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## ON TESTING FOR HIGH-DIMENSIONAL WHITE NOISE

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Testing for white noise is a classical yet important problem in statistics, especially for diagnostic checks in time series modeling and linear regression. For high-dimensional time series in the sense that the dimension  $p$  is large in relation to the sample size  $T$ , the popular omnibus tests including the multivariate Hosking and Li-McLeod tests are extremely conservative, leading to substantial power loss. To develop more relevant tests for high-dimensional cases, we propose a portmanteau-type test statistic which is the sum of squared singular values of the first  $q$  lagged sample autocovariance matrices. It, therefore, encapsulates all the serial correlations (upto the time lag  $q$ ) within and across all component series. Using the tools from random matrix theory and assuming both  $p$  and  $T$  diverge to infinity, we derive the asymptotic normality of the test statistic under both the null and a specific VMA(1) alternative hypothesis. As the actual implementation of the test requires the knowledge of three characteristic constants of the population cross-sectional covariance matrix and the value of the fourth moment of the standardized innovations, non trivial estimations are proposed for these parameters and their integration leads to a practically usable test. Extensive simulation confirms the excellent finite-sample performance of the new test with accurate size and satisfactory power for a large range of finite  $(p, T)$  combinations, therefore ensuring wide applicability in practice. In particular, the new tests are consistently superior to the traditional Hosking and Li-McLeod tests.

**1. Introduction.** Testing for white noise is an important problem in statistics. It is indispensable in diagnostic checking for linear regression and linear time series modeling in particular. The surge of recent interests in modeling high-dimensional time series adds a further challenge: diagnostic checking demands the testing for high-dimensional white noise in the sense that the dimension of time series is comparable to or even greater than the sample size (i.e., the observed length of the time series). One prominent example showing the need for diagnostic checking in high-dimensional time series concerns the vector autoregressive model, which has a large literature. When the dimension is large, most existing works regularize the fitted models by Lasso [7, 18, 20, 32], Dantzig penalization [17], banded autocovariances [8], or banded auto-coefficient matrices [14]. However, none of them have developed any residual-based diagnostic tools. Another popular approach is to represent high-dimensional time series by lower-dimensional fac-

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tors. See for example, [35–37], [12, 13], [1], [24] and [10]. Again, there is a pertinent need to develop appropriate tools for checking the validity of the fitted factor models through careful examination of the residuals.

There are several well-established white noise tests for univariate time series [25]. Some of them have been extended for testing vector time series [19, 26, 29]. However, these methods are designed for the cases where the dimension of the time series is small or relatively small compared to the sample size. For the purpose of model diagnostic checking, the so-called omnibus tests are often adopted which are designed to detect any forms of departure from white noise. The celebrated Box-Pierce portmanteau test and its variations are the most popular omnibus tests. The fact that the Box-Pierce test and its variations are asymptotically distribution-free and  $\chi^2$ -distributed under the null hypothesis makes them particularly easy to use in practice. However, it is well known in the literature that the slow convergence to their asymptotic null distributions is particularly pronounced in multivariate cases. On the other hand, testing for high-dimensional time series is still in an infancy stage. To our best knowledge, the only available methods are [11] and [38].

To appreciate the challenge in testing for a high-dimensional white noise, we refer to an example reported in Section 3.1 below where say, we have to check the residuals from a fitted multivariate volatility for a portfolio containing  $p = 50$  stocks using their daily returns over a period of one semester. The length of the returns time series is then approximately  $T = 100$ . Table 2 shows that the two variants of the multivariate portmanteau test, namely the Hosking and Li-McLeod tests, all have actual sizes around 0.1%, instead of the nominal level of 5%. These omnibus tests are thus extremely conservative and they will not be able to detect an eventual misfitting of the volatility model.

The above example illustrates the following fact, which is now better understood: many popular tools in multivariate statistics are severely challenged by the emergence of high-dimensional data, and they need to be *re-examined* or *corrected*. Recent advances in high-dimensional statistics demonstrate that feasible and quality solutions to these high-dimensional challenges can be obtained by exploiting tools of random matrix theory via a precise spectral analysis of large sample covariance or sample autocovariance matrices. For a review on such progress, we refer to [22], [30] and monograph [41]. In particular, asymptotic results found in this context using random matrix theory exhibit fast convergence rates, and hence provide satisfactory finite sample approximation for data analysis.

This paper proposes a new method for testing high-dimensional white noise. The test statistic encapsulates the serial correlations within and across all component series. Precisely, the statistic is the sum of the squared singular values of several lagged sample autocovariance matrices. Using random matrix theory, asymptotic normality for the test statistics under the null is established under the Marčenko-Pastur asymptotic regime where  $p$  and  $T$  are large and comparable. Next, original methods are proposed for estimation of a few parameters in the limiting distribution in order to get a fully implementable version of the test. The asymptotic power of the test under a specific alternative of first-order vector moving average process (VMA(1)) has also been derived. Extensive simulation demonstrates excellent behavior of the proposed tests for a wide array of combinations of  $(p, T)$ , with accurate size and satisfactory power. In this paper, we also explore the reasons why the popular multivariate Hosking and Li-McLeod tests are no longer reliable

when the dimension is large in relation to the sample size.

The rest of the paper is organized as follows. Section 2 presents the main contributions of the paper. A new high-dimensional test for white noise is introduced, its asymptotic distributions under both the null and the VMA(1) alternative hypothesis are established. Section 3 reports extensive Monte-Carlo experiments which assess the finite sample behavior of the tests. Whenever possible, comparison is made with the popular Hosking and Li-McLeod tests. Numerical evidence also indicates that the new test is more powerful than that of [11]. Section 4 collects all the technical proofs.

**2. A test for high-dimensional white noise.** Let  $\mathbf{x}_1, \dots, \mathbf{x}_T$  be observations from a  $p \times 1$  complex-valued linear process of the form

$$\mathbf{x}_t = \sum_{l \geq 0} A_l \mathbf{z}_{t-l},$$

where  $A_l$  are  $p \times p$  coefficient matrices,  $\{\mathbf{z}_t\}$  is a sequence of  $p$ -dimensional random vectors such that, if the coordinates of  $\mathbf{z}_t$  are  $\{z_{it}\}$ , then the two dimensional array  $\{z_{it} : 1 \leq i \leq p, t \geq 1\}$  of variables are i.i.d. satisfying the moment conditions  $\mathbb{E}z_{it} = 0$ ,  $\mathbb{E}|z_{it}|^2 = 1$  and  $\mathbb{E}|z_{it}|^4 = \nu_4 < \infty$ . Hence  $\mathbb{E}\mathbf{x}_t = \mathbf{0}$ , and  $\Sigma_\tau \equiv \text{Cov}(\mathbf{x}_{t+\tau}, \mathbf{x}_t)$  depends on  $\tau$  only. Note that  $\Sigma_0 = \text{var}(\mathbf{x}_t)$  is the population covariance matrix of the time series. The goal is to test the null hypothesis

$$(2.1) \quad H_0 : \mathbf{x}_t = A_0 \mathbf{z}_t$$

where  $A_0$  is unknown. This in fact tests the independence instead of linear independence (i.e.  $\Sigma_\tau = 0$  for all  $\tau \neq 0$ ), which is however a common practice in the literature of white noise tests. Throughout the paper, the complex adjoint of a matrix (or vector)  $A$  is denoted by  $A^*$ . For  $\tau \geq 1$ , let  $\widehat{\Sigma}_\tau$  be the *lag  $\tau$  sample autocovariance matrix*

$$\widehat{\Sigma}_\tau = \frac{1}{T} \sum_{t=1}^T \mathbf{x}_t \mathbf{x}_{t-\tau}^*.$$

where by convention  $\mathbf{x}_t = \mathbf{x}_{T+t}$  when  $t \leq 0$ . Under the null hypothesis,  $\mathbb{E}(\widehat{\Sigma}_\tau) = 0$  for  $\tau \neq 0$ , and a natural test statistic is the sum of squared singular values of the first  $q$  lagged sample autocovariance matrices:

$$G_q = \sum_{\tau=1}^q \text{Tr}(\widehat{\Sigma}_\tau^* \widehat{\Sigma}_\tau) = \sum_{\tau=1}^q \sum_j \alpha_{\tau,j}^2,$$

where  $\{\alpha_{\tau,j}\}$  are the singular values of  $\widehat{\Sigma}_\tau$ , and  $\text{Tr}$  denotes the trace operation for square matrices. We reject the null hypothesis  $H_0$  for large values of  $G_q$ .

Notice that the setting here allows for complex-valued observations: this is important for applications in areas such as signal processing where signal time series are usually complex-valued. However, for the sake of presentation, we mostly focus on the real-valued case in the subsequent sections. Directions on how the tests can be extended to accommodate complex-valued observations will be given in the last Section 2.4.

2.1. *High dimensional asymptotics.* We adopt the so-called *Marčenko-Pastur regime* for asymptotic analysis, i.e. we assume  $c_p = p/T \rightarrow c > 0$  when  $p, T \rightarrow \infty$ . This asymptotic framework has been widely employed in the literature on high-dimensional statistics, see, [22], [30], also monograph [41] and the references within. Most of the results in this area concern sample covariance matrices only. However our test statistic  $G_q$  is based on the sample autocovariance matrices, which is much less studied; see [28] and [9].

As a main contribution of the paper, we characterize the asymptotic distribution of  $G_q$  in this high-dimensional setting when the observations are real-valued. We introduce the following limits whenever they exist: for  $\ell \geq 1$ ,

$$(2.2) \quad s_\ell = \lim_{p \rightarrow \infty} \frac{1}{p} \text{Tr}(\Sigma_0^\ell), \quad s_{d,\ell} = \lim_{p \rightarrow \infty} \frac{1}{p} \text{Tr}(D^\ell(\Sigma_0)),$$

where  $D(A)$  denotes the diagonal matrix consisting of the main diagonal elements of  $A$  (here the  $d$  in the index is a reminder of this diagonal structure).

**THEOREM 2.1.** *Let  $q \geq 1$  be a fixed integer, and the following assertions hold.*

1.  $\{\mathbf{z}_t\}$  is a sequence of real-valued independent  $p \times 1$  random vectors with independent components  $\mathbf{z}_t = (z_{it})$  satisfying  $\mathbb{E}z_{it} = 0$ ,  $\mathbb{E}z_{it}^2 = 1$  and  $\mathbb{E}z_{it}^4 = \nu_4 < \infty$ ;
2.  $\{\Sigma_0\}$  is a sequence of  $p \times p$  semi-positive definite matrices with bounded spectral norm such that the limits  $\{s_1, s_2\}$  and  $\{s_{d,2}\}$  exist;
3. (*Marčenko-Pastur regime*). The dimension  $p$  and the sample size  $T$  grow to infinity in a related way such that  $c_p := p/T \rightarrow c > 0$ .

Then when  $\mathbf{x}_t = \Sigma_0^{1/2} \mathbf{z}_t$ , the limiting distribution of the test statistic  $G_q$  is

$$(2.3) \quad G_q - qT c_p^2 s_1^2 \xrightarrow{d} \mathcal{N}(0, \sigma^2(c)),$$

where

$$(2.4) \quad \sigma^2(c) = 2qc^2 s_2^2 + 4q^2 c^3 (\nu_4 - 3) s_1^2 s_{d,2} + 8q^2 c^3 s_1^2 s_2.$$

The proof of this theorem is given in Section 4. It's worth mentioning here that in [9], they considered a simpler case when  $\Sigma_0 = \mathbf{I}_p$ ,  $q = 1$  and  $p = T$  with Gaussian population distribution, which is consistent with the results above.

Let  $Z_\alpha$  be the upper- $\alpha$  quantile of the standard normal distribution at level  $\alpha$ . Based on Theorem 2.1, we obtain a procedure for testing the null hypothesis in (2.1) as follows.

$$(2.5) \quad \text{Reject } H_0 \text{ if } \{G_q - qT c_p^2 s_1^2 > Z_\alpha \sigma(c)\}.$$

The illustration in Section 3 indicates that the test above is much more powerful than some classical alternatives, especially when the dimension  $p$  is growing linearly with the sample size  $T$ . The power of this test is gained from gathering together the serial correlations from the first  $q$  lags within and across all  $p$  component series; see the definition of  $G_q$ . Also note that the

asymptotic mean of  $G_q$  is  $qTc_p^2s_1^2$ , which grows linearly with  $T$  (and  $p$ ), while its asymptotic variance  $\sigma^2(c)$  is a constant. This implies that even for moderately large  $T$ , departure from white noise in the first  $q$  lags of the autocovariance matrices is likely to result in a large and different mean, which will be a large multiple standard deviation away from  $qTc_p^2s_1^2$  since the standard deviation  $\sigma(c)$  is constant.

However the test  $G_q$  in (2.5) is not yet practically usable as it depends on (i) three characteristic constants,  $s_1$ ,  $s_2$  and  $s_{d,2}$  of the (population) cross-sectional covariance matrix  $\Sigma_0$  and (ii) the fourth moment  $\nu_4$  of the innovations  $\{\mathbf{z}_t\}$ . These issues are addressed below.

*2.2. Estimation of the covariance characteristics  $s_1$  and  $s_2$ .* If the cross-sectional covariance matrix  $\Sigma_0$  is known, reasonable approximations of these characteristics are readily calculated from  $\Sigma_0$ . By Slutsky's Theorem, these estimates can substitute for the true ones in the asymptotic variance  $\sigma^2(c)$  and the centering term  $qTc_p^2s_1^2$ . The test (2.5) still applies.

However, the population covariance matrix  $\Sigma_0$  is in general unknown and the situation becomes challenging as estimating a general  $\Sigma_0$  is somehow out of reach without specific assumptions on its structure. Luckily, as observed previously, we only need consistent estimates of the three characteristics. First of all, in the setting of Theorem 2.1 and under the null, it is not difficult to find consistent estimators for these characteristics, thus a consistent estimator of the limiting variance  $\sigma^2(c)$ . The situation is much more intricate for the centering term  $qTc_p^2s_1^2$ . Suppose  $\hat{s}_1^2$  is a consistent estimator of  $s_1^2$ . Plugging it into the centering term leads to

$$(2.6) \quad G_{q,1} := G_q - qTc_p^2\hat{s}_1^2 = \{G_q - qTc_p^2s_1^2\} + qTc_p^2\{s_1^2 - \hat{s}_1^2\}.$$

Because of the multiplication by  $T$  here, the asymptotic distribution would remain the same only if the estimation error  $\{\hat{s}_1^2 - s_1^2\}$  is of order  $o_p(1/T)$ . This is however not the case and in general the error is exactly of the order  $O_p(1/T)$  and  $T\{\hat{s}_1^2 - s_1^2\}$  converges to some other normal distribution.

Our method is as follows. First we establish the joint asymptotic distribution of  $G_q - qTc_p^2s_1^2$  and  $p\{\hat{s}_1^2 - s_1^2\}$  for a natural estimator  $\hat{s}_1^2$ . This result extends Theorem 2.1 which addresses the statistic  $G_q - qTc_p^2s_1^2$  only. Next, the asymptotic null distribution of the ‘‘feasible’’ test statistic  $G_{q,1}$  is readily obtained as a simple consequence.

Precisely, consider the sample covariance matrix  $\widehat{\Sigma}_0 = \frac{1}{T} \sum_{t=1}^T \mathbf{x}_t \mathbf{x}_t^*$  and define the natural estimators of  $s_1$  and  $s_2$  as

$$\hat{s}_1 = \frac{1}{p} \text{Tr}(\widehat{\Sigma}_0), \quad \hat{s}_2 = \frac{1}{p} \text{Tr}(\widehat{\Sigma}_0^2).$$

**THEOREM 2.2.** *Assume the same conditions as in Theorem 2.1, then when  $\mathbf{x}_t = \Sigma_0^{1/2} \mathbf{z}_t$ , we have*

$$\begin{pmatrix} p(\hat{s}_1^2 - s_1^2) \\ G_q - qTc_p^2s_1^2 \end{pmatrix} \xrightarrow{d}$$

$$\mathcal{N}_2 \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4c(\nu_4 - 3)s_1^2 s_{d,2} + 8cs_1^2 s_2 & 4qc^2(\nu_4 - 3)s_1^2 s_{d,2} + 8qc^2 s_1^2 s_2 \\ 4qc^2(\nu_4 - 3)s_1^2 s_{d,2} + 8qc^2 s_1^2 s_2 & \sigma^2(c) \end{pmatrix} \right),$$

where the variance  $\sigma^2(c)$  is given in (2.4).

The proof of this theorem is relegated to Section 4.

Applying Theorem 2.2 to the decomposition (2.6), the following proposition establishes the asymptotic null distribution of the feasible statistic  $G_{q,1}$ . Second order terms of the mean and variance of  $G_{q,1}$  are also provided to improve finite sample performance.

**PROPOSITION 2.1.** *Assume the same conditions as in Theorem 2.2 and the observations are real-valued, we have*

$$(2.7) \quad G_{q,1} = G_q - qTc_p^2 \hat{s}_1^2 \xrightarrow{d} \mathcal{N}(0, \xi^2(c)),$$

where  $\xi^2(c) = 2qc^2 s_2^2$ . Meanwhile,

$$\begin{aligned} \mathbb{E}(G_{q,1}) &= -\frac{q}{T^2} \left( 2\text{Tr}(\Sigma_0^2) + (\nu_4 - 3)\text{Tr}(D^2(\Sigma_0)) \right), \quad \mathbb{E}(\hat{s}_1) = \frac{1}{p} \text{Tr}(\Sigma_0), \\ \text{Var}(G_{q,1}) &= \frac{2q}{T^2} \text{Tr}^2(\Sigma_0^2) + \frac{q}{T^3} \left( 2\text{Tr}(\Sigma_0^2) + (\nu_4 - 3)\text{Tr}(D^2(\Sigma_0)) \right)^2 + o\left(\frac{1}{T}\right), \\ \mathbb{E}(\hat{s}_2) &= \frac{1}{p} \text{Tr}(\Sigma_0^2) + \frac{1}{pT} \text{Tr}^2(\Sigma_0) + \frac{1}{pT} \left( \text{Tr}(\Sigma_0^2) + (\nu_4 - 3)\text{Tr}(D^2(\Sigma_0)) \right). \end{aligned}$$

Now we aim at consistent estimates for the unknown quantity  $s_2$  in the asymptotic variance  $\xi^2(c)$ . It is well known that almost surely ([2]),

$$\hat{s}_1 \rightarrow s_1, \quad \hat{s}_2 \rightarrow s_2 + cs_1^2.$$

Therefore  $\tilde{s}_2 = \hat{s}_2 - c_p \hat{s}_1^2$  is a strongly consistent estimator of  $s_2$ .

In summary, when  $\Sigma_0$  is unknown, we obtain a procedure for testing the null hypothesis of white noise (2.1) as follows:

$$(2.8) \quad \text{Reject } H_0 \text{ if } \{G_q - qTc_p^2 \hat{s}_1^2 > Z_\alpha \tilde{\xi}\}$$

where  $\tilde{\xi}^2 = 2qc_p^2 \tilde{s}_2^2$ .

**2.3. Finite sample correction and estimation for non-Gaussian innovations.** Although the test procedure (2.8) is already practically usable, it can be further improved by finite sample corrections provided in Proposition 2.1 which are especially useful for non-Gaussian population where  $\nu_4 \neq 3$ . To this goal, it remains to obtain a consistent estimate for (i) the covariance characteristic

$$s_{d,2} = \frac{1}{p} \sum_{i=1}^p d_i^2 = \frac{1}{p} \text{Tr}(D^2(\Sigma_0)),$$



where  $d_i = \Sigma_{0,ii}$  is the  $i$ th diagonal element of  $\Sigma_0$ , and (ii) the fourth moment  $\nu_4$  of the innovations.

(i) *Estimation of  $s_{d,2}$ .* By its very definition,  $d_i$  can be consistently estimated by its sample counterpart

$$\tilde{d}_i = \frac{1}{T} \sum_{t=1}^T x_{it}^2.$$

It follows that a consistent estimator for  $s_{d,2}$  is simply  $\tilde{s}_{d,2} = p^{-1} \sum_{i=1}^p \tilde{d}_i^2$ .

(ii) *Estimation of  $\nu_4$ .* This is again a non trivial problem which has not been touched yet in the literature (to our best knowledge). In order to get rid of the role of the unknown cross-sectional covariance matrix  $\Sigma_0$ , we adopt the following splitting strategy: the original data  $\{\mathbf{x}_t, t = 1, \dots, T\}$  are split into two halves of length  $T_1$  and  $T_2$ , respectively ( $T = T_1 + T_2$ ). Define the two corresponding sample cross-sectional covariance matrices

$$(2.9) \quad S_{n,1} = \frac{1}{T_1} \sum_{t=1}^{T_1} \mathbf{x}_t \mathbf{x}_t^*, \quad S_{n,2} = \frac{1}{T_2} \sum_{t=1}^{T_2} \mathbf{x}_{t+T_1} \mathbf{x}_{t+T_1}^*.$$

This yields the corresponding  $F$ -ratio, or Fisher matrix,  $F_n = S_{n,1}^{-1} S_{n,2}$ . Observe that this matrix does not depend on the value of the cross-sectional covariance  $\Sigma_0$  so that in what follows we can assume  $\Sigma_0 = I$ .

Let  $(\lambda_j)_{1 \leq j \leq p}$  be the eigenvalues of  $F_n$ . Define  $K$  test functions  $f_k(x) = \log(a_k + b_k x)$  where  $(a_k, b_k)_{1 \leq k \leq K}$  are some positive constants. For each  $k$ , we have an eigenvalue statistic of the Fisher matrix

$$X_{T,k} = f_k(\lambda_1) + \dots + f_k(\lambda_p) - p \int f_k(x) dF_{c_{p,1}, c_{p,2}}(x),$$

where  $c_{p,i} = p/T_i$  ( $i = 1, 2$ ) and  $F_{c,c'}$  is the limiting Wachter distribution with index  $(c, c')$ , see formula (3.1) in [42]. It is proved on page 452 of the reference, when  $p, T_1, T_2$  grow proportionally to infinity,

$$(2.10) \quad X_{T,k} = u_{T,k} + v_{T,k} \nu_4 + \varepsilon_{T,k},$$

where  $\{u_{T,k}, v_{T,k}\}$  are constants depending on  $\{c_{p,i}\}$  and  $(a_k, b_k)$ , and  $\{\varepsilon_{T,k}\}$  is a centered and asymptotically Gaussian error. Then the least squares estimator of  $\nu_4$  using the above regression model leads to a consistent estimate, say  $\hat{\nu}_4$  for the unknown parameter.

Under the null hypothesis, the observations are independent. We may repeat this estimation procedure, say  $B$  times, by taking  $B$  random splits of the initial sample. The final estimate of  $\nu_4$  is then taken to be the average of the estimates  $\{\hat{\nu}_{4,b}\}_{1 \leq b \leq B}$ .

Finally we can implement the following test procedure with finite sample correction for the null hypothesis of white noise (2.1):

$$(2.11) \quad \text{Reject } H_0 \text{ if } \left\{ G_{q,1}^* = G_q - qT c_p^2 \hat{s}_1^2 + \frac{1}{T} \cdot q c_p (2\tilde{s}_2 + (\hat{\nu}_4 - 3)\tilde{s}_{d,2}) > Z_\alpha \hat{\xi} \right\}$$



where

$$\hat{\xi}^2 = 2qc_p^2 \tilde{s}_2^2 + \frac{1}{T} \cdot qc_p^2 (2\tilde{s}_2 + (\hat{\nu}_4 - 3)\tilde{s}_{d,2})^2$$

with the above estimator  $\hat{\nu}_4$  for the fourth moment.

*2.4. Tests when the observations are complex-valued.* To proceed, we first define  $\mathbf{x}_t = \Sigma_0^{1/2} \mathbf{z}_t$  where  $\mathbf{z}_t$  is a *proper* complex random vector, and  $\Sigma_0^{1/2}$  is such that  $\Sigma_0^{1/2}$  is Hermitian with  $\Sigma_0 = \Sigma_0^{1/2} (\Sigma_0^{1/2})^*$  (*Properness* of a complex random vector  $\mathbf{z}_t$  means that  $\mathbb{E}(\mathbf{z}_t \mathbf{z}_t^T) = 0$ ). We immediately have

$$0 = \mathbb{E}(\mathbf{z}_t \mathbf{z}_t^T) = \mathbb{E}(z_{it}^2) I_p,$$

so that  $\mathbb{E}(z_{it}^2) = 0$  for all  $i = 1, \dots, p$  and  $t = 1, \dots, T$ . It also implies that  $b = |\mathbb{E}(z_{it}^2)|^2 = 0$ . Since  $\mathbf{x}_t = \Sigma_0^{1/2} \mathbf{z}_t$ , we have

$$\mathbb{E}(\mathbf{x}_t \mathbf{x}_t^T) = \mathbb{E}(\Sigma_0^{1/2} \mathbf{z}_t \mathbf{z}_t^T \Sigma_0^{T/2}) = 0,$$

so that we are also assuming an observed vector  $\mathbf{x}_t$  is proper.

From Corollary 4.1, since  $b = 0$  from the properness of  $\mathbf{z}_t$ , the asymptotic covariance of  $G_q$  is then

$$\text{Var}(G_q) \rightarrow qc^2 s_2^2 + 4q^2 c^3 s_1^2 [(\nu_4 - 2)s_{d,2} - s_2' + 2s_{r,2}],$$

where  $s_2' = \lim_{p \rightarrow \infty} \text{Tr}(\Sigma_0 \Sigma_0^T)/p$ ,  $s_{r,2} = \lim_{p \rightarrow \infty} \text{Tr}(\Re^2(\Sigma_0))/p$ , with  $\Re(A) = (\Re(a_{ij}))$ , the matrix of the real parts of all entries in  $A$ .

Using Lemma A.1, defining  $\Im(A) = (\Im(a_{ij}))$  to be the matrix of the imaginary parts of all entries in  $A$ , we have

$$\begin{aligned} 2\text{Tr}(\Re^2(\Sigma_0)) - \text{Tr}(\Sigma_0 \Sigma_0^T) &= 2\text{Tr}(\Sigma_0 \Re(\Sigma_0)) - \text{Tr}(\Sigma_0 (\Re(\Sigma_0) - i\Im(\Sigma_0))) \\ &= \text{Tr}(\Sigma_0 (\Re(\Sigma_0) + i\Im(\Sigma_0))) = \text{Tr}(\Sigma_0^2), \end{aligned}$$

so that  $2s_{r,2} - s_2' = s_2$ . The asymptotic variance for  $G_q$  is then

$$\text{Var}(G_q) \rightarrow \sigma^2(c) = qc^2 s_2^2 + 4q^2 c^3 s_1^2 [(\nu_4 - 2)s_{d,2} + s_2],$$

which can be estimated consistently using the estimators suggested in Section 2.2.

*2.5. Testing power of  $G_{q,1}$ .* In this section, we look into the power function of the tests when an alternative hypothesis  $H_1$  is specified. Here we assume that under  $H_1$ , the observations  $\mathbf{x}_1, \dots, \mathbf{x}_T$  follows from a  $p \times 1$  real-valued  $p$ -dimensional first-order vector moving average process, VMA(1) in short, of the form

$$(2.12) \quad H_1 : \mathbf{x}_t = A_0 \mathbf{z}_t + A_1 \mathbf{z}_{t-1},$$

where  $A_0, A_1$  are  $p \times p$  coefficient matrices. Now we only consider the asymptotic behavior of our test statistic  $G_q$  and  $G_{q,1}$  when  $q = 1$  since higher order autocorrelations of  $\mathbf{x}_t$  are null under both  $H_0$  and  $H_1$ .

Denote

$$\tilde{\Sigma}_0 = A_0^* A_0, \quad \tilde{\Sigma}_1 = A_1^* A_1, \quad \tilde{\Sigma}_{01} = A_0^* A_1,$$

we characterize the joint limiting distribution of  $\hat{s}_1^2$  and  $G_1$  under the VMA(1) alternative (2.12) as follows.

**THEOREM 2.3.** *Assume that*

1.  $\{\mathbf{z}_t\}$  is a sequence of real-valued independent  $p \times 1$  random vectors with independent components  $\mathbf{z}_t = (z_{it})$  satisfying  $\mathbb{E}z_{it} = 0$ ,  $\mathbb{E}z_{it}^2 = 1$  and  $\mathbb{E}z_{it}^4 = \nu_4 < \infty$ ;
2.  $\widetilde{\Sigma}_0$ ,  $\widetilde{\Sigma}_1$  and  $\widetilde{\Sigma}_{01}\widetilde{\Sigma}_{01}^*$  all have bounded spectral norm and for integers  $i, j, k \geq 0$ ,  $1 \leq i+j+k \leq 4$ , the limits  $\lim_{T \rightarrow \infty} \frac{1}{T} \text{Tr}(\widetilde{\Sigma}_0^i \widetilde{\Sigma}_1^j \widetilde{\Sigma}_{01}^k)$  exist;
3. (Marčenko-Pastur regime). The dimension  $p$  and the sample size  $T$  grow to infinity in a related way such that  $c_p := p/T \rightarrow c > 0$ .

Then under the VMA(1) alternative (2.12), the joint limiting distribution of the  $G_1$  and  $\hat{s}_1^2$  is

$$\begin{pmatrix} \sigma_G^2 & \sigma_{GS} \\ \sigma_{GS} & \sigma_S^2 \end{pmatrix}^{-1/2} \begin{pmatrix} G_1 - \mu_G \\ T c_p^2 \hat{s}_1^2 - \mu_S \end{pmatrix} \xrightarrow{d} \mathcal{N}_2(\mathbf{0}, I_2),$$

where

$$\begin{aligned} \mu_G &= \frac{1}{T} \text{Tr}^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) + \text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1) + \frac{2}{T} \text{Tr}^2(\widetilde{\Sigma}_{01}) \\ &\quad + \frac{1}{T} \left[ \text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1) + (\nu_4 - 3) \text{Tr}(D(\widetilde{\Sigma}_0)D(\widetilde{\Sigma}_1)) \right], \\ \mu_S &= \frac{1}{T} \text{Tr}^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) + \frac{4}{T^2} \text{Tr}(\widetilde{\Sigma}_{01}\widetilde{\Sigma}_{01}^*) \\ &\quad + \frac{1}{T^2} \left[ 2\text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)^2 + (\nu_4 - 3) \text{Tr}(D^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)) \right], \end{aligned}$$

$$\begin{aligned} \sigma_S^2 &= \frac{4}{T^3} \text{Tr}^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) \left[ 2\text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)^2 + (\nu_4 - 3) \text{Tr}(D^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)) \right] \\ &\quad + \frac{16}{T^3} \text{Tr}^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) \text{Tr}(\widetilde{\Sigma}_{01}\widetilde{\Sigma}_{01}^*) + R_n, \end{aligned}$$

and

$$\begin{aligned} \sigma_G^2 &= \frac{4}{T^3} \text{Tr}^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) \left[ 2\text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)^2 + (\nu_4 - 3) \text{Tr}(D^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)) \right] \\ &\quad + \frac{8}{T^2} \text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) \left[ 2\text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1 (\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)) + (\nu_4 - 3) \text{Tr}(D(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1)D(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)) \right] \\ &\quad + \frac{2}{T^2} \text{Tr}^2(\widetilde{\Sigma}_0^2 + \widetilde{\Sigma}_1^2) + \frac{6}{T^2} \text{Tr}^2(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1) + \frac{4}{T} \left[ 2\text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1)^2 + (\nu_4 - 3) \text{Tr}(D^2(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1)) \right] \\ &\quad + \frac{8}{T^2} \text{Tr}(\widetilde{\Sigma}_{01}\widetilde{\Sigma}_{01}^*) \text{Tr}(\widetilde{\Sigma}_0^2 + \widetilde{\Sigma}_1^2) + \frac{16}{T^2} \text{Tr}(\widetilde{\Sigma}_{01}\widetilde{\Sigma}_1) \text{Tr}(\widetilde{\Sigma}_{01}\widetilde{\Sigma}_0) \\ &\quad + \frac{16}{T^2} \text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) \left[ \text{Tr}(\widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_{01} \widetilde{\Sigma}_0) + \text{Tr}(\widetilde{\Sigma}_{01}\widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_1) \right] \\ &\quad + \frac{16}{T^2} \text{Tr}(\widetilde{\Sigma}_{01}) \left[ \text{Tr}(\widetilde{\Sigma}_0^2 \widetilde{\Sigma}_{01}^*) + \text{Tr}(\widetilde{\Sigma}_1^2 \widetilde{\Sigma}_{01}) + 2\text{Tr}(\widetilde{\Sigma}_1 \widetilde{\Sigma}_{01} \widetilde{\Sigma}_0) \right] \\ &\quad + \frac{4}{T} \text{Tr}(\widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_{01} \widetilde{\Sigma}_0^2 + \widetilde{\Sigma}_{01}\widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_1^2 + 2\widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_1 \widetilde{\Sigma}_{01} \widetilde{\Sigma}_0) \end{aligned}$$

$$\begin{aligned}
& + \frac{16}{T^3} \text{Tr}^2(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^*) + \frac{16}{T^3} \text{Tr}^2(\tilde{\Sigma}_{01}) \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)^2 \\
& + \frac{32}{T^3} \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \text{Tr}(\tilde{\Sigma}_{01}) \text{Tr}(\tilde{\Sigma}_{01}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \\
& + \frac{4}{T} \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^* \tilde{\Sigma}_{01}^* \tilde{\Sigma}_{01}) + \frac{12}{T^2} \text{Tr}^2(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^*) + \frac{16}{T^2} \text{Tr}(\tilde{\Sigma}_{01}) \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^* \tilde{\Sigma}_{01}^*) \\
& + \frac{16}{T^3} \text{Tr}^2(\tilde{\Sigma}_{01}) \left[ \text{Tr}(\tilde{\Sigma}_{01})^2 + 2 \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^*) + (\nu_4 - 3) \text{Tr}(D^2(\tilde{\Sigma}_{01})) \right] + \frac{8}{T^2} \text{Tr}^2(\tilde{\Sigma}_1 \tilde{\Sigma}_{01}) \\
& + \frac{16}{T^3} \text{Tr}(\tilde{\Sigma}_{01}) \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \left[ 2 \text{Tr}(\tilde{\Sigma}_{01}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) + (\nu_4 - 3) \text{Tr}(D(\tilde{\Sigma}_{01})D(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \right] \\
& + \frac{8}{T^2} \text{Tr}^2(\tilde{\Sigma}_0 \tilde{\Sigma}_{01}) + \frac{16}{T^2} \text{Tr}(\tilde{\Sigma}_{01}) \left[ 2 \text{Tr}(\tilde{\Sigma}_0 \tilde{\Sigma}_1 \tilde{\Sigma}_{01}) + (\nu_4 - 3) \text{Tr}(D(\tilde{\Sigma}_0 \tilde{\Sigma}_1)D(\tilde{\Sigma}_{01})) \right] + R_n,
\end{aligned}$$

$$\begin{aligned}
\sigma_{GS} & = \frac{4}{T^3} \text{Tr}^2(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \left[ 2 \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)^2 + (\nu_4 - 3) \text{Tr}(D^2(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \right] \\
& + \frac{4}{T^2} \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \left[ 2 \text{Tr}(\tilde{\Sigma}_0 \tilde{\Sigma}_1(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) + (\nu_4 - 3) \text{Tr}(D(\tilde{\Sigma}_0 \tilde{\Sigma}_1)D(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \right] \\
& + \frac{8}{T^2} \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \left[ \text{Tr}(\tilde{\Sigma}_{01}^* \tilde{\Sigma}_{01} \tilde{\Sigma}_0) + \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^* \tilde{\Sigma}_1) \right] + \frac{16}{T^3} \text{Tr}^2(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^*) \\
& + \frac{8}{T^3} \text{Tr}(\tilde{\Sigma}_{01}) \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \left[ 2 \text{Tr}(\tilde{\Sigma}_{01}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) + (\nu_4 - 3) \text{Tr}(D(\tilde{\Sigma}_{01})D(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \right] \\
& + \frac{16}{T^3} \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \text{Tr}(\tilde{\Sigma}_{01}) \text{Tr}(\tilde{\Sigma}_{01}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) + R_n.
\end{aligned}$$

Here the  $R_n$ 's, possibly different, represent remainders which have smaller orders than the other terms listed in  $\sigma_S^2$ ,  $\sigma_G^2$  and  $\sigma_{GS}$ , respectively.

The proof of this theorem is relegated to Section 4. Similarly, applying Theorem 2.3 to the decomposition (2.6), the following proposition establishes the asymptotic distribution of our test statistic  $G_{q,1}$  under the VMA(1) alternative (2.12) when  $q = 1$ .

**PROPOSITION 2.2.** *Assume the same conditions as in Theorem 2.3, when  $\mathbf{x}_t = A_0 \mathbf{z}_t + A_1 \mathbf{z}_{t-1}$  and the observables are real-valued, we have*

$$(2.13) \quad \sigma_{G_{1,1}}^{-1} \left( G_1 - T c_p^2 \hat{s}_1^2 - \mu_{G_{1,1}} \right) \xrightarrow{d} \mathcal{N}(0, 1),$$

where

$$\begin{aligned}
\mu_{G_{1,1}} & = \text{Tr}(\tilde{\Sigma}_0 \tilde{\Sigma}_1) + \frac{2}{T} \text{Tr}^2(\tilde{\Sigma}_{01}) + \frac{1}{T} \left[ \text{Tr}(\tilde{\Sigma}_0 \tilde{\Sigma}_1) + (\nu_4 - 3) \text{Tr}(D(\tilde{\Sigma}_0)D(\tilde{\Sigma}_1)) \right] \\
& \quad - \frac{4}{T^2} \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^*) - \frac{1}{T^2} \left[ 2 \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)^2 + (\nu_4 - 3) \text{Tr}(D^2(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \right], \\
\sigma_{G_{1,1}}^2 & = \frac{2}{T^2} \text{Tr}^2(\tilde{\Sigma}_0^2 + \tilde{\Sigma}_1^2) + \frac{4}{T} \left[ 2 \text{Tr}(\tilde{\Sigma}_0 \tilde{\Sigma}_1)^2 + (\nu_4 - 3) \text{Tr}(D^2(\tilde{\Sigma}_0 \tilde{\Sigma}_1)) \right] \\
& \quad + \frac{6}{T^2} \text{Tr}^2(\tilde{\Sigma}_0 \tilde{\Sigma}_1) + \frac{8}{T^2} \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^*) \text{Tr}(\tilde{\Sigma}_0^2 + \tilde{\Sigma}_1^2) + \frac{16}{T^2} \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_1) \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_0)
\end{aligned}$$

$$\begin{aligned}
& + \frac{16}{T^2} \text{Tr}(\widetilde{\Sigma}_{01}) \left[ \text{Tr}(\widetilde{\Sigma}_0^2 \widetilde{\Sigma}_{01}^*) + \text{Tr}(\widetilde{\Sigma}_1^2 \widetilde{\Sigma}_{01}) + 2 \text{Tr}(\widetilde{\Sigma}_1 \widetilde{\Sigma}_{01} \widetilde{\Sigma}_0) \right] \\
& + \frac{4}{T} \text{Tr}(\widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_{01} \widetilde{\Sigma}_0^2 + \widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_1^2 + 2 \widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_1 \widetilde{\Sigma}_{01} \widetilde{\Sigma}_0) + \frac{16}{T^3} \text{Tr}^2(\widetilde{\Sigma}_{01}) \text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)^2 \\
& + \frac{4}{T} \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}) + \frac{12}{T^2} \text{Tr}^2(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^*) + \frac{16}{T^2} \text{Tr}(\widetilde{\Sigma}_{01}) \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_{01}^*) \\
& + \frac{16}{T^3} \text{Tr}^2(\widetilde{\Sigma}_{01}) \left[ \text{Tr}(\widetilde{\Sigma}_{01})^2 + 2 \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^*) + (\nu_4 - 3) \text{Tr}(D^2(\widetilde{\Sigma}_{01})) \right] + \frac{8}{T^2} \text{Tr}^2(\widetilde{\Sigma}_1 \widetilde{\Sigma}_{01}) \\
& + \frac{8}{T^2} \text{Tr}^2(\widetilde{\Sigma}_0 \widetilde{\Sigma}_{01}) + \frac{16}{T^2} \text{Tr}(\widetilde{\Sigma}_{01}) \left[ 2 \text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1 \widetilde{\Sigma}_{01}) + (\nu_4 - 3) \text{Tr}(D(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1) D(\widetilde{\Sigma}_{01})) \right] + R_n.
\end{aligned}$$

Here  $R_n$  represents a remainder of smaller order than the other terms listed in  $\sigma_{G_{1,1}}^2$ .

Notice that if  $A_1 = \mathbf{0}$ ,  $\widetilde{\Sigma}_1 = 0$  and  $\widetilde{\Sigma}_{01} = 0$ . Then Theorem 2.3 and Proposition 2.2 reduce to Theorem 2.2 and Proposition 2.1, respectively.

Acturally, under the VMA(1) alternative (2.12) with  $q = 1$ , we have almost surely,  $\tilde{\xi} = \sqrt{2}c_p \tilde{s}_2 \rightarrow \xi_0$  as  $p, T \rightarrow \infty$ , where

$$(2.14) \quad \xi_0 = \lim_{T \rightarrow \infty} \sqrt{2} \left[ \frac{1}{T} \text{Tr}(\widetilde{\Sigma}_0^2 + \widetilde{\Sigma}_1^2) + \frac{2}{T} \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^*) + \frac{2}{T^2} \text{Tr}^2(\widetilde{\Sigma}_{01}) \right].$$

With Proposition 2.1 and 2.2, the power function of the test (2.8) is then easily derived.

**PROPOSITION 2.3.** *Assume the same conditions as in Theorem 2.3, then under  $H_1 : \mathbf{x}_t = A_0 \mathbf{z}_t + A_1 \mathbf{z}_{t-1}$ , the power function*

$$\beta_\alpha = \Pr(G_1 - Tc_p^2 \hat{s}_1^2 > Z_\alpha \tilde{\xi} \mid H_1) \rightarrow \Pr\left(Z > Z_\alpha \frac{\xi_0}{\widetilde{\sigma}_{G_{1,1}}} - \frac{\widetilde{\mu}_{G_{1,1}}}{\widetilde{\sigma}_{G_{1,1}}}\right), \text{ as } p, T \rightarrow \infty,$$

where  $Z$  represents a standard normal random variable,  $Z_\alpha$  is the upper- $\alpha$  quantile of the standard normal distribution,  $\widetilde{\mu}_{G_{1,1}}$  and  $\widetilde{\sigma}_{G_{1,1}}$  are limits of  $\mu_{G_{1,1}}$  and  $\sigma_{G_{1,1}}$  as  $T \rightarrow \infty$ .

In fact, under  $H_1$ , when  $\widetilde{\Sigma}_0$  and  $\widetilde{\Sigma}_1$  have bounded spectral norm, both  $\widetilde{\sigma}_{G_{1,1}}$  and  $\xi_0$  are of order  $O(1)$  and  $0 < \frac{\xi_0}{\widetilde{\sigma}_{G_{1,1}}} \leq 1$ , while the leading order term of  $\widetilde{\mu}_{G_{1,1}}$  is

$$\lim_{T \rightarrow \infty} \text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1) = \lim_{T \rightarrow \infty} \text{Tr}(A_1 A_0^* A_0 A_1^*) > 0.$$

Consequently,

Case 1. If  $\text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1)$  diverges as  $T \rightarrow \infty$ , then the power function  $\beta_\alpha \rightarrow 1$ ;

Case 2. If  $\text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1)$  is of order  $\Omega(1)$  (bounded from below and above), then the power function  $\beta_\alpha$  converges to the constant  $\beta = \Pr\left(Z > Z_\alpha \frac{\xi_0}{\widetilde{\sigma}_{G_{1,1}}} - \frac{\widetilde{\mu}_{G_{1,1}}}{\widetilde{\sigma}_{G_{1,1}}}\right)$  and  $\alpha \leq \beta \leq 1$ .

Therefore, as expected the asymptotic power of the test (2.8) under the VMA(1) alternative (2.12) depends on the eigenstructure of the coefficient matrix  $A_1$ . To illustrate, assume that (i)  $A_0A_0^*$  is of rank  $r_{0p} \sim rp$  for some constant  $0 < r \leq 1$ ; (ii)  $A_0A_0^*$  is of rank  $1 \ll r_{1p} \ll p$ , for example  $r_{1p} \sim r' \log p$  for some constant  $r' > 0$ , and that the non-null eigenvalues of both matrices are of order  $\Omega(1)$ . Then  $\text{Tr}(\widetilde{\Sigma}_0\widetilde{\Sigma}_1) \sim r''r_{1p} \rightarrow \infty$  for some constant  $r'' > 0$ , and the asymptotic power is equal to 1 (Case 1). If instead,  $r_{1p} = \Omega(1)$ , then the asymptotic power can be smaller than 1 (Case 2). Both situations correspond to a low-rank alternative for  $A_1$ , with exploding ranks in Case 1 and constant order ranks in Case 2.

Finally, as here the alternative is a VMA(1), one would expect that  $G_{q,1}$  with  $q > 1$  might have smaller power than  $G_{1,1}$ . This is indeed true because  $\widetilde{\mu}_{G_{q,1}}$  remains the same with  $\widetilde{\mu}_{G_{1,1}}$  under  $H_1$ , while  $\widetilde{\sigma}_{G_{q,1}}$  is larger than  $\widetilde{\sigma}_{G_{1,1}}$  and  $\xi_0$  increases with  $q$  as well.

**3. Simulation experiments.** Most of the experiments of this section are designed in order to compare the test procedures in (2.5) and (2.8) based on the statistics  $G_q$  and  $G_{q,1}$ , with two well known classical white noise tests, namely the Hosking test [19] and the Li-McLeod test [26].

To introduce the Hosking and Li-McLeod tests and using their notations, consider a  $p$ -dimensional VARMA( $u, v$ ) process of the form

$$\mathbf{x}_t - \Phi_1\mathbf{x}_{t-1} - \cdots - \Phi_u\mathbf{x}_{t-u} = \mathbf{a}_t - \Theta_1\mathbf{a}_{t-1} - \cdots - \Theta_v\mathbf{a}_{t-v},$$

where  $\mathbf{a}_t$  is a  $p$ -dimensional white noise with mean zero and variance  $\Sigma$ . Since  $\mathbf{x}_t$  is observed, with an initial guess of  $u$  and  $v$ , by assuming  $\mathbf{a}_t$  to be Gaussian, estimation of parameters  $\{\Phi, \Theta\}$  is conducted by the method of maximum likelihood. The initial estimates of  $u$  and  $v$  are further refined at the diagnostic checking stage based on the autocovariance matrices  $\hat{C}_\tau$  of the residuals  $\{\hat{\mathbf{a}}_t\}$ :

$$\hat{C}_\tau = \frac{1}{T} \sum_{t=\tau+1}^T \hat{\mathbf{a}}_t \hat{\mathbf{a}}_{t-\tau}^*, \quad \tau = 0, 1, 2, \dots$$

Hosking [19] proposed the portmanteau statistic

$$\widetilde{Q}_q = T^2 \sum_{\tau=1}^q \frac{1}{T-\tau} \text{Tr}(\hat{C}_\tau^* \hat{C}_0^{-1} \hat{C}_\tau \hat{C}_0^{-1}),$$

while Li et al. [26] recommended the use of the statistic

$$Q_q^* = T \sum_{\tau=1}^q \text{Tr}(\hat{C}_\tau^* \hat{C}_0^{-1} \hat{C}_\tau \hat{C}_0^{-1}) + \frac{p^2 q(q+1)}{2T}.$$

When  $\{\mathbf{x}_t\}$  follows a VARMA( $u, v$ ) model, both  $\widetilde{Q}_q$  and  $Q_q^*$  converge to  $\chi^2(p^2(q-u-v))$  distribution as  $T \rightarrow \infty$ , while the dimension  $p$  remains fixed.

To compare with our multi-lag  $q$  test statistics  $G_q$  and  $G_{q,1}$  when  $\Sigma_0$  is either known or unknown, we set  $u = v = 0$ . All tests use 5% significance level and the critical regions of the three tests are as follows:

- (i)  $G_q$  when all the limiting parameters are known as defined in (2.5) with  $\alpha = 5\%$ ;

- (ii)  $G_{q,1}$  with estimated limiting parameters as defined in (2.8) with  $\alpha = 5\%$ ;
- (iii) Hosking test:  $\left\{ \widetilde{Q}_q > \chi_{0.05, qp^2}^2 \right\}$ ;
- (iv) Li-McLeod test:  $\left\{ Q_q^* > \chi_{0.05, qp^2}^2 \right\}$ .

Here  $Z_{0.05}$  and  $\chi_{0.05, m}^2$  denote the upper-5% quantile of the standard normal distribution and the chi-squared distribution with degrees of freedom  $m$ , respectively. Empirical statistics are obtained using 2000 independent replicates.

3.1. *Empirical sizes.* The data is generated as  $\mathbf{x}_t = \Sigma_0^{1/2} \mathbf{z}_t$ , with  $\mathbf{z}_t$ ,  $t = 1, \dots, T$  being independent and identically distributed. We adopt diverse settings for  $\mathbf{z}_t$  and  $\Sigma_0$  respectively to compare the sizes of four test statistics.

As for  $\mathbf{z}_t$ , we use two models to represent different distributions for  $\mathbf{z}_t$ :

- (I)  $\mathbf{z}_t \sim N_p(\mathbf{0}, \mathbf{I}_p)$ , i.i.d.  $t = 1, \dots, T$ ;
- (II)  $\mathbf{z}_t$  with i.i.d. components  $z_{it} \sim \text{Gamma}(4, 0.5) - 2$ ,  $i = 1, \dots, p$ ,  $t = 1, \dots, T$ ,  $\mathbb{E}(z_{it}) = 0$ ,  $\text{Var}(z_{it}) = 1$ ,  $v_4(z_{it}) = 4.5$ ;

As for  $\Sigma_0$ , we use two different models as follows.

- (III)  $\Sigma_0 = \mathbf{I}_p$ ;
- (IV)  $\Sigma_0 = \frac{4}{p} A_0 A_0^*$ ,  $A_0$  is  $p \times p$  matrix with entries  $a_{ij} \sim U(-1, 1)$  i.i.d..

Table 2 compares the sizes of the four tests for two different  $q$  when  $\Sigma_0 = \mathbf{I}_p$ . Cases when  $p > T$  are not considered here since  $\widetilde{Q}_q$  and  $Q_q^*$  are not applicable then.

The main information from Table 2 is that classical test procedures derived using large sample scheme, namely by letting the sample size  $T \rightarrow \infty$  while the dimension  $p$  remains fixed, are heavily biased when the dimension  $p$  is in fact not negligible with respect to the sample size. To be more precise, these biases are clearly present when the dimension-to-sample ratio  $p/T$  is not “small enough”, say greater than 0.1. Such high-dimensional traps for classical procedures have already been reported in other testing problems, see for example [4] and [40]. Here we observe that the empirical sizes of the Hosking and the Li-McLeod tests quickly degenerate to 0 as the ratio  $p/T$  increases from 0.1 to 0.5. In other words, the critical values from their  $\chi_{qp^2}^2$  asymptotic limits are seemingly *too large*. On the other hand, the statistics  $G_q$  and  $G_{q,1}$  have reasonable sizes when compared to the 5% nominal level across all the tested  $(p, T)$  combinations. Various  $(p, T)$  combinations are accommodated to testify the adaptability of our test statistics,  $G_q$  and  $G_{q,1}$ . Test sizes in both high and low dimension cases are shown in Table 5. It can be seen that both  $G_q$  and  $G_{q,1}$  attain the nominal level accurately under various scenarios.

3.2. *Empirical powers and adjusted powers.* In this section, we compare the empirical powers of the tests by assuming that  $\mathbf{x}_t = \Sigma_0^{1/2} \mathbf{y}_t$ ,  $\mathbf{y}_t$  follows a vector autoregressive process of order 1,

$$\mathbf{x}_t = \Sigma_0^{1/2} \mathbf{y}_t, \mathbf{y}_t = A \mathbf{y}_{t-1} + \mathbf{z}_t,$$

where  $A = a \mathbf{I}_p$ ,  $\mathbf{z}_t \sim N_p(\mathbf{0}, \mathbf{I}_p)$  being independent of each other for  $t = 1, \dots, T$ . Here we assign  $a = 0.1$  and apply two classic test procedures,  $\widetilde{Q}_q$  and  $Q_q^*$ , to get the power values as in Table 1.

From Table 2 we know that the two classic tests become seriously biased when the dimension  $p$  is large compared to the sample size  $T$ . Their sizes approach zero when  $p/T$  becomes larger. From Table 1, we see that due to the biased critical values used in  $\widetilde{Q}_q$  and  $Q_q^*$  as shown in Table 2, their powers are driven downward. This is particularly severe when the ratio  $p/T$  is larger than 0.5.

In Table 3, we compare the *intrinsic powers* of all four test procedures when  $\Sigma_0 = I_p$ . Namely, we empirically find the 95 percentiles of  $\widetilde{Q}_q$  and  $Q_q^*$  under the null and use these values as the corrected critical value for the power comparison. It is interesting to observe that after such correction, both  $\widetilde{Q}_q$  and  $Q_q^*$  show very reasonable powers which all increase to 1 when the dimension and the sample size increases. Our test statistics  $G_q$  and  $G_{q,1}$  also maintain comparably high power in all the tested  $(p, T)$  combinations. Table 6 demonstrates the feasibility of our test statistics under both high and low dimension cases. Interestingly enough,  $G_{q,1}$  shows slightly better power than  $G_q$  under the present AR(1) alternative which is not intuitive. Comparison with the Hosking and the Li-McLeod tests sheds new light on the superiority of our test statistics in both low and high dimensional cases.

3.3. *Why both the Hosking and the Li-McLeod tests fail in high dimension.* The experiments here are designed to explore the reasons behind the failure of the Hosking and the Li-McLeod tests in high dimension. For the test statistics  $\widetilde{Q}_q$  and  $Q_q^*$  as well as our test statistic  $\phi_\tau$ , we consider their empirical mean, variance and the 95% quantile, say  $\theta_{emp}$ , with their theoretical values predicted by their respective asymptotic distributions (denoted as  $\theta_{theo}$ ). As for the two classical tests, we have often observed very large discrepancy between these values so it is more convenient to report the corresponding relative errors  $(\theta_{theo} - \theta_{emp})/\theta_{emp}$  (in percentage). This is done in Table 4. It clearly appears from this table that for both statistics  $\widetilde{Q}_q$  and  $Q_q^*$ , the traditional asymptotic theory severely overestimated their variances, that is their empirical means are close to the degree of freedom  $p^2(q - u - v)$  of the asymptotic chi-squared distribution while their empirical variances are much smaller than  $2p^2(q - u - v)$  as suggested by the same chi-squared limit. This leads to an inflated 95th percentiles which, although in a lesser proportion, is enough to create a high downward-bias in the empirical sizes of these two classical tests with high-dimensional data; See Table 2.

3.4. *Comparison with other test statistics.* In this section, we compare our test statistics with some others in recent literature. Chang, Yao and Zhou [11] proposed an omnibus test for vector white noise using the maximum absolute autocorrelations and cross-correlations of the component series. Let

$$\widehat{\Gamma}(k) = \{\widehat{\rho}_{ij}(k)\}_{1 \leq i, j \leq p} = \text{diag}\{\widehat{\Sigma}(0)\}^{-1/2} \widehat{\Sigma}(k) \text{diag}\{\widehat{\Sigma}(0)\}^{-1/2}$$

be the sample autocorrelation matrix at lag  $k$ , where  $\widehat{\Sigma}(k) = \frac{1}{T} \sum_{t=1}^{T-k} \mathbf{x}_{t+k} \mathbf{x}_t^*$ . Their test statistic  $T_n$  is defined as

$$T_n = \max_{1 \leq k \leq q} T_{n,k},$$

where  $T_{n,k} = \max_{1 \leq i, j \leq p} T^{1/2} |\widehat{\rho}_{ij}(k)|$ . Another test statistic  $T_n^*$  is defined in the same manner as  $T_n$ , only that the time series principal component analysis proposed by Chang *et al.* [10] is applied to the data  $\{\mathbf{x}_t\}$  first.



Here we fix  $p = 20$ ,  $T = 100$  and adopt the spherical AR(1) process for power comparison, i.e.  $\mathbf{x}_t = \Sigma_0^{1/2} \mathbf{y}_t$ ,  $\mathbf{y}_t = A\mathbf{y}_{t-1} + \mathbf{z}_t$ ,  $A = a\mathbf{I}_p$ , where  $\mathbf{z}_t$  and  $\Sigma_0$  follow different combinations of settings. Power values of all the five test statistics, i.e.  $G_q$ ,  $G_{q,1}$ ,  $G_{q,1}^*$ ,  $T_n$  and  $T_n^*$ , are compared when VAR coefficient  $a$  grows from 0 to 0.5. Here  $G_{q,1}^*$  is our test statistic with finite sample correction as demonstrated in (2.11). Empirical statistics are obtained using 2000 independent replicates. Results are shown in Fig. 1. Notice that on these displays,  $G_{q,1}$  and  $G_{q,1}^*$  coincide almost everywhere showing a high accuracy of the parameter estimates used in  $G_{q,1}^*$ .

It can be seen that our test statistics show better performance under this spherical AR(1) model setting. Designed via Frobenius norm of sample autocovariance matrices, the strength of our test statistics are fully demonstrated in such VAR(1) settings. While  $T_n$  and  $T_n^*$  are more adapted to settings where majority coordinates of the test sequence  $\mathbf{x}_t$  or their linear transformations remain to be white noise, see the model settings in Chang, Yao and Zhou [11]. Moreover, it can be seen that test size of  $T_n$  is a little biased when  $p = 20$ ,  $T = 100$ . Actually, such bias appears to be more significant when we increase the dimension-to-sample ratio  $p/T$  to a relative higher level, say 0.5. On the contrary, our test statistics maintain the nominal level accurately in both low and high dimensional settings.  $T_n^*$  shows very resilient powerful performance while it is quite time-consuming due to its relatively complicated bootstrap procedures. All in all, our test statistics  $G_q$ ,  $G_{q,1}$  and  $G_{q,1}^*$  provide very satisfactory alternatives for high dimensional diagnostic checking.

**3.5. Performance under VMA(1) model.** In this section we compare performance of the tests when  $\mathbf{x}_t$  follows a vector moving average process of order 1, i.e.

$$\mathbf{x}_t = A_0\mathbf{z}_t + A_1\mathbf{z}_{t-1}.$$

We use different settings for  $\mathbf{z}_t$  and  $A_0, A_1$  respectively to compare our test statistic  $G_q$  as defined in (2.5) and  $G_{q,1}$  in (2.8) under nominal level  $\alpha = 5\%$ .

As for  $\mathbf{z}_t$ , we use the same two models as defined in (I) and (II) in Section 3.1. As for  $A_0$  and  $A_1$ , we use two different models as follows.

(V)  $A_0 = \mathbf{I}_p$  and  $A_1 = a\mathbf{I}_p$ ,  $0 < a < 1$ .

(VI)  $A_0 = \mathbf{I}_p$  and for  $0 < r < 1$ , take  $d = [pr]$ , here  $[\cdot]$  means to take the closest integer to the given value.  $A_1 = \left(\frac{4}{p}E_0E_0^*\right)^{1/2}$ , where  $E_0$  is  $p \times d$  matrix with entries  $e_{ij} \sim U(-1, 1)$  i.i.d., thus  $\text{rank}(A_1) \leq d < p$ .

To evaluate the performance of our test statistics  $G_q$  and  $G_{q,1}$  under VMA(1) models, we assign  $a = 0.07$  and  $r = 0.01$ ,  $d = \max(1, [pr])$  respectively for Scenario (V) and (VI). Testing power of  $G_q$  and  $G_{q,1}$  are shown in Table 7 for  $q = 1$  under various  $(p, T)$  combinations. The asymptotic power  $\beta(G_{1,1})$  of the test statistic  $G_{1,1}$  derived in Proposition 2.3 are also listed for comparison. All empirical results are obtained using 2000 independent replicates.

Similarly as in Section 3.4, we further compare our test statistics with others, i.e.  $T_n$  and  $T_n^*$  in Chang, Yao and Zhou [11] under the VMA(1) settings. Here we fix  $p = 20$ ,  $T = 100$  and let  $\mathbf{x}_t = A_0\mathbf{z}_t + A_1\mathbf{z}_{t-1}$  where  $A_1$  follows model (V) or (VI) and  $\mathbf{z}_t$  is either Gaussian or Non-Gaussian. Power values of all the five test statistics, i.e.  $G_q$ ,  $G_{q,1}$ ,  $G_{q,1}^*$ ,  $T_n$  and  $T_n^*$ , are compared under model (V) and (VI) separately. Figure 2 shows the results under model (V) when coefficient  $a$

of  $A_1$  grows from 0 to 0.5 (top rows), and for model (VI) when parameter  $r$  varies from 0 to 0.5 (bottom rows). All results are based on 2000 independent experiments.

From Table 7, it can be seen that our test statistics  $G_1$  and  $G_{1,1}$  consistently show reasonable powers for various  $(p, T)$  combinations under both VMA(1) model settings. Especially  $G_{1,1}$  performs surprisingly well under VMA model (VI) even when  $d$  ( $\text{rank}(A_1)$ ) is very small. Meanwhile the empirical power of  $G_{1,1}$  is consistent with the asymptotic values  $\beta(G_{1,1})$  derived in Proposition 2.3. As for comparison with  $T_n$  and  $T_n^*$  in Figure 2, our test statistics in general show better performance under VMA(1) model settings. The test sizes of  $T_n$  and  $T_n^*$  are a little biased when  $p = 20$ ,  $T = 100$ , especially for non-Gaussian cases. While our test statistics maintain the nominal level accurately and uphold higher detection power even when the signals are relatively weak.

#### 4. Proofs.

4.1. *Proof of Theorem 2.1.* To derive the null distribution of  $G_q$  when  $\mathbf{x}_t = \Sigma_0^{1/2} \mathbf{z}_t$ , we looked into the Free probability and moment method proposed by Bhattacharjee and Bose [9]. In Section 4.2.3 of Bhattacharjee and Bose [9], they have proved the following result:

PROPOSITION 4.1. *Let  $\Pi := \Pi(\widehat{\Sigma}_\tau, \widehat{\Sigma}_\tau^* : \tau \geq 0)$  be a symmetric polynomial in  $\{\widehat{\Sigma}_\tau, \widehat{\Sigma}_\tau^* : \tau \geq 0\}$ ,*

$$\sigma_\Pi^2 = \lim \mathbb{E}(\text{Tr}(\Pi) - \mathbb{E}(\text{Tr}(\Pi)))^2.$$

They have,

$$\lim \mathbb{E}(\text{Tr}(\Pi) - \mathbb{E}(\text{Tr}(\Pi)))^k = \begin{cases} 0, & \text{if } k = 2d - 1, \\ \left(\prod_{l=1}^d (2d - 2l + 1)\right) \sigma_\Pi^{2d}, & \text{if } k = 2d. \end{cases}$$

therefore, as  $p, T \rightarrow \infty$ ,  $c_p = p/T \rightarrow c \in (0, \infty)$ ,

$$\text{Tr}(\Pi) - \mathbb{E}\text{Tr}(\Pi) \xrightarrow{d} \mathcal{N}(0, \sigma_\Pi^2).$$

Since  $G_q$  is a symmetric polynomial in  $\{\widehat{\Sigma}_\tau, \widehat{\Sigma}_\tau^* : \tau \geq 0\}$ , its asymptotic normality directly results from the proposition above. It remains to determine its first two moments in order to get the null distribution. This is done in the following corollary which is a direct consequence of moment calculations presented in the appendix, Section A.

COROLLARY 4.1. *Let the assumptions for  $\mathbf{z}_t$  in Theorem 2.1 hold. Under the framework  $p/T \rightarrow c > 0$ , assume that  $\|\Sigma_0\| = O(1)$ . Then as  $p, T \rightarrow \infty$ ,*

$$\begin{aligned} \mathbb{E}(G_q) &\sim qp^2 s_1^2 / T, \\ \text{Var}(G_q) &\rightarrow qc^2 (s_2^2 + b^2 (s_2')^2) \\ &\quad + 4q^2 c^3 (v_4 - b - 2) s_1^2 s_{d,2} + 8q^2 c^3 s_1^2 s_{r,2} + 4q^2 c^3 (b - 1) s_1^2 s_2', \end{aligned}$$

where  $s_2' = \lim_{p \rightarrow \infty} \text{Tr}(\Sigma_0 \Sigma_0^T) / p$ ,  $s_{r,2} = \lim_{p \rightarrow \infty} \text{Tr}(\mathfrak{K}^2(\Sigma_0)) / p$ .

If the  $z_{it}$ 's are real, then  $\Sigma_0$  is real symmetric and  $b = 1$ ,  $s'_2 = s_{r,2} = s_2$ . The asymptotic formula for  $\text{Var}(G_q)$  then reduces to

$$2qc^2s_2^2 + 4q^2c^3(\nu_4 - 3)s_1^2s_{d,2} + 8q^2c^3s_1^2s_2,$$

which further reduces to  $2qc^2s_2^2 + 8q^2c^3s_1^2s_2$  if all the  $z_{it}$ 's are Gaussian.

**4.2. Proof of Theorem 2.2.** The proof of Theorem 2.2 is similar to that of Theorem 2.1, while in this proof we only consider the real value cases. Both  $G_q$  and  $p(\hat{s}_1^2 - s_1^2)$  are symmetric polynomials in  $\{\widehat{\Sigma}_\tau, \widehat{\Sigma}_\tau^* : \tau \geq 0\}$ , thus the asymptotic normality of any linear combinations of these two statistics have been proven by Proposition 4.1. We can directly calculate the first two order moments and covariance of these two statistics to obtain the joint limiting distribution. By directly conducting moment calculations in the appendix, Section A, we have the following proposition.

**PROPOSITION 4.2.** *Let the assumptions for  $\mathbf{z}_t$  in Theorem 2.1 hold. Under the framework  $p/T \rightarrow c > 0$ , assume that  $\|\Sigma_0\| = O(1)$ . Then as  $p, T \rightarrow \infty$ ,*

$$\begin{aligned} \mathbb{E}(p\hat{s}_1^2) &= \frac{1}{p}\text{Tr}^2(\Sigma_0) + \frac{1}{pT}\left(2\text{Tr}(\Sigma_0^2) + (\nu_4 - 3)\text{Tr}(D^2(\Sigma_0))\right), \\ \text{Var}(p\hat{s}_1^2) &= \frac{8}{p^2T}\text{Tr}(\Sigma_0^2)\text{Tr}^2(\Sigma_0) + \frac{4}{p^2T}(\nu_4 - 3)\text{Tr}^2(\Sigma_0)\text{Tr}(D(\Sigma_0)) + o\left(\frac{1}{T}\right), \\ \mathbb{E}(\hat{s}_2) &= \frac{1}{p}\text{Tr}(\Sigma_0^2) + \frac{1}{pT}\text{Tr}^2(\Sigma_0) + \frac{1}{pT}\left(\text{Tr}(\Sigma_0^2) + (\nu_4 - 3)\text{Tr}(D^2(\Sigma_0))\right), \\ \mathbb{E}(G_q) &= \frac{q}{T}\text{Tr}^2(\Sigma_0), \quad \text{Var}(G_q) = \frac{4q^2}{T^3}\text{Tr}^2(\Sigma_0)\left(2\text{Tr}(\Sigma_0^2) + (\nu_4 - 3)\text{Tr}(D^2(\Sigma_0))\right) \\ &\quad + \frac{2q}{T^2}\text{Tr}^2(\Sigma_0^2) + \frac{q}{T^3}\left(2\text{Tr}(\Sigma_0^2) + (\nu_4 - 3)\text{Tr}(D^2(\Sigma_0))\right)^2 + o\left(\frac{1}{T}\right), \\ \text{Cov}(G_q, p\hat{s}_1^2) &= \frac{4q}{pT^2}\text{Tr}^2(\Sigma_0)\left(2\text{Tr}(\Sigma_0^2) + (\nu_4 - 3)\text{Tr}(D^2(\Sigma_0))\right) + o\left(\frac{1}{T}\right). \end{aligned}$$

Results in Theorem 2.2 and Proposition 2.1 naturally follows from Proposition 4.2. The proof of Proposition 4.2 is postponed to Section A.

**4.3. Proof of Theorem 2.3.** The proof of Theorem 2.3 is similar to that of Theorem 2.2 while the calculations are more complicated. When  $\mathbf{x}_t = A_0\mathbf{z}_t + A_1\mathbf{z}_{t-1}$ , both  $G_q$  and  $p(\hat{s}_1^2 - s_1^2)$  are still symmetric polynomials in  $\{\widehat{\Sigma}_\tau, \widehat{\Sigma}_\tau^* : \tau \geq 0\}$ , thus the asymptotic normality of any linear combinations of these two statistics have been proven by Proposition 4.1. We can directly calculate the first two order moments and covariance of these two statistics to obtain the joint limiting distribution.

To elucidate the calculations of moments, we implement the following decompositions on both  $G_q$  and  $qTc_p^2\hat{s}_1^2$  when  $\mathbf{x}_t = A_0\mathbf{z}_t + A_1\mathbf{z}_{t-1}$  for  $q = 1$ . Actually,

$$G_1 = \frac{1}{T^2} \sum_{s,t=1}^T (A_0\mathbf{z}_s + A_1\mathbf{z}_{s-1})^* (A_0\mathbf{z}_t + A_1\mathbf{z}_{t-1}) (A_0\mathbf{z}_{t-1} + A_1\mathbf{z}_{t-2})^* (A_0\mathbf{z}_{s-1} + A_1\mathbf{z}_{s-2})$$

$$\begin{aligned}
&= G(I) + G(II) + G(III), \\
Tc_p^2 \hat{s}_1^2 &= \frac{1}{T^3} \sum_{s,t=1}^T (A_0 \mathbf{z}_s + A_1 \mathbf{z}_{s-1})^* (A_0 \mathbf{z}_s + A_1 \mathbf{z}_{s-1}) (A_0 \mathbf{z}_t + A_1 \mathbf{z}_{t-1})^* (A_0 \mathbf{z}_t + A_1 \mathbf{z}_{t-1}) \\
&= S(I) + S(II) + S(III),
\end{aligned}$$

where

$$\begin{aligned}
G(I) &= \frac{1}{T^2} \sum_{s,t=1}^T (\mathbf{z}_s^* A_0^* A_0 \mathbf{z}_t \mathbf{z}_{t-1}^* A_0^* A_0 \mathbf{z}_{s-1} + \mathbf{z}_{s-1}^* A_1^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{t-2}^* A_1^* A_1 \mathbf{z}_{s-2} \\
&\quad + \mathbf{z}_s^* A_0^* A_0 \mathbf{z}_t \mathbf{z}_{t-2}^* A_1^* A_1 \mathbf{z}_{s-2} + \mathbf{z}_{s-1}^* A_1^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{t-1}^* A_0^* A_0 \mathbf{z}_{s-1}), \\
G(II) &= \frac{1}{T^2} \sum_{s,t=1}^T (\mathbf{z}_s^* A_0^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{t-1}^* A_0^* A_0 \mathbf{z}_{s-1} + \mathbf{z}_{s-1}^* A_1^* A_0 \mathbf{z}_t \mathbf{z}_{t-1}^* A_0^* A_0 \mathbf{z}_{s-1} \\
&\quad + \mathbf{z}_{s-1}^* A_1^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{t-1}^* A_0^* A_1 \mathbf{z}_{s-2} + \mathbf{z}_{s-1}^* A_1^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{t-2}^* A_1^* A_0 \mathbf{z}_{s-1} \\
&\quad + \mathbf{z}_s^* A_0^* A_0 \mathbf{z}_t \mathbf{z}_{t-2}^* A_1^* A_0 \mathbf{z}_{s-1} + \mathbf{z}_s^* A_0^* A_0 \mathbf{z}_t \mathbf{z}_{t-1}^* A_0^* A_1 \mathbf{z}_{s-2} \\
&\quad + \mathbf{z}_s^* A_0^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{t-2}^* A_1^* A_1 \mathbf{z}_{s-2} + \mathbf{z}_{s-1}^* A_1^* A_0 \mathbf{z}_t \mathbf{z}_{t-2}^* A_1^* A_1 \mathbf{z}_{s-2}), \\
G(III) &= \frac{1}{T^2} \sum_{s,t=1}^T (\mathbf{z}_s^* A_0^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{t-1}^* A_0^* A_1 \mathbf{z}_{s-2} + \mathbf{z}_{s-1}^* A_1^* A_0 \mathbf{z}_t \mathbf{z}_{t-2}^* A_1^* A_0 \mathbf{z}_{s-1} \\
&\quad + \mathbf{z}_s^* A_0^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{t-2}^* A_1^* A_0 \mathbf{z}_{s-1} + \mathbf{z}_{s-1}^* A_1^* A_0 \mathbf{z}_t \mathbf{z}_{t-1}^* A_0^* A_1 \mathbf{z}_{s-2}), \\
S(I) &= \frac{1}{T^3} \sum_{s,t=1}^T (\mathbf{z}_t^* A_0^* A_0 \mathbf{z}_t \mathbf{z}_s^* A_0^* A_0 \mathbf{z}_s + \mathbf{z}_{t-1}^* A_1^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{s-1}^* A_1^* A_1 \mathbf{z}_{s-1} \\
&\quad + \mathbf{z}_t^* A_0^* A_0 \mathbf{z}_t \mathbf{z}_{s-1}^* A_1^* A_1 \mathbf{z}_{s-1} + \mathbf{z}_{t-1}^* A_1^* A_1 \mathbf{z}_{t-1} \mathbf{z}_s^* A_0^* A_0 \mathbf{z}_s), \\
S(II) &= \frac{1}{T^2} \sum_{s,t=1}^T (\mathbf{z}_t^* A_0^* A_1 \mathbf{z}_{t-1} \mathbf{z}_s^* A_0^* A_0 \mathbf{z}_s + \mathbf{z}_{t-1}^* A_1^* A_0 \mathbf{z}_t \mathbf{z}_s^* A_0^* A_0 \mathbf{z}_s \\
&\quad + \mathbf{z}_{t-1}^* A_1^* A_1 \mathbf{z}_{t-1} \mathbf{z}_s^* A_0^* A_1 \mathbf{z}_{s-1} + \mathbf{z}_{t-1}^* A_1^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{s-1}^* A_1^* A_0 \mathbf{z}_s \\
&\quad + \mathbf{z}_t^* A_0^* A_0 \mathbf{z}_t \mathbf{z}_{s-1}^* A_1^* A_0 \mathbf{z}_s + \mathbf{z}_t^* A_0^* A_0 \mathbf{z}_t \mathbf{z}_s^* A_0^* A_1 \mathbf{z}_{s-1} \\
&\quad + \mathbf{z}_t^* A_0^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{s-1}^* A_1^* A_1 \mathbf{z}_{s-1} + \mathbf{z}_{t-1}^* A_1^* A_0 \mathbf{z}_t \mathbf{z}_{s-1}^* A_1^* A_1 \mathbf{z}_{s-1}), \\
S(III) &= \frac{1}{T^2} \sum_{s,t=1}^T (\mathbf{z}_t^* A_0^* A_1 \mathbf{z}_{t-1} \mathbf{z}_s^* A_0^* A_1 \mathbf{z}_{s-1} + \mathbf{z}_{t-1}^* A_1^* A_0 \mathbf{z}_t \mathbf{z}_{s-1}^* A_1^* A_0 \mathbf{z}_s \\
&\quad + \mathbf{z}_t^* A_0^* A_1 \mathbf{z}_{t-1} \mathbf{z}_{s-1}^* A_1^* A_0 \mathbf{z}_s + \mathbf{z}_{t-1}^* A_1^* A_0 \mathbf{z}_t \mathbf{z}_s^* A_0^* A_1 \mathbf{z}_{s-1}),
\end{aligned}$$

By conducting similar moment calculations in the appendix, Section A, we have the following proposition.

PROPOSITION 4.3. *Let the assumptions in Theorem 2.3 hold, as  $p, T \rightarrow \infty$ ,  $p/T \rightarrow c > 0$ , we have*

$$\begin{aligned}\mathbb{E}(G(I)) &= \frac{1}{T} \left[ \text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1) + (\nu_4 - 3) \text{Tr}(D(\widetilde{\Sigma}_0)D(\widetilde{\Sigma}_1)) \right] \\ &\quad + \frac{1}{T} \text{Tr}^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) + \text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1), \\ \mathbb{E}(G(II)) &= 0, \quad \mathbb{E}(G(III)) = \frac{2}{T} \text{Tr}^2(\widetilde{\Sigma}_{01}), \\ \mathbb{E}(S(I)) &= \frac{1}{T^2} \left[ 2\text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)^2 + (\nu_4 - 3) \text{Tr}(D^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)) \right] \\ &\quad + \frac{1}{T} \text{Tr}^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1), \\ \mathbb{E}(S(II)) &= 0, \quad \mathbb{E}(S(III)) = \frac{4}{T^2} \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^*),\end{aligned}$$

and

$$\begin{aligned}\text{Var}(G(I)) &= \frac{4}{T^3} \text{Tr}^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) \left[ 2\text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)^2 + (\nu_4 - 3) \text{Tr}(D^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)) \right] \\ &\quad + \frac{8}{T^2} \text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) \left[ 2\text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1 (\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)) + (\nu_4 - 3) \text{Tr}(D(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1) D(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)) \right] \\ &\quad + \frac{2}{T^2} \text{Tr}^2(\widetilde{\Sigma}_0^2 + \widetilde{\Sigma}_1^2) + \frac{6}{T^2} \text{Tr}^2(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1) + \frac{4}{T} \left[ 2\text{Tr}(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1)^2 + (\nu_4 - 3) \text{Tr}(D^2(\widetilde{\Sigma}_0 \widetilde{\Sigma}_1)) \right] + R_n,\end{aligned}$$

$$\begin{aligned}\text{Var}(G(III)) &= \frac{4}{T} \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_{01}) + \frac{12}{T^2} \text{Tr}^2(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^*) + \frac{16}{T^2} \text{Tr}(\widetilde{\Sigma}_{01}) \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_{01}^*) \\ &\quad + \frac{16}{T^3} \text{Tr}^2(\widetilde{\Sigma}_{01}) \left[ \text{Tr}(\widetilde{\Sigma}_{01})^2 + 2\text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^*) + (\nu_4 - 3) \text{Tr}(D^2(\widetilde{\Sigma}_{01})) \right] + R_n,\end{aligned}$$

$$\begin{aligned}\text{Var}(G(II)) &= \frac{8}{T^2} \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^*) \text{Tr}(\widetilde{\Sigma}_0^2 + \widetilde{\Sigma}_1^2) + \frac{16}{T^2} \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_1) \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_0) \\ &\quad + \frac{16}{T^2} \text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) \left[ \text{Tr}(\widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_{01} \widetilde{\Sigma}_0) + \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_1) \right] \\ &\quad + \frac{16}{T^2} \text{Tr}(\widetilde{\Sigma}_{01}) \left[ \text{Tr}(\widetilde{\Sigma}_0^2 \widetilde{\Sigma}_{01}^*) + \text{Tr}(\widetilde{\Sigma}_1^2 \widetilde{\Sigma}_{01}) + 2\text{Tr}(\widetilde{\Sigma}_1 \widetilde{\Sigma}_{01} \widetilde{\Sigma}_0) \right] \\ &\quad + \frac{4}{T} \text{Tr}(\widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_{01} \widetilde{\Sigma}_0^2 + \widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_1^2 + 2\widetilde{\Sigma}_{01}^* \widetilde{\Sigma}_1 \widetilde{\Sigma}_{01} \widetilde{\Sigma}_0) \\ &\quad + \frac{16}{T^3} \text{Tr}^2(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) \text{Tr}(\widetilde{\Sigma}_{01} \widetilde{\Sigma}_{01}^*) + \frac{16}{T^3} \text{Tr}^2(\widetilde{\Sigma}_{01}) \text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)^2 \\ &\quad + \frac{32}{T^3} \text{Tr}(\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1) \text{Tr}(\widetilde{\Sigma}_{01}) \text{Tr}(\widetilde{\Sigma}_{01} (\widetilde{\Sigma}_0 + \widetilde{\Sigma}_1)) + R_n,\end{aligned}$$

$$\text{Cov}(G(I), G(III)) = \frac{4}{T^2} \text{Tr}^2(\widetilde{\Sigma}_0 \widetilde{\Sigma}_{01}) + \frac{4}{T^2} \text{Tr}^2(\widetilde{\Sigma}_1 \widetilde{\Sigma}_{01})$$

$$\begin{aligned}
& + \frac{8}{T^3} \text{Tr}(\tilde{\Sigma}_{01}) \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \left[ 2\text{Tr}(\tilde{\Sigma}_{01}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) + (\nu_4 - 3)\text{Tr}(D(\tilde{\Sigma}_{01})D(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \right] \\
& + \frac{8}{T^2} \text{Tr}(\tilde{\Sigma}_{01}) \left[ 2\text{Tr}(\tilde{\Sigma}_0 \tilde{\Sigma}_1 \tilde{\Sigma}_{01}) + (\nu_4 - 3)\text{Tr}(D(\tilde{\Sigma}_0 \tilde{\Sigma}_1)D(\tilde{\Sigma}_{01})) \right] + R_n, \\
& \text{Cov}(G(I), G(II)) = o(1), \quad \text{Cov}(G(II), G(III)) = o(1);
\end{aligned}$$

$$\begin{aligned}
\text{Var}(S(I)) &= \frac{4}{T^3} \text{Tr}^2(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \left[ 2\text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)^2 + (\nu_4 - 3)\text{Tr}(D^2(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \right] + R_n, \\
\text{Var}(S(II)) &= \frac{16}{T^3} \text{Tr}^2(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^*) + R_n, \\
\text{Var}(S(III)) &= \frac{32}{T^4} \text{Tr}^2(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^*), \quad \text{Cov}(S(I), S(II)) = o(1), \\
& \text{Cov}(S(I), S(III)) = o(1), \quad \text{Cov}(S(II), S(III)) = o(1);
\end{aligned}$$

$$\begin{aligned}
\text{Cov}(G(I), S(I)) &= \frac{4}{T^3} \text{Tr}^2(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \left[ 2\text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)^2 + (\nu_4 - 3)\text{Tr}(D^2(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \right] \\
& + \frac{4}{T^2} \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \left[ 2\text{Tr}(\tilde{\Sigma}_0 \tilde{\Sigma}_1 (\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) + (\nu_4 - 3)\text{Tr}(D(\tilde{\Sigma}_0 \tilde{\Sigma}_1)D(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \right] + R_n, \\
\text{Cov}(G(II), S(II)) &= \frac{8}{T^2} \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \left[ \text{Tr}(\tilde{\Sigma}_{01}^* \tilde{\Sigma}_{01} \tilde{\Sigma}_0) + \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^* \tilde{\Sigma}_1) \right] \\
& + \frac{16}{T^3} \text{Tr}^2(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \text{Tr}(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^*) + \frac{16}{T^3} \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \text{Tr}(\tilde{\Sigma}_{01}) \text{Tr}(\tilde{\Sigma}_{01}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) + R_n, \\
\text{Cov}(G(III), S(I)) &= \frac{8}{T^3} \text{Tr}(\tilde{\Sigma}_{01}) \text{Tr}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1) \left[ 2\text{Tr}(\tilde{\Sigma}_{01}(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \right. \\
& \quad \left. + (\nu_4 - 3)\text{Tr}(D(\tilde{\Sigma}_{01})D(\tilde{\Sigma}_0 + \tilde{\Sigma}_1)) \right] + R_n, \\
& \text{Cov}(G(III), S(III)) = 0, \quad \text{Cov}(G(I), S(II)) = o(1), \quad \text{Cov}(G(I), S(III)) = 0, \\
& \text{Cov}(G(II), S(I)) = o(1), \quad \text{Cov}(G(II), S(III)) = o(1), \quad \text{Cov}(G(III), S(II)) = o(1).
\end{aligned}$$

Here the  $R_n$ 's are possibly different: they represent remainder terms with smaller orders than the others listed in each variance covariance items.

Theorem 2.3 naturally follows from Proposition 4.3.

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TABLE 1  
Power performance of the Hosking test  $\tilde{Q}_q$  and the Li-McLeod test  $Q_q^*$ .

$p$	$T$	$p/T$	$\tilde{Q}_q$		$Q_q^*$	
			$q = 1$	$q = 3$	$q = 1$	$q = 3$
10	100	0.1	0.0952	0.0952	0.0914	0.0926
20	200	0.1	0.2392	0.1994	0.2362	0.1958
40	400	0.1	0.6638	0.5410	0.6622	0.5380
60	600	0.1	0.9406	0.8452	0.9404	0.8448
100	1000	0.1	1	0.9982	1	0.9982
50	100	0.5	0.0014	0.0060	0.0014	0.0052
100	200	0.5	0.0036	0.0208	0.0030	0.0194
200	400	0.5	0.0330	0.2022	0.0328	0.1994
300	600	0.5	0.1156	0.6348	0.1138	0.6306
500	1000	0.5	0.5816	0.9974	0.5806	0.9972
80	100	0.8	0	0	0	0
160	200	0.8	0	0	0	0
320	400	0.8	0	0	0	0
480	600	0.8	0	0	0	0
800	1000	0.8	0.0004	0.0038	0.0004	0.0032
90	100	0.9	0	0	0	0
180	200	0.9	0	0	0	0
360	400	0.9	0	0	0	0
540	600	0.9	0	0	0	0
900	1000	0.9	0	0	0	0

TABLE 2  
*Empirical sizes for our tests  $G_q$  and  $G_{q,1}$ , the Hosking test  $\tilde{Q}_q$  and the Li-McLeod test  $Q_q^*$ .*

$p$	$T$	$p/T$	$G_q$		$G_{q,1}$		$\tilde{Q}_q$		$Q_q^*$	
			$q = 1$	$q = 3$	$q = 1$	$q = 3$	$q = 1$	$q = 3$	$q = 1$	$q = 3$
5	1000	0.005	0.0630	0.0615	0.0610	0.0645	0.0490	0.0478	0.0488	0.0476
10	2000	0.005	0.0630	0.0580	0.0615	0.0575	0.0492	0.0440	0.0492	0.0436
25	5000	0.005	0.0520	0.0470	0.0575	0.0535	0.0498	0.0528	0.0498	0.0528
40	8000	0.005	0.0565	0.0395	0.0540	0.0430	0.0508	0.0520	0.0508	0.0520
10	1000	0.01	0.0740	0.0565	0.0675	0.0570	0.0472	0.0468	0.0470	0.0464
20	2000	0.01	0.0500	0.0555	0.0540	0.0540	0.0502	0.0530	0.0502	0.0530
50	5000	0.01	0.0455	0.0555	0.0450	0.0580	0.0488	0.0498	0.0488	0.0498
80	8000	0.01	0.0500	0.0490	0.0510	0.0520	0.0464	0.0406	0.0464	0.0404
50	1000	0.05	0.0375	0.0495	0.0410	0.0475	0.0408	0.0466	0.0408	0.0466
100	2000	0.05	0.0570	0.0525	0.0560	0.0515	0.0432	0.0414	0.0432	0.0414
250	5000	0.05	0.0500	0.0480	0.0495	0.0500	0.0456	0.0436	0.0456	0.0434
400	8000	0.05	0.0410	0.0480	0.0455	0.0505	0.0418	0.0410	0.0418	0.0410
10	100	0.1	0.0570	0.0555	0.0555	0.0570	0.0300	0.0400	0.0280	0.0362
40	400	0.1	0.0560	0.0590	0.0575	0.0525	0.0362	0.0342	0.0358	0.0338
60	600	0.1	0.0465	0.0585	0.0550	0.0595	0.0340	0.0340	0.0340	0.0338
100	1000	0.1	0.0515	0.0500	0.0435	0.0480	0.0370	0.0268	0.0366	0.0264
50	100	0.5	0.0520	0.0465	0.0480	0.0520	0.0006	0.0018	0.0006	0.0018
200	400	0.5	0.0400	0.0415	0.0505	0.0545	0.0010	0.0004	0.0010	0.0004
300	600	0.5	0.0390	0.0480	0.0455	0.0480	0.0002	0.0008	0.0002	0.0008
500	1000	0.5	0.0470	0.0470	0.0430	0.0545	0	0	0	0
90	100	0.9	0.0555	0.0580	0.0460	0.0455	0	0	0	0
360	400	0.9	0.0475	0.0520	0.0535	0.0405	0	0	0	0
540	600	0.9	0.0535	0.0550	0.0550	0.0540	0	0	0	0
900	1000	0.9	0.0495	0.0505	0.0545	0.0515	0	0	0	0

TABLE 3  
Adjusted powers of the Hosking test  $\tilde{Q}_q$  and the Li-McLeod test  $Q_q^*$  as compared to powers of our tests  $G_q$  and  $G_{q,1}$   
when  $\Sigma_0 = \mathbf{I}_p$

$p$	$T$	$p/T$	$a$	$G_q$		$G_{q,1}$		$\tilde{Q}_q$		$Q_q^*$	
				$q = 1$	$q = 3$	$q = 1$	$q = 3$	$q = 1$	$q = 3$	$q = 1$	$q = 3$
5	1000	0.005	0.05	0.4865	0.2820	0.4900	0.2820	0.4290	0.2535	0.4290	0.2540
10	2000	0.005	0.05	0.9235	0.6080	0.9305	0.6355	0.9010	0.6495	0.9010	0.6500
25	5000	0.005	0.05	1	0.9985	1	0.9995	1	0.9985	1	0.9985
40	8000	0.005	0.05	1	1	1	1	1	1	1	1
10	1000	0.01	0.05	0.4965	0.2650	0.5145	0.2850	0.4715	0.2450	0.4715	0.2450
20	2000	0.01	0.05	0.9310	0.6425	0.9375	0.6855	0.9315	0.6530	0.9315	0.6530
50	5000	0.01	0.05	1	0.9995	1	1	1	0.9990	1	0.9990
80	8000	0.01	0.05	1	1	1	1	1	1	1	1
50	1000	0.05	0.05	0.5360	0.2825	0.6010	0.3800	0.4795	0.2910	0.4795	0.2905
100	2000	0.05	0.05	0.9670	0.6780	0.9865	0.8365	0.9470	0.7450	0.9470	0.7460
250	5000	0.05	0.05	1	1	1	1	1	1	1	1
400	8000	0.05	0.05	1	1	1	1	1	1	1	1
10	100	0.1	0.1	0.1960	0.1360	0.2235	0.1695	0.1392	0.1218	0.1392	0.1214
40	400	0.1	0.1	0.8435	0.5380	0.9355	0.8050	0.7082	0.5968	0.7082	0.5988
60	600	0.1	0.1	0.9905	0.8140	0.9990	0.9830	0.9598	0.8808	0.9598	0.8816
100	1000	0.1	0.1	1	0.9940	1	1	1	0.9992	1	0.9992
50	100	0.5	0.1	0.2110	0.1610	0.3810	0.4660	0.1004	0.1376	0.1004	0.1380
200	400	0.5	0.1	0.9245	0.7665	0.9990	1	0.4012	0.7708	0.4012	0.7708
300	600	0.5	0.1	0.9975	0.9725	1	1	0.6626	0.9746	0.6626	0.9748
500	1000	0.5	0.1	1	1	1	1	0.9666	1	0.9666	1
90	100	0.9	0.1	0.2505	0.1920	0.5970	0.7620	0.1384	0.0992	0.1384	0.1002
360	400	0.9	0.1	0.9705	0.9080	1	1	0.7138	0.5172	0.7138	0.5176
540	600	0.9	0.1	1	1	1	1	0.9496	0.8368	0.9496	0.8368
900	1000	0.9	0.1	1	1	1	1	0.9998	0.9966	0.9998	0.9966

TABLE 4  
*Relative errors for the mean, variance and 95 percentile for the Hosking test statistic  $\tilde{Q}_q$  and the Li-McLeod test statistic  $Q_q^*$  (with  $q = 3$ )*

$p$	$T$	$p/T$	$\tilde{Q}_q$			$Q_q^*$		
			Mean	Variance	95% Quantile	Mean	Variance	95% Quantile
10	100	0.1	0.234%	19.976%	1.366%	0.234%	24.922%	1.547%
20	200	0.1	0.067%	30.862%	0.993%	0.067%	33.526%	1.049%
40	400	0.1	-0.015%	22.057%	0.253%	-0.015%	23.286%	0.265%
60	600	0.1	0.000%	21.457%	0.162%	0.000%	22.269%	0.162%
100	1000	0.1	0.007%	20.666%	0.125%	0.007%	21.153%	0.125%
50	100	0.5	0.041%	267.179%	1.322%	0.041%	282.546%	1.354%
100	200	0.5	0.007%	284.025%	0.655%	0.007%	291.875%	0.662%
200	400	0.5	0.000%	289.080%	0.330%	0.000%	292.998%	0.330%
300	600	0.5	0.000%	297.059%	0.222%	0.000%	299.734%	0.222%
500	1000	0.5	0.000%	296.364%	0.134%	0.000%	297.941%	0.134%
80	100	0.8	0.010%	1742.257%	1.289%	0.005%	1820.096%	1.300%
160	200	0.8	0.000%	2020.024%	0.655%	0.000%	2063.959%	0.657%
320	400	0.8	0.000%	2214.386%	0.332%	0.000%	2237.811%	0.332%
480	600	0.8	0.001%	2266.151%	0.223%	0.001%	2282.093%	0.223%
800	1000	0.8	0.000%	2348.823%	0.137%	0.000%	2358.701%	0.137%
90	100	0.9	0.004%	5382.234%	1.292%	0.000%	5618.993%	1.297%
180	200	0.9	0.000%	6906.920%	0.657%	0.000%	7053.897%	0.658%
360	400	0.9	0.000%	8110.500%	0.332%	0.000%	8195.108%	0.332%
540	600	0.9	0.000%	8705.234%	0.222%	0.000%	8764.569%	0.222%
900	1000	0.9	0.000%	9170.563%	0.133%	0.000%	9208.205%	0.133%

TABLE 5  
 Test sizes of our tests  $G_q$  and  $G_{q,1}$

			Gaussian (I)				Non-Gaussian (II)			
$p$	$T$	$p/T$	$G_q$		$G_{q,1}$		$G_q$		$G_{q,1}$	
			$q = 1$	$q = 3$	$q = 1$	$q = 3$	$q = 1$	$q = 3$	$q = 1$	$q = 3$
5	500	0.01	0.0500	0.0545	0.0465	0.0485	0.0650	0.0655	0.0655	0.0540
10	1000	0.01	0.0565	0.0420	0.0575	0.0400	0.0515	0.0615	0.0600	0.0575
20	2000	0.01	0.0545	0.0570	0.0515	0.0525	0.0610	0.0595	0.0600	0.0510
25	500	0.05	0.0550	0.0570	0.0630	0.0510	0.0570	0.0645	0.0520	0.0565
50	1000	0.05	0.0520	0.0515	0.0510	0.0455	0.0500	0.0485	0.0495	0.0455
100	2000	0.05	0.0565	0.0410	0.0545	0.0355	0.0500	0.0595	0.0440	0.0530
100	100	1	0.0515	0.0545	0.0565	0.0520	0.0515	0.0520	0.0395	0.0420
200	200	1	0.0540	0.0460	0.0485	0.0395	0.0475	0.0495	0.0450	0.0520
400	400	1	0.0570	0.0565	0.0505	0.0450	0.0385	0.0420	0.0505	0.0510
200	100	2	0.0530	0.0480	0.0560	0.0380	0.0560	0.0545	0.0370	0.0420
400	200	2	0.0480	0.0500	0.0510	0.0420	0.0545	0.0515	0.0470	0.0390
800	400	2	0.0505	0.0485	0.0480	0.0520	0.0475	0.0470	0.0405	0.0445
(III)										
5	500	0.01	0.0630	0.0715	0.0585	0.0665	0.0670	0.0560	0.0650	0.0585
10	1000	0.01	0.0680	0.0645	0.0695	0.0580	0.0555	0.0540	0.0545	0.0565
20	2000	0.01	0.0590	0.0545	0.0575	0.0540	0.0655	0.0520	0.0635	0.0560
25	500	0.05	0.0510	0.0545	0.0505	0.0505	0.0635	0.0590	0.0595	0.0580
50	1000	0.05	0.0435	0.0425	0.0475	0.0405	0.0550	0.0555	0.0535	0.0465
100	2000	0.05	0.0480	0.0460	0.0470	0.0420	0.0600	0.0460	0.0595	0.0520
100	100	1	0.0500	0.0525	0.0455	0.0455	0.0545	0.0485	0.0595	0.0530
200	200	1	0.0510	0.0530	0.0530	0.0505	0.0495	0.0460	0.0480	0.0520
400	400	1	0.0535	0.0495	0.0530	0.0390	0.0450	0.0440	0.0510	0.0520
200	100	2	0.0550	0.0545	0.0480	0.0605	0.0480	0.0485	0.0415	0.0450
400	200	2	0.0470	0.0485	0.0540	0.0525	0.0545	0.0525	0.0460	0.0520
800	400	2	0.0415	0.0505	0.0450	0.0495	0.0480	0.0490	0.0510	0.0495
(IV)										

TABLE 6  
*Test power of our tests  $G_q$  and  $G_{q,1}$*

				Gaussian (I)				Non-Gaussian (II)			
$p$	$T$	$p/T$	$a$	$G_q$		$G_{q,1}$		$G_q$		$G_{q,1}$	
				$q = 1$	$q = 3$	$q = 1$	$q = 3$	$q = 1$	$q = 3$	$q = 1$	$q = 3$
5	500	0.01	0.05	0.2355	0.1535	0.2500	0.1540	0.2485	0.1475	0.2465	0.1505
10	1000	0.01	0.05	0.5280	0.2770	0.5335	0.2935	0.5135	0.2645	0.5265	0.2930
20	2000	0.01	0.05	0.9460	0.6620	0.9495	0.6995	0.9355	0.6010	0.9500	0.6670
25	500	0.05	0.05	0.2260	0.1300	0.2485	0.1770	0.2315	0.1395	0.2585	0.1810
50	1000	0.05	0.05	0.5410	0.2800	0.5995	0.3785	0.5105	0.2495	0.5960	0.3750
100	2000	0.05	0.05	0.9580	0.6550	0.9815	0.8275	0.9500	0.5895	0.9805	0.8385
100	100	1	0.1	0.2615	0.2205	0.6170	0.8190	0.2100	0.1750	0.6165	0.8285
200	200	1	0.1	0.6010	0.4720	0.9870	0.9995	0.4460	0.3370	0.9865	1
400	400	1	0.1	0.9745	0.9230	1	1	0.9025	0.7875	1	1
200	100	2	0.1	0.3275	0.2710	0.9375	0.9980	0.2420	0.2135	0.9390	0.9995
400	200	2	0.1	0.7415	0.6745	1	1	0.5715	0.4830	1	1
800	400	2	0.1	0.9995	0.9930	1	1	0.9710	0.9350	1	1
5	500	0.01	0.05	0.2540	0.1680	0.2590	0.1700	0.2355	0.1505	0.2450	0.1615
10	1000	0.01	0.05	0.4650	0.2870	0.4730	0.2850	0.4650	0.2885	0.4825	0.2970
20	2000	0.01	0.05	0.8750	0.6170	0.8815	0.6285	0.8880	0.5980	0.8950	0.6190
25	500	0.05	0.05	0.2580	0.1630	0.2555	0.1710	0.2475	0.1415	0.2655	0.1750
50	1000	0.05	0.05	0.5215	0.2650	0.5525	0.3110	0.5165	0.2575	0.5450	0.3270
100	2000	0.05	0.05	0.9450	0.6500	0.9555	0.7320	0.9345	0.6240	0.9635	0.7405
100	100	1	0.1	0.2145	0.1690	0.3700	0.4695	0.1970	0.1470	0.3765	0.4495
200	200	1	0.1	0.4910	0.3470	0.8335	0.9005	0.4355	0.2935	0.8430	0.9150
400	400	1	0.1	0.9205	0.7690	1	1	0.8655	0.6735	1	1
200	100	2	0.1	0.2450	0.2035	0.6255	0.8115	0.2240	0.1745	0.6425	0.8235
400	200	2	0.1	0.5815	0.4790	0.9915	1	0.5000	0.3770	0.9880	1
800	400	2	0.1	0.9705	0.9205	1	1	0.9425	0.8525	1	1

(III)

(IV)

TABLE 7  
 Test power of our tests  $G_1$  and  $G_{1,1}$  under  $VMA(1)$

				Gaussian (I)			Non-Gaussian (II)		
$p$	$T$	$p/T$	$a$	$G_1$	$G_{1,1}$	$\beta(G_{1,1})$	$G_1$	$G_{1,1}$	$\beta(G_{1,1})$
10	200	0.05	0.07	0.2085	0.2260	0.2144	0.1865	0.1990	0.2159
20	400	0.05	0.07	0.4135	0.4410	0.4530	0.3805	0.4315	0.4548
40	800	0.05	0.07	0.8350	0.8985	0.8903	0.7910	0.8885	0.8910
20	200	0.1	0.07	0.1830	0.2120	0.2235	0.1755	0.2165	0.2250
40	400	0.1	0.07	0.3915	0.4925	0.5015	0.3605	0.4800	0.5034
80	800	0.1	0.07	0.8480	0.9395	0.9372	0.7995	0.9485	0.9377
50	100	0.5	0.07	0.1185	0.1705	0.1790	0.1070	0.1730	0.1804
100	200	0.5	0.07	0.2070	0.3820	0.3958	0.1600	0.3850	0.3977
200	400	0.5	0.07	0.4940	0.8395	0.8521	0.3660	0.8400	0.8531
100	100	1	0.07	0.1305	0.2670	0.2754	0.1120	0.2715	0.2771
200	200	1	0.07	0.2540	0.6605	0.6485	0.1925	0.6470	0.6502
400	400	1	0.07	0.5520	0.9900	0.9903	0.4110	0.9895	0.9904
200	100	2	0.07	0.1510	0.4990	0.5157	0.1225	0.5000	0.5177
400	200	2	0.07	0.3005	0.9480	0.9500	0.2385	0.9500	0.9504
800	400	2	0.07	0.7310	1	0.9999	0.5500	1	0.9999

(V)

$p$	$T$	$p/T$	$r$	$G_1$	$G_{1,1}$	$\beta(G_{1,1})$	$G_1$	$G_{1,1}$	$\beta(G_{1,1})$
10	200	0.05	0.01	0.9955	0.9995	0.9838	1	1	0.9884
20	400	0.05	0.01	1	1	0.9995	1	1	0.9994
40	800	0.05	0.01	1	1	0.9999	1	1	0.9999
20	200	0.1	0.01	0.9700	0.9935	0.9705	0.9875	0.9980	0.9815
40	400	0.1	0.01	0.9995	1	0.9970	0.9980	1	0.9979
80	800	0.1	0.01	1	1	0.9999	0.9995	1	0.9999
50	100	0.5	0.01	0.0530	0.0445	0.0500	0.0615	0.0510	0.0500
100	200	0.5	0.01	0.3255	0.5855	0.6185	0.2765	0.5925	0.6155
200	400	0.5	0.01	0.6080	0.9390	0.9439	0.5150	0.9565	0.9544
100	100	1	0.01	0.1135	0.1910	0.2132	0.1110	0.2575	0.2759
200	200	1	0.01	0.2200	0.5665	0.5690	0.1780	0.5450	0.5770
400	400	1	0.01	0.5110	0.9755	0.9709	0.3810	0.9620	0.9628
200	100	2	0.01	0.0910	0.2430	0.2553	0.1020	0.2660	0.2772
400	200	2	0.01	0.1625	0.5785	0.5972	0.1320	0.5575	0.5917
800	400	2	0.01	0.3695	0.9755	0.9714	0.2615	0.9785	0.9746

(VI)



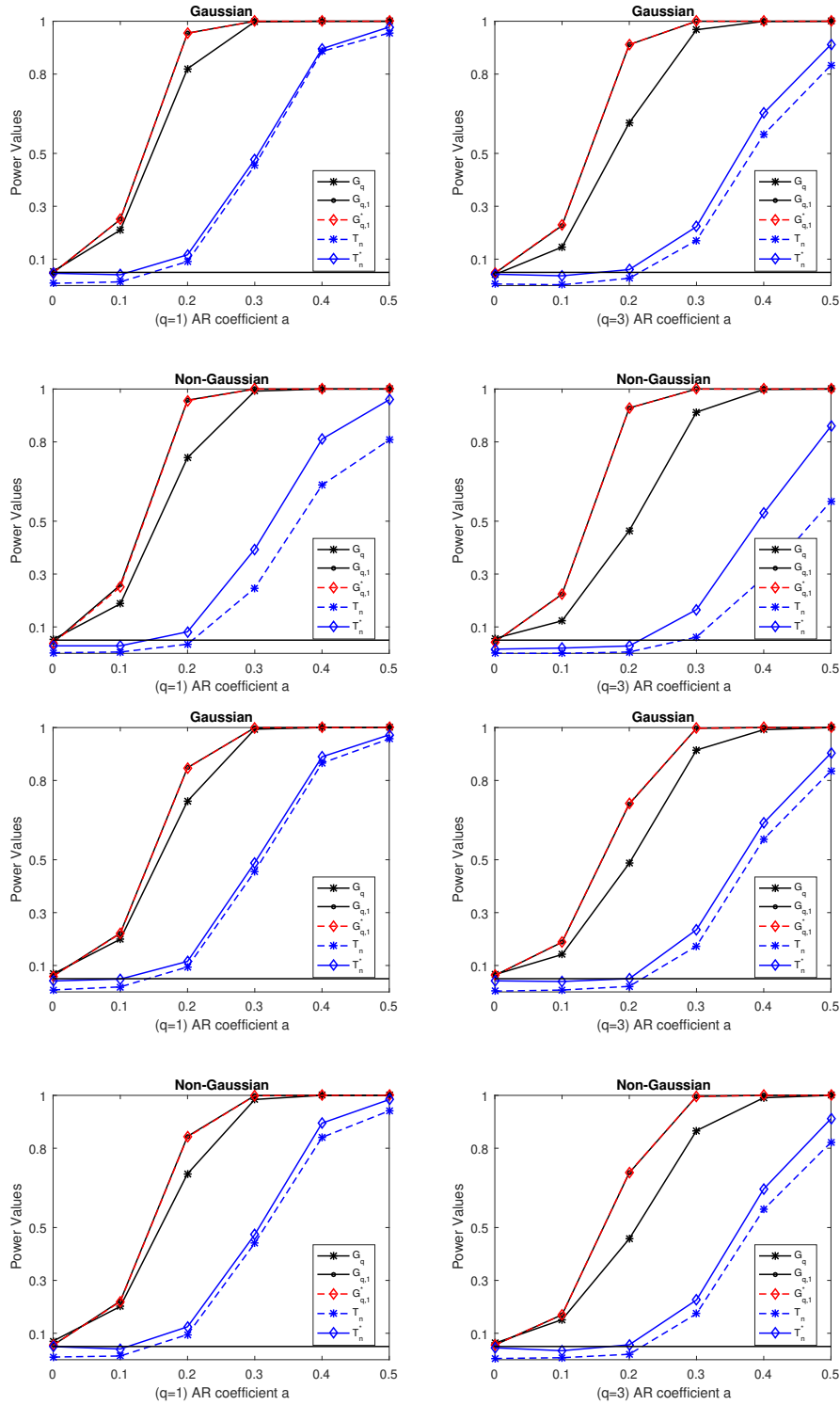


Fig 1: Power comparison under VAR(1) with  $(p, T) = (20, 100)$ . Left column with  $q = 1$  and right column with  $q = 3$ . First two rows under alternative model (III); last two rows under alternative model (IV).

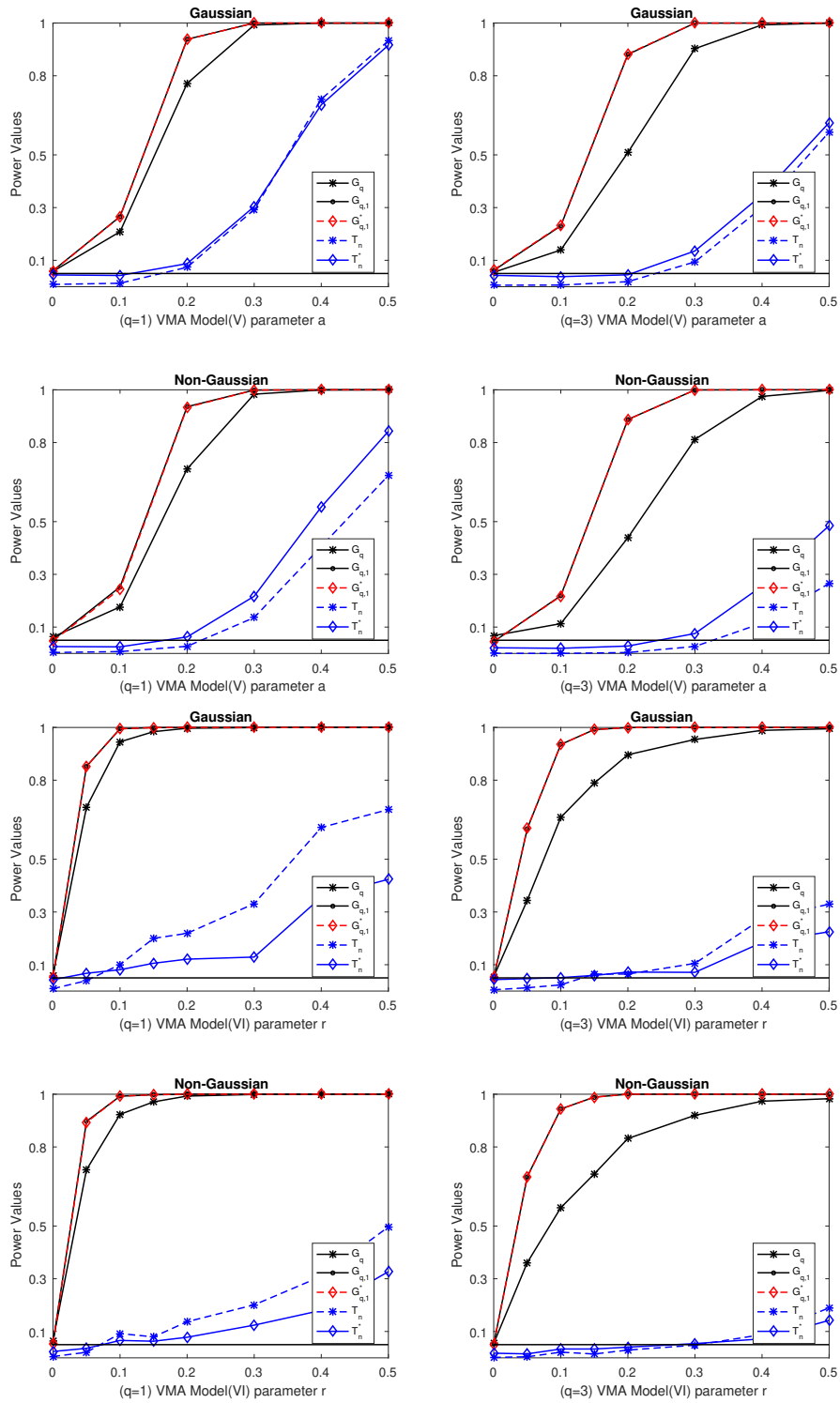


Fig 2: Power comparison under VMA(1) with  $(p, T) = (20, 100)$ . Left column with  $q = 1$  and right column with  $q = 3$ . First two rows under alternative model (V); last two rows under alternative model (VI).