ideal in* A. *Then*

(1) $R := \mathbb{K} + I$ *is a unital algebra with* $\mathbf{1}_R = \mathbf{1}_A$;
(2) $U_n(R) = U_n(A) \cap R^n$.
(3) $\operatorname{bsr} R \le \operatorname{bsr} A$.

If, additionally, A *is a commutative unital Banach algebra over* $\mathbb{K}$ *and* I *a proper closed ideal in* A, *then*

(4) $B := \mathbb{K} + I$ *is a Banach algebra and* $\operatorname{tsr} B \le \operatorname{tsr} A$.

Proof. (1) Just use that $(\alpha + f)(\beta + g) = \alpha\beta + h$ for $f, g, h \in I$ and $\alpha, \beta \in \mathbb{K}$. It is clear that $\mathbf{1}_A = \mathbf{1}_R$. We denote this unit by $\mathbf{1}$.

(2) Let $\boldsymbol{f} = (f_1, \dots, f_n) \in U_n(A) \cap R^n$. We first deal with the case $n = 1$. Then there is $\alpha_1 \in \mathbb{K}$, $F_1 \in I$ and $x_1 \in A$ such that $f_1 = \alpha_1 + F_1$ and

$$(\alpha_1 + F_1)x_1 = \mathbf{1}.$$

Now $\alpha_1 \ne 0$, because otherwise $F_1 = f_1 \in I \cap A^{-1}$; a contradiction to the assumption that I is a proper ideal. Since $F_1 x_1 \in I$, we deduce that $\alpha_1 x_1 = \mathbf{1} - F_1 x_1 \in R$. Hence $x_1 \in R$ (because $\alpha_1 \ne 0$), and so $f_1 \in U_1(R)$.

Now let n be arbitrary. Then at least one of the elements f_j does not belong to I, because otherwise $A = \sum_{j=1}^n f_j A \subseteq I$, and so I would no longer be proper. Say $f_1 \notin I$. Since $f_1 \in R$, we have $f_1 = \alpha_1 + F_1$ with $\alpha_1 \in \mathbb{K}$ and $F_1 \in I$, where once again $\alpha_1 \ne 0$.

Consider now the tuple

$$\boldsymbol{F} := (\alpha_1 + F_1, F_1 f_2, \dots, F_1 f_n) = (f_1, F_1 f_2, \dots, F_1 f_n).$$

Then its components cannot belong to a joint maximal ideal of A. In fact, suppose that there would exist a maximal ideal M such that $f_1 \in M$ and $F_1 f_j \in M$ for each j. Then $F_1 \in M$, because otherwise, due to the primeness of maximal ideals, every f_j would belong to M, which is a contradiction to $\boldsymbol{f} \in U_n(A)$. But $\alpha_1 + F_1 \in M$, too. So $\alpha_1 \in M$; which is impossible, since $\alpha_1 \ne 0$. We conclude that $\boldsymbol{F} \in U_n(A) \cap R^n$. Hence there exist $x_j \in A$ such that

$$x_1(\alpha_1 + F_1) + \sum_{j=2}^n x_j (F_1 f_j) = \mathbf{1}.$$

Where as usual $\mathbb{K}$ is identified with $\mathbb{K} \cdot \mathbf{1}_A$.

Since $h_j := x_j F_1 \in I \subseteq R$, we obtain

$$r := x_1(\alpha_1 + F_1) \in R,$$

and so, due to $\alpha_1 x_1 = r - x_1 F_1 \in R + I \subseteq R$, $x_1 \in R$. We conclude that $\sum_{j=1}^n h_j f_j = \mathbf{1}$ (where $h_1 := x_1$). In other words, $\boldsymbol{f} \in U_n(R)$. Since the inclusion $U_n(R) \subseteq U_n(A) \cap R^n$, is obvious we obtain assertion (2).

(3) We may assume that $n := \operatorname{bsr} A < \infty$. Let $(f_1, \ldots, f_n, h)$ be an invertible tuple in R.

Case 1: Suppose there is j_0 such that $f_{j_0} = \alpha + F$ for some $F \in I$ and $\alpha \in \mathbb{K}, \alpha \neq 0$. Then $(f_1, \ldots, f_n, Fh)$ is an invertible tuple in R, too. In fact, suppose that the ideal $J = f_1 R + \cdots + f_n R + Fh R$ is contained in a maximal ideal M of R. Since M is prime, either F or h is in M. But, by our hypothesis, h can't; so F is in M. But then $\alpha = f_{j_0} - F \in M$; a contradiction.

Since A has stable rank n, there exist $x_j \in A$ such that

$$(f_1 + x_1 Fh, \ldots, f_n + x_n Fh)$$

is an invertible tuple in A^n. But $x_j F \in I \subseteq R$. Since by (2), $U_n(R) = R^n \cap U_n(A)$, we conclude that the tuple $(f_1, \cdots, f_n, h)$ is reducible in R.

Case 2: If all the f_j are in I, then necessarily $h \notin I$. Hence it is easily checked that $(f_1 + h, f_2, \ldots, f_n, h)$ is an invertible tuple in R^{n+1}. Note that $f_1 + h \notin I$; so we have the situation of the first case (for $j_0 = 1$). Thus there are $y_j \in R$ such that

$$(f_1 + h + y_1 h, f_2 + y_2 h, \ldots, f_n + y_n h) = (f_1 + (y_1 + \mathbf{1})h, f_2 + y_2 h, \ldots, f_n + y_n h)$$

is an invertible tuple in R^n. We deduce that $(f_1, \ldots, f_n, h)$ is reducible in R.

(4) It is straightforward to check that B is closed in A. For the remaining assertion, we may assume that $n := \operatorname{tsr} A < \infty$. Let $\boldsymbol{f} := (f_1, \ldots, f_n) \in B^n$ and $\varepsilon > 0$.

Case 1: If there is j_0 such that $f_{j_0} = \alpha + F$ for some $F \in I$ and $\alpha \in \mathbb{K}, \alpha \neq 0$, then

$$(f_1, \ldots, f_n, F) \in U_{n+1}(B) \subseteq U_{n+1}(A).$$

By Theorem D, there is $\boldsymbol{v} \in U_n(A)$ and $\boldsymbol{y} \in A^n$ such that $||\boldsymbol{v} - \boldsymbol{f}|| < \varepsilon$ and $\boldsymbol{v} = \boldsymbol{f} + \boldsymbol{y} F$. But $\boldsymbol{y} F \in I^n$; so $\boldsymbol{v} \in B^n$. Due to (2), $\boldsymbol{v} \in U_n(B)$.

Case 2: If all the elements f_j belong to I then, for each $\varepsilon > 0$, the n-tuple $(f_1 + \varepsilon, \ldots, f_n) \in B^n$ satisfies the condition of Case 1. Hence, there is $\boldsymbol{v} \in U_n(B)$ such that $||\boldsymbol{v} - \boldsymbol{f}|| < 2\varepsilon$. Hence $U_n(B)$ is dense in B^n. □

2. The algebras $\mathrm{AP}_{\mathbb{Q}}$ and $\mathrm{AP}_{\mathbb{Q}^+}$

To study the algebra $\mathrm{AP}_{\mathbb{Q}^+}$ we need the following result(s) given in [7] and [13] that allow us to consider the functions in $\mathrm{AP}_{\mathbb{Q}^+}$, or more generally $\mathrm{AP}^+ := \mathrm{AP}_{\mathbb{R}^+}$, as boundary values of bounded holomorphic functions in the upper half plane

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Theorem E *Let* $\mathrm{AP}^+_{\mathrm{hol}}$ *denote the uniform closure in* $C_b(\mathbb{C}^+,\mathbb{C})$ *of the set of all functions of the form*

$$q(z)=\sum_{j=1}^{n} a_j e^{i\lambda_j z},$$

where $a_j\in\mathbb{C}, \lambda_j\in\mathbb{R}^+$, *and* $n\in\mathbb{N}^*$. *Then*

(1) $\mathrm{AP}^+_{\mathrm{hol}}$ *is a closed subalgebra of* $H^\infty(\mathbb{C}^+)$.
(2) *Every function* $f\in\mathrm{AP}^+_{\mathrm{hol}}$ *has a continuous extension,* f^*, *to the boundary* $\mathbb{R}$ *of* $\mathbb{C}^+$.
(3) $f^*\in\mathrm{AP}^+$ *and* $||f||_{\mathbb{C}^+}:=\sup_{z\in\mathbb{C}^+}|f(z)|=||f^*||_\infty$.
(4) AP^+ *is isomorphic isometric to* $\mathrm{AP}^+_{\mathrm{hol}}$.
(5) *If* $g\in\mathrm{AP}^+$, *then its Poisson-integral*

$$[g](z):=\int_{\mathbb{R}} P_y(x-t)g(t)\,dt,\quad z=x+iy\in\mathbb{C}^+$$

belongs to $\mathrm{AP}^+_{\mathrm{hol}}$ *and* $[g]^*=g$.
(6) *The Poisson operator* $\mathrm{AP}\to C(\mathbb{C}^+,\mathbb{C}), f\mapsto[f]$ *is multiplicative on* AP^+.
(7) AP^+ *is the set of functions in* AP *that admit a* bounded *holomorphic extension to* $\mathbb{C}^+$.
(8) $\mathbb{C}^+$ *can be embedded in the spectrum (maximal ideal space) of* $\mathrm{AP}_{\mathbb{Q}^+}$ *(and* AP^+*), via the functionals* $\delta_z: f\mapsto[f](z)$ *and the Gelfand transform* $\hat{f}$ *restricted to these functionals coincides with this Poisson extension.*

Now we are ready to prove Theorem 1.3.

$$\boxed{\operatorname{tsr}\mathrm{AP}_{\mathbb{Q}}=1.}$$

Let $f\in\mathrm{AP}_{\mathbb{Q}}$. According to Theorem B, let p be a generalized trigonometric polynomial with $\sigma(p)\subseteq\mathbb{Q}$ such that $||f-p||_\infty<\varepsilon/3$, say

$$p(t)=\sum_{j=1}^{N} a_j e^{ir_j t},$$

where $a_j\in\mathbb{C}$ and $r_j=s_j/m\in\mathbb{Q}$ with $s_j\in\mathbb{Z}$ and $m\in\mathbb{N}^*$, $(j=1,\dots,N)$. Associate with p the rational function

$$P(w)=\sum_{j=1}^{N} a_j w^{s_j}\in C(\mathbb{T}).$$

Then $P(e^{it/m})=p(t)$. Since $\operatorname{tsr}C(\mathbb{T})=1$ (see [16]), there exists $G\in C(\mathbb{T})$, $G\neq 0$ on $\mathbb{T}$ such that $||G-P||_{\mathbb{T}}<\varepsilon/3$. Say $|G|\geq\delta>0$ on $\mathbb{T}$.

Using Weierstrass' approximation theorem, there is a polynomial $Q(\xi,\eta)\in\mathbb{C}[\xi,\eta]$ such that

$$\max_{w\in\mathbb{T}}|Q(w,\overline{w})-G(w)|<\min\{\delta/2,\varepsilon/3\}.$$

Now

$$q(t):=Q(e^{it/m},e^{-it/m})\in\mathrm{AP}_{\mathbb{Q}}$$

and $||q - f||_\infty < \varepsilon$. Moreover, $|q| \geq \delta/2$ on $\mathbb{R}$. Hence, by Theorem [13, Proposition 2.4], $q \in \mathrm{AP}_{\mathbb{Q}}^{-1}$. Thus $\operatorname{tsr} \mathrm{AP}_{\mathbb{Q}} = 1$.

$$\boxed{\operatorname{tsr} \mathrm{AP}_{\mathbb{Q}^+} = 2.}$$

We first unveil a function f that cannot be uniformly approximated by functions invertible in $\mathrm{AP}_{\mathbb{Q}^+}$, showing that $\operatorname{tsr} \mathrm{AP}_{\mathbb{Q}^+} \geq 2$. Just take $f(t) = e^{it} - e^{-1}$ and note that the Poisson extension of $f(t)$ has the form $f(z) = e^{iz} - e^{-1}$, a holomorphic function that vanishes at $z_0 = i$. By Hurwitz's Theorem, f cannot be uniformly approximated on $\mathbb{C}^+$ by holomorphic functions having no zeros on $\mathbb{C}^+$. Theorem E (3), (8) now imply that f is the desired function.

Next we show that $\operatorname{tsr} \mathrm{AP}_{\mathbb{Q}^+} \leq 2$. Let $(f, g) \in (\mathrm{AP}_{\mathbb{Q}^+})^2$. By Theorem B, for $\varepsilon > 0$, there is a pair of generalized trigonometric polynomials $(p, q) \in (\mathrm{AP}_{\mathbb{Q}^+})^2$ such that $||p - f||_\infty + ||g - q||_\infty < \varepsilon/2$. Now

$$p(t) = \sum_{j=1}^{N} a_j e^{ir_j t} \text{ and } q(t) = \sum_{j=1}^{N} b_j e^{ir'_j t},$$

where $a_j, b_j \in \mathbb{C}$ and $r_j, r'_j \in \mathbb{Q}^+$. We may write $r_j = m_j/m$ and $r'_j = m'_j/m$ for $m_j, m'_j, m \in \mathbb{N}$, $m \neq 0$.

Associate with (p, q) the polynomials (P, Q) defined as

$$P(z) = \sum_{j=1}^{N} a_j z^{m_j} \text{ and } Q(z) = \sum_{j=1}^{N} b_j z^{m'_j}.$$

Then

$$P(e^{it/m}) = p(t) \text{ and } Q(e^{it/m}) = q(t).$$

Choose $0 < \varepsilon' < \varepsilon/2$ so that $\varepsilon' \notin P(Z(Q))$, where $Z(Q)$ is the zero-set of Q. Then $(P - \varepsilon', Q)$ is a pair of polynomials having no common zero. Hence there is another pair (P', Q') of polynomials in $\mathbb{C}[z]$ such that $P'(P - \varepsilon') + Q'Q = 1$. If we let

$$p'(t) = P'(e^{it/m}) \text{ and } q'(t) = Q'(e^{it/m})$$

then p' and q' belong to $\mathrm{AP}_{\mathbb{Q}^+}$ and $p'(p - \varepsilon') + q'q = 1$. Since

$$||p - \varepsilon' - f|| + ||q - g|| < \varepsilon' + \varepsilon/2 < \varepsilon,$$

we have approximated (f, g) by an invertible pair $(p - \varepsilon', q) \in (\mathrm{AP}_{\mathbb{Q}^+})^2$. Hence $\operatorname{tsr} \mathrm{AP}_{\mathbb{Q}^+} \leq 2$.

$$\boxed{\operatorname{bsr} \mathrm{AP}_{\mathbb{Q}} = 1.}$$

This is obvious due to the inequality $\operatorname{bsr} A \leq \operatorname{tsr} A$ for any Banach algebra.

$$\boxed{\operatorname{bsr} \mathrm{AP}_{\mathbb{Q}^+} = 1.}$$

Let (f, g) be an invertible pair in $\mathrm{AP}_{\mathbb{Q}^+} \subseteq \mathrm{AP}^+$. Without loss of generality, we may assume that $||f||_{\mathbb{R}} \leq 1/2$ and $||g||_{\mathbb{R}} \leq 1/2$. Since $\mathbb{C}^+$ can be viewed of as part of the spectrum of $\mathrm{AP}_{\mathbb{Q}^+}$ (Theorem E), we have

$$\inf_{z \in \mathbb{C}^+} |\hat{f}(z)| + |\hat{g}(z)| =: \delta > 0,$$

where $\hat{f}$ is the Gelfand transform (or equivalently) the Poisson extension of f. According to Theorem B, let (p, q) be a pair of generalized trigonometric polynomials with $\sigma(p) \cup \sigma(q) \subseteq \mathbb{Q}^+$, such that

$$||p - f||_{\mathbb{R}} + ||q - g||_{\mathbb{R}} < \varepsilon := \min\left\{\frac{\delta}{2}, \frac{1}{2}, \frac{1}{2C^2}\right\},$$

where $C = C(\delta/2)$ is the constant from Corollary 1.7. Without loss of generality, we may assume that $C > 1$. Now, if $f(t) = e^{i\lambda t}$, $\lambda \geq 0$, then $\hat{f}(z) = e^{i\lambda z}$. Since p and q are bounded on $\mathbb{R}$, the maximum principle for holomorphic functions therefore implies that

$$||\hat{p} - \hat{f}||_{\mathbb{C}^+} + ||\hat{q} - \hat{g}||_{\mathbb{C}^+} < \varepsilon$$

as well. Moreover, on $\mathbb{C}^+$,

$$|\hat{p}(z)| + |\hat{q}(z)| \geq |\hat{f}| + |\hat{g}| - \big(|\hat{f} - \hat{p}| + |\hat{g} - \hat{q}|\big) \geq \delta - \delta/2 = \delta/2.$$

Next we use the special structure of the functions $\hat{p}$ and $\hat{q}$. We deduce from the facts $p \in \mathrm{AP}_{\mathbb{Q}^+}$ and $q \in \mathrm{AP}_{\mathbb{Q}^+}$ that

$$\hat{p}(z) = \textstyle\sum_{j=1}^N a_j e^{i(s_j/m)z} \text{ for some } a_j \in \mathbb{C},\ s_j \in \mathbb{N} \text{ and } m \in \mathbb{N}^*,$$

and

$$\hat{q}(z) = \textstyle\sum_{j=1}^N b_j e^{i(k_j/m)z} \text{ for some } b_j \in \mathbb{C},\ k_j \in \mathbb{N} \text{ and } m \in \mathbb{N}^*.$$

Next we consider for $z \in \mathbb{C}$ the entire function $h(z) = e^{(i/m)z}$. It is straightforward to check that h maps $\mathbb{C}^+ \cup \mathbb{R}$ onto $\overline{\mathbb{D}} \setminus \{0\}$. Using the (non-injective) variable transformation $w = h(z)$, we see that the polynomials

$$P(w) = \textstyle\sum_{j=1}^N a_j w^{s_j} \text{ and } Q(w) = \sum_{j=1}^N b_j w^{k_j}$$

have the following properties:

i) $|P(w)| + |Q(w)| \geq \delta/2$ for all $w \in \overline{\mathbb{D}}$;
ii) $||P||_{\mathbb{D}} \leq 1$ and $||Q||_{\mathbb{D}} \leq 1$.

By Corollary 1.7, there exists $\chi \in \mathbb{C}[z]$ such that

$$\varphi := P + \chi Q \text{ is invertible in } W^+(\mathbb{D})$$

and

$$||\chi||_{\mathbb{D}} + ||\varphi||_{\mathbb{D}} + ||\varphi^{-1}||_{\mathbb{D}} \leq C$$

where $C > 1$ is the previously introduced constant. Let $\Phi(z) := \varphi(h(z))$. Since $h(t) = e^{(i/m)t} \in \mathrm{AP}_{\mathbb{Q}^+}$, we see that

$$\Phi|_{\mathbb{R}} \in (\mathrm{APW}_{\mathbb{Q}^+})^{-1}.$$

In particular, $(\Phi|_{\mathbb{R}})^{-1} \in \mathrm{AP}_{\mathbb{Q}^+}$. With $K(z) := \chi(h(z)) \in \mathrm{AP}^{\mathrm{trig}}_{\mathbb{Q}^+}$, we obtain

$$\Phi = \hat{p} + K\,\hat{q}.$$

Now, on $\mathbb{R}$,

$$f + Kg = \Phi + (f - p) + (g - q)\,K = \Phi \cdot \Big(1 - \Phi^{-1}(p - f) - \Phi^{-1}(q - g)\,K\Big).$$

But

$$||\Phi^{-1}(p - f) + \Phi^{-1}(q - g)\,K||_{\mathbb{R}} \leq C\frac{1}{2C^2} + C\frac{1}{2C^2}C < 1.$$

Hence, by a basic Banach-algebraic property, the restriction of the function $\Phi^{-1}(p-f)+\Phi^{-1}(q-g)\,K$ to $\mathbb{R}$ is invertible in $AP_{\mathbb{Q}^+}$. Thus, by letting $k:=K|_{\mathbb{R}}$, we conclude that $f+kg$ is invertible in $\mathrm{AP}^+_{\mathbb{Q}}$. Consequently, $\operatorname{bsr} AP_{\mathbb{Q}^+}=1$.

3. The algebras AP_Λ where Λ is a subgroup of $\mathbb{Z}$ or a semigroup in $\mathbb{N}$

For matter of completeness we recall the following nice result, forming part of common knowledge. Part (1) e.g. is in [1, p.45].

Theorem 3.1. *Let Λ be a subgroup of $\mathbb{Z}$. Then the following assertions hold:*

(1) $\Lambda=m\mathbb{Z}$ *for some* $m\in\mathbb{N}$.
(2) AP_Λ *is a self-adjoint algebra; that is,* $f\in\mathrm{AP}_\Lambda$ *implies* $\overline{f}\in\mathrm{AP}_\Lambda$.
(3) *If* $\varphi(x)=x^m$, *then* AP_Λ *is isometrically isomorphic to*

$$\{f\circ\varphi: f\in C(\mathbb{R},\mathbb{C}), f \text{ is } 2\pi\text{-periodic}\}.$$

(4) $\operatorname{bsr}\mathrm{AP}_\Lambda=\operatorname{tsr}\mathrm{AP}_\Lambda=1$.

Proof. (1) Let Λ be a subgroup $\mathbb{Z}$. We may assume that $1\notin\Lambda$, otherwise $\Lambda=\mathbb{Z}$. Let $m>0$ be the smallest positive element in Λ. Take $k\in\Lambda$. Due to the Euclidean algorithm, $k=qm+r$ with $0\leq r<m$. Since $k-qm\in\Lambda$ (as it is a group), $r\in\Lambda$. This is a contradiction to the minimality of m if $r\neq 0$. Thus $\Lambda=m\,\mathbb{Z}$.

(2) If $e^{i\lambda t}\in\mathrm{AP}_\Lambda$, then $e^{-i\lambda t}\in\mathrm{AP}_\Lambda$ (since Λ is a group). Hence AP_Λ is self-adjoint.

(3) By (1), $\Lambda=m\mathbb{Z}$. By Theorem B, each $f\in\mathrm{AP}_\Lambda$ is the uniform limit of trigonometric polynomials of the form $p(t)=\sum_{j=-N}^N a_je^{ijmt}$. Since by Weierstrass' Theorem, each continuous 2π-periodic function is a limit of genuine trigonometric polynomials $\sum_{j=-k}^k c_je^{ijt}$, we conclude that AP_Λ is isometrically isomorphic to

$$R=\{f\circ\varphi: f\in C(\mathbb{R},\mathbb{C}), f \text{ is } 2\pi\text{-periodic}\}.$$

(4) Since $\operatorname{tsr} C(\mathbb{T},\mathbb{C})=\lfloor\frac{\dim\mathbb{T}}{2}\rfloor+1=1$, we may approximate $p(z)=\sum_{j=-N}^N a_jz^j$ on $\mathbb{T}$ by such a function $q(z)=\sum_{j=-N}^N b_jz^j$ having no zeros on $\mathbb{T}$. Hence $q(e^{itm})$ is uniformly close on $\mathbb{R}$ to $p(e^{itm})$ which itself approximates $f(t)$. □

It remains to prove Theorem 1.4.

Proof. (1) By assumption, let Λ be a sub-semigroup of $\mathbb{N}$ such that $\gcd\Lambda=1$. We may assume that $1\notin\Lambda$, otherwise $\Lambda=\mathbb{N}$. In order to show that $S:=\mathbb{N}\setminus\Lambda$ is finite, we claim that there is $n_1,\dots,n_m\in\Lambda$ such that $\gcd(n_1,\dots,n_m)=1$. Write $\Lambda=\{n_1,n_2,\dots\}$ in an increasing order and let $d_k:=\gcd\{n_1,n_2,\dots,n_k\}$. Then $d_{k+1}\leq d_k$. Thus (d_k) must eventually be stationary, say $d_k=d$ for $k\geq k_0$. Then d divides all the elements in Λ and so $d=1$. This proves the claim.

Using the Euclidean algorithm, there are $s_j\in\mathbb{Z}$ such that $1=\sum_{j=1}^m s_jn_j$. Not all s_j can be positive. By re-enumerating the data here, we may assume that

$s_1, \dots, s_r \geq 0$ and $s_{r+1}, \dots, s_m < 0$. Hence

$$\underbrace{\sum_{j=1}^{r} s_j n_j}_{:=p} - \underbrace{\sum_{j=r+1}^{m} |s_j| n_j}_{:=q} = 1.$$

Note that $p, q \in \Lambda$. Now we have

$$\begin{array}{rcll} 1 + m_1 q & = & p + (m_1 - 1)q \in \Lambda & \forall m_1 \geq 1 \\ 2 + m_2 q & = & 2p + (m_2 - 2)q \in \Lambda & \forall m_2 \geq 2 \\ \vdots & & & \\ (q-1) + m_{q-1} q & = & (q-1)p + \big(m_{q-1} - (q-1)\big)q \in \Lambda & \forall m_{q-1} \geq q - 1. \end{array}$$

Thus, $j + kq \in \Lambda$ for all $k \geq q$ and $j = 0, 1, \dots, q-1$, and we conclude that $S = \mathbb{N} \setminus \Lambda$ is finite.

(2) Next we prove that AP_Λ is isometrically isomorphic to

$$A_S := \{f \in A(\mathbb{D}) : f^{(j)}(0) = 0 \text{ for } j \in S\}.$$

This is an easy consequence of the fact that the map

$$\Phi_\Lambda : \begin{cases} A_S & \to \mathrm{AP}_\Lambda \\ f & \mapsto f(e^{it}) \end{cases}$$

assigning to the disk-algebra function f its boundary function is an isometric isomorphism. The latter is due to the denseness in A_S of the polynomials whose coefficients associated with the monomials z^n are zero whenever $n \in S$.

(3) In view of (2) it suffices to determine the stable ranks (which are invariant under isometric isomorphisms) for the algebra A_S. Let us first give a concrete example to illustrate the method. If $\Lambda = 3\mathbb{N} + 5\mathbb{N}$, then $S = \{1, 2, 4, 7\}$. Now each $f \in A_S$ writes as

$$f(z) = a_0 + a_3 z^3 + a_5 z^5 + a_6 z^6 + z^8 g = a_0 + z^3(a_3 + z^2(a_5 + z(a_6 + z^2 g))).$$

for some $g \in A(\mathbb{D})$. Note that the powers in the second representation are the successive differences of the powers in the first representation. Thus

$$A_S = \mathbb{C} + z^3(\mathbb{C} + z^2(\mathbb{C} + z(\mathbb{C} + z^2 A(\mathbb{D})))).$$

Since $z^k A(\mathbb{D})$ is a closed ideal in $A(\mathbb{D})$, we get the conclusion $\mathrm{bsr}\, A_S = 1$ from Theorem 1.8. Since $z^3 \in A_S$, $\mathrm{tsr}\, A_S$ cannot be equal to one, because by Hurwitz's Lemma the uniform limit of zero-free functions is either zero-free or constant 0. Thus, by Theorem 1.8 again, $2 \leq \mathrm{tsr}\, A_S \leq \mathrm{tsr}\, A(\mathbb{D}) = 2$. The general case for an arbitrary semigroup Λ of $\mathbb{N}$ should be clear by now. Here are the details. First note that $1 \in S$, but not 0. We may assume that $\Lambda \neq \mathbb{N}$, as otherwise we know $\mathrm{bsr}\, AP_{\mathbb{N}} = 1$ and $\mathrm{tsr}\, AP_{\mathbb{N}} = 2$.

Let $\mathbb{N}\setminus\Lambda = \{d_0, \dots, d_r\}$, where $d_0 < d_1 < \cdots < d_r$. As $\Lambda \neq \mathbb{N}$, we have $d_0 = 1$. Given $f \in A_S$, we can write, analogously to the above example,

$$\begin{aligned} f(z) &= a_0 + a_1 z^{d_1+1} + \cdots + a_{r-1} z^{d_{r-1}+1} + z^{d_r+1} g \\ &= a_0 + z^{d_1+1}\Big(a_1 + \cdots \big(a_{r-2} + z^{d_{r-1}-d_{r-2}}\,\big(a_{r-1} + z^{d_r-d_{r-1}} g\big)\,\big)\Big), \end{aligned}$$

for some complex numbers $a_0, \cdots, a_r$ and some $g \in A(\mathbb{D})$. Thus

$$A_S = \mathbb{C} + z^{d_1+1}\Big(\mathbb{C} + \cdots \big(\mathbb{C} + z^{d_{r-2}-d_{r-1}}(\mathbb{C} + z^{d_r-d_{r-1}} A(\mathbb{D}))\big)\Big).$$

Since $z^{d_r+1}A(\mathbb{D})$ is a closed ideal in $A(\mathbb{D})$, we get the conclusion $\operatorname{bsr} A_S = 1$ from Theorem 1.8. As $z^{d_r+1} \in A_S$, we see that $\operatorname{tsr} A_S$ cannot be 1, thanks to Hurwitz's Lemma, since the uniform limit of zero-free functions (converging to z^{d_r+1}) must be either zero-free or constant. So by Theorem 1.8 again, $2 \leq \operatorname{tsr} A_S \leq \operatorname{tsr} A(\mathbb{D}) = 2$. □

As an amusing byproduct we obtain the following: Why the vector spaces $\{f \in A(\mathbb{D}) : f'(0) = 0\}$ and $\{f \in A(\mathbb{D}) : f^{(j)}(0) = 0\}$ for $j = 1, 2, 4, 7$ are algebras, whereas $\{f \in A(\mathbb{D}) : f''(0) = 0\}$ is not an algebra? Well, this is just a consequence of the fact that $\{2\}$ is not the complement of a semigroup in $\mathbb{N}$, whereas $\{1\} = \mathbb{N} \setminus (2\mathbb{N} + 3\mathbb{N})$ and $\{1, 2, 4, 7\} = \mathbb{N} \setminus (3\mathbb{N} + 5\mathbb{N})$ is. We have the following result:

Proposition 3.2. *Let $I \subseteq \mathbb{N}^*$ be a finite set. Then the vector space*

$$\{f \in A(\mathbb{D}) : f^{(j)}(0) = 0 \text{ for } j \in I\}$$

is an algebra if and only if I is the complement of a semigroup Λ in $\mathbb{N}$ with $\gcd \Lambda = 1$.

4. Epilog

In view of the results in this paper and in [13], we conjecture that the following holds:

Conjecture 4.1. Let Λ be an additive sub-semigroup of $\mathbb{R}^+$, but not a group, for which $N := \dim_{\mathbb{Q}}[\Lambda] < \infty$ (the dimension of the $\mathbb{Q}$-vector space generated by Λ). Then

(1) $\operatorname{bsr} AP_\Lambda = \left\lfloor \frac{N}{2} \right\rfloor + 1$;

(2) $\operatorname{tsr} AP_\Lambda = N + 1$.

If Λ is a group with $N := \dim_{\mathbb{Q}}[\Lambda] < \infty$, then we guess that

$$\operatorname{bsr} AP_\Lambda = \operatorname{tsr} AP_\Lambda = \left\lfloor \frac{N}{2} \right\rfloor + 1.$$

The support for this conjecture comes from the results above and the following result that was established in [13]:

Theorem F *Suppose that* $\Lambda_0 = \{\lambda_1, \dots, \lambda_N\}$ *is a set of* $\mathbb{Q}$*-linearly independent, positive reals. Let*

$$\Lambda_1 := \Big\{\sum_{j=1}^{N} s_j\lambda_j : s_j \in \mathbb{N}\Big\}$$

and

$$\Lambda_2 = \Big\{\sum_{j=1}^{N} s_j\lambda_j : s_j \in \mathbb{Z}\Big\}.$$

Then

$$A_1 := \mathrm{AP}_{\Lambda_1} = \{f \in \mathrm{AP} : \sigma(f) \subseteq \Lambda_1\}$$

is a uniformly closed subalgebra of AP^+ *that is isomorphic isometric to* $A(\mathbb{D}^N)$ *and*

$$A_2 := \mathrm{AP}_{\Lambda_2} = \{f \in \mathrm{AP} : \sigma(f) \subseteq \Lambda_2\}$$

is a uniformly closed subalgebra of AP that is isomorphic isometric to $C(\mathbb{T}^N, \mathbb{C})$*. In particular, by* [9],

$$\mathrm{bsr}\, A_1 = \mathrm{bsr}\, A(\mathbb{D}^N) = \left\lfloor \frac{N}{2} \right\rfloor + 1, \quad \mathrm{tsr}\, A_1 = \mathrm{tsr}\, A(\mathbb{D}^N) = N + 1,$$

$$\mathrm{bsr}\, A_2 = \mathrm{bsr}\, C(\mathbb{T}^N, \mathbb{C}) = \left\lfloor \frac{N}{2} \right\rfloor + 1 \text{ and } \mathrm{tsr}\, A_2 = \mathrm{tsr}\, C(\mathbb{T}^N, \mathbb{C}) = \left\lfloor \frac{N}{2} \right\rfloor + 1.$$

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