SIAM J. FINANCIAL MATH.
Vol. 10, No. 1, pp. 37–67



# Interbank Clearing in Financial Networks with Multiple Maturities*

Michael Kusnetsov† and Luitgard Anna Maria Veraart†

**Abstract.** We consider the problem of systemic risk assessment in interbank networks in which interbank liabilities can have multiple maturities. In particular, we allow for both short-term and long-term interbank liabilities. We develop a clearing mechanism for the interbank liabilities to deal with the default of one or more market participants. Our approach generalizes the clearing approach for the single maturity setting proposed by Eisenberg and Noe [*Management Sci.*, 47 (2001), pp. 236–249]. Our clearing mechanism focuses on the vector of each bank's liquid assets at each maturity date and develops a fixed-point formulation of this vector for a given set of defaulted banks. Our formulation is consistent with the main stylized principles of insolvency law. We show that in the context of multiple maturities, specifying a set of defaulted banks is challenging. We propose two approaches to overcome this challenge: First, we propose an algorithmic approach for defining the default set and show that this approach leads to a well-defined liquid asset vector for all financial networks with multiple maturities. Second, we propose a simpler functional approach which leads to a functional liquid asset vector which need not exist but under a regularity condition does exist and coincides with the algorithmic liquid asset vector. Our analysis permits construction of simple dynamic models and furthermore demonstrates that systemic risk can be underestimated by single maturity models.

**Key words.** systemic risk, default, multiple maturities, clearing, financial network

**AMS subject classifications.** 90B10, 91G80

**DOI.** 10.1137/18M1180542

## 1. Introduction.

Since the financial crisis of 2007–2008 there has been a rapid expansion of literature which aims to explain bank failure in interconnected financial systems; see, e.g., [25] for a recent overview. One main modeling aim is to find a suitable contagion mechanism that describes how losses can spread through a financial network. The ultimate objective of such an analysis is to assess the degree of *systemic* risk in a financial network and use this to make informed policy decisions to increase financial stability.

One approach to assessing systemic risk in financial networks is to derive *clearing* cash flows between financial institutions and to study which market participants default during the clearing process. Such clearing payments represent the actual payments made by the market participants and are constructed such that they obey certain stylized principles of contract and insolvency law.

We contribute to this area of research by proposing an extension of the clearing approach

*Received by the editors April 12, 2018; accepted for publication (in revised form) November 7, 2018; published electronically January 17, 2019.

http://www.siam.org/journals/sifin/10-1/M118054.html

**Funding:** The first author gratefully acknowledges financial support by the EPSRC Centre for Doctoral Training in Financial Computing & Analytics funded by the Engineering and Physical Sciences Research Council (EPSRC), UK.

†Department of Mathematics, London School of Economics and Political Science, Houghton Street, London WC2A 2AE, UK (m.kusnetsov@lse.ac.uk, l.veraart@lse.ac.uk).

first developed by Eisenberg and Noe [14] from financial networks with only one maturity date to networks with multiple maturity dates. In practice, financial networks do consist of liabilities with different maturity dates. When the clearing process is triggered at the first maturity date long-term debt must not be ignored. We develop clearing mechanisms that account for long-term debt in a way that is consistent with the main principles of insolvency law. This approach is also extended to a multiperiod model that can be used as a basis for a full dynamic model of systemic risk.

Typically bank default models assume, as, e.g., proposed in [14], three stylized principles of insolvency law which are common to many jurisdictions. These are the principles of *limited liability*, which says that a financial institution never pays more than it has, *absolute priority* of debt claims, implying that all outstanding debt has to be completely paid off first before shareholders can be considered, and *proportionality*. The principle of proportionality states that the total value of assets paid out in this case is distributed between all the creditors in proportion to the size of their nominal claims.

A crucial nuance of the principle of proportionality is that all liabilities, including future liabilities, are required to be treated equally for the purposes of proportional distribution to creditors. For example, the UK Insolvency Service Technical Manual stipulates that: *A creditor may prove for a debt where payment would have become due at a date later than the insolvency proceedings [...] and it is only because the company [...] has entered into insolvency proceedings that the debt is claimed by the creditor in advance of its due payment date. Where this occurs, the creditor is entitled to the dividend equally with others [...]* [33, Chapter 36A, section 48].

Our model explicitly incorporates this important feature. This contrasts with single maturity models where it is assumed that assets of defaulting banks are distributed to creditors proportionally to the short-term liabilities only. The failure to account for future liabilities in calculating the proportional distributions leads to an incomplete view of systemic risk in financial systems. We show that two financial systems with the same overall interbank liabilities but different maturity profiles can lead to different clearing outcomes. In particular, it follows that uncertainty about maturity profiles of banks' portfolios is a distinct source of systemic risk that is unaccounted for in single maturity models. Our approach can be used in an analysis of systemic risk to evaluate the effect of such maturity profile uncertainty.

This paper makes four main contributions. First, in section 2 we introduce the notion of an equilibrium achieved by clearing the financial markets at the first maturity date and accounting for long-term liabilities which are due beyond the first maturity date (Definition 2.3). We also show that in contrast to the single maturity setting, developing a notion of default in a multiple maturity setting is challenging. A key insight that emerges out of this observation is that characterizing the set of banks in default is an integral part of the solution to the clearing problem. This is in contrast to much of the literature, where default sets are treated as secondary quantities derived from the clearing cash flows. In particular, we show in Lemma 3.17 and Remark 3.20 that under a mild assumption financial systems have at most a finite number of clearing solutions, each uniquely determined by a corresponding default set.

Our second contribution, in section 3, is to introduce two possible approaches to clearing at the first maturity date. We show that these two approaches—algorithmic (Definition 3.1) and functional (Definition 3.3)—solve the general equilibrium problem in Propositions 3.2 and

3.5. In section 3.3 we describe how the algorithmic approach extends the functional approach, which in turn extends the Eisenberg and Noe [14] model. Construction of clearing solutions under both approaches is addressed in section 3.4.

Our third contribution is to show that the functional approach, used in much of the literature in a single maturity setting, is problematic in a multiple maturity setting. In particular, we elucidate the importance of monotonicity in clearing problems. In general, under the functional approach, the clearing function is not monotone and may not have a fixed-point solution. Nevertheless, we show in section 3.2 that a simple condition, the Monotonicity Condition, Definition 3.7, is sufficient to ensure the existence of a solution.

Finally, we highlight some applications of the algorithmic approach. In section 3.5 we apply the algorithmic approach to demonstrate how single maturity models can underestimate systemic risk. In section 4, we discuss the evolution of the financial system after clearing at the first maturity. In particular, in section 4.3, we describe a simple multiperiod extension of our model. Such an extension then captures both the multimaturity and multiperiod aspects and therefore is a basis for a full dynamic model of financial systems.

The remainder of this section provides a summary of the current literature and how it relates to the multiple maturity clearing problem that we consider here.

**1.1. Literature review.** The role of complexity and contagion in financial networks has been studied by numerous authors, e.g., [1], [20], [5], and [12]. There has been an increasing recognition that there are in fact multiple channels through which network complexity can give rise to systemic risk. Bisias et al. [6], for example, provide a wide-ranging overview.

In most studies it is assumed that the financial network itself is observable. We will also make this assumption here. Under incomplete information, network reconstruction methods could be applied first; see, e.g., the Bayesian approach proposed in [22], [21], and the references therein.

We focus on one specific channel of contagion, namely the domino effect which arises when complex networks of debt obligations are cleared. This places our work at the intersection of two strands of literature. The first focuses on contagion and domino effects; see, e.g., [9], [34], [30], [17], [11], [23], and [15]. The second investigates clearing, typically in the context of central counterparty clearing in over-the-counter (OTC) markets. Some contributions from this latter strand include [10], [13], [8], and [2].

Our paper presents a generalization of the classic static single maturity approach that originates with [14]. While the model in [14] was concerned primarily with payment systems, the key ideas have been adapted by numerous authors to model systemic risk in a financial system. In this stream of literature, an interbank system is modeled as a directed graph with weighted edges. The nodes of this graph correspond to systemically significant banks which are endowed with initial assets. Each edge represents an outstanding debt owed by the bank at the tail of the edge to the bank at the head of edge. The weights correspond to the nominal values of the debt. A central question is of *clearing* the financial system, that is, calculating the actual amounts that banks transfer to each other in satisfaction of their nominal obligations. This question is particularly pertinent when a shock is applied to the asset side of their balance sheets, which may cause some banks to default.

The key findings include the existence and construction of clearing solutions and the conditions for their uniqueness. These results rely on a number of simplifying assumptions on clearing, which subsequent authors have attempted to relax. Thus Hurd [27] clarifies the role that the external liabilities play, Rogers and Veraart [31] investigate the effect of liquidation costs, while Elsinger [16] incorporates cross-holdings and different seniorities of debt. The combined effect of cross-holdings and bankruptcy costs is investigated in Weber and Weske [36]. All these extensions are single period models and hence assume a single maturity for the liabilities.

Glasserman and Young [24] provide an alternative interpretation of clearing as dynamic re-valuation of bank assets by the market. Since in many extensions the uniqueness of clearing solutions is lost, this interpretation is particularly interesting in the systemic risk context as different solutions can be given meaningful interpretation in terms of alternative valuations. Veraart [35] follows this approach and investigates the effect of predefault contagion, i.e., contagion that can be triggered prior to the actual default event due to distress and mark-to-market losses. The notions of distress and time-dependent valuation are also developed in Barucca et al. [4].

Recent papers (e.g., [7], [19], [3]) have developed multiperiod models. The model in Capponi and Chen [7] has a "central bank" node and random interbank liabilities. In particular, it highlights the distinction between illiquid and insolvent banks which arises whenever liabilities can become due at different times. This model focuses on the role of liquidity injection policies by the central bank and only tangentially analyzes the differences in the default behavior that arise from this generalization. Meanwhile, Ferrara et al. [19] describe how a multiperiod system can be cleared simultaneously for every period. Similarly, Banerjee, Bernstein, and Feinstein [3] consider both a discrete and a continuous-time dynamic extension of the Eisenberg and Noe [14] model.

While these models generalize the single period aspect of [14], they remain fundamentally single maturity models. Future liabilities are only revealed one period at a time and are not considered as long-term debt at the short-term maturity date, but are rather considered as new short-term debt that started at a later point in time. The clearing mechanism they consider therefore corresponds effectively to a repeated application of a single maturity clearing algorithm.

Sonin and Sonin [32] provide a dynamic solution approach to the static [14] setting but again do not account for a multiple maturity structure as we do in our paper.

In contrast, our model accounts for long-term debt before short-term debt is cleared and settled. In practice, banks have instruments of many maturities in their portfolio, and therefore it is important to account for this feature. To the best of our knowledge, our contribution is the first attempt to explicitly account for multiple maturities in a manner consistent with the insolvency rules.

Related work is the approach by Feinstein [18], who considers an extension of the single network approach by Eisenberg and Noe [14] to a multilayered financial network to study contagion in multiple asset classes. This approach could also be applied to a multiperiod or multimaturity setting.

## 2. Clearing in financial systems with multiple maturities.

**2.1. The financial market.** We consider a financial market consisting of $N$ banks with indices in $\mathcal{N} = \{1, \ldots, N\}$. Banks have liabilities to each other and to external entities which are due at two different maturity dates $0 < T_1 < T_2$. We will later show that we can easily generalize our model to more than two maturities. Hence, time $t = 0$ represents the starting point of the analysis, and we model what happens at the two maturity dates $t \in \{T_1, T_2\}$. We assume that all liabilities of the same maturity have the same seniority.

Each bank's liabilities for some maturity can be represented by a *liability matrix*. Together with vectors representing the banks' cash assets, these are sufficient to describe the financial system at $t = T_1$. These and other related concepts are summarized in Definition 2.1. In the following we denote by $\mathbf{1}$ the $n$-dimensional vector of ones.

Definition 2.1 (financial system).

1. *A matrix* $M \in \mathbb{R}_+^{N\times N}$ *is called a* liability matrix *if, for all* $i \in \mathcal{N}$, $M_{ii} = 0$.
2. *A* financial system *is given by the tuple* $(a, L^{(s)}, L^{(l)}; \gamma)$, *where* $L^{(s)}$, $L^{(l)}$ *are liability matrices with maturity dates* $T_1$ *and* $T_2$, *respectively, and* $a \in \mathbb{R}_+^N$, $\gamma \in [0, 1]$.
   *We will refer to the following quantities:*
   - *the* cash assets $a$;
   - *the* short-term, long-term, *and* overall liability matrices $L^{(s)}$, $L^{(l)}$, *and* $L := L^{(s)} + L^{(l)}$, *respectively;*
   - *the* short-term, long-term, *and* overall total nominal liability *vectors* $\bar{L}^{(s)} := L^{(s)}\mathbf{1}$, $\bar{L}^{(l)} := L^{(l)}\mathbf{1}$, *and* $\bar{L} := \bar{L}^{(s)} + \bar{L}^{(l)}$, *respectively;*
   - *the* short-term, long-term, *and* overall interbank asset vectors $\bar{A}^{(s)} := (L^{(s)})^\top\mathbf{1}$, $\bar{A}^{(l)} := (L^{(l)})^\top\mathbf{1}$, *and* $\bar{A} := (L)^\top\mathbf{1}$, *respectively;*
   - *the* short-term *and* overall relative liability matrices $\Pi^{(s)}$ *and* $\Pi$, *respectively, which are given by* $\Pi^{(s)}_{ij} = \frac{L^{(s)}_{ij}}{\bar{L}^{(s)}_i}$ *and* $\Pi_{ij} = \frac{L_{ij}}{\bar{L}_i}$ *for all* $i, j \in \mathcal{N}$ *if* $\bar{L}^{(s)}_i > 0$ *(respectively,* $\bar{L}_i > 0$*) and* $\Pi^{(s)}_{ij} = 0$ *(respectively,* $\Pi_{ij} = 0$*) otherwise;*
   - *the* bankruptcy cost parameter $\gamma$.

**Table 1**

*Initial stylized balance sheet at* $t = 0$ *of bank* $i \in \mathcal{N}$.

| Assets | Liabilities |
|---|---|
| • Cash assets: $a_i$ | • Short-term interbank liabilities: $\bar{L}^{(s)}_i = \sum_{j=1}^N L^{(s)}_{ij}$ |
| • Short-term interbank loans: $\bar{A}^{(s)}_i = \sum_{j=1}^N L^{(s)}_{ji}$ | • Long-term interbank liabilities: $\bar{L}^{(l)}_i = \sum_{j=1}^N L^{(l)}_{ij}$ |
| • Long-term interbank loans: $\bar{A}^{(l)}_i = \sum_{j=1}^N L^{(l)}_{ji}$ | • Equity: $E_i$ |

Thus, given a matrix $M$ of liabilities of some maturity, a bank $i$ has an outstanding liability of that maturity to bank $j$ if $M_{ij} > 0$ and the nominal value of this liability is given by $M_{ij}$. If $M_{ij} = 0$, then $i$ does not owe anything to $j$ and in particular $M$ has a zero diagonal since we assume banks do not owe anything to themselves. The $i$th row sum of $M$ then gives the total nominal value of liabilities of each bank of the relevant maturity, and the $i$th column sum gives the total nominal value of assets of that maturity.

Table 1 shows the stylized balance sheet at time $t = 0$ of bank $i \in \mathcal{N}$ where the equity is defined as $E_i := a_i + \bar{A}_i^{(s)} + \bar{A}_i^{(l)} - \bar{L}_i^{(s)} - \bar{L}_i^{(l)}$.

*Remark* 2.2. The set of banks $\mathcal{N}$ is assumed to contain a "sink node," e.g., in this paper $N \in \mathcal{N}$. This node has no cash assets or liabilities. However, other banks may well have liabilities to the sink node. These represent banks' liabilities external to the interbank market, but for ease of reference we refer to all entries of the liability matrices as "interbank" liabilities. In [16] it is pointed out that in order to use a sink node in this manner external liabilities need to be treated as having the same seniority as interbank liabilities; this is indeed our assumption in this paper.

**2.2. General equilibrium.** In this paper we formulate a characterization of an equilibrium achieved by clearing the market at the first maturity date that is based on the requirements of the UK insolvency rules as outlined in [33], which can be heuristically summarized as follows:

- Banks are not required to make any payments either in excess of the total value of their liquidated assets or in excess of the total amount they owe across all maturities.
- Conversely, shareholders are not permitted to retain any value of the defaulting banks as long as any part of any creditor's outstanding claims remains.
- Such claims include both short-term and long-term liabilities, which are treated with the same priority within the same seniority class.
- A bank that is liquidated under the insolvency rules ceases to exist and cannot recover even if liquidators recover sufficient assets to fully compensate all creditors.

Suppose we are at the first maturity date $t = T_1$ and suppose some banks with indices in $\mathcal{D} \subseteq \mathcal{N}$ are in default at $t = T_1$. We postpone the discussion on the cause of these defaults to section 2.3. We will now determine a clearing equilibrium at $t = T_1$.

We start by considering the case where a bank $j$ does not default, i.e., $j \in \mathcal{N} \setminus \mathcal{D}$. Then it pays its short-term nominal obligations $\bar{L}_j^{(s)}$ in full; in particular, it pays $L_{ji}^{(s)}$ to every bank $i$. Next, we consider a bank $j$ that defaults, i.e., $j \in \mathcal{D}$. Bank $j$ is liable to pay its creditors all of its available liquid asset resources, denoted by $v_j$, subject to two constraints. First, since default is costly and lawyers and other service providers need to be paid, only a fraction $\gamma \in [0, 1]$ of its liquid asset resources reaches its creditors. Second, we now need to consider both its short-term and its long-term liabilities. In general, $\bar{L}_j \geq \bar{L}_j^{(s)}$ and if $j$ has any long-term liabilities, then $\bar{L}_j > \bar{L}_j^{(s)}$. We assume that the creditors are not entitled to more than the overall total liabilities $\bar{L}_j$.

Finally, we also need to model what is permitted to happen to the long-term interbank assets $\bar{A}_j^{(l)}$ of a bank $j$ at or prior to the first maturity date. In practice they can be auctioned to provide additional cash to satisfy the total liabilities, especially if the bank $j$ is attempting to avoid being in default. In the interests of model parsimony and tractability, we refrain from modeling an auction mechanism here and take a reduced approach where, following an auction, the bank $j$ would have a further amount $R\bar{A}_j^{(l)}$ of liquid assets where $R \geq 0$ is the recovery rate. See, for example, [7] for an example of an auction mechanism. We discuss this further in section 4.

Provided the auction of long-term assets does not take place concurrently with the clearing process, the two can be separated in time. In other words, we assume that the auctions take place before the clearing at the first maturity date and any auction proceeds are already incorporated into the cash assets $a_j$ of the bank $j$ by the time the clearing process commences. Therefore, we make this explicit by making a further modeling assumption that, during clearing, $R = 0$. The situation can become significantly more complex if the auctions can take place concurrently with the clearing process. To keep the model tractable, the assumption that $R = 0$ then also allows us to avoid dealing with such a case.

We therefore need to determine the liquid asset resources $v$ that each bank has at time $t = T_1$. We characterize $v$ in terms of a fixed-point problem for a given financial system $(a, L^{(s)}, L^{(l)}; \gamma)$. Note that in this paper $\mathbf{0}$ denotes the vector of zeros which in Definition 2.3 below corresponds to an $N$-dimensional vector.

Definition 2.3. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system and $\mathcal{D} \subseteq \mathcal{N}$. Define $\Psi(\cdot; \mathcal{D}) : [\mathbf{0}, a + \bar{A}] \to [\mathbf{0}, a + \bar{A}]$ where $[\mathbf{0}, a + \bar{A}] \subset \mathbb{R}^N_+$ and, for each $i \in \mathcal{N}$,*

$$\Psi_i(v; \mathcal{D}) := a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}} L^{(s)}_{ji} + \sum_{j \in \mathcal{D}} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j).$$

*We refer to any vector $v \in [\mathbf{0}, a + \bar{A}]$ satisfying $v = \Psi(v; \mathcal{D})$ as a* general liquid asset vector *with respect to $\mathcal{D}$.*

*Remark* 2.4. Note that, indeed, $0 \le \Psi(v; \mathcal{D})_i \le a_i + \bar{A}_i$ for all $v$ and $i$. This follows directly from the fact that for each $i, j \in \mathcal{N}$ and $v \in \mathbb{R}^N_+$, $\Pi_{ji}(\bar{L}_j \wedge \gamma v_j) \le \Pi_{ji}\bar{L}_j = L_{ji}$. Therefore, since $L^{(s)}_{ji} \le L_{ji}$ for all $i, j \in \mathcal{N}$, we have that $\Psi(v; \mathcal{D})_i \le a_i + \sum_{j \in \mathcal{N}} L_{ji} = a_i + \bar{A}_i$.

Importantly, the set $[\mathbf{0}, a + \bar{A}]$ forms a complete lattice under the componentwise ordering of $\mathbb{R}^N_+$.

Definition 2.3 defines the liquid asset vector with respect to a default set $\mathcal{D}$. In the following we discuss properties of the default set $\mathcal{D}$ before we propose two approaches to define it in subsection 3.1.

**2.3. Identification of default.** Most models based on the Eisenberg and Noe [14] framework define default by checking whether some value is less than the total nominal short-term liabilities $\bar{L}^{(s)}$. This leads to the following general definition.

Definition 2.5. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system of bank $\mathcal{N}$ with the total nominal short-term liabilities vector $\bar{L}^{(s)}$. We define the function $D$ by setting, for each vector $x \in \mathbb{R}^N_+$,*

$$D(x) := \{i \in \mathcal{N} \mid x_i < \bar{L}^{(s)}_i\}. \tag{1}$$

This allows us to define *fundamental defaults*, i.e., defaults that occur even if everyone is assumed to satisfy their payment obligations. The *fundamental default set* is given by

$$\mathcal{F} := D(a + \bar{A}^{(s)}) = \left\{ i \in \mathcal{N} \mid a_i + \sum_{j \in \mathcal{N}} L^{(s)}_{ji} < \bar{L}^{(s)}_i \right\}.$$

Fundamental defaults can be read off directly from the stylized balance sheet. It is reasonable to assume that any default set $\mathcal{D}$ satisfies $\mathcal{F} \subseteq \mathcal{D}$. Furthermore, it is reasonable to assume that $\mathcal{F} = \emptyset$ implies $\mathcal{D} = \emptyset$.

Nevertheless, $\mathcal{F}$ is too small to be a suitable choice for the default set $\mathcal{D}$. Not all defaults are fundamental defaults. A bank may have interbank assets whose book value is sufficient but contingent on its counterparties avoiding default. If some of the counterparties default, this would cause the market value of assets to be adjusted down, making the bank illiquid and thus triggering its default. This type of default is known as a *contagious default* and is well established as one of the key drivers of systemic risk. These contagious defaults cannot be directly determined from the stylized balance sheet.

To capture some of these contagious defaults, we can ask whether some bank $i$ is *illiquid* in the sense that its liquid assets $v_i$ are insufficient for it to meet its own short-term liabilities in full. The set of such illiquid banks is then given by $D(v)$. We would expect that for any default set $\mathcal{D}$ one should have $D(v) \subseteq \mathcal{D}$. As with the fundamental defaults, the converse is not necessarily true. Since default changes the rules of distribution between counterparties, it may be the case that after a bank defaults, its liquid assets exceed its short-term liabilities. However, default is an absorbing state, and, once defaulted, a bank cannot recover. Thus $D(v)$ may also be too small to be a suitable choice for the default set $\mathcal{D}$.

Combining these considerations leads to the necessary condition on the default set $\mathcal{D}$:

$$\text{(1) } \mathcal{D} \supseteq \mathcal{F} \cup D(v), \quad \text{(2) } \mathcal{F} = \emptyset \Rightarrow \mathcal{D} = \emptyset. \tag{2}$$

## 3. Clearing at the first maturity.

### 3.1. Algorithmic and functional approaches to defining default.

In the following we introduce two particular approaches to formalize the notion of default and hence to define the default set $\mathcal{D}$, which we refer to as *the algorithmic approach* and *the functional approach*. In section 3.2 we will discuss the conditions under which these approaches are well defined and ensure existence of liquid asset vectors. Alternative definitions of a default set are also possible, but we will not investigate them further here.

#### 3.1.1. Algorithmic approach.

In the algorithmic approach we will *start* by providing an algorithm which outputs a vector and a set, which we *define* as a liquid asset vector and a default set.

It is similar in spirit to the *Fictitious Default Algorithm* (FDA) developed in [14], but in contrast to the FDA we use it to *define* default and the liquid asset vector and do not just use it as a convenient computational tool to calculate a predefined quantity of interest.

We consider a fixed financial system $(a, L^{(s)}, L^{(l)}; \gamma)$ and make the crucial modeling assumption that *default is an absorbing state*. In particular, we assume that once a bank enters the default set, it will stay there. Furthermore, a bank enters the default set if and only if the value of its liquid assets is less than its total short-term liabilities. Algorithm 1 formalizes this idea.

Thus, for a given financial system $(a, L^{(s)}, L^{(l)}; \gamma)$ Algorithm 1 computes a vector $v^*$ and a set $\mathcal{D}^*$ which will correspond to a liquid asset vector with respect to the default set $\mathcal{D}^*$.

Definition 3.1. *Let $\mathcal{D}^*$ and $v^*$ be the outputs of Algorithm 1. We refer to*

- $\mathcal{D}^*$ *as the* algorithmic default set*; and*
- $v^*$ *as the* algorithmic liquid asset vector with respect to $\mathcal{D}^*$.

**Algorithm 1** Algorithmic definition of the default set

1: Set $\mathcal{D}^{(0)} = \emptyset$, $v^{(0)} = a + \bar{A}^{(s)}$, $n = 1$.
2: Set
$$\mathcal{D}^{(n)} = \mathcal{D}^{(n-1)} \cup D(v^{(n-1)}).$$
3: **if** $\mathcal{D}^{(n)} = \mathcal{D}^{(n-1)}$ **then**
4: **return** $\mathcal{D}^* = \mathcal{D}^{(n-1)}$ and $v^* = v^{(n-1)}$.
5: **else**
6: determine the greatest fixed point $v^{(n)}$ satisfying
$$v^{(n)} = \Psi(v^{(n)}; \mathcal{D}^{(n)}), \tag{3}$$
where $\Psi$ is defined in Definition 2.3.
7: **end if**
8: Set $n = n + 1$ and go to 2.

Proposition 3.2. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system, and let $\mathcal{D}^*$ and $v^*$ be the outputs of Algorithm 1. Then, the algorithmic liquid asset vector $v^*$ is a general liquid asset vector with respect to $\mathcal{D}^*$.*

Since Proposition 3.2 follows directly from the definition, we omit the proof.

The algorithmic approach incorporates the intuition of default sets discussed in section 2.3. Namely, it ensures that default is an absorbing state and that the necessary criteria on the default set $\mathcal{D}^*$ specified in (2) are satisfied. To see that the latter claim is true, consider that $\mathcal{F} = D(a + \bar{A}^{(s)}) = D(v^{(0)}) = \mathcal{D}^{(1)} \subseteq \mathcal{D}^*$. Furthermore, if $\mathcal{F} = \emptyset = \mathcal{D}^{(0)}$, then Algorithm 1 terminates with $\mathcal{D}^* = \emptyset$ and $v^* = a + \bar{A}^{(s)} = \Psi(v^*; \emptyset)$.

The other key intuition behind the algorithmic approach is that it views the clearing process as a dynamic process that proceeds in several rounds. It starts with the assumption that initially the default set is empty, and then it computes in every round the best possible outcome for the financial system based on the given (absorbing) default set by finding the greatest fixed point.

The algorithmic approach therefore introduces an ordering of financial institution, depending on the round in which they default. This ordering depends on the initial default set (in our case the empty set which corresponds to no defaults). One could consider modifications of the algorithm with different (initial) default sets, but it would be less clear what the output of the algorithm represents. Intuitively, we think of the solution returned by the algorithmic approach as a best-case outcome similar to the greatest clearing vector in [14], since we start with no defaults and in every round compute the greatest fixed point rather than just any fixed points to keep the number of additional defaults minimal in every step of the algorithm.

**3.1.2. Functional approach.** We will argue in the following sections that the algorithmic approach is a more general approach that works for any financial system with multiple maturities. However, it is instructive to consider why the more conventional route along the lines of [14] is problematic in the multiple maturity setting. To this end we consider an alternative approach where the default set is characterized as a closed-form function $D(v)$ of the liquid asset vector.

Definition 3.3. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system. Define $\tilde{\Psi} : [\mathbf{0}, a+\bar{A}] \to [\mathbf{0}, a+\bar{A}]$, where*

$$\tilde{\Psi}_i(v) := a_i + \sum_{j \in \mathcal{N} \setminus D(v)} L_{ji}^{(s)} + \gamma \sum_{j \in D(v)} \Pi_{ji} v_j. \tag{4}$$

*We refer to any vector $v \in [\mathbf{0}, a+\bar{A}]$ satisfying $v = \tilde{\Psi}(v)$ as a* functional liquid asset vector *and the set $D(v)$ as a* functional default set.

Proposition 3.4. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system. Then*

$$\tilde{\Psi}(v) = \Psi(v; D(v)) \quad \forall v \in [\mathbf{0}, a+\bar{A}].$$

The following proposition is a direct corollary to the definitions and Proposition 3.4 and provides the link between functional and general liquid asset vectors.

Proposition 3.5. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system, and let $v$ be a functional liquid asset vector. Then $v$ is a general liquid asset vector with respect to $D(v)$.*

We will show that, in contrast to the algorithmic liquid asset vector, which exists for all financial systems, a functional liquid asset vector need not exist in a multiple maturity setting.

**3.2. Existence of liquid asset vectors.** To see that the algorithmic liquid asset vector and the algorithmic default set are well defined and exist for any financial system, consider the following theorem.

Theorem 3.6. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system. Then, the greatest solution to the fixed-point problem (3) exists and lies in $[\mathbf{0}, a+\bar{A}]$. Furthermore, Algorithm 1 terminates after a finite number of steps.*

The proof of Theorem 3.6 and all subsequent results can be found in Appendix A unless indicated otherwise. We will discuss the *construction* of the algorithmic liquid asset vector in section 3.4.

The functional liquid asset vector does not exist for all financial systems. There is a sufficient (but not necessary) monotonicity condition, however, that guarantees existence of a functional liquid asset vector.

Definition 3.7 (Monotonicity Condition). *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system, with short-term and overall relative liability matrices $\Pi^{(s)}$ and $\Pi$, respectively. We refer to a financial system as satisfying* Monotonicity Condition 3.7 *if and only if*

$$\Pi_{ij}^{(s)} \geq \gamma \Pi_{ij} \; \forall i, j \in \mathcal{N}.$$

From a financial point of view, Monotonicity Condition 3.7 just asserts that, for any bank $i$ in the system, it is guaranteed that if it defaults, it does not pay a larger proportion of its liquid assets to any bank $j$ in the system than its original proportion of short-term liabilities to this particular bank $j$. In this sense no bank benefits from the default of another bank in the system.

From a mathematical point of view, Monotonicity Condition 3.7 is a sufficient condition for the function $\tilde{\Psi}$ being nondecreasing. Furthermore, it highlights the fact that the distinction between $\Pi^{(s)}$ and $\Pi$ in our model is a crucial element that is missing in single maturity models.

*Remark* 3.8. Note that networks in which $L_{ij}^{(s)} = 0$ and $L_{ij}^{(l)} > 0$ for some $i, j$ will never satisfy Monotonicity Condition 3.7. Furthermore, if $\gamma = 1$, Monotonicity Condition 3.7 implies $\Pi^{(s)} = \Pi$.

*Remark* 3.9. Suppose $L^{(l)} = \mathbf{Z}$, where $\mathbf{Z}$ is a zero matrix. Then the short-term and overall nominal liabilities vectors $\bar{L}^{(s)}$ and $\bar{L}$ are equal, and hence so are the short-term and overall relative liability matrices $\Pi^{(s)}$ and $\Pi$. Thus Monotonicity Condition 3.7 is always satisfied if $L^{(l)} = \mathbf{Z}$.

Theorem 3.10 (sufficient conditions for the existence of a functional liquid asset vector). *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system.*

1. *If $\tilde{\Psi}$ is nondecreasing, then there exist functional liquid asset vectors $v^-$ (the* least functional liquid asset vector*) and $v^+$ (the* greatest functional liquid asset vector*) such that for any functional liquid asset vector $v$ we have that $v^- \leq v \leq v^+$.*
2. *If Monotonicity Condition 3.7 is satisfied, then the function $\tilde{\Psi}$ is nondecreasing. In particular, the greatest and least functional liquid assets vectors exist.*

In practice, checking whether $\tilde{\Psi}$ is nondecreasing can be quite cumbersome, whereas checking whether Monotonicity Condition 3.7 is satisfied is straightforward.

The following proposition demonstrates that Monotonicity Condition 3.7 is not a necessary condition, but neither is it a redundant condition.

Proposition 3.11.

1. *There exists a financial system that does not satisfy Monotonicity Condition 3.7 for which a functional liquid asset vector exists.*
2. *There also exists a financial system that does not satisfy Monotonicity Condition 3.7 for which no functional liquid asset vector exists.*

**3.3. Relationship between clearing models.** In this section we look at the relationship between several clearing models. In particular, we show that the algorithmic approach is indeed a proper generalization of the functional approach, which in turn generalizes the models of [14] and [31].

We introduce a new algorithm, Algorithm 2, which can be used to construct a functional liquid asset vector under Monotonicity Condition 3.7. We then show that under Monotonicity Condition 3.7 Algorithm 1 is reduced to Algorithm 2. Therefore, the algorithmic liquid asset vector and the algorithmic default set coincide with the functional liquid asset vector and the functional default set under Monotonicity Condition 3.7.

The only difference between Algorithm 1 and Algorithm 2 is in step 2, when the new default set is defined. Algorithm 2 only considers banks in default which in the current round have fewer liquid assets than nominal short term liabilities. Algorithm 1 makes the absorbing property of default explicit in the definition by additionally always keeping those banks in the default set that have defaulted in one of the previous rounds of the algorithm.

**Algorithm 2** Functional approach to define the default set and the liquid asset vector under Monotonicity Condition 3.7

1: Set $\mathcal{D}^{(0)} = \emptyset$, $v^{(0)} = a + \bar{A}^{(s)}$, $n = 1$.
2: Set

$$\mathcal{D}^{(n)} = D(v^{(n-1)}) = \{i \in \mathcal{N} \mid v_i^{(n-1)} < \bar{L}_i^{(s)}\}.$$

3: **if** $\mathcal{D}^{(n)} = \mathcal{D}^{(n-1)}$ **then**
4: **return** $\tilde{\mathcal{D}}^* = \mathcal{D}^{(n-1)}$ and $\tilde{v}^* = v^{(n-1)}$.
5: **else**
6: determine the greatest fixed point $v^{(n)}$ satisfying

$$v^{(n)} = \Psi(v^{(n)}; \mathcal{D}^{(n)}), \tag{5}$$

where $\Psi$ is defined in Definition 2.3.
7: **end if**
8: Set $n = n + 1$ and go to 2.

From the definition of Algorithm 2 and Proposition 3.4 we immediately get the following result.

Proposition 3.12. *Let $\tilde{\mathcal{D}}^*$ and $\tilde{v}^*$ be the output of Algorithm 2. Then, $\tilde{\mathcal{D}}^* = D(\tilde{v}^*)$, and hence $\tilde{\mathcal{D}}^*$ is a functional default set and $\tilde{v}^*$ is a functional liquid asset vector.*

Theorem 3.13. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system satisfying Monotonicity Condition 3.7. Then*

(i) *Algorithm 2 produces a monotone sequence of vectors $(v^{(n)})_{n\geq 0}$ such that $v^{(n)} \leq v^{(n-1)} \leq a + \bar{A}^{(s)}$ $\forall n \geq 1$ and a monotone sequence of sets $(\mathcal{D}^{(n)})_{n\geq 0}$ such that $\mathcal{D}^{(n-1)} \subseteq \mathcal{D}^{(n)}$ $\forall n \geq 1$. In particular, $\mathcal{D}^{(n)} = D(v^{(n-1)})$ $\forall n \geq 1$.*
(ii) *Algorithms 1 and 2 coincide.*
(iii) *The output of Algorithm 2 satisfies $\tilde{v}^* = v^+$.*

The assumption of Monotonicity Condition 3.7 is crucial. Without it Algorithm 2 can fail to terminate.

Proposition 3.14. *There exists a financial system not satisfying Monotonicity Condition 3.7 such that the sequence of vectors $\left(v^{(n)}\right)_{n\geq 0}$ constructed in Algorithm 2 is not monotone and Algorithm 2 does not terminate.*

By Remark 3.9, a functional liquid asset vector exists for any financial system $(a, L^{(s)}, \mathbf{Z}; \gamma)$ where $\mathbf{Z}$ is a zero matrix. In fact, the system then reduces to a special case of the model in [31] where the parameters modeling the default costs in [31] denoted by $\alpha, \beta$ are all the same and equal to $\gamma$, i.e., $\gamma = \alpha = \beta$. Proposition 3.15 formalizes this relationship.

Proposition 3.15. *Let $(a, L^{(s)}, \mathbf{Z}; \gamma)$ be a financial system where $\mathbf{Z}$ is a zero matrix.*

1. *Let $v$ be a functional liquid asset vector. Let $q$ be a vector defined by*

$$q_i = \begin{cases} \bar{L}_i^{(s)} & \text{if } i \in \mathcal{N} \setminus D(v), \\ \gamma v_i & \text{if } i \in D(v), \end{cases}$$

*for each $i \in \mathcal{N}$. Then $q$ is a clearing vector in the sense of [31], i.e., $q$ solves the fixed-point problem*

$$q_i = \begin{cases} \bar{L}_i & \text{if } a_i + \sum_{j\in\mathcal{N}} \Pi_{ji} q_j \geq \bar{L}_j, \\ \gamma a_i + \gamma \sum_{j\in\mathcal{N}} \Pi_{ji} q_j & \text{if } a_i + \sum_{j\in\mathcal{N}} \Pi_{ji} q_j < \bar{L}_j. \end{cases} \tag{6}$$

2. *Let $q$ be a clearing vector in the sense of [31], i.e., a solution of (6). Then $v = a + \Pi^\top q$ is a functional liquid asset vector.*

If $\gamma = 1$, then $(a, L^{(s)}, \mathbf{Z}; 1)$ is effectively a (single maturity) financial system as defined in [14], as the following proposition demonstrates.

Proposition 3.16. *Let $(a, L^{(s)}, \mathbf{Z}; 1)$ be a financial system where $\mathbf{Z}$ is a zero matrix.*

1. *Let $v$ be a functional liquid asset vector. Let $p := \bar{L}^{(s)} \wedge v$. Then $p$ is a clearing vector in the sense of [14], i.e., $p$ solves the fixed-point problem*

$$p = \bar{L}^{(s)} \wedge (a + \Pi^\top p). \tag{7}$$

2. *Let $p$ be a clearing vector in the sense of [14], i.e., a solution of (7). Then $v = a + \Pi^\top p$ is a functional liquid asset vector.*

**3.4. Construction of liquid asset vectors.** One of the questions we postponed answering was how to construct the liquid asset vectors (and hence default sets) using Algorithms 1 and 2 given that this requires us to compute a solution to the fixed-point problems (3) and (5), respectively.

In the statements and proofs of the results in this section we use the following notation for (sub)vectors and (sub)matrices. For a vector $v \in \mathbb{R}_+^{|\mathcal{N}|}$ and some nonempty index set $\mathcal{A} \subseteq \mathcal{N}$, $v_{\mathcal{A}} \in \mathbb{R}_+^{|\mathcal{A}|}$ denotes the vector given componentwise by $(v_{\mathcal{A}})_i = v_i$ for all $i \in \mathcal{A}$. Similarly, for another nonempty index set $\mathcal{B} \subseteq \mathcal{N}$ and a matrix $M \in \mathbb{R}_+^{|\mathcal{N}|\times|\mathcal{N}|}$, $M_{\mathcal{AB}} \in \mathbb{R}_+^{|\mathcal{A}|\times|\mathcal{B}|}$ denotes the matrix given componentwise by $(M_{\mathcal{AB}})_{ij} = M_{ij}$ for all $i \in \mathcal{A}$ and $j \in \mathcal{B}$. Furthermore, for $n \in \mathbb{N}$ we denote by $\mathrm{I}$ the $n \times n$ identity matrices and by $\mathbf{1}$ the $n$-dimensional vector of ones.

In both fixed-point problems, for each $n$, the relevant set $\mathcal{D}^{(n)}$ is fixed. This leads to the following general lemma, which we will use to construct the solutions to these fixed-point problems.

Lemma 3.17. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system, $\mathcal{D} \subseteq \mathcal{N}$ some fixed set of $m := |\mathcal{D}|$ banks, and $b \in \mathbb{R}_+^m$ some vector. Suppose that*

1. *$\gamma < 1$ or*
2. *$b_i > 0 \; \forall\, i \in \mathcal{D}$.*

*Then the system of $m$ linear equations $x_i = b_i + \gamma \sum_{j\in\mathcal{D}} \Pi_{ji} x_j$ for all $i \in \mathcal{D}$ has a unique nonnegative solution.*

We can now state the result on how to construct the functional liquid asset vector.

Proposition 3.18. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system satisfying Monotonicity Condition 3.7 such that $a_i > 0$ for all $i \in \mathcal{N}$. Then, for each $n$, the fixed-point problem (5) in Algorithm 2 has a unique nonnegative solution given by*

$$v_i^{(n)} = \begin{cases} x_i & \text{if } i \in \mathcal{D}^{(n)}, \\ a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(n)}} L_{ji}^{(s)} + \gamma \sum_{j \in \mathcal{D}^{(n)}} \Pi_{ji} x_j & \text{if } i \in \mathcal{N} \setminus \mathcal{D}^{(n)}, \end{cases}$$

*where $x = (\mathrm{I} - \gamma(\Pi_{\mathcal{D}^{(n)}\mathcal{D}^{(n)}})^\top)^{-1}(a_{\mathcal{D}^{(n)}} + (L^{(s)}_{\mathcal{L}^{(n)}\mathcal{D}^{(n)}})^\top \mathbf{1}_{\mathcal{L}^{(n)}})$ and $\mathcal{L}^{(n)} := \mathcal{N} \setminus \mathcal{D}^{(n)}$.*

We now turn to the algorithmic approach. First, note that for $\gamma = 0$ the fixed-point problem (3) in Algorithm 1 is trivial since, for each $n$ in Algorithm 1, we have $v^{(n)} = a + (L^{(s)}_{\mathcal{L}^{(n)}\mathcal{N}})^\top \mathbf{1}_{\mathcal{L}^{(n)}}$. Thus $v^{(n)}$ is explicitly fixed, and no fixed point needs to be found. For $\gamma > 0$, the key observation is that, for each $n$ in Algorithm 1, the banks in the set $\mathcal{D}^{(n)}$ can be treated as a financial system in their own right. Moreover, such a financial system satisfies Monotonicity Condition 3.7, and hence we can apply Proposition 3.18 to construct the fixed point satisfying fixed-point problem (3) in Algorithm 1.

Proposition 3.19. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system such that $a_i > 0$ for all $i \in \mathcal{N}$ and $\gamma > 0$. For each $n$ in Algorithm 1 with $\mathcal{D}^{(n)} \neq \emptyset$ we can construct a financial system $S_n$ of $|\mathcal{D}^{(n)}| + 1$ banks such that $S_n$ satisfies Monotonicity Condition 3.7 and $v^{(n)}$, the solution to the fixed-point problem (3), is given by*

$$v_i^{(n)} = \begin{cases} x_i & \text{if } i \in \mathcal{D}^{(n)}, \\ a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(n)}} L_{ji}^{(s)} + \sum_{j \in \mathcal{D}^{(n)}} \Pi_{ji} (\bar{L}_j \wedge \gamma x_j) & \text{if } i \in \mathcal{N} \setminus \mathcal{D}^{(n)}, \end{cases}$$

*where $x$ is the greatest functional liquid asset vector of $S_n$.*

The precise form of the system $S_n$ is given in the proof of Proposition 3.19 in Appendix A.

*Remark* 3.20. We showed in Proposition 3.15 that clearing in the model of [31] can be formulated in terms of the functional liquid asset vector. In that paper it was observed that, unlike in [14], even when $a > 0$ the clearing vectors are not necessarily unique, and therefore the same observation must hold of functional liquid asset vectors.

One interesting consequence of Lemma 3.17 is that it implies that there is at most a finite number of functional liquid asset vectors for any given financial system with $a > 0$. This follows from the fact that there is only a finite number of possible default sets and for each such possible default set there is at most one $v$ satisfying Definition 3.3.

**3.5. Uncertainty of the maturity profile.** The ability to construct algorithmic liquid asset vectors and default sets for any financial system allows us to demonstrate that the maturity profile of a financial system has a substantial impact on which banks can default.

Proposition 3.21. *There exists a financial system $S_1 = (a, L^{(s)}, L^{(l)}; \gamma)$ with the algorithmic default set $\mathcal{D}_1^*$ such that the financial system $S_2 := (a, L^{(s)} + L^{(l)}, \mathbf{Z}; \gamma)$, where $\mathbf{Z}$ is a zero matrix, has the algorithmic default set $\mathcal{D}_2^*$ that satisfies $\mathcal{D}_2^* \subsetneq \mathcal{D}_1^*$.*

In Proposition 3.21 the system $S_2$ has the same overall interbank liabilities as $S_1$, but all the interbank liabilities are now short-term liabilities. The proposition shows that if we treat all maturities as if they are the same, then we could end up with the financial system $S_2$ in which fewer banks default than if we account for the different maturity dates as in $S_1$. Therefore, this shows that approximating multiple maturity systems by single maturity systems can underestimate the severity of the risk of default. More generally, any uncertainty about the maturity profile in a financial system is itself a potential source of systemic risk.

This observation is particularly pertinent because in practice regulators do not have precise information about the banks' maturity profiles. Typically regulatory reports group liabilities into broad categories without recording the exact maturity dates. According to [29], in the UK, "Banks report exposures with breakdown by the maturity of the instrument" and "Categories of maturities are: open; less than 3 months; between 3 months and 1 year; between 1 year and 5 years; and more than 5 years. Derivatives are not reported with a maturity breakdown." It is therefore an open question whether these five categories are a sufficient representation of the maturity profile in the UK financial system for the purposes of assessing systemic risk.

## 4. Financial system after the first clearing.

**4.1. Stylized balance sheet after clearing at the first maturity date.** Let us denote the financial system $(a, L^{(s)}, L^{(l)}; \gamma)$ that we have been considering so far by $S(0)$ to indicate that it represents the system at time $t = 0$, prior to clearing at $t = T_1$. Following clearing at $t = T_1$, using the algorithmic approach described above, we obtain the algorithmic liquid asset vector and the algorithmic default set, which we now denote by $v^*(T_1)$ and $\mathcal{D}^*(T_1)$. This allows us to formulate a new financial system $S(T_1) := (a(T_1), L^{(s)}(T_1), L^{(l)}(T_1); \gamma)$ of banks in some set $\mathcal{N}(T_1) \subseteq \mathcal{N}$ after clearing at $t = T_1$. The banks that defaulted as part of the clearing at $t = T_1$ are no longer a part of the financial system, and so

$$\mathcal{N}(T_1) = \mathcal{N} \setminus \mathcal{D}^*(T_1). \tag{8}$$

Note that the sink node $N \in \mathcal{N}$ does not default as it has no liabilities and hence $N \in \mathcal{N}(T_1)$. We assume that the only changes between $t = 0$ and $t = T_1$ are attributable exclusively to the clearing process itself. Thus the new cash assets $a(T_1)$ are just the liquid assets of banks in $\mathcal{N}(T_1)$ less their payments at $T_1$. Since the banks that do not make their full payments at $T_1$ default and are not in $\mathcal{N}(T_1)$, it follows that for all $i \in \mathcal{N}(T_1)$,

$$a(T_1)_i = v^*(T_1)_i - \bar{L}_i^{(s)}. \tag{9}$$

At maturity date $T_1$, a typical surviving bank in $\mathcal{N}(T_1)$ may have had outstanding long-term liabilities both to banks in $\mathcal{D}^*(T_1)$ that defaulted at $T_1$ and to banks in $\mathcal{N}(T_1)$ that did not. As between the surviving banks in $\mathcal{N}(T_1)$, the new short-term liabilities at $T_1$ are just the remaining liabilities that were not due at $T_1$. Thus for all $i, j \in \mathcal{N}(T_1)$ such that $i, j \neq N$

$$L^{(s)}(T_1)_{ij} = L_{ij}^{(l)}. \tag{10}$$

The outstanding liabilities of surviving banks in $\mathcal{N}(T_1)$ to the defaulting banks in $\mathcal{D}^*(T_1)$ may comprise, for example, long-term interbank assets that the defaulting banks were not able to liquidate in time to avert the default. The surviving banks do not escape those liabilities by virtue of the defaults. There is, however, the question of who now owns these liabilities and thus to whom are they owed. In reality, such liabilities are assets of the banks in $\mathcal{D}^*(T_1)$ and these assets typically would be redistributed by liquidation administrators, likely through an auction. Such an auction would then determine who becomes their new owner. However, as discussed earlier in the paper, modeling such auctions is outside the scope of this paper, and we refer the reader to [7] for a model of an auction in this context.

We address the problem of who acquires the long-term assets of defaulting institutions by just assuming that all defaulting banks sell their long-term interbank assets to the sink node $N$. This assumption keeps the model clear and does not require arbitrary choices of who else in the network would be willing to acquire these assets.

The important consequence of this transaction is that, coupled with (10) above, we can now complete the characterization of the new short-term liability matrix $L^{(s)}(T_1)$. As before, we continue with the assumption that the sink node has no liabilities. Hence for all $i, j \in \mathcal{N}(T_1)$

$$L^{(s)}(T_1)_{iN} = L^{(l)}_{iN} + \sum_{k \in \mathcal{D}^*(T_1)} L^{(l)}_{ik}, \tag{11}$$

$$L^{(s)}(T_1)_{Nj} = L^{(l)}(T_1)_{Nj} = 0. \tag{12}$$

In particular, it follows that $\bar{L}^{(s)}(T_1)_i = \sum_{j \in \mathcal{N}(T_1) \cup \mathcal{D}^*(T_1)} L^{(l)}_{ij} = \bar{L}^{(l)}_i$ for all $i \in \mathcal{N}(T_1)$.

Furthermore, since these are the only liabilities of banks $\mathcal{N}(T_1)$ at $t = T_1$, we also have that for all $i, j \in \mathcal{N}(T_1)$ there are no new long-term liabilities:

$$L^{(l)}(T_1)_{ij} = 0. \tag{13}$$

The following proposition confirms that we have indeed constructed a new financial system.

Proposition 4.1. *Let $\mathcal{N}(T_1)$ be a set given in (8). The tuple $S(T_1) := (a(T_1), L^{(s)}(T_1), L^{(l)}(T_1); \gamma)$ satisfying (9)–(13) is a financial system.*

The stylized balance sheet of each bank except the sink node in this new financial system is given in Table 2. The sink node in the new financial system has no cash assets or short-term interbank liabilities, and hence $E(T_1)_N = \bar{A}^{(s)}(T_1)_N$. Its short-term interbank loans are given by $\bar{A}^{(s)}(T_1)_N = \sum_{j \in \mathcal{N}(T_1)} L^{(l)}_{jN} + \sum_{j \in \mathcal{N}(T_1)} \sum_{k \in \mathcal{D}^*(T_1)} L^{(l)}_{jk}$.

**Table 2**
*Stylized balance sheet at $t = T_1$ of bank $i \in \mathcal{N}(T_1) \setminus \{N\}$ after clearing.*

| Assets | Liabilities |
|---|---|
| • Cash assets: $a(T_1)_i = v^*(T_1)_i - \bar{L}^{(s)}_i$ | • Short-term interbank liabilities: $\bar{L}^{(s)}(T_1)_i = \sum_{j \in \mathcal{N}} L^{(l)}_{ij}$ |
| • Short-term interbank loans: $\bar{A}^{(s)}(T_1)_i = \sum_{j \in \mathcal{N}(T_1)} L^{(l)}_{ji}$ | • Equity: $E(T_1)_i = a(T_1)_i + \bar{A}^{(s)}(T_1)_i - \bar{L}^{(s)}(T_1)_i$ |

**4.2. Clearing at the second maturity date.** The financial system $S(T_1)$, described in section 4.1, can be cleared again by the application of Algorithm 1. In fact, by Remark 3.9,

$S(T_1)$ satisfies Monotonicity Condition 3.7 and so can be cleared by the application of the simpler Algorithm 2. Moreover, by Propositions 3.16 and 3.15, we can see that at the last maturity the financial system is reducible to the familiar models of [14] and [31].

Let $\tilde{v}^*(T_2)$ and $\tilde{D}^*(T_2)$ be the output of Algorithm 2 applied to the financial system $S(T_1)$. Then, after clearing at $t = T_2$, we obtain a new financial system $S(T_2)$ consisting of banks in the set $\mathcal{N}(T_2) := \mathcal{N}(T_1) \setminus \tilde{D}^*(T_2)$. Since the banks in $\mathcal{N}(T_2)$ have only cash assets and no liabilities, this system is given by $S(T_2) := (a(T_2), \mathbf{Z}, \mathbf{Z}; \gamma)$. Thus $S_2$ is characterized by the cash assets given by

$$a(T_2)_i = \tilde{v}^*(T_2)_i - \bar{L}^{(s)}(T_1) \quad \forall i \in \mathcal{N}(T_2).$$

We also have that $\bar{A}^{(s)}(T_2) = \bar{L}^{(s)}(T_2) = \mathbf{0}$ and hence $a(T_2)_i = E(T_2)_i$ for all $i \in \mathcal{N}(T_2)$. Moreover, no further clearing of $S(T_2)$ is necessary.

**4.3. Extension to more than two maturity dates.** So far we have focused on financial systems with at most two maturities. However, provided we track the precise maturity profile of all the liabilities amalgamated in the long-term liability matrix $L^{(l)}$, we can readily extend our modeling framework to $n > 2$ maturity dates $0 < T_1 < T_2 < \cdots < T_n$.

We write $L^{(T_i)} \in \mathbb{R}_+^{N \times N}$ for the matrix containing all interbank liabilities maturing at $T_i$, $i \in \{1, \ldots, n\}$. We then consider an $n$-maturity financial system as a tuple $S = (a, L^{(T_1)}, L^{(T_2)}, \ldots, L^{(T_n)}; \gamma)$. At $t = 0$ we can define a 2-maturity financial system $S(0) := (a, L^{(s)}, L^{(l)}; \gamma)$ given by $L^{(s)} := L^{(T_1)}$ and $L^{(l)}_{ij} := \sum_{\tau=2}^{n} L^{(T_\tau)}_{ij}$ for all $i, j \in \mathcal{N}$. Then clearing the $n$-maturity financial system $S$ at time $t = T_1$ reduces to clearing the 2-maturity financial system $S(0)$ at time $t = T_1$ using Algorithm 1 and, using the methodology similar to the one described in section 4.1, produces a new 2-maturity financial system $S(T_1) := (a(T_1), L^{(s)}(T_1), L^{(l)}(T_1); \gamma)$.

The new liquid assets vector $a(T_1)$ is as in (9) and only the definition of the new short-term and new long-term interbank liability matrices change so that the liabilities maturing at $t = T_2$ become the new short-term liabilities and all liabilities maturing at $t \geq T_3$ are aggregated into the new long-term liabilities. Thus we obtain, for all $i, j \in \mathcal{N}(T_1)$ with $i, j \neq N$, that

$$\begin{aligned}
&L^{(s)}(T_1)_{ij} = L^{(T_2)}_{ij}, \\
&L^{(s)}(T_1)_{iN} = L^{(T_2)}_{iN} + \sum_{k \in \mathcal{D}^*(T_1)} L^{(T_2)}_{ik}, \\
&L^{(l)}(T_1)_{ij} = \sum_{\tau=3}^{n} L^{(T_\tau)}_{ij}, \\
&L^{(l)}(T_1)_{iN} = \sum_{\tau=3}^{n} L^{(T_\tau)}_{iN} + \sum_{k \in \mathcal{D}^*(T_1)} \sum_{\tau=3}^{n} L^{(T_\tau)}_{ik}, \\
&L^{(s)}(T_1)_{Nj} = L^{(l)}(T_1)_{Nj} = 0.
\end{aligned}$$

Similarly, we can clear $S(T_2)$ using our methodology for two maturities and then repeat this approach until we reach the point $t = T_{n-1}$, where, for all $i, j \in \mathcal{N}(T_1)$ with $i, j \neq N$,

$$\begin{aligned}
&L^{(s)}(T_{n-1})_{iN} = L_{ij}^{(T_n)},\\
&L^{(s)}(T_{n-1})_{Nj} = 0,\\
&L^{(s)}(T_{n-1})_{iN} = L_{ij}^{(T_n)} + \sum_{k \in \mathcal{D}^*(T_{n-1})} L_{ik}^{(T_n)},
\end{aligned}$$

and $L^{(l)}(T_{n-1}) = \mathbf{Z}$.

This system can now be cleared using Algorithm 2, analogously to what we did in section 4.2. In the end we obtain the last financial system $S(T_n) := (a(T_n), \mathbf{Z}, \mathbf{Z}; \gamma)$ such that $a(T_n) = E(T_n)$ and no further clearing is necessary.

## 5. Conclusion.

This paper has developed a rigorous clearing framework for interbank networks with multiple maturities. We have shown that a vector of clearing cash flows (a vector of liquid assets, in our case) on its own is not sufficient to fully describe the clearing framework. A suitable definition of the set of banks in default is needed. This does not arise naturally from the description of the stylized balance sheets and must be specified as part of the model. We discussed the necessary conditions on such a default set. These conditions are not sufficient, and we considered the algorithmic approach and the functional approach as two possible approaches to specifying default.

The functional default set corresponds to the definitions that have been used in previous literature and has a simple functional representation. It does not have an absorbing property, and, as a consequence, a liquid asset vector using the functional default set may not exist for every financial system. On the other hand, the algorithmic default set has a more complex algorithmic definition that guarantees that default is an absorbing state. Therefore, the algorithmic liquid asset vector can be found for any financial system. We proposed Algorithm 1, which produces a sequence of vectors that converges to the algorithmic liquid asset vector. This sequence of vectors is not in general monotone, but the absorption property of the default sets ensures that the algorithm converges in a finite number of steps.

The functional approach has a number of uses despite restrictions on the existence of functional liquid asset vectors. We have shown that for certain types of financial systems the algorithmic approach reduces to the functional approach. Furthermore, we have shown that the functional approach reduces to the models in [14] and [31] if only one maturity is considered. In addition, we have shown that functional liquid asset vectors can be used in the construction of clearing solutions under the algorithmic approach. For these reasons the properties of functional liquid asset vectors are important. We have shown that under a regularity condition functional liquid asset vectors can be characterized as fixed points and a greatest functional liquid asset vector and a least functional liquid asset vector exist. We have also shown that functional liquid asset vectors are in general not unique, but under a mild condition we could show that there can be at most one such vector corresponding to any given default set.

We have illustrated two key applications of Algorithm 1. We demonstrated that the default risk of a bank depends in a nontrivial manner on the precise maturity profile of its liabilities. Relying on the assumption that all interbank liabilities have the same maturity can lead to an inaccurate assessment of risks. Our clearing approach provides a rigorous tool to incorporate different maturities in the clearing process. We also showed how to extend the model to a multiperiod one by describing a settlement mechanism which characterizes the stylized balance sheets of the surviving banks after clearing.

There are many directions in which one could extend this line of research. The most ambitious extension of the multiple maturity model would be to develop a full dynamic model of interbank networks. The multiperiod approach in section 4.3 provides a solid basis for this. The next steps would involve developing a control theory by deciding on a set of actions that financial institutions can choose from as they move forward in time. Examples of such actions could, for example, be new borrowing or lending activities. For such dynamic models one could then also include stochastic dynamics for some of the quantities of interest.

## Appendix A. Appendix.

*Proof of Proposition* 3.4. Let $v \in \mathbb{R}^N_+$; then for all $j \in D(v) = \{i \in \mathcal{N} \mid v_i < \bar{L}^{(s)}_i\}$ it holds that $\gamma v_j < \bar{L}^{(s)}_j \leq \bar{L}_j$ and hence $\bar{L}_j \wedge \gamma v_j = \gamma v_j$. Hence for all $i \in \mathcal{N}$

$$\begin{aligned}\Psi(v, D(v))_i &= a_i + \sum_{j \in \mathcal{N}\setminus D(v)} L^{(s)}_{ji} + \sum_{j \in D(v)} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j)\\ &= a_i + \sum_{j \in \mathcal{N}\setminus D(v)} L^{(s)}_{ji} + \sum_{j \in D(v)} \Pi_{ji}\gamma v_j\\ &= \tilde{\Psi}_i(v).\end{aligned}$$ ■

In order to prove Theorems 3.6 and 3.10, we need the following lemma.

Lemma A.1. *Let $S = (a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system and $d : \mathbb{R}^N_+ \to \mathcal{P}(\mathcal{N})$ some function, where $\mathcal{P}$ denotes the power set. Let $\Psi^d : [\mathbf{0}, a + \bar{A}] \to [\mathbf{0}, a + \bar{A}]$ be the function given by $x \mapsto \Psi(x; d(x))$ for all $x \in [\mathbf{0}, a + \bar{A}]$.*

1. *Suppose $d \equiv \mathcal{D}$, i.e., $d(x) = \mathcal{D}$ for all $x \in [\mathbf{0}, a + \bar{A}]$ and some fixed $\mathcal{D} \subseteq \mathcal{N}$. Then $\Psi^d = \Psi(\cdot; \mathcal{D})$ and $\Psi^d$ is nondecreasing; i.e., for all $x', x \in [\mathbf{0}, a + \bar{A}]$ with $x' \leq x$ we have that $\Psi^d(x') \leq \Psi^d(x)$.*
2. *Suppose $d = D$, i.e., $d(x) = D(x) = \{i \in \mathcal{N} \mid x_i < \bar{L}^{(s)}_i\}$ for all $x \in [\mathbf{0}, a + \bar{A}]$ and suppose that $S$ satisfies Monotonicity Condition 3.7. Then $\Psi^d = \tilde{\Psi}$ and $\Psi^d$ is nondecreasing; i.e., for all $x', x \in [\mathbf{0}, a + \bar{A}]$ with $x' \leq x$ we have that $\Psi^d(x') \leq \Psi^d(x)$.*

*Proof of Lemma* A.1.

1. Suppose $d \equiv \mathcal{D}$ for some fixed $\mathcal{D} \subseteq \mathcal{N}$. Then, for each $x \in [\mathbf{0}, a + \bar{A}]$, $\Psi^d(x) = \Psi(x; d(x)) = \Psi(x; \mathcal{D})$, and hence $\Psi^d = \Psi(\cdot; \mathcal{D})$.
   Let $x', x \in \mathbb{R}_+$ with $x' \leq x$. Define $E(x') := \{i \in \mathcal{D} \mid \gamma x'_i < \bar{L}_i\}$ and, similarly, $E(x)$. Since $\gamma x'_i \leq \gamma x_i$ for all $i \in \mathcal{N}$, we see that $E(x) \subseteq E(x') \subseteq \mathcal{N}$. Then, for each $i \in \mathcal{N}$,

we have

$$\begin{aligned}
\Psi_i^d(x') = \Psi_i(x';\mathcal{D}) &= a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}} \Pi_{ji}(\bar{L}_j \wedge \gamma x_j') \\
&= a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}\setminus E(x')} \Pi_{ji}\bar{L}_j + \gamma \sum_{j\in E(x')} \Pi_{ji}x_j' \\
&= a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}\setminus E(x')} L_{ji} + \gamma \sum_{j\in E(x')} \Pi_{ji}x_j' \\
&= a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}\setminus E(x)} L_{ji} + \gamma \sum_{j\in E(x)} \Pi_{ji}x_j' \\
&\quad + \sum_{j\in E(x')\setminus E(x)} (\Pi_{ji}\gamma x_j' - L_{ji}) \\
&\leq a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}\setminus E(x)} L_{ji} + \gamma \sum_{j\in E(x)} \Pi_{ji}x_j' \\
&\leq a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}\setminus E(x)} L_{ji} + \gamma \sum_{j\in E(x)} \Pi_{ji}x_j \\
&= \Psi_i(x;\mathcal{D}) = \Psi_i^d(x).
\end{aligned}$$

The first inequality (on the sixth line) follows since $\gamma x_j' < \bar{L}_j$ for $j \in E(x')$ and hence $\Pi_{ji}\gamma x_j' - L_{ji} \leq \Pi_{ji}\bar{L}_j - L_{ji} = 0$. The second inequality (on the seventh line) follows since $x' \leq x$ by assumption. Therefore, $\Psi^d$ is nondecreasing.

2. Suppose $d = D$. Then, for each $x \in [\mathbf{0}, a + \bar{A}]$, $\Psi^d(x) = \Psi(x; d(x)) = \Psi(x; D(x))$ and hence, by Proposition 3.4, $\Psi^d = \tilde{\Psi}$.
Again, let $x', x \in \mathbb{R}_+$ with $x' \leq x$. Note that $D(x) \subseteq D(x') \subseteq \mathcal{N}$. Then, for each $i \in \mathcal{N}$, we have

$$\begin{aligned}
\Psi_i^d(x') = \tilde{\Psi}_i(x') &= a_i + \sum_{j\in\mathcal{N}\setminus D(x')} L_{ji}^{(s)} + \gamma \sum_{j\in D(x')} \Pi_{ji}x_j' \\
&= a_i + \sum_{j\in\mathcal{N}\setminus D(x')} L_{ji}^{(s)} + \gamma \sum_{j\in D(x')\setminus D(x)} \Pi_{ji}x_j' + \gamma \sum_{j\in D(x)} \Pi_{ji}x_j' \\
&\leq a_i + \sum_{j\in\mathcal{N}\setminus D(x')} L_{ji}^{(s)} + \sum_{j\in D(x')\setminus D(x)} \Pi_{ji}^{(s)} x_j' + \gamma \sum_{j\in D(x)} \Pi_{ji}x_j' \\
&\leq a_i + \sum_{j\in\mathcal{N}\setminus D(x')} L_{ji}^{(s)} + \sum_{j\in D(x')\setminus D(x)} \Pi_{ji}^{(s)} \bar{L}_j^{(s)} + \gamma \sum_{j\in D(x)} \Pi_{ji}x_j \\
&= a_i + \sum_{j\in\mathcal{N}\setminus D(x')} L_{ji}^{(s)} + \sum_{j\in D(x')\setminus D(x)} L_{ji}^{(s)} + \gamma \sum_{j\in D(x)} \Pi_{ji}x_j \\
&= a_i + \sum_{j\in\mathcal{N}\setminus D(x)} L_{ji}^{(s)} + \gamma \sum_{j\in D(x)} \Pi_{ji}x_j \\
&= \tilde{\Psi}_i(x) = \Psi_i^d(x).
\end{aligned}$$

The first inequality (on the third line) follows due to Monotonicity Condition 3.7 and the fact that $\gamma \leq 1$. The second inequality (on the fourth line) follows because $x' \leq x$ by assumption and $x'_j < \bar{L}_j^{(s)}$ for all $j \in D(x')$. Therefore $\Psi^d$ is nondecreasing. ∎

*Proof of Theorem* 3.6. For each $n$, $\mathcal{D}^{(n)}$ depends on $v^{(n-1)}$ but not on $v^{(n)}$. Therefore, by Lemma A.1, $\Psi(\cdot;\mathcal{D}^{(n)})$ is nondecreasing and by Remark 2.4 is a mapping from a complete lattice to itself. Hence, by the Tarski–Knaster theorem, $\Psi(\cdot;\mathcal{D}^{(n)})$ has the greatest fixed point, which lies within the image of $\Psi(\cdot;\mathcal{D}^{(n)})$, i.e., in $[\mathbf{0}, a+\bar{A}]$. In Algorithm 1 this fixed point is denoted $v^{(n)}$. Hence whenever $\mathcal{D}^{(n)}$ is well defined, $\mathcal{D}^{(n+1)}$ is also well defined until the algorithm terminates.

In particular, $(\mathcal{D}^{(n)})_{n\geq 0}$ is a well-defined and, by construction, increasing sequence of subsets of the finite set $\mathcal{N}$. Hence there exists the least $n$ such that $\mathcal{D}^{(n)} = \mathcal{D}^{(n-1)}$, and so Algorithm 1 terminates after $n$ iterations. ∎

*Proof of Theorem* 3.10.

1. The result follows directly by the application of the Tarski–Knaster theorem since $\tilde{\Psi}$ is nondecreasing by assumption and it is a mapping from a complete lattice to itself by Remark 2.4.
2. Since $\tilde{\Psi}$ is nondecreasing by Lemma A.1, the result follows directly from part 1 of this theorem. ∎

*Proof of Proposition* 3.11.

1. We first provide one example of a financial system in which the functional liquid asset vector exists even though Monotonicity Condition 3.7 is not satisfied.
   Let $(a, L^{(s)}, L^{(l)}; 1)$ be a financial system of three banks where
$$a = \begin{pmatrix} 1 \\ 98 \\ 10 \end{pmatrix}, \quad L^{(s)} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 98 \\ 0 & 0 & 0 \end{pmatrix}, \quad L^{(l)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 2 & 2 \\ 3 & 0 & 99 \\ 0 & 0 & 0 \end{pmatrix}.$$
   Then,
$$\bar{L}^{(s)} = \begin{pmatrix} 4 \\ 100 \\ 0 \end{pmatrix}, \quad \bar{L} = \begin{pmatrix} 4 \\ 102 \\ 0 \end{pmatrix}, \quad \Pi^{(s)} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{50} & 0 & \frac{49}{50} \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{34} & 0 & \frac{33}{34} \\ 0 & 0 & 0 \end{pmatrix}.$$
   In particular, we see that Monotonicity Condition 3.7 is not satisfied because $\Pi_{21}^{(s)} = \frac{1}{50} < \frac{1}{34} = \Pi_{21}$. Nevertheless, it can be verified that $(v_1, v_2, v_3)^\top = (3\frac{63}{67}, 99\frac{65}{67}, 109\frac{1}{2})^\top \approx (3.94, 99.97, 109.5)^\top$ is a functional liquid asset vector.
   One can check that in this example the function $\tilde{\Psi}$ is nondecreasing even though Monotonicity Condition 3.7 is not satisfied.
2. Next, we provide an example of a financial system in which Monotonicity Condition 3.7 is not satisfied and a functional liquid asset vector does not exist.
   We construct an example with three banks in which only bank 1 is in fundamental default. We set up the network such that this leads to a contagious default of bank 2, which is asset rich. We introduce long-term liabilities in such a way that once bank 2

defaults, it repays a much larger proportion of its debt to bank 1 than if it were not in default. This leads to bank 1 being able to pay more than $\bar{L}_1^{(s)}$.
Let $(a, L^{(s)}, L^{(l)}; 1)$ be a financial system of three banks where

$$a = \begin{pmatrix} 1 \\ 98 \\ 10 \end{pmatrix}, \quad L^{(s)} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 98 \\ 0 & 0 & 0 \end{pmatrix}, \quad L^{(l)} = \begin{pmatrix} 0 & 2 & 2 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 4 & 4 \\ 102 & 0 & 98 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then,

$$\bar{L}^{(s)} = \begin{pmatrix} 4 \\ 100 \\ 0 \end{pmatrix}, \quad \bar{L} = \begin{pmatrix} 8 \\ 200 \\ 0 \end{pmatrix}, \quad \Pi^{(s)} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{50} & 0 & \frac{49}{50} \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{51}{100} & 0 & \frac{49}{100} \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that Monotonicity Condition 3.7 is not satisfied since, for example, $\Pi_{21}^{(s)} = \frac{1}{50} < \frac{51}{100} = \Pi_{21}$. Hence, if bank 2 defaults, it repays a larger proportion to bank 1 than if it survives. We show in the following that no functional liquid asset vector exists.
According to Definition 3.3, bank 3 can never default since it does not have any short-term (or indeed any) liabilities. In particular, since $\tilde{\Psi}$ is nonnegative, we have that $\{i \in \mathcal{N} \mid \tilde{\Psi}(v)_i < 0\} = \emptyset$ for any $v$. Hence we need to consider four cases.

**All banks survive.** Suppose there exists a functional liquid asset vector $v$, such that $D(v) = \emptyset$. Hence, $v_i \geq \bar{L}_i^{(s)}$ for all $i$. Then, for all $i \in \mathcal{N}$,

$$v_i = a_i + \sum_{j \in \mathcal{N}} L_{ji}^{(s)}.$$

Consider $i = 1$. Then $v_1 = 1 + 2 = 3 < 4 = \bar{L}_1^{(s)}$, implying that $1 \in D(v)$ and therefore contradicting the assumption that $D(v) = \emptyset$.

**Only bank 1 defaults.** Suppose there exists a functional liquid asset vector $v$, such that $D(v) = \{i : v_i < \bar{L}_i^{(s)}\} = \{1\}$. Then,

$$v_1 = a_1 + \sum_{j \in \{2,3\}} L_{ji}^{(s)} = 1 + 2 + 0 = 3 < 4 = \bar{L}_1^{(s)},$$
$$v_2 = a_2 + L_{32}^{(s)} + \Pi_{12} v_1 = 98 + 0 + \frac{1}{2}3 = 99\frac{1}{2} < 100 = \bar{L}_2^{(s)}.$$

Hence $2 \in D(v)$, contradicting the assumption that $D(v) = \{i \in \mathcal{D} : v_i < \bar{L}_i^{(s)}\} = \{1\}$.

**Only bank 2 defaults.** Suppose there exists a functional liquid asset vector $v$, such that $D(v) = \{i \in \mathcal{D} : v_i < \bar{L}_i^{(s)}\} = \{2\}$. Then,

$$v_2 = a_2 + \sum_{j \in \{1,3\}} L_{ji}^{(s)} = 98 + 2 + 0 = 100 = \bar{L}_2^{(s)}.$$

Hence $2 \notin D(v)$, contradicting our assumption.

**Both bank 1 and bank 2 default.** Suppose there exists a functional liquid asset vector $v$, such that $D(v) = \{i : v_i < \bar{L}_i^{(s)}\} = \{1, 2\}$. Then,

$$v_1 = a_1 + L_{31}^{(s)} + \Pi_{21} v_2 = 1 + 0 + \frac{51}{100} \cdot v_2,$$
$$v_2 = a_2 + L_{32}^{(s)} + \Pi_{12} v_1 = 98 + \frac{1}{2} \cdot v_1.$$

We then obtain that $(1 - \frac{1 \cdot 51}{2 \cdot 100}) v_1 = 1 + 98 \frac{51}{100}$ and hence $v_1 \approx 68.43 > 4 = \bar{L}_1^{(s)}$. Therefore, $1 \notin D(v)$, contradicting our assumption.

Hence, in all cases we get a contradiction, and therefore no functional liquid asset vector exists. ∎

*Proof of Theorem* 3.13. The proof uses arguments similar to those in [31, Proof of Theorem 3.7].

(i) We prove that $v^{(n)} \leq v^{(n-1)} \leq a + \bar{A}^{(s)}$ $\forall n \geq 1$ and $\mathcal{D}^{(n)} = D(v^{(n-1)})$ $\forall n \geq 1$ by induction.
Note that for all $n$ and $j \in \mathcal{N}$ we have $\bar{L}_j \wedge \gamma v_j^{(n)} \leq \gamma v_j^{(n)} \leq v_j^{(n)}$. Furthermore, for all $n$ and $j \in D(v^{(n)})$ we also have $v_j^{(n)} < \bar{L}_j^{(s)}$. Therefore, by Monotonicity Condition 3.7, for all $n$, $i \in \mathcal{N}$ and $j \in D(v^{(n)})$ we have that

$$\Pi_{ji}(\bar{L}_j \wedge \gamma v^{(n)})_j \leq \Pi_{ji}^{(s)} \bar{L}_j^{(s)} = L_{ji}^{(s)}. \tag{14}$$

Now let $n = 1$. Then by the definition of the algorithm $\mathcal{D}^{(1)} = \mathcal{D}^{(0)} \cup D(v^{(0)}) = \emptyset \cup D(v^{(0)}) = D(v^{(0)})$. Next we show that $v^{(1)} \leq v^{(0)} = a + \bar{A}^{(s)}$.
By (14), for all $i \in \mathcal{N}$, we have

$$\begin{aligned}
\Psi_i(v^{(0)}; \mathcal{D}^{(1)}) &= a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(1)}} L_{ji}^{(s)} + \sum_{j \in \mathcal{D}^{(1)}} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(0)}) \\
&\leq a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(1)}} L_{ji}^{(s)} + \sum_{j \in \mathcal{D}^{(1)}} L_{ji}^{(s)} \\
&= a_i + \sum_{j \in \mathcal{N}} L_{ji}^{(s)} = a_i + \bar{A}_i^{(s)} = v_i^{(0)}.
\end{aligned}$$

By Lemma A.1(1), $\Psi(\cdot; \mathcal{D}^{(1)})$ is nondecreasing, and so

$$0 \leq \Psi^{k+1}(v^{(0)}; \mathcal{D}^{(1)}) \leq \Psi^k(v^{(0)}; \mathcal{D}^{(1)}) \leq v^{(0)} = a + \bar{A}^{(s)}$$

for all $k$ where $\Psi^k$ is a $k$-fold composition of $\Psi$. Since this sequence is bounded from below by zero, the limit $v^{(1)} := \lim_{k \to \infty} \Psi^k(v^{(0)}; \mathcal{D}^{(1)})$ exists and solves $v^{(1)} = \Psi(v^{(1)}; \mathcal{D}^{(1)})$.
Induction hypothesis: Suppose for an $n \in \mathbb{N}$ it holds that

$$\begin{aligned}
\mathcal{D}^{(n)} &= D(v^{(n-1)}), \\
v^{(n)} &\leq v^{(n-1)} \leq v^{(0)} = a + \bar{A}^{(s)}.
\end{aligned}$$

We show that

$$\begin{aligned}\mathcal{D}^{(n+1)} &= D(v^{(n)}),\\ v^{(n+1)} &\leq v^{(n)} \leq v^{(0)} = a + \bar{A}^{(s)}.\end{aligned}$$

We start with the default sets

$$\mathcal{D}^{(n+1)} = \mathcal{D}^{(n)} \cup D(v^{(n)}) \overset{\text{ind. hyp. part 1}}{=} D(v^{(n-1)}) \cup D(v^{(n)}) \overset{\text{ind. hyp. part 2}}{=} D(v^{(n)}).$$

Next we consider the vector

$$v^{(n+1)} = \Psi(v^{(n+1)}; \mathcal{D}^{(n+1)}) = \Psi(v^{(n+1)}; D(v^{(n)})).$$

Then, by (14), for all $i \in \mathcal{N}$, we have

$$\begin{aligned}&\Psi_i(v^{(n)}; \mathcal{D}^{(n+1)})\\
&= a_i + \sum_{j \in \mathcal{N} \setminus \underbrace{\mathcal{D}^{(n+1)}}_{D(v^{(n)})}} L_{ji}^{(s)} + \sum_{j \in \mathcal{D}^{(n+1)}} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(n)})\\
&= a_i + \sum_{j \in \mathcal{N} \setminus D(v^{(n)})} L_{ji}^{(s)} + \sum_{j \in D(v^{(n-1)})} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(n)})\\
&\quad + \sum_{j \in D(v^{(n)}) \setminus D(v^{(n-1)})} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(n)})\\
&\leq a_i + \sum_{j \in \mathcal{N} \setminus D(v^{(n)})} L_{ji}^{(s)} + \sum_{j \in D(v^{(n-1)})} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(n)}) + \sum_{j \in D(v^{(n)}) \setminus D(v^{(n-1)})} L_{ji}^{(s)}\\
&= a_i + \sum_{j \in \mathcal{N} \setminus \underbrace{D(v^{(n-1)})}_{\mathcal{D}^{(n)}}} L_{ji}^{(s)} + \sum_{j \in \underbrace{D(v^{(n-1)})}_{\mathcal{D}^{(n)}}} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(n)})\\
&= \Psi_i(v^{(n)}; \mathcal{D}^{(n)}) = v_i^{(n)} \leq a_i + \bar{A}_i^{(s)}.\end{aligned}$$

Again, as before one can show by Lemma A.1 that

$$0 \leq \Psi^{k+1}(v^{(n)}; \mathcal{D}^{(n+1)}) \leq \Psi^{k}(v^{(n)}; \mathcal{D}^{(n+1)}) \leq v^{(n)} \overset{\text{ind. hyp. part 2}}{\leq} a + \bar{A}^{(s)}$$

for all $k$. Therefore, the sequence $\left(\Psi^k(v^{(n)}; \mathcal{D}^{(n+1)})\right)_{k \geq 0}$ decreases monotonically to the limit $\lim_{k\to\infty} \Psi^k(v^{(n)}; \mathcal{D}^{(n+1)}) \geq 0$, which we denote by $v^{(n+1)}$. In particular, this limit satisfies $v^{(n+1)} = \Psi(v^{(n+1)}; \mathcal{D}^{(n+1)})$ and $v^{(n+1)} \leq v^{(n)} \leq v^{(0)}$.

(ii) Since the only difference between the two algorithms is the definition of the default sets in step 2 and we have just proved in (i) that the default sets coincide, both algorithms are indeed identical under Monotonicity Condition 3.7.

(iii) By Proposition 3.12 and maximality of $v^+$, we have that $\tilde{v}^* \le v^+$, and so the result follows as soon as we show that $\tilde{v}^* \ge v^+$. Since $\tilde{v}^* = v^{(n)}$ for some $n$, we proceed by induction to show that $v^{(n)} \ge v^+$ for all $n$.
First, observe that by Monotonicity Condition 3.7 we have for each $i \in \mathcal{N}$

$$\begin{aligned} v_i^+ = \tilde{\Psi}_i(v^+) &= a_i + \sum_{j\in\mathcal{N}\setminus D(v^+)} L_{ji}^{(s)} + \sum_{j\in D(v^+)} \underbrace{\gamma\Pi_{ji}}_{\le \Pi_{ji}^{(s)}} \underbrace{v_j^+}_{<\bar{L}_j^{(s)}} \\ &\le a_i + \sum_{j\in\mathcal{N}\setminus D(v^+)} L_{ji}^{(s)} + \sum_{j\in D(v^+)} \underbrace{\Pi_{ji}^{(s)}\bar{L}_j^{(s)}}_{=L_{ji}^{(s)}} \\ &= a_i + \sum_{j\in\mathcal{N}} L_{ji}^{(s)} = a_i + \bar{A}_i^{(s)} = v_i^{(0)}. \end{aligned}$$

Hence $v^{(0)} \ge v^+$. Now suppose that $v^{(n)} \ge v^+$ for some $n$. We then show that $v^{(n+1)} \ge v^+$.
By the induction hypothesis and Lemma A.1 it follows that $\Psi(v^{(n)}; \mathcal{D}^{(n+1)}) \ge \Psi(v^+; \mathcal{D}^{(n+1)})$. We will also show that $\Psi(v^+; \mathcal{D}^{(n+1)}) \ge v^+$. Therefore, by Lemma A.1, $\Psi^k(v^{(n)}; \mathcal{D}^{(n+1)}) \ge v^+$ for all $k \ge 0$. But we showed in the proof of Theorem 3.13.(i) that the sequence $\left(\Psi^k(v^{(n)}; \mathcal{D}^{(n+1)})\right)_{k\ge 0}$ decreases monotonically to its limit $v^{(n+1)}$. Therefore, $v^{(n+1)} \ge v^+$, which completes the induction.
It remains to show that, indeed, $\Psi(v^+; \mathcal{D}^{(n+1)}) \ge v^+$ given the induction hypothesis above. This follows by Monotonicity Condition 3.7 as follows. Note that $\mathcal{D}^{(n+1)} = D(v^{(n)}) \subseteq D(v^+)$ since $v^{(n)} \ge v^+$ by the induction hypothesis. Moreover, for $j \in D(v^{(n)})$ we also have that $\gamma v_j^+ \le v_j^{(n)} < \bar{L}_j^{(s)} \le \bar{L}_j$, and so $\Pi_{ji}(\bar{L}_j \wedge \gamma v_j^+) = \gamma \Pi_{ji} v_j^+$. Then, for any $i \in \mathcal{N}$ we have that

$$\begin{aligned} \Psi_i(v^+; \mathcal{D}^{(n+1)}) &= a_i + \sum_{j\in\mathcal{N}\setminus D(v^{(n)})} L_{ji}^{(s)} + \sum_{j\in D(v^{(n)})} \gamma\Pi_{ji} v_j^+ \\ &= a_i + \sum_{j\in\mathcal{N}\setminus D(v^+)} L_{ji}^{(s)} + \sum_{j\in D(v^+)} \gamma\Pi_{ji} v_j^+ \\ &\quad + \sum_{j\in D(v^+)\setminus D(v^{(n)})} \left( L_{ji}^{(s)} - \underbrace{\gamma\Pi_{ji}}_{\le \Pi_{ji}^{(s)}} \underbrace{v_j^+}_{<\bar{L}_j^{(s)}} \right) \\ &\ge a_i + \sum_{j\in\mathcal{N}\setminus D(v^+)} L_{ji}^{(s)} + \sum_{j\in D(v^+)} \gamma\Pi_{ji} v_j^+ \\ &= \tilde{\Psi}_i(v^+) = v_i^+. \end{aligned}$$

■

*Proof of Proposition* 3.14. Let $(a, L^{(s)}, L^{(l)}; 1)$ be as in the proof of Proposition 3.11(2), where, as mentioned above, Monotonicity Condition 3.7 fails. Algorithm 2 would fail to terminate since the sequences $v^{(n)}$ and $D(v^{(n)})$ would evolve as follows:

$$\begin{aligned}
v^{(0)} &= (3, 100, 110), & D(v^{(0)}) &= \{1\},\\
v^{(1)} &= (3, 99.5, 109.5), & D(v^{(1)}) &= \{1, 2\},\\
v^{(2)} &\approx (68.43, 132.21, 93.43), & D(v^{(2)}) &= \emptyset,\\
v^{(3)} &= (3, 100, 110), & D(v^{(3)}) &= \{1\},\\
&\dots
\end{aligned}$$

and it is clear that this sequence would not terminate. ■

*Proof of Proposition* 3.15. Since $L^{(l)} = \mathbf{Z}$, we have that $\bar{L} = \bar{L}^{(s)}$ and $\Pi = \Pi^{(s)}$.

1. For $i \in \mathcal{N}$ we have

$$\begin{aligned}
v_i &= a_i + \sum_{j \in \mathcal{N} \setminus D(v)} \underbrace{L^{(s)}_{ji}}_{= \Pi_{ji} \underbrace{\bar{L}^{(s)}_j}_{= q_j}} + \sum_{j \in D(v)} \Pi_{ji} \underbrace{\gamma v_j}_{= q_j}\\
&= a_i + \sum_{j \in \mathcal{N}} \Pi_{ji} q_j.
\end{aligned}$$

   Hence, $D(v) = \{i \in \mathcal{N} \mid a_i + \sum_{j \in \mathcal{N}} \Pi_{ji} q_j < \bar{L}_i\}$. Hence, for all $i \in D(v)$

$$q_i = \gamma v_i = \gamma a_i + \gamma \sum_{j \in \mathcal{N}} \Pi_{ji} q_j,$$

   and for all $i \in \mathcal{N} \setminus D(v)$ we have that $q_i = \bar{L}^{(s)}_i = L_i$. Hence, $q$ satisfies the fixed-point equation (6).

2. Let $q$ be a solution to (6). We show that $v = a + \Pi^\top q$ is a functional liquid asset vector, i.e., $\tilde{\Psi}(v) = v$. Note that $D(v) = \{i \in \mathcal{N} \mid a_i + \sum_{j \in \mathcal{N}} \Pi_{ji} q_j < \bar{L}_i\}$. Therefore, for all $i \in \mathcal{N}$

$$\begin{aligned}
\tilde{\Psi}_i(v) &= a_i + \sum_{j \in \mathcal{N} \setminus D(v)} \underbrace{L^{(s)}_{ji}}_{= \Pi_{ji} \underbrace{\bar{L}^{(s)}_j}_{= q_j}} + \sum_{j \in D(v)} \Pi_{ji} \underbrace{\gamma v_j}_{= q_j}\\
&= a_i + \sum_{j \in \mathcal{N}} \Pi_{ji} q_j = v_i.
\end{aligned}$$

■

*Proof of Proposition* 3.16. Since $L^{(l)} = \mathbf{Z}$, we have that $L = L^{(s)}$, $\bar{L} = \bar{L}^{(s)}$, and $\Pi = \Pi^{(s)}$. The result follows directly from Proposition 3.15 with $\gamma = 1$.

1. Let $v$ be a functional liquid asset vector and $D(v) = \{i \in \mathcal{N} \mid v_i < \bar{L}^{(s)}_i\}$. Hence, with $\gamma = 1$ in Proposition 3.15, $q = \bar{L}^{(s)} \wedge v$. Furthermore, the fixed-point equation (6) simplifies to $q = \bar{L}^{(s)} \wedge a + \Pi^\top q$, which is exactly (7), and hence the result follows.

2. Similarly, since the fixed-point equations (6) and (7) coincide for $\gamma = 1$ the result follows directly from Proposition 3.15. ■

The following lemma is used in the proof of Lemma 3.17 below.

Lemma A.2. *Suppose* $\Pi \in \mathbb{R}_+^{N\times N}$ *is a row-substochastic matrix,* $0 \le \rho \le 1$ *its spectral radius, and* $0 \le \gamma \le 1$ *a constant.*

(i) *If* $\gamma < 1$ *or* $\rho < 1$*, then the matrix* $(\mathrm{I} - \gamma\Pi^\top)$ *is invertible and* $(\mathrm{I} - \gamma\Pi^\top)^{-1}$ *is nonnegative.*

(ii) *If* $\gamma = 1$ *and* $\rho = 1$*, then there exists a set* $\mathcal{C} \subseteq \mathcal{N}$ *such that for all* $i \in \mathcal{C}$ *we have that* $\sum_{j\in\mathcal{C}} \Pi_{ij} = 1$*.*

*Proof.* (i) If $\gamma = 0$, then $\left(\mathrm{I} - \gamma\Pi^\top\right) = \mathrm{I}$, which is clearly invertible with a nonnegative inverse. So we assume that $0 < \gamma$.
Since $\rho$ is the spectral radius of $\Pi$, it is also the spectral radius of $\Pi^\top$. Since $\Pi$ is a row-substochastic matrix, we have that $\rho \le 1$. As $\Pi^\top$ is nonnegative, standard results for M-matrices (see, for example, Theorem 2.5.3.2 and 2.5.3.17 in [26]) imply that $(\alpha\mathrm{I} - \Pi^\top)$ is invertible with a nonnegative inverse if and only if $\alpha > \rho$. Set $\alpha = \gamma^{-1} > 0$. If $\gamma < 1$, then $\alpha > 1 \ge \rho$, and if $\gamma = 1$ but $\rho < 1$, then $\alpha = 1 > \rho$. Hence $\left(\mathrm{I} - \gamma\Pi^\top\right) = \alpha^{-1}\left(\alpha\mathrm{I} - \Pi^\top\right)$ is invertible with a nonnegative inverse.

(ii) As a standard result in the theory of finite-state Markov chains (see, for example, Theorem 2.1 in [28]), the number of sets $\mathcal{C} \subseteq \mathcal{N}$ satisfying the property that for all $i \in \mathcal{C}$ $\sum_{j\in\mathcal{C}} \Pi_{ij} = 1$ is equal to the multiplicity of the eigenvalue 1 of $\Pi$. Since $\rho = 1$ by assumption, the multiplicity must be at least 1, and hence at least one such set $\mathcal{C}$ must exist. ■

*Proof of Lemma* 3.17. The system of $m$ linear equations has a unique solution, $x \in \mathbb{R}_+^m$, if it can be expressed as

$$x = \left(\mathrm{I} - \gamma\left(\Pi_{\mathcal{D}\mathcal{D}}\right)^\top\right)^{-1} b,$$

where $(\mathrm{I} - \gamma\left(\Pi_{\mathcal{D}\mathcal{D}}\right)^\top)$ is invertible.

We note that $\Pi_{\mathcal{D}\mathcal{D}}$ is a row-substochastic matrix. By Lemma A.2.(i), we only need to consider the case where $\gamma = 1$ and the spectral radius of $\Pi_{\mathcal{D}\mathcal{D}}$ is exactly 1. In this case, by Lemma A.2.(ii), there is a set $\mathcal{C} \subseteq \mathcal{D}$ such that $\sum_{j\in\mathcal{C}} \Pi_{ij} = 1$ for each $i \in \mathcal{C}$. By assumption, if $\gamma = 1$, then $b > 0$, and so

$$\begin{aligned} x_i &= b_i + \sum_{j\in\mathcal{D}} \Pi_{ji} x_j \\ &\ge b_i + \sum_{j\in\mathcal{C}} \Pi_{ji} x_j > \sum_{j\in\mathcal{C}} \Pi_{ji} x_j. \end{aligned}$$

By summing $x_i$ for all $i \in \mathcal{C}$, we arrive at the contradiction

$$\sum_{i\in\mathcal{C}} x_i > \sum_{j\in\mathcal{C}} x_j \left(\sum_{j\in\mathcal{C}} \Pi_{ji}\right) = \sum_{j\in\mathcal{C}} x_j.$$

Thus $\gamma < 1$ or $\rho < 1$, and so $(\mathrm{I} - \gamma(\Pi_{\mathcal{D}^{(n)}\mathcal{D}^{(n)}})^\top)$ is invertible and $x$ is the well defined and unique solution to the system of linear equations.

Nonnegativity of $x$ follows by Lemma A.2.(i). ■

*Proof of Proposition* 3.18. By Theorem 3.13, $\mathcal{D}^{(n)} \subseteq \mathcal{D}^{(n+1)} = D(v^{(n)})$, and, under Monotonicity Condition 3.7, $v^{(n)}$ is a fixed point of $\Psi(\cdot; \mathcal{D}^{(n)})$. Then for all $j \in \mathcal{D}^{(n)}$ we have that $\bar{L}_j \wedge \gamma v_j^{(n)} = \gamma v_j^{(n)}$. Therefore, the fixed-point problem (5) in Algorithm 2 is in fact a system of linear equations:

$$v_i^{(n)} = \Psi_i(v^{(n)}; \mathcal{D}^{(n)}) = a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(n)}} L_{ji}^{(s)} + \gamma \sum_{j \in \mathcal{D}^{(n)}} \Pi_{ji} v_j^{(n)} \tag{15}$$

for $i \in \mathcal{N}$. Moreover, it is sufficient to consider (15) only for $i \in \mathcal{D}^{(n)}$. Indeed, if $x \in \mathbb{R}_+^m$, where $m := |\mathcal{D}^{(n)}|$, is some such solution, then we can simply set $v_i^{(n)} := x_i$ for $i \in \mathcal{D}^{(n)}$ and $v_i^{(n)} := a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(n)}} L_{ji}^{(s)} + \gamma \sum_{j \in \mathcal{D}^{(n)}} \Pi_{ji} x_j$ for $i \in \mathcal{N} \setminus \mathcal{D}^{(n)}$.

Setting $b_i := a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(n)}} L_{ji}^{(s)}$ for each $i \in \mathcal{D}^{(n)}$, we note that $b_i \geq a_i > 0$ for all $i \in \mathcal{D}^{(n)}$. Therefore, by Lemma 3.17, $x$ is a unique solution to the system of linear equations (15) for $i \in \mathcal{D}^{(n)}$. In particular, letting $\mathcal{L}^{(n)} := \mathcal{N} \setminus \mathcal{D}^{(n)}$, we can write

$$x = \left(\mathrm{I} - \gamma \left(\Pi_{\mathcal{D}^{(n)}\mathcal{D}^{(n)}}\right)^\top\right)^{-1} b,$$

where $(\mathrm{I} - \gamma(\Pi_{\mathcal{D}^{(n)}\mathcal{D}^{(n)}})^\top)$ is invertible and $b = a_{\mathcal{D}^{(n)}} + (L^{(s)}_{\mathcal{L}^{(n)}\mathcal{D}^{(n)}})^\top \mathbf{1}_{\mathcal{L}^{(n)}}$.

Nonnegativity of $v^{(n)}$ then follows by Lemma 3.17 and the fact that $\Psi$ is nondecreasing (Lemma A.1). ■

*Proof of Proposition* 3.19. To simplify the notation we set $m := |\mathcal{D}^{(n)}|$ and in this proof assume that whenever, for some $i$, we let $1 \leq i \leq m$, that means that $i \in \mathcal{D}^{(n)}$. In this context, if $i = m + 1$, then $i \notin \mathcal{D}^{(n)}$. Moreover, we set $\mathcal{L}^{(n)} := \mathcal{N} \setminus \mathcal{D}^{(n)}$ and let $b \in \mathbb{R}_+^{m+1}$, $\Lambda^{(s)} \in \mathbb{R}_+^{(m+1)\times(m+1)}$ be given by

$$b = \left(\begin{array}{c} a_1 + \sum_{j \in \mathcal{L}^{(n)}} L_{j1}^{(s)} \\ \vdots \\ a_m + \sum_{j \in \mathcal{L}^{(n)}} L_{jm}^{(s)} \\ \hline 0 \end{array}\right), \quad \Lambda^{(s)} = \left(\begin{array}{ccc|c} L_{11} & \dots & L_{1m} & \frac{\bar{L}_1}{\gamma} - \sum_{k=1}^m L_{1k} \\ \vdots & & & \vdots \\ L_{m1} & \dots & L_{mm} & \frac{\bar{L}_m}{\gamma} - \sum_{k=1}^m L_{mk} \\ \hline 0 & \dots & 0 & 0 \end{array}\right).$$

It is clear that $\mathbf{Z}$, the $(m + 1) \times (m + 1)$ zero matrix, is a liability matrix. To see that $\Lambda^{(s)}$ is a liability matrix, we need to check that the last column is nonnegative and all other properties follow immediately from the definition. For all $i \in \{1, \dots, m\}$ we have $\gamma \sum_{k=1}^m L_{ik} \leq \sum_{k=1}^m L_{ik} \leq \sum_{k=1}^N L_{ik} = \bar{L}_i$. Since $\bar{L}_i \geq \gamma \sum_{k=1}^m L_{ik} \Leftrightarrow \frac{\bar{L}_i}{\gamma} - \sum_{k=1}^m L_{ik} \geq 0$, the last column is indeed nonnegative.

So we define a financial system $S_n := (b, \Lambda^{(s)}, \mathbf{Z}; 1)$ on the set of $m + 1$ banks containing $\mathcal{D}^{(n)}$.

Since $S_n$ has no long-term liabilities, we denote both the short-term and overall total nominal liabilities vector of $S_n$ by $\bar{\Lambda}$ and we immediately see that $\bar{\Lambda}_i = \frac{1}{\gamma} \bar{L}_i$ for $1 \leq i \leq m$

and $\bar{\Lambda}_{m+1} = 0$. Moreover, the short-term and overall relative liability matrices of $S_n$ are also the same. Denoting them by $\Theta^{(s)}$ and $\Theta$, respectively, we have that $\Theta^{(s)} = \Theta \geq 1 \cdot \Theta$, and so Monotonicity Condition 3.7 is satisfied. Note that for $1 \leq i, j \leq m$ we have

$$\Theta_{ij} = \frac{\Lambda_{ij}}{\bar{\Lambda}_i} = \frac{\gamma L_{ij}}{\bar{L}_i} = \gamma \Pi_{ij}.$$

Suppose that $x \in \mathbb{R}_+^{m+1}$ is some functional liquid asset vector of $S_n$ with respect to

$$D(x) = \{i \in \{1, \dots, m, m+1\} \mid x_i < \bar{\Lambda}_i\} = \{i \in \mathcal{D}^{(n)} \mid \gamma x_i < \bar{L}_i\},$$

where we used the convention that the $m$ elements of $\mathcal{D}^{(n)}$ are labeled by $1, \dots, m$, and the last equality holds because $\bar{\Lambda}_{m+1} = 0$ and hence the index $m+1$ will never be in the default set.

Since $x$ is a functional liquid asset vector, we have that $x_i \leq \bar{\Lambda}_i = \frac{1}{\gamma}\bar{L}_i$ and hence $\Theta_{ji} x_j \leq L_{ji}$ for $i, j \in \mathcal{D}^{(n)}$. Moreover, $\Lambda^{(s)}_{m+1 i} = 0$ for all $i \in \mathcal{D}^{(n)}$ and hence we have for each $i \in \mathcal{D}^{(n)}$

$$\begin{aligned} x_i = \tilde{\Psi}_i(x) &= b_i + \sum_{j \in \mathcal{D}^{(n)} \setminus D(x)} \Lambda^{(s)}_{ji} + 1 \cdot \sum_{j \in D(x)} \Theta_{ji} x_j \\ &= a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(n)}} L^{(s)}_{ji} + \sum_{j \in \mathcal{D}^{(n)} \setminus D(x)} L_{ji} + \sum_{j \in D(x)} \gamma \Pi_{ji} x_j \\ &= a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(n)}} L^{(s)}_{ji} + \sum_{j \in \mathcal{D}^{(n)} \setminus D(x)} \Pi_{ji} \bar{L}_j + \gamma \sum_{j \in D(x)} \Pi_{ji} x_j \\ &= a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(n)}} L^{(s)}_{ji} + \sum_{j \in \mathcal{D}^{(n)}} \Pi_{ji} (\bar{L}_j \wedge \gamma x_j) \\ &= \Psi_i(x; \mathcal{D}^{(n)}). \end{aligned} \tag{16}$$

Then, we set

$$v_i^{(n)} = \begin{cases} x_i & \text{for } i \in \mathcal{D}^{(n)}, \\ a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(n)}} L^{(s)}_{ji} + \sum_{j \in \mathcal{D}^{(n)}} \Pi_{ji}(\bar{L}_j \wedge \gamma x_j) & \text{for } i \in \mathcal{N} \setminus \mathcal{D}^{(n)}. \end{cases} \tag{17}$$

Note that $\Psi(v^{(n)}; \mathcal{D}^{(n)})$ does not depend on $v_i^{(n)}$ for $i \in \mathcal{N} \setminus \mathcal{D}^{(n)}$. Hence, from (16) we immediately see that $v_i^{(n)} = \Psi_i(v^{(n)}; \mathcal{D}^{(n)})$ for all $i \in \mathcal{D}^{(n)}$.

Furthermore, for all $i \in \mathcal{N} \setminus \mathcal{D}^{(n)}$ we have by (17) that $v_i^{(n)} = \Psi_i(v^{(n)}; \mathcal{D}^{(n)})$. Hence we have shown that $\Psi((v^{(n)}); \mathcal{D}^{(n)}) = v^{(n)}$. ∎

*Proof of Proposition* 3.21. Let $S_1 = (a, L^{(s)}, L^{(l)}; 1)$ denote the financial system introduced in the proof of Proposition 3.11(2) and also used in the proof of Proposition 3.14 above. In Algorithm 1, using the construction in Proposition 3.19, the sequences $v^{(n)}$ and $D(v^{(n)})$ would evolve as follows:

$$v^{(0)} = (3, 100, 110), \qquad D(v^{(0)}) = \{1\},$$
$$v^{(1)} = \left(3, 99\frac{1}{2}, 109\frac{1}{2}\right), \qquad D(v^{(1)}) = \{1, 2\},$$
$$v^{(2)} = \left(53\frac{1}{50}, 102, 65\frac{1}{25}\right), \qquad D(v^{(2)}) = \{2\}.$$

Thus we conclude that $v^* = v^{(2)}$ and $\mathcal{D}_1^* = \{1, 2\}$.

Now let $S_2 = (a, L^{(s)} + L^{(l)}, \mathbf{Z}; 1)$. Then we can verify that the vector $v^*$, obtained above, is also the unique functional liquid asset vector of $S_2$ with the functional default set $D(v^*) = \{2\}$. By Remark 3.9, $S_2$ satisfies Monotonicity Condition 3.7, and hence by Theorem 3.13 $\mathcal{D}_2^* := \{2\}$ is the algorithmic default set of $S_2$. ■

*Proposition of* 4.1. We need to show that $a(T_1)$ is nonnegative and $L^{(s)}(T_1)$ and $L^{(l)}(T_1)$ are liability matrices.

By construction of Algorithm 1 $v^*(T_1) = \Psi(v^*(T_1); \mathcal{D}^*(T_1))$ such that $D(v^*(T_1)) \subseteq \mathcal{D}^*(T_1)$. Suppose there is some $i \in \mathcal{N}(T_1)$ such that $v^*(T_1)_i < \bar{L}_i^{(s)}$. Then $i \in D(v^*(T_1))$, and so $i \notin \mathcal{N}(T_1)$. Hence, for all $i \in \mathcal{N}(T_1)$, $a(T_1)_i = v^*(T_1)_i - \bar{L}_i^{(s)} \geq 0$.

The fact that $L^{(s)}(T_1)$ and $L^{(l)}(T_1)$ are liability matrices follows from the definitions since it is immediately clear that they are nonnegative matrices with zero diagonals. ■

## REFERENCES


[1] F. Allen and D. Gale, *Financial contagion*, J. Political Economy, 108 (2000), pp. 1–33.

[2] H. Amini, D. Filipović, and A. Minca, *Systemic Risk and Central Clearing Counterparty Design*, Swiss Finance Institute Research Paper 13-34, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2275376, 2015.

[3] T. Banerjee, A. Bernstein, and Z. Feinstein, *Dynamic Clearing and Contagion in Financial Networks*, preprint, https://arxiv.org/abs/1801.02091, 2018.

[4] P. Barucca, M. Bardoscia, F. Caccioli, M. D'Errico, G. Visentin, S. Battiston, and G. Caldarelli, *Network valuation in financial systems*, preprint, SSRN, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2795583, 2016.

[5] S. Battiston, D. Delli Gatti, M. Gallegati, B. Greenwald, and J. E. Stiglitz, *Liaisons dangereuses: Increasing connectivity, risk sharing, and systemic risk*, J. Econom. Dynam. Control, 36 (2012), pp. 1121–1141.

[6] D. Bisias, M. Flood, A. W. Lo, and S. Valavanis, *A survey of systemic risk analytics*, Ann. Rev. Financial Econom., 4 (2012), pp. 255–296.

[7] A. Capponi and P.-C. Chen, *Systemic risk mitigation in financial networks*, J. Econom. Dynam. Control, 58 (2015), pp. 152–166.

[8] A. Capponi, W. A. Cheng, and S. Rajan, *Firm Capital Dynamics in Centrally Cleared Markets*, preprint, SSRN, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2542684, 2014.

[9] R. Cifuentes, G. Ferrucci, and H. S. Shin, *Liquidity risk and contagion*, J. European Econom. Assoc., 3 (2005), pp. 556–566.

[10] R. Cont and T. Kokholm, *Central clearing of OTC derivatives: Bilateral vs. multilateral netting*, Statist. Risk Model., 31 (2014), pp. 3–22.

[11] R. Cont, A. Moussa, and E. B. Santos, *Network structure and systemic risk in banking systems*, in Handbook of Systemic Risk, J.-P. Fouque and J. A. Langsam, eds., Cambridge University Press, Cambridge, UK, 2013, pp. 327–368.

[12] A. David and A. Lehar, *Imperfect renegotiations in interbank financial networks*, Management Sci., to appear, https://doi.org/10.1287/mnsc.2017.2869.

[13] D. Duffie, M. Scheicher, and G. Vuillemey, *Central clearing and collateral demand*, J. Financial Econom., 116 (2015), pp. 237–256.

[14] L. Eisenberg and T. H. Noe, *Systemic risk in financial systems*, Management Sci., 47 (2001), pp. 236–249.

[15] M. Elliott, B. Golub, and M. O. Jackson, *Financial networks and contagion*, Amer. Econom. Rev., 104 (2014), pp. 3115–3153.

[16] H. Elsinger, *Financial Networks, Cross Holdings, and Limited Liability*, Oesterreichische Nationalbank Working Paper 156, 2011.

[17] H. Elsinger, A. Lehar, and M. Summer, *Network models and systemic risk assessment*, in Handbook on Systemic Risk, J.-P. Fouque and J. A. Langsam, eds., Cambridge University Press, Cambridge, UK, 2013, pp. 287–305.

[18] Z. Feinstein, *Obligations with Physical Delivery in a Multi-Layered Financial Network*, preprint, https://arxiv.org/abs/1702.07936, 2017.

[19] G. Ferrara, S. Langfield, Z. Liu, and T. Ota, *Systemic Illiquidity in the Interbank Network*, Tech. report 586, Bank of England, 2016.

[20] P. Gai, A. Haldane, and S. Kapadia, *Complexity, concentration and contagion*, J. Monetary Econom., 58 (2011), pp. 453–470.

[21] A. Gandy and L. A. M. Veraart, *A Bayesian methodology for systemic risk assessment in financial networks*, Management Sci., 63 (2017), pp. 4428–4446, https://doi.org/10.1287/mnsc.2016.2546.

[22] A. Gandy and L. A. M. Veraart, *Adjustable network reconstruction with applications to CDS exposures*, J. Multivariate Anal., to appear, https://doi.org/10.1016/j.jmva.2018.08.011.

[23] C.-P. Georg, *The effect of the interbank network structure on contagion and common shocks*, J. Banking Finance, 37 (2013), pp. 2216–2228.

[24] P. Glasserman and H. P. Young, *How likely is contagion in financial networks?*, J. Banking Finance, 50 (2014), pp. 383–399.

[25] P. Glasserman and H. P. Young, *Contagion in financial networks*, J. Econom. Lit., 54 (2016), pp. 779–831.

[26] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1991.

[27] T. Hurd, *Contagion! The Spread of Systemic Risk in Financial Networks*, Springer, New York, 2016.

[28] S. Karlin and H. E. Taylor, *A Second Course in Stochastic Processes*, Elsevier, New York, 1981.

[29] S. Langfield, Z. Liu, and T. Ota, *Mapping the UK interbank system*, J. Banking Finance, 45 (2014), pp. 288–303.

[30] M. Liu, M. Feng, and J. Staum, *Systemic risk components in a network model of contagion*, IIE Trans., 48 (2012), pp. 501–510.

[31] L. C. G. Rogers and L. A. M. Veraart, *Failure and rescue in an interbank network*, Management Sci., 59 (2013), pp. 882–898.

[32] I. M. Sonin and K. Sonin, *Banks as Tanks: A Continuous-Time Model of Financial Clearing*, preprint, https://arxiv.org/abs/1705.05943, 2017.

[33] The Insolvency Service, *The Insolvency Service Technical Manual*, UK Government, 2010. Available at https://www.insolvencydirect.bis.gov.uk/technicalmanual/Ch25-36/Chapter%2036A/Ch%2036%20Intro.htm.

[34] C. Upper, *Simulation methods to assess the danger of contagion in interbank markets*, J. Financial Stability, 7 (2011), pp. 111–125.

[35] L. A. M. Veraart, *Distress and Default Contagion in Financial Networks*, preprint, SSRN, https://ssrn.com/abstract=3029702, 2017.

[36] S. Weber and K. Weske, *The joint impact of bankruptcy costs, fire sales and cross-holdings on systemic risk in financial networks*, Probab. Uncertain. Quant. Risk, 2 (2017), 9.