Appendixes

APPENDIX A: TECHNICAL LEMMAS

Hereafter, for a matrix $A = (a_{ij})$, define $\Re(A) = (\Re(a_{ij}))$ and $\Im(A) = (\Im(a_{ij}))$, where $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of the complex number z.

LEMMA A.1. *Let A be Hermitian. Then* $\mathrm{Tr}(A\Re(A)) = \mathrm{Tr}(\Re^2(A))$.

PROOF. Let a_{ij} be the (i, j)th element of A. Then

$$\mathrm{Tr}(A\Re(A)) = \sum_{j,k} a_{jk}\Re(a_{kj}) = \sum_{j,k} \Re^2(a_{kj}) - i\sum_{j,k} \Re(a_{kj})\Im(a_{kj}) = \mathrm{Tr}(\Re^2(A)),$$

where we used $\Re(a_{jk}) = \Re(a_{kj})$ and $\Im(a_{jk}) = -\Im(a_{kj})$. □

LEMMA A.2. *Let A be Hermitian and $\mathbf{z}_t$ has independent components with identical second and fourth order moments. Furthermore, $\mathbb{E}(z_{it}) = 0$, $\mathbb{E}|z_{it}|^2 = 1$, $\mathbb{E}|z_{it}|^4 = \nu_4 < \infty$ and $b = |\mathbb{E}(z_{it}^2)|^2$. Then*

$$\mathbb{E}(\mathbf{z}_t\mathbf{z}_t^* A\mathbf{z}_t\mathbf{z}_t^*) = \mathrm{Tr}(A)I_p + (\nu_4 - b - 2)\mathrm{diag}(A) + 2\Re(A) + (b-1)A^T.$$

If A is symmetric, then by denoting the kth moment of z_{it} as $\mathbb{E}(z^k)$, $k = 2, 4$,

$$\mathbb{E}(\mathbf{z}_t\mathbf{z}_t^T A\mathbf{z}_t\mathbf{z}_t^T) = \mathbb{E}^2(z^2)\mathrm{Tr}(A)I_p + (\mathbb{E}(z^4) - 3\mathbb{E}^2(z^2))\mathrm{diag}(A) + 2\mathbb{E}^2(z^2)A.$$

PROOF. The (i, j)th entry of $\mathbb{E}(\mathbf{z}_t\mathbf{z}_t^* A\mathbf{z}_t\mathbf{z}_t^*)$ is

$$\begin{aligned}(\mathbb{E}(\mathbf{z}_t\mathbf{z}_t^* A\mathbf{z}_t\mathbf{z}_t^*))_{ij} &= \mathbb{E}\Big(\sum_{k,\ell} z_{it}z_{kt}^* a_{k\ell}z_{\ell t}z_{jt}^*\Big) = \sum_k a_{kk}\mathbb{E}(z_{it}|z_{kt}|^2 z_{jt}^*) + \sum_{k\neq\ell} a_{k\ell}\mathbb{E}(z_{it}z_{kt}^* z_{\ell t}z_{jt}^*) \\ &= \begin{cases} a_{ij} + a_{ji}\mathbb{E}(z_{it}^2)\mathbb{E}((z_{jt}^*)^2), & i \neq j; \\ (\nu_4 - 1)a_{ii} + \mathrm{Tr}(A), & i = j \end{cases} = \begin{cases} 2\Re(a_{ij}) + a_{ji}(|\mathbb{E}(z_{it}^2)|^2 - 1), & i \neq j; \\ (\nu_4 - 1)a_{ii} + \mathrm{Tr}(A), & i = j. \end{cases}\end{aligned}$$

This completes the proof of the first part. For the second part,

$$\begin{aligned}(\mathbb{E}(\mathbf{z}_t\mathbf{z}_t^T A\mathbf{z}_t\mathbf{z}_t^T))_{ij} &= \mathbb{E}\Big(\sum_{k,\ell} z_{it}z_{kt}a_{k\ell}z_{\ell t}z_{jt}\Big) = \sum_k a_{kk}\mathbb{E}(z_{kt}^2 z_{it}z_{jt}) + \sum_{k\neq\ell} a_{k\ell}\mathbb{E}(z_{it}z_{kt}z_{\ell t}z_{jt}) \\ &= \begin{cases} (a_{ij} + a_{ji})\mathbb{E}^2(z^2), & i \neq j; \\ (\mathbb{E}(z^4) - \mathbb{E}^2(z^2))a_{ii} + \mathbb{E}^2(z^2)\mathrm{Tr}(A), & i = j \end{cases} \\ &= \begin{cases} 2a_{ij}\mathbb{E}^2(z^2), & i \neq j; \\ (\mathbb{E}(z^4) - \mathbb{E}^2(z^2))a_{ii} + \mathbb{E}^2(z^2)\mathrm{Tr}(A), & i = j \end{cases}\end{aligned}$$

This completes the proof of the lemma. □

LEMMA A.3. *Let the assumptions for $\mathbf{z}_t$ in Lemma A.2 hold. Then for $s \neq t$ and $\tau = 1, \ldots, q$,*

$$
\begin{aligned}
V_1 &= \mathbb{E}(\mathbf{z}_t^* \Sigma_0 \mathbf{z}_t) = \mathrm{Tr}(\Sigma_0),\\
V_2 &= \mathbb{E}(\mathbf{z}_t^* \Sigma_0 \mathbf{z}_t)^2 = \mathrm{Tr}^2(\Sigma_0) + (\nu_4 - b - 2)\mathrm{Tr}(\mathrm{diag}^2(\Sigma_0)) + 2\mathrm{Tr}(\Re^2(\Sigma_0)) + (b-1)\mathrm{Tr}(\Sigma_0 \Sigma_0^T),\\
V_3 &= \mathbb{E}(\mathbf{z}_t^* \Sigma_0 \mathbf{z}_s)^2 = b\mathrm{Tr}(\Sigma_0 \Sigma_0^T),\\
V_3' &= \mathbb{E}|\mathbf{z}_t^* \Sigma_0 \mathbf{z}_s|^2 = \mathrm{Tr}(\Sigma_0^2),\\
V_4 &= \mathbb{E}((\mathbf{z}_t^* \Sigma_0 \mathbf{z}_{t+\tau})^2 (\mathbf{z}_t^* \Sigma_0 \mathbf{z}_{t-\tau})^2)\\
&= b^2 \mathrm{Tr}^2(\Sigma_0 \Sigma_0^T) + (\mathbb{E}(\bar{z}^4)\mathbb{E}^2(z^2) - 3b^2)\mathrm{Tr}(\mathrm{diag}^2(\Sigma_0 \Sigma_0^T)) + 2b^2 \mathrm{Tr}(\Sigma_0 \Sigma_0^T)^2,\\
V_4' &= \mathbb{E}(|\mathbf{z}_{t+\tau}^* \Sigma_0 \mathbf{z}_t|^2 |\mathbf{z}_t^* \Sigma_0 \mathbf{z}_{t-\tau}|^2)\\
&= \mathrm{Tr}^2(\Sigma_0^2) + (\nu_4 - b - 2)\mathrm{Tr}(\mathrm{diag}^2(\Sigma_0^2)) + 2\mathrm{Tr}(\mathrm{Re}^2(\Sigma_0^2)) + (b-1)\mathrm{Tr}(\Sigma_0^2 (\Sigma_0^2)^T).
\end{aligned}
$$

PROOF. We have

$$V_1 = \mathrm{Tr}[\mathbb{E}(\mathbf{z}_t \mathbf{z}_t^*)\Sigma_0] = \mathrm{Tr}(\Sigma_0).$$

For V_2, using Lemma A.1 and the first part of Lemma A.2,

$$
\begin{aligned}
V_2 &= \mathrm{Tr}[\mathbb{E}(\mathbf{z}_t \mathbf{z}_t^* \Sigma_0 \mathbf{z}_t \mathbf{z}_t^*)\Sigma_0]\\
&= \mathrm{Tr}\left(\mathrm{Tr}(\Sigma_0)\Sigma_0 + (\nu_4 - b - 2)\mathrm{diag}(\Sigma_0)\Sigma_0 + 2\Re(\Sigma_0)\Sigma_0 + (b-1)\Sigma_0^T \Sigma_0\right)\\
&= \mathrm{Tr}^2(\Sigma_0) + (\nu_4 - b - 2)\mathrm{Tr}(\mathrm{diag}^2(\Sigma_0)) + 2\mathrm{Tr}(\Re^2(\Sigma_0)) + (b-1)\mathrm{Tr}(\Sigma_0 \Sigma_0^T).
\end{aligned}
$$

Also,

$$V_3 = \mathbb{E}(\mathbf{z}_t^* \Sigma_0 \mathbf{z}_s \mathbf{z}_s^T \Sigma_0^T \bar{\mathbf{z}}_t) = \mathrm{Tr}(\mathbb{E}(\bar{\mathbf{z}}_t \mathbf{z}_t^*)\Sigma_0 \mathbb{E}(\mathbf{z}_s \mathbf{z}_s^T)\Sigma_0^T) = b\mathrm{Tr}(\Sigma_0 \Sigma_0^T).$$

For V_3',

$$V_3' = \mathbb{E}(\mathbf{z}_t^* \Sigma_0 \mathbf{z}_s \mathbf{z}_s^* \Sigma_0 \mathbf{z}_t) = \mathrm{Tr}(\mathbb{E}(\mathbf{z}_t \mathbf{z}_t^*)\Sigma_0 \mathbb{E}(\mathbf{z}_s \mathbf{z}_s^*)\Sigma_0) = \mathrm{Tr}(\Sigma_0^2).$$

Using the second part of Lemma A.2 with $A = \Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^T \Sigma_0^T$ and z replaced by $\bar{z}$,

$$
\begin{aligned}
V_4 &= \mathbb{E}(\mathbf{z}_t^* \Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^T \Sigma_0^T \bar{\mathbf{z}}_t \mathbf{z}_t^* \Sigma_0 \mathbf{z}_{t-\tau} \mathbf{z}_{t-\tau}^T \Sigma_0^T \bar{\mathbf{z}}_t) = \mathrm{Tr}(\mathbb{E}(\bar{\mathbf{z}}_t \bar{\mathbf{z}}_t^T \Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^T \Sigma_0^T \bar{\mathbf{z}}_t \bar{\mathbf{z}}_t^T)\Sigma_0 \mathbb{E}(\mathbf{z}_{t-\tau} \mathbf{z}_{t-\tau}^T)\Sigma_0^T)\\
&= \mathbb{E}(z^2)\mathrm{Tr}\Big(\mathbb{E}^2(\bar{z}^2)\mathrm{Tr}(\mathbb{E}(\Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^T \Sigma_0^T))\Sigma_0 \Sigma_0^T + (\mathbb{E}(\bar{z}^4) - 3\mathbb{E}^2(\bar{z}^2))\mathrm{diag}(\mathbb{E}(\Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^T \Sigma_0^T))\Sigma_0 \Sigma_0^T\\
&\qquad + 2\mathbb{E}^2(\bar{z}^2)\mathbb{E}(\Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^T \Sigma_0^T)\Sigma_0 \Sigma_0^T\Big)\\
&= b^2 \mathrm{Tr}^2(\Sigma_0 \Sigma_0^T) + (\mathbb{E}(\bar{z}^4)\mathbb{E}^2(z^2) - 3b^2)\mathrm{Tr}(\mathrm{diag}^2(\Sigma_0 \Sigma_0^T)) + 2b^2 \mathrm{Tr}(\Sigma_0 \Sigma_0^T)^2.
\end{aligned}
$$

Finally, using the first part of Lemma A.2 with $A = \Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^* \Sigma_0$,

$$
\begin{aligned}
V_4' &= \mathbb{E}(\mathbf{z}_t^* \Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^* \Sigma_0 \mathbf{z}_t \mathbf{z}_t^* \Sigma_0 \mathbf{z}_{t-\tau} \mathbf{z}_{t-\tau}^* \Sigma_0 \mathbf{z}_t) = \mathrm{Tr}(\mathbb{E}(\mathbf{z}_t \mathbf{z}_t^* \Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^* \Sigma_0 \mathbf{z}_t \mathbf{z}_t^*)\Sigma_0 \mathbb{E}(\mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^*)\Sigma_0)\\
&= \mathrm{Tr}\Big(\mathrm{Tr}(\mathbb{E}(\Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^* \Sigma_0))\Sigma_0^2 + (\nu_4 - b - 2)\mathrm{diag}(\mathbb{E}(\Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^* \Sigma_0))\Sigma_0^2\\
&\qquad + 2\Re(\mathbb{E}(\Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^* \Sigma_0))\Sigma_0^2 + (b-1)(\mathbb{E}(\Sigma_0 \mathbf{z}_{t+\tau} \mathbf{z}_{t+\tau}^* \Sigma_0))^T \Sigma_0^2\Big)\\
&= \mathrm{Tr}^2(\Sigma_0^2) + (\nu_4 - b - 2)\mathrm{Tr}(\mathrm{diag}^2(\Sigma_0^2)) + 2\mathrm{Tr}(\Re^2(\Sigma_0^2)) + (b-1)\mathrm{Tr}(\Sigma_0^2 (\Sigma_0^2)^T).
\end{aligned}
$$

The proof is now complete. □

LEMMA A.4. *Let the assumptions for* $\mathbf{z}_t$ *in Lemma A.2 hold. Then* $G_q = \sum_{\tau=1}^{q} \mathrm{Tr}(\widehat{\Sigma}_\tau \widehat{\Sigma}_\tau^*)$ *has expectation and variance given by*

$$\mathbb{E}(G_q) = qV_1^2/T,$$

$$\mathrm{Var}(G_q) = \frac{q(V_2 - V_1^2)^2 + 2q(\Re(V_4) + V_4') + q(T-2)(V_3^2 + (V_3')^2)}{T^3} + \frac{4q^2V_1^2(V_2 - V_1^2)}{T^3}.$$

PROOF. Write

$$G_q = \sum_{\tau=1}^{q} Q_\tau, \quad \text{with} \quad Q_\tau = \frac{1}{T^2}\sum_{t,s=1}^{T} \mathbb{E}(\mathbf{x}_s^*\mathbf{x}_t\mathbf{x}_{t-\tau}^*\mathbf{x}_{s-\tau}).$$

Since $\mathbf{x}_t = \Sigma_0^{1/2}\mathbf{z}_t$ with the $\mathbf{z}_t$'s independent of each other, we have

$$\mathbb{E}(Q_\tau) = \frac{1}{T}\mathbb{E}(\mathbf{x}_t^*\mathbf{x}_t\mathbf{x}_{t-\tau}^*\mathbf{x}_{t-\tau}) = \frac{1}{T}\mathbb{E}^2(\mathbf{x}_t^*\mathbf{x}_t) = \frac{\mathrm{Tr}^2(\Sigma_0)}{T} = V_1^2/T.$$

The value of $\mathbb{E}(G_q)$ follows.

To evaluate $\mathbb{E}(Q_\tau^2)$, observe that

$$\mathbb{E}(Q_\tau^2) = \frac{1}{T^4}\sum_{t_1,t_2,s_1,s_2=1}^{T} \mathbb{E}(\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{t_1}\mathbf{z}_{t_1-\tau}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{s_2}^*\Sigma_0\mathbf{z}_{t_2}\mathbf{z}_{t_2-\tau}^*\Sigma_0\mathbf{z}_{s_2-\tau}).$$

Let $E(t_1, s_1, t_2, s_2)$ be the expectation on the right hand side above. We detail the cases where this expectation is non-zero below.

(I) $s_1 = t_1$, $s_2 = t_2$. Sub-cases:

i. $t_1 = t_2 (= t)$:
$T^{-4}\sum_{t=1}^{T} E(t,t,t,t) = \frac{1}{T^4}\sum_{t=1}^{T}\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)^2\mathbb{E}(\mathbf{z}_{t-\tau}^*\Sigma_0\mathbf{z}_{t-\tau})^2 = \frac{1}{T^4}\sum_{t=1}^{T} V_2^2 = V_2^2/T^3$;

ii. $t_1 = t_2 - \tau (= t)$:
$T^{-4}\sum_{t=1}^{T} E(t,t,t+\tau,t+\tau) = \frac{1}{T^4}\sum_{t=1}^{T}\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)^2\mathbb{E}(\mathbf{z}_{t+\tau}^*\Sigma_0\mathbf{z}_{t+\tau})\mathbb{E}(\mathbf{z}_{t-\tau}^*\Sigma_0\mathbf{z}_{t-\tau}) = V_2V_1^2/T^3$.

iii. $t_1 = t_2 + \tau (= t)$:
$T^{-4}\sum_{t=1}^{T} E(t,t,t-\tau,t-\tau) = V_2V_1^2/T^3$.

iv. Otherwise:

$$T^{-4}\sum_{s\neq t,t+\tau,t-\tau} E(t,t,s,s) = \frac{1}{T^4}\sum_{s\neq t,t+\tau,t-\tau}^{T}\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)\mathbb{E}(\mathbf{z}_{t-\tau}^*\Sigma_0\mathbf{z}_{t-\tau})\mathbb{E}(\mathbf{z}_s^*\Sigma_0\mathbf{z}_s)\mathbb{E}(\mathbf{z}_{s-\tau}^*\Sigma_0\mathbf{z}_{s-\tau})$$
$$= (T^2 - 3T)V_1^4/T^4.$$

(II) $s_1 = s_2, t_1 = t_2$. Sub-cases (not overlapping with (I))

i. $s_1 = t_1 + \tau (= t)$:
$T^{-4}\sum_{t=1}^{T} E(t-\tau,t,t-\tau,t) = \frac{1}{T^4}\sum_{t=1}^{T}\mathbb{E}[(\mathbf{z}_{t-2\tau}^*\Sigma_0\mathbf{z}_{t-\tau})^2(\mathbf{z}_t^*\Sigma_0\mathbf{z}_{t-\tau})^2] = \bar{V}_4/T^3$.

ii. $s_1 = t_1 - \tau (= t)$:
$T^{-4} \sum_{t=1}^{T} E(t+\tau, t, t+\tau, t) = V_4/T^3$.

iii. Otherwise:
$T^{-4} \sum_{s \neq t+\tau, t-\tau} E(t, s, t, s) = (T^2 - 2T)V_3^2/T^4$.

(III) $s_1 = t_2, s_2 = t_1$. Sub-cases (not overlapping with (I) and (II))

i. $s_2 = t_2 + \tau (= t)$:
$T^{-4} \sum_{t=1}^{T} E(t, t-\tau, t-\tau, t) = V_4'/T^3$.

ii. $s_2 = t_2 - \tau (= t)$:
$T^{-4} \sum_{t=1}^{T} E(t, t+\tau, t+\tau, t) = V_4'/T^3$.

iii. Otherwise:
$T^{-4} \sum_{s \neq t+\tau, t-\tau} E(t, s, s, t) = (T^2 - 2T)(V_3')^2/T^4$.

With the above,

$$\mathbb{E}(Q_\tau^2) = \frac{V_2^2 + 2V_1^2 V_2 + (T-3)V_1^4}{T^3} + \frac{2\Re(V_4) + (T-2)V_3^2}{T^3} + \frac{2V_4' + (T-2)(V_3')^2}{T^3}, \text{ so that}$$

$$\mathrm{Var}(Q_\tau) = \frac{V_2^2 + 2V_1^2 V_2 - 3V_1^4 + 2(\Re(V_4) + V_4') + (T-2)(V_3^2 + (V_3')^2)}{T^3}.$$

For $k \neq \ell$ and both k, ℓ are non-zero, consider

$$\begin{aligned}
\mathbb{E}(Q_k Q_\ell) &= \frac{1}{T^4} \sum_{t_1, s_1, t_2, s_2 = 1}^{T} \mathbb{E}(\mathbf{z}_{s_1}^* \Sigma_0 \mathbf{z}_{t_1} \mathbf{z}_{t_1-k}^* \Sigma_0 \mathbf{z}_{s_1-k} \mathbf{z}_{s_2}^* \Sigma_0 \mathbf{z}_{t_2} \mathbf{z}_{t_2-\ell}^* \Sigma_0 \mathbf{z}_{s_2-\ell}) \\
&= \frac{1}{T^4} \sum_{t,s=1}^{T} \mathbb{E}(\mathbf{z}_s^* \Sigma_0 \mathbf{z}_s \mathbf{z}_{s-k}^* \Sigma_0 \mathbf{z}_{s-k} \mathbf{z}_t^* \Sigma_0 \mathbf{z}_t \mathbf{z}_{t-\ell}^* \Sigma_0 \mathbf{z}_{t-\ell}) \\
&= \frac{4V_2V_1^2}{T^3} + \frac{(T^2-4T)V_1^4}{T^4},
\end{aligned}$$

where the last equality used the fact that when $t = s, t = s-k, s = t-\ell$ or $s-k = t-\ell$, the resulting expectation in the summation in the second equality is $V_2V_1^2$, otherwise it is V_1^4. Hence

$$\mathrm{Cov}(Q_k, Q_\ell) = \frac{4V_1^2(V_2 - V_1^2)}{T^3}.$$

With the above, we have

$$\begin{aligned}
\mathrm{Var}(G_q) &= \sum_{\tau=1}^{q} \mathrm{Var}(Q_\tau) + \sum_{k \neq \ell} \mathrm{Cov}(Q_k, Q_\ell) \\
&= q\mathrm{Var}(Q_\tau) + q(q-1)\mathrm{Cov}(Q_k, Q_\ell) \\
&= \frac{q(V_2 - V_1^2)^2 + 2q(\Re(V_4) + V_4') + q(T-2)(V_3^2 + (V_3')^2)}{T^3} + \frac{4q^2V_1^2(V_2 - V_1^2)}{T^3}.
\end{aligned}$$

This completes the proof of the lemma. □

LEMMA A.5. *Assume that Σ_0 is semipositive definite and $\mathbf{z}_t$ has real-valued independent components with identical eight order moments, $\mathbb{E}(z_{it}) = 0$, $\mathbb{E}|z_{it}|^2 = 1$, $\mathbb{E}|z_{it}|^k = \nu_k < \infty$, $k = 1, \cdots, 8$. Let ι denotes a $p \times 1$ vector with unit elements and $\odot$ denote Hadamard product, i.e. for any two given matrices A, B with same size, $(A \odot B)_{ij} = A_{ij}B_{ij}$, then*

$$
\begin{aligned}
V_5 &= \mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)^3 = \mathrm{Tr}^3(\Sigma_0) + 6\mathrm{Tr}(\Sigma_0)\mathrm{Tr}(\Sigma_0^2) + 8\mathrm{Tr}(\Sigma_0^3)\\
&\quad + (\nu_4 - 3)\left[3\mathrm{Tr}\left(D^2(\Sigma_0)\right)\mathrm{Tr}(\Sigma_0) + 4\mathrm{Tr}\left(D(\Sigma_0^2)\Sigma_0\right) + 8\mathrm{Tr}\left(D(\Sigma_0)\Sigma_0^2\right)\right]\\
&\quad + \nu_3^2\left[4\mathrm{Tr}\left(\Sigma_0(\Sigma_0 \odot \Sigma_0)\right) + 2\iota^* D(\Sigma_0)\Sigma_0 D(\Sigma_0)\iota + 4\iota^* D^2(\Sigma_0)\Sigma_0\iota\right]\\
&\quad + \left(\nu_6 - 10\nu_3^2 - 15(\nu_4 - 3) - 15\right)\mathrm{Tr}\left(\Sigma_0\left(\Sigma_0 \odot \Sigma_0\right)\right),\\
V_6 &= \mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)^4 = \mathrm{Tr}^4(\Sigma_0) + 12\mathrm{Tr}(\Sigma_0^2)\mathrm{Tr}^2(\Sigma_0) + 12\mathrm{Tr}^2(\Sigma_0^2) + 32\mathrm{Tr}(\Sigma_0)\mathrm{Tr}(\Sigma_0^3)\\
&\quad + 48\mathrm{Tr}(\Sigma_0^4) + (\nu_4 - 3)\left\{6\mathrm{Tr}^2(\Sigma_0)\mathrm{Tr}(\Sigma_0 \odot \Sigma_0) + 12\mathrm{Tr}(\Sigma_0^2)\mathrm{Tr}(\Sigma_0 \odot \Sigma_0)\right.\\
&\quad \left. + 48\mathrm{Tr}(\Sigma_0)\mathrm{Tr}(\Sigma_0 \odot \Sigma_0^2) + 48\mathrm{Tr}(D^2(\Sigma_0^2)) + 96\mathrm{Tr}(D(\Sigma_0)\Sigma_0^3)\right\}\\
&\quad + (\nu_4 - 3)^2\left\{3\mathrm{Tr}^2(\Sigma_0 \odot \Sigma_0) + 24\iota^* D(\Sigma_0)\left(\Sigma_0 \odot \Sigma_0\right)D(\Sigma_0)\iota + 8\iota^*\left(\Sigma_0 \odot \Sigma_0 \odot \Sigma_0 \odot \Sigma_0\right)\iota\right\}\\
&\quad + \left(\nu_6 - 15(\nu_4 - 3) - 10\nu_3^2 - 15\right)\left\{4\mathrm{Tr}(\Sigma_0)\mathrm{Tr}(\Sigma_0 \odot \Sigma_0 \odot \Sigma_0) + 24\mathrm{Tr}\left(\Sigma_0 \odot \Sigma_0 \odot (\Sigma_0^2)\right)\right\}\\
&\quad + 2\nu_3^2\left\{12\iota^*\left(D(\Sigma_0)\Sigma_0 D(\Sigma_0)\right)\iota \cdot \mathrm{Tr}(\Sigma_0) + 24\iota^*\left(D(\Sigma_0)\Sigma_0^2 D(\Sigma_0)\right)\iota\right.\\
&\quad \left. + 8\iota^*\left(\Sigma_0 \odot \Sigma_0 \odot \Sigma_0\right)\iota \cdot \mathrm{Tr}(\Sigma_0) + 48\iota^*(\Sigma_0 \odot \Sigma_0)\Sigma_0 D(\Sigma_0)\iota + 48\mathrm{Tr}\left(\Sigma_0^2\left(\Sigma_0 \odot \Sigma_0\right)\right)\right\}\\
&\quad + 2\nu_3(\nu_5 - 10\nu_3)\left\{12\iota^*\left(D(\Sigma_0)\Sigma_0 D^2\left(\Sigma_0\right)\right)\iota + 16\iota^*\left(\Sigma_0 \odot \Sigma_0 \odot \Sigma_0\right)D(\Sigma_0)\iota\right\} +\\
&\quad \left(\nu_8 - 28\nu_6 + 210(\nu_4 - 3) - 35(\nu_4 - 3)^2 - 56\nu_3(\nu_5 - 10\nu_3) + 315\right)\mathrm{Tr}\left(\Sigma_0 \odot \Sigma_0 \odot \Sigma_0 \odot \Sigma_0\right),\\
V_7 &= \mathbb{E}\left(\mathbf{z}_s^*\Sigma_0\mathbf{z}_{s+\tau}\mathbf{z}_{s+\tau}^*\Sigma_0\mathbf{z}_{s+\tau}\mathbf{z}_s^*\Sigma_0\mathbf{z}_{s-\tau}\mathbf{z}_{s-\tau}^*\Sigma_0\mathbf{z}_{s-\tau}\right) = \nu_3^2 \cdot \iota^*\left(D(\Sigma_0)\Sigma_0^2 D(\Sigma_0)\right)\iota.
\end{aligned}
$$

PROOF. Moments of quadratic forms are well studied in the fields of econometrics and statistics. Specifically, there have been long interest in deriving $\mathbb{E}\left(\prod_{i=1}^n Q_i\right)$, where $Q_i = y^*A_iy$, A_i are $p \times p$ non-stochastic symmetric matrices and y is an $p \times 1$ random vector with mean μ and identity covariance matrix. y^* represents transpose of y. Both V_5 and V_6 are moments of quadratic forms as a special case where all $A_i's$ equal to Σ_0, thus we can write down the results by directly referring to calculations and techniques presented in previous works of [39] and [6]. As for V_7,

$$
\begin{aligned}
V_7 &= \mathbb{E}\left(\mathbf{z}_{s+\tau}^*\Sigma_0\mathbf{z}_{s+\tau}\mathbf{z}_{s+\tau}^*\right)\Sigma_0\mathbb{E}(\mathbf{z}_s\mathbf{z}_s^*)\Sigma_0\mathbb{E}\left(\mathbf{z}_{s-\tau}\mathbf{z}_{s-\tau}^*\Sigma_0\mathbf{z}_{s-\tau}\right)\\
&= \nu_3^2 \cdot \iota^*\left(D(\Sigma_0)\Sigma_0^2 D(\Sigma_0)\right)\iota,
\end{aligned}
$$

because according to [39], $\mathbb{E}\left(\mathbf{z}_t\mathbf{z}_t^*\Sigma_0\mathbf{z}_t\right) = \nu_3\left(\mathbf{I}_p \odot \Sigma_0\right)\iota$.

□

LEMMA A.6. *Assume the same conditions as in Lemma A.5, then $p\hat{s}_1^2 = \frac{1}{p}\mathrm{Tr}^2(\Sigma_0)$ has expectation and variance given by*

$$
\mathbb{E}(p\hat{s}_1^2) = \frac{V_1^2}{p} - \frac{1}{pT}\left(V_1^2 - V_2\right),
$$

$$Var(p\hat{s}_1^2) = \frac{1}{p^2T^3}V_6 + \left(\frac{4}{p^2T^2} - \frac{4}{p^2T^3}\right)V_1V_5 + \left(\frac{2}{p^2T^2} - \frac{3}{p^2T^3}\right)V_2^2 + \left(\frac{4}{p^2T} - \frac{16}{p^2T^2} + \frac{12}{p^2T^3}\right)V_1^2V_2 + \left(-\frac{4}{p^2T} + \frac{10}{p^2T^2} - \frac{6}{p^2T^3}\right)V_1^4.$$

The covariance between G_q and $p\hat{s}_1^2$ is

$$Cov\left(G_q, p\hat{s}_1^2\right) = \left(\frac{4q}{pT^2} - \frac{10q}{pT^3}\right)V_1^2\left(V_2 - V_1^2\right) - \frac{4q}{pT^3}V_1^4 + \frac{2q}{pT^3}V_1V_5 + \frac{2q}{pT^3}V_2^2 + \frac{4q}{pT^3}V_7.$$

PROOF. Write

$$p\hat{s}_1^2 = \frac{1}{pT^2}\sum_{s,t=1}^{T}\mathbf{x}_t^*\mathbf{x}_t\mathbf{x}_s^*\mathbf{x}_s,$$

since $\mathbf{x}_t = \Sigma_0^{1/2}\mathbf{z}_t$ with $\mathbf{z}_t's$ independent of each other,

$$\mathbb{E}\left(p\hat{s}_1^2\right) = \frac{1}{pT^2}\sum_{t=1}^{T}\mathbb{E}\left(\mathbf{x}_t^*\mathbf{x}_t\mathbf{x}_t^*\mathbf{x}_t\right) + \frac{1}{pT^2}\sum_{t\neq s}\mathbb{E}(\mathbf{x}_t^*\mathbf{x}_t)\mathbb{E}(\mathbf{x}_s^*\mathbf{x}_s) = \frac{1}{pT}V_2 + \left(\frac{1}{p} - \frac{1}{pT}\right)V_1^2.$$

To evaluate $\mathbb{E}(p^2\hat{s}_1^4)$, observe that

$$\mathbb{E}(p^2\hat{s}_1^4) = \frac{1}{p^2T^4}\sum_{t_1,t_2,s_1,s_2=1}^{T}\mathbb{E}\left(\mathbf{z}_{t_1}^*\Sigma_0\mathbf{z}_{t_1}\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1}\mathbf{z}_{t_2}^*\Sigma_0\mathbf{z}_{t_2}\mathbf{z}_{s_2}^*\Sigma_0\mathbf{z}_{s_2}\right).$$

Denote $\mathbb{E}\left(\mathbf{z}_{t_1}^*\Sigma_0\mathbf{z}_{t_1}\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1}\mathbf{z}_{t_2}^*\Sigma_0\mathbf{z}_{t_2}\mathbf{z}_{s_2}^*\Sigma_0\mathbf{z}_{s_2}\right)$ by $F(t_1, s_1, t_2, s_2)$, we detail the cases where $F(t_1, s_1, t_2, s_2)$ is non-zero as follows.

(I) $t_1 = s_1 = t_2 = s_2$:

$$\frac{1}{p^2T^4}\sum_{t=1}^{T}F(t,t,t,t) = \frac{T}{p^2T^4}\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)^4 = \frac{1}{p^2T^3}V_6;$$

(II) $t_1 = s_1 = t_2 \neq s_2$:

$$\frac{4}{p^2T^4}\sum_{t_1\neq t_2}^{T}F(t_1,t_1,t_1,t_2) = \frac{4(T^2-T)}{p^2T^4}\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)^3 = \left(\frac{4}{p^2T^2} - \frac{4}{p^2T^3}\right)V_1V_5;$$

(III) $t_1 = s_1 \neq t_2 = s_2$:

$$\frac{3}{p^2T^4}\sum_{t_1\neq t_2}^{T} F(t_1,t_1,t_2,t_2) = \frac{3(T^2-T)}{p^2T^4}\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)^2\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)^2$$
$$= \left(\frac{3}{p^2T^2} - \frac{3}{p^2T^3}\right)V_2^2;$$

(IV) $t_1 = s_1 \neq t_2 \neq s_2$:

$$\frac{6}{p^2T^4}\sum_{t_1\neq t_2\neq s_2}^{T} F(t_1,t_1,t_2,s_2) = \frac{6T(T-1)(T-2)}{p^2T^4}\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)^2\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)$$
$$= \left(\frac{6}{p^2T} - \frac{18}{p^2T^2} + \frac{12}{p^2T^3}\right)V_2V_1^2;$$

(V) Otherwise, $t_1 \neq s_1 \neq t_2 \neq s_2$:

$$\frac{1}{p^2T^4}\sum_{t_1\neq s_1\neq t_2\neq s_2}^{T} F(t_1,s_1,t_2,s_2) = \frac{T^4 - 6T(T-1)(T-2) - 7T(T-1) - T}{p^2T^4}\left(\mathbb{E}(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t)\right)^4$$
$$= \left(\frac{1}{p^2} - \frac{6}{p^2T} + \frac{11}{p^2T^2} - \frac{6}{p^2T^3}\right)V_1^4.$$

With the above,

$$\mathbb{E}(p\hat{s}_1^2) = \frac{V_1^2}{p} - \frac{1}{pT}\left(V_1^2 - V_2\right),$$
$$\mathrm{Var}(p\hat{s}_1^2) = \frac{1}{p^2T^3}V_6 + \left(\frac{4}{p^2T^2} - \frac{4}{p^2T^3}\right)V_1V_5 + \left(\frac{2}{p^2T^2} - \frac{3}{p^2T^3}\right)V_2^2$$
$$+ \left(\frac{4}{p^2T} - \frac{16}{p^2T^2} + \frac{12}{p^2T^3}\right)V_1^2V_2 + \left(-\frac{4}{p^2T} + \frac{10}{p^2T^2} - \frac{6}{p^2T^3}\right)V_1^4.$$

As for covariance between G_q and $p\hat{s}_1^2$, since

$$G_q = \sum_{\tau=1}^{q} Q_\tau,\ Q_\tau = \frac{1}{T^2}\sum_{t,s=1}^{T}\mathbb{E}(\mathbf{x}_s^*\mathbf{x}_t\mathbf{x}_{t-\tau}^*\mathbf{x}_{s-\tau}),\ \mathbb{E}(Q_\tau) = V_1^2/T,$$
$$\mathrm{Cov}\left(G_q, p\hat{s}_1^2\right) = q\left(\mathbb{E}(Q_\tau\cdot p\hat{s}_1^2) - \mathbb{E}(Q_\tau)\mathbb{E}(p\hat{s}_1^2)\right),$$

we only need to consider

$$\mathbb{E}(Q_\tau\cdot p\hat{s}_1^2) = \frac{1}{pT^4}\sum_{t_1,t_2,s_1,s_2=1}^{T}\mathbb{E}\left(\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{t_1}\mathbf{z}_{t_1-\tau}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{t_2}^*\Sigma_0\mathbf{z}_{t_2}\mathbf{z}_{s_2}^*\Sigma_0\mathbf{z}_{s_2}\right).$$

In the following we detail the cases where the expectation on the right hand side above is non-zero.

(I) $s_1 = t_1$: $\mathbb{E}\left(\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{t_1}\mathbf{z}_{t_1-\tau}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{t_2}^*\Sigma_0\mathbf{z}_{t_2}\mathbf{z}_{s_2}^*\Sigma_0\mathbf{z}_{s_2}\right) = F(t_1, t_1-\tau, t_2, s_2)$. Sub-cases:

(i) $t_1 = t_2 = s_2$:
$$\frac{T}{pT^4}F(t, t-\tau, t, t) = \frac{1}{pT^3}V_1V_5;$$

(ii) $t_1 - \tau = t_2 = s_2$:
$$\frac{T}{pT^4}F(t, t-\tau, t-\tau, t-\tau) = \frac{1}{pT^3}V_1V_5;$$

(iii) $t_1 = t_2,\ t_1 - \tau = s_2$:
$$\frac{T}{pT^4}F(t, t-\tau, t, t-\tau) = \frac{1}{pT^3}V_2^2;$$

(iv) $t_1 - \tau = t_2\ , t_1 = s_2$:
$$\frac{T}{pT^4}F(t, t-\tau, t-\tau, t) = \frac{1}{pT^3}V_2^2;$$

(v) $t_1 = t_2,\ s_2 \neq t_1,\ s_2 \neq t_1 - \tau$:
$$\frac{T(T-2)}{pT^4}F(t_1, t_1-\tau, t_1, s_2) = \left(\frac{1}{pT^2} - \frac{2}{pT^3}\right)V_2V_1^2;$$

(vi) $t_1 = s_2,\ t_2 \neq t_1,\ t_2 \neq t_1 - \tau$:
$$\frac{T(T-2)}{pT^4}F(t_1, t_1-\tau, t_2, t_1) = \left(\frac{1}{pT^2} - \frac{2}{pT^3}\right)V_2V_1^2;$$

(vii) $t_1 - \tau = t_2,\ s_2 \neq t_1 - \tau,\ s_2 \neq t_1$:
$$\frac{T(T-2)}{pT^4}F(t_1, t_1-\tau, t_1-\tau, s_2) = \left(\frac{1}{pT^2} - \frac{2}{pT^3}\right)V_2V_1^2;$$

(viii) $t_1 - \tau = s_2,\ t_2 \neq t_1,\ t_2 \neq t_1 - \tau$:
$$\frac{T(T-2)}{pT^4}F(t_1, t_1-\tau, t_2, t_1-\tau) = \left(\frac{1}{pT^2} - \frac{2}{pT^3}\right)V_2V_1^2;$$

(ix) $t_2 = s_2,\ t_2 \neq t_1,\ t_2 \neq t_1 - \tau$:
$$\frac{T(T-2)}{pT^4}F(t_1, t_1-\tau, t_2, t_2) = \left(\frac{1}{pT^2} - \frac{2}{pT^3}\right)V_2V_1^2;$$

(x) Otherwise, $t_1 \neq t_1 - \tau \neq t_2 \neq s_2$:
$$\frac{T^3 - 4T - 5T(T-2)}{pT^4}F(t_1, t_1-\tau, t_2, s_2) = \left(\frac{1}{pT} - \frac{5}{pT^2} + \frac{6}{pT^3}\right)V_1^4;$$

(II) $s_1 \neq t_1, s_1 = t_1 - \tau$: $\quad \mathbb{E}\left(\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1+\tau}\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{t_2}^*\Sigma_0\mathbf{z}_{t_2}\mathbf{z}_{s_2}^*\Sigma_0\mathbf{z}_{s_2}\right)$.
Subcases:

(i) $t_2 = s_1 - \tau, s_2 = s_1 + \tau$:

$$\frac{T}{pT^4}\mathbb{E}\left(\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1+\tau}\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{s_1-\tau}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{s_1+\tau}^*\Sigma_0\mathbf{z}_{s_1+\tau}\right) = \frac{1}{pT^3}V_7;$$

(ii) $s_2 = s_1 - \tau, t_2 = s_1 + \tau$:

$$\frac{T}{pT^4}\mathbb{E}\left(\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1+\tau}\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{s_1+\tau}^*\Sigma_0\mathbf{z}_{s_1+\tau}\mathbf{z}_{s_1-\tau}^*\Sigma_0\mathbf{z}_{s_1-\tau}\right) = \frac{1}{pT^3}V_7;$$

(III) $s_1 \neq t_1, t_1 = s_1 - \tau$: $\mathbb{E}\left(\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{s_1-2\tau}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{t_2}^*\Sigma_0\mathbf{z}_{t_2}\mathbf{z}_{s_2}^*\Sigma_0\mathbf{z}_{s_2}\right)$.
Subcases:

(i) $t_2 = s_1, s_2 = s_1 - 2\tau$:

$$\frac{T}{pT^4}\mathbb{E}\left(\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{s_1-2\tau}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1}\mathbf{z}_{s_1-2\tau}^*\Sigma_0\mathbf{z}_{s_1-2\tau}\right) = \frac{1}{pT^3}V_7;$$

(ii) $t_2 = s_1 - 2\tau, s_2 = s_1$:

$$\frac{T}{pT^4}\mathbb{E}\left(\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{s_1-2\tau}^*\Sigma_0\mathbf{z}_{s_1-\tau}\mathbf{z}_{s_1-2\tau}^*\Sigma_0\mathbf{z}_{s_1-2\tau}\mathbf{z}_{s_1}^*\Sigma_0\mathbf{z}_{s_1}\right) = \frac{1}{pT^3}V_7;$$

With all the above, we have, the covariance between Q_τ and $p\hat{s}_1^2$ is

$$\begin{aligned}\mathrm{Cov}\left(Q_\tau, p\hat{s}_1^2\right) &= \left(\frac{4}{pT^2} - \frac{10}{pT^3}\right)V_1^2\left(V_2 - V_1^2\right) - \frac{4}{pT^3}V_1^4 + \frac{2}{pT^3}V_1V_5 \\ &+ \frac{2}{pT^3}V_2^2 + \frac{4}{pT^3}V_7.\end{aligned}$$

This completes the proof of the lemma.

□

Proof of Proposition 4.2.

Proof. Considering Σ_0 is with bounded spectral norm, by implementing the results in Lemma A.3 and Lemma A.5 to Lemma A.4 and Lemma A.6, we can evaluate the order of each term and select terms of orders $O(1)$ and $O(\frac{1}{T})$. Therefore, the leading order terms of $\mathbb{E}(p\hat{s}_1^2)$, $\mathrm{Var}(p\hat{s}_1^2)$, $\mathbb{E}(G_q)$, $\mathrm{Var}(G_q)$ and $\mathrm{Cov}(G_q, p\hat{s}_1^2)$ can be selected out accordingly. As for $\mathbb{E}(\hat{s}_2)$,

$$\begin{aligned}\mathbb{E}(\hat{s}_2) &= \frac{1}{pT^2}\sum_{t_1,t_2=1}^{T}\mathbb{E}\left(\mathbf{z}_{t_1}\Sigma_0\mathbf{z}_{t_2}\mathbf{z}_{t_2}^*\Sigma_0\mathbf{z}_{t_1}\right) \\ &= \frac{1}{pT^2}\sum_{t=1}^{T}\mathbb{E}\left(\mathbf{z}_t^*\Sigma_0\mathbf{z}_t\right)^2 + \frac{1}{pT^2}\sum_{t\neq s}\mathbb{E}|\mathbf{z}_t^*\Sigma_0\mathbf{z}_s|^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{pT} V_2 + \left(\frac{1}{p} - \frac{1}{pT} \right) V_3' \\
&= \frac{1}{p} \mathrm{Tr}(\Sigma_0^2) + \frac{1}{pT} \mathrm{Tr}^2(\Sigma_0) + \frac{1}{pT} \left(\mathrm{Tr}(\Sigma_0^2) + (\nu_4 - 3) \mathrm{Tr}(D^2(\Sigma_0)) \right).
\end{aligned}
$$

This completes the proof of Proposition 4.2. □

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