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Article (Published version)
(Refereed)

Original citation:
Gapeev, Pavel V. and Al Motairi, Hessah (2018) Perpetual American defaultable options in models with random dividends and partial information. Risks, 64 (4). ISSN 2227-9091

DOI: https://doi.org/10.3390/risks6040127 risks

MDPI

Article

Perpetual American Defaultable Options in Models with Random Dividends and Partial Information

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Received: 30 December 2017; Accepted: 25 October 2018; Published: 6 November 2018

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Abstract: We present closed-form solutions to the perpetual American dividend-paying put and call option pricing problems in two extensions of the Black–Merton–Scholes model with random dividends under full and partial information. We assume that the dividend rate of the underlying asset price changes its value at a certain random time which has an exponential distribution and is independent of the standard Brownian motion driving the price of the underlying risky asset. In the full information version of the model, it is assumed that this time is observable to the option holder, while in the partial information version of the model, it is assumed that this time is unobservable to the option holder. The optimal exercise times are shown to be the first times at which the underlying risky asset price process hits certain constant levels. The proof is based on the solutions of the associated free-boundary problems and the applications of the change-of-variable formula.

Keywords: perpetual American options; random dividends; optimal stopping problem; Brownian motion; hidden Markov chain; filtering estimate; innovation process; free-boundary problem; a change-of-variable formula with local time on surfaces

MSC: primary 60G40; 34K10; 91B70; secondary 60J60; 60J75; 91B28

1. Introduction

The main aim of this paper is to present closed-form solutions to the discounted optimal stopping problems of Equations (11) and (12) for the processes (S, Θ) and (S, Π) defined in Equations (1) and (2) and (3)–(5). These problems are related to the option pricing theory in mathematical finance, where the process S can describe the price of the underlying risky asset (e.g., a stock) on a financial market, while the process Θ reflects the current state of the economy, and Π represents its filtering estimate based on the underlying risky asset price observations. The values of Equations (11) and (12) can therefore be interpreted as the rational (or no-arbitrage) ex-dividend prices of perpetual American dividend-paying put and call options in certain extensions of the Black–Merton–Scholes model with random dividends under full and partial information (see, e.g., Shiryaev 1999, chp. VIII, sct. 2a; Peskir and Shiryaev 2006, chp. VII, sct. 25; or Detemple 2006 for an extensive overview of other related results in the area).

The models of financial markets in which the parameter values are switching according to the dynamics of continuous-time Markov chains have recently been considered in the literature. Guo (2001) and Guo and Zhang (2004) obtained closed-form solutions to the perpetual American lookback and put option pricing problems in an extension of the Black–Merton–Scholes model in which both the drift and volatility coefficients of the underlying asset price process are switching between two constant values, according to the change in the state of the observable continuous-time Markov chain.

Jobert and Rogers (2006) considered the perpetual American put option problem within an extension of that model to the case of several states for the Markov chain and solved the corresponding problem with finite expiry numerically. Dalang and Hongler (2004) presented a complete and essentially explicit solution to a similar problem in a model with a two-state Markov chain and no diffusion part. Jiang and Pistorius (2008) extended these results and studied the perpetual American put option problem within the framework of an exponential jump-diffusion model with observable dynamics of regime-switching behaving parameters. A similar model for the pricing of European options, in which the underlying dividend process is given by a diffusion process with Markov-modulated coefficients, was considered by Di Graziano and Rogers (2009) (see also other related references therein).

In this paper, we consider an extension of the Black–Merton–Scholes model in which the dividend rate changes from one constant value to another at some random time which has an exponential distribution under a risk-neutral (or martingale) probability measure. We reduce the original perpetual American option pricing problems to optimal stopping problems for two-dimensional continuous-time Markov processes and derive the closed-form solutions of the associated free-boundary problems. In the version of the model with full information, it is assumed that the time of change is observable and the indicator of the occurrence of the change represents a continuous-time Markov chain with two states, so that the original optimal stopping problem is equivalent to a free-boundary problem with a system of two ordinary differential equations which is solvable in a closed form. In the version of the model with partial information, it is assumed that the time of change is unobservable and the filtering estimate of the indicator of the occurrence of the change represents a continuous diffusion process, so that the original optimal stopping problem is equivalent to a free-boundary problem with a partial differential equation of parabolic (or degenerate elliptic) type. Optimal stopping games with various information flows were recently studied by Gapeev and Rodosthenous (2018) in the framework of such models with random dividends under full and partial information.

It turns out that the filtering estimate of the indicator of the occurrence of the change in the dividend rate of the underlying risky asset process represents the posterior probability of the occurrence of a change point in the associated quickest detection problems (see, e.g., Shiryaev 1978, chp. IV, sct. 4; and Peskir and Shiryaev 2006, chp. VI, sct. 22). Furthermore, because of the specific structure of the considered posterior probability process, the underlying risky asset price process becomes a Markovian sufficient statistic in the associated initially two-dimensional optimal stopping problem. The property of reduction of the dimension of the space of sufficient statistics was earlier observed in certain optimal stopping problems arising in quickest detection theory. Shiryaev (1964) and Poor (1998) proved that the weighted likelihood ratio process turns out to be a one-dimensional Markovian sufficient statistic in the quickest detection problem for sequences of i.i.d. observations with exponential delay penalty. This idea was further applied by Beibel (2000) for the solution of the appropriate problem of detecting a change in the drift rate of an observable Wiener process as a generalised parking problem.

In this paper, due to the specific structure of the stochastic differential equation for the posterior probability process and its contribution to the partial differential operator, the associated parabolic-type free-boundary problem becomes equivalent to an ordinary one which is solvable in a closed form. Bayraktar and Dayanik (2006) recognised such a property from the structure of the partial differential-difference equation in the free-boundary problem associated with the Bayesian problem of detecting a change in the constant intensity rate of an observable Poisson process with the exponential delay penalty. More recently, Gapeev and Shiryaev (2013) applied similar techniques for the solution of the parabolic-type free-boundary problem associated with the Bayesian problem of detecting a change in the drift rate function of an observable diffusion process within the same delay penalty framework.

The paper is organised as follows. In Section 2, we formulate the optimal stopping problems for two-dimensional Markov processes related to the rational pricing of the perpetual American dividend-paying options in the extensions of the Black–Merton–Scholes model with random dividends described above under full and partial information. In Section 3, we derive closed-form solutions of the associated free-boundary problems for the value functions and the optimal stopping boundaries in the both versions of the model, with full and partial information. In Section 4, by applying the change-of-variable formula with local times on surfaces from Peskir (2007), we verify that the solutions of the resulting free-boundary problems provides the solutions of the original optimal stopping problems. The main results of the paper are stated in Propositions 1 and 2.

2. Formulation of the Problems

In this section, we introduce the setting and notation of the two-dimensional optimal stopping problems, which are related to the pricing of perpetual American dividend-paying put and call options, and formulate the associated free-boundary problems.

2.1. The Model

Let us consider a probability space $(\Omega, \mathcal{G}, P)$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$ and a random time θ with the conditionally exponential distribution $P(\theta = 0) = \pi$, $P(\theta > t \,|\, \theta > 0) = e^{-\lambda t}$, for some $\lambda > 0$ and $\pi \in [0,1]$ fixed (B and θ are supposed to be independent). Assume that there exists a process $S = (S_t)_{t \geq 0}$ given by

$$S_t = s \, \exp\Big(\big(r - \delta_0 - \sigma^2/2\big)\, t - (\delta_1 - \delta_0)\,(t - \theta)^+ + \sigma\, B_t \Big) \tag{1}$$

which solves the stochastic differential equation:

$$dS_t = \big(r - \delta_0 - (\delta_1 - \delta_0)\, \Theta_t\big)\, S_t\, dt \; + \sigma\, S_t\, dB_t \quad (S_0 = s) \tag{2}$$

where $s > 0$ is fixed, and $r > 0$, $\delta_i > 0$, for $i = 0, 1$, and $\sigma > 0$ are some given constants. Here, we set $\Theta_t = I(\theta \leq t)$, for all $t \geq 0$, where $I(\cdot)$ denotes the indicator function. In this case, the process $\Theta = (\Theta_t)_{t \geq 0}$ is a continuous-time Markov chain with the initial distribution $\{1 - \pi, \pi\}$, the transition-probability matrix $\{e^{-\lambda t}, 1 - e^{-\lambda t}; 0, 1\}$, and the intensity-matrix $\{-\lambda, \lambda; 0, 0\}$, for all $t \geq 0$, and some $\pi \in [0,1]$ and $\lambda > 0$ fixed. Suppose that the process S describes the price of a risky asset on a financial market, where r is the riskless interest rate, δ_i, for $i = 0, 1$, are the dividend rates paid to the asset holders, and σ is the volatility rate. We may also assume that Θ reflects the behavior of the market state, towards 0 when the market is in the so-called "good" state or towards 1 when the market is in the so-called "bad" state.

It is shown by means of standard arguments (see, e.g., Liptser and Shiryaev [1977] 2001, chp. IX; or Elliott et al. 1995, chp. VIII) that the asset price process S from Equations (1) and (2) admits the representation:

$$dS_t = \big(r - \delta_0 - (\delta_1 - \delta_0)\, \Pi_t\big)\, S_t\, dt + \sigma\, S_t\, d\overline{B}_t \quad (S_0 = s), \tag{3}$$

and the filtering estimate $\Pi = (\Pi_t)_{t \geq 0}$ defined by $\Pi_t = E[\Theta_t \,|\, \mathcal{F}_t] \equiv P(\theta \leq t \,|\, \mathcal{F}_t)$ solves the stochastic differential equation:

$$d\Pi_t = \lambda\, (1 - \Pi_t)\, dt - \frac{\delta_1 - \delta_0}{\sigma}\, \Pi_t (1 - \Pi_t)\, d\overline{B}_t \quad (\Pi_0 = \pi) \tag{4}$$

for some $(s, \pi) \in (0, \infty) \times [0,1]$ fixed. It follows from the result of (Liptser and Shiryaev [1977] 2001, Theorem 8.3) that the innovation process $\overline{B} = (\overline{B}_t)_{t \geq 0}$ defined by

$$\overline{B}_t = \int_0^t \frac{dS_u}{\sigma S_u} - \frac{1}{\sigma} \int_0^t \big(r - \delta_0 - (\delta_1 - \delta_0)\, \Pi_u\big)\, du \tag{5}$$

is a standard Brownian motion under the probability measure P with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$, according to P. Lévy's characterisation theorem (see, e.g., Liptser and Shiryaev [1977] 2001, Theorem 4.1; and Revuz and Yor 1999, chp. IV, Theorem 3.6). It can be verified that (S, Π) is a (time-homogeneous strong) Markov process under P with respect to its natural filtration $(\mathcal{F}_t)_{t \geq 0}$ as a unique strong solution of the system of stochastic differential equations in Equations (3) and (4) (see, e.g., Øksendal 1998, Theorem 7.2.4).

The main purpose of the present paper is to compute the values of the following optimal stopping problems:

$$U_j^*(s,\pi) = \sup_{\tau} E\big[e^{-r\tau}\, G_j(S_\tau)\, I(\tau < \theta) + e^{-r\theta}\,(\varphi_j + \psi_j\, S_\theta)\, I(\theta \le \tau) + (1 - e^{-r(\tau \wedge \theta)})\, \nu_j\big] \tag{6}$$

and

$$V_j^*(s,\pi) = \sup_{\zeta} E\big[e^{-r\zeta}\, G_j(S_\zeta)\, I(\zeta < \theta) + e^{-r\theta}\,(\varphi_j + \psi_j\, S_\theta)\, I(\theta \le \zeta) + (1 - e^{-r(\zeta \wedge \theta)})\, \nu_j\big] \tag{7}$$

with $G_1(s) = K_1 - s$ and $G_2(s) = s - K_2$, for all $s > 0$ and every $j = 1,2$. The supremum in Equation (6) is taken over all stopping times τ of the natural filtration $(\mathcal{G}_t)_{t\ge 0}$ of the process (S,Θ), while the supremum in Equation (7) is taken over all stopping times ζ of the natural filtration $(\mathcal{F}_t)_{t\ge 0}$ of the process S. Since we assume that the initial probability measure P is a martingale martingale measure (see, e.g., Shiryaev 1999, chp. VII, sct. 3g), the values of Equations (6) and (7) provide the rational (or no-arbitrage) ex-dividend prices of the perpetual American put and call options under full and partial information, respectively. Here, $K_j > 0$ is the strike price, $\varphi_j + \psi_j S$, is a (linear) recovery, and $\varkappa_j$ is the rate of promised continuously paid dividends, for some $\varphi_j > 0$, $\psi_j > 0$, and $\nu_j > 0$, and every $j = 1,2$. Contingent claims of European-type (finite-time horizon) with such a payoff and dividend structure were described in (Bielecki and Rutkowski 2004, sct. 2.1).

2.2. The Optimal Stopping Problems

It is shown by means of standard arguments that the value functions in Equations (6) and (7) admit the representations

$$U_j^*(s,\pi) = \sup_{\tau} E\bigg[e^{-r\tau}\, G_j(S_\tau)\,(1-\Theta_\tau) + \int_0^\tau e^{-rt}\,(\eta_j + \varkappa_j\, S_t)\,(1-\Theta_t)\, dt\bigg] \tag{8}$$

and

$$V_j^*(s,\pi) = \sup_{\zeta} E\bigg[e^{-r\zeta}\, G_j(S_\zeta)\,(1-\Pi_\zeta) + \int_0^\zeta e^{-rt}\,(\eta_j + \varkappa_j\, S_t)\,(1-\Pi_t)\, dt\bigg] \tag{9}$$

with $\eta_j = \lambda\varphi_j + \nu_j$ and $\varkappa_j = \lambda\psi_j$, for $j = 1,2$. Using the tower property for conditional expectations and taking into account the fact that the supremum in Equation (8) is taken over at all stopping times τ with respect to the filtration $(\mathcal{G}_t)_{t\ge 0}$, we may conclude that the value functions in Equation (8) can be expressed as

$$U_j^*(s,\pi) = U_j^*(s,0)\,(1-\pi) + U_j^*(s,1)\,\pi \tag{10}$$

for all $\pi \in [0,1]$ and every $j = 1,2$. In this respect, the problems in Equation (8) can be reduced to the optimal stopping problems for the (time-homogeneous strong) Markov process $(S,\Theta) = (S_t,\Theta_t)_{t\ge 0}$ given by

$$U_j^*(s,i) = \sup_{\tau} E_{s,i}\bigg[e^{-r\tau}\, G_j(S_\tau)\,(1-\Theta_\tau) + \int_0^\tau e^{-rt}\,(\eta_j + \varkappa_j\, S_t)\,(1-\Theta_t)\, dt\bigg], \tag{11}$$

while the problems in Equation (9) can be reduced to the optimal stopping problems for the (time-homogeneous strong) Markov process $(S,\Pi) = (S_t,\Pi_t)_{t\ge 0}$ given by

$$V_j^*(s,\pi) = \sup_{\zeta} E_{s,\pi}\bigg[e^{-r\zeta}\, G_j(S_\zeta)\,(1-\Pi_\zeta) + \int_0^\zeta e^{-rt}\,(\eta_j + \varkappa_j\, S_t)\,(1-\Pi_t)\, dt\bigg] \tag{12}$$

for $j = 1,2$. Here, we denote by $E_{s,i}$ the expectation under the assumption that the (two-dimensional) process (S, Θ) with the representation in Equation (2) starts at $(s,i) \in (0,\infty) \times \{0,1\}$, while $E_{s,\pi}$ denotes the expectation under the assumption that the (two-dimensional) process (S, Π) solving the stochastic differential equations in Equations (3) and (4) starts at $(s,\pi) \in (0,\infty) \times [0,1]$.

By means of standard applications of Itô's formula (see, e.g., Liptser and Shiryaev [1977] 2001, Theorem 4.4) to the process $e^{-rt}G_j(S_t)(1-\Theta_t)$ and $e^{-rt}G_j(S_t)(1-\Pi_t)$, for $j = 1,2$, we obtain the representations

$$\begin{aligned} e^{-rt}\, G_j(S_t)\,(1-\Theta_t) &= G_j(s)\,(1-i) \\ &+ (-1)^j \int_0^t e^{-ru}\,\big((r+\lambda)\,K_j - (\delta_0+\lambda)\,S_u\big)\,(1-\Theta_u)\,du \;+ N_t \end{aligned} \tag{13}$$

when the process Θ starts at the state $i \in \{0,1\}$, and

$$\begin{aligned} e^{-rt}\, G_j(S_t)\,(1-\Pi_t) &= G_j(s)\,(1-\pi) \\ &+ (-1)^j \int_0^t e^{-ru}\,\big((r+\lambda)\,K_j - (\delta_0+\lambda)\,S_u\big)\,(1-\Pi_u)\,du \;+ \overline{N}_t \end{aligned} \tag{14}$$

where the processes $N = (N_t)_{t\ge0}$ and $\overline{N} = (\overline{N}_t)_{t\ge0}$ defined by

$$N_t = (-1)^j \int_0^t e^{-ru}\, G_j'(S_u)\,(1-\Theta_u)\,\sigma\, S_u\, dB_u - \int_0^t e^{-ru}\, G(S_u)\,\big(d\Theta_u - \lambda\,(1-\Theta_u)\,du\big) \tag{15}$$

and

$$\overline{N}_t = (-1)^j \int_0^t e^{-ru}\, G_j'(S_u)\,(1-\Pi_u)\,\sigma\, S_u\, d\overline{B}_u + \int_0^t e^{-ru}\, G_j(S_u)\,\frac{\delta_1-\delta_0}{\sigma}\,\Pi_u(1-\Pi_u)\,d\overline{B}_u \tag{16}$$

are square integrable martingales under the probability measures $P_{s,i}$ and $P_{s,\pi}$, for each $(s,i) \in (0,\infty) \times \{0,1\}$ and $(s,\pi) \in (0,\infty) \times [0,1]$, respectively. Hence, applying Doob's optional sampling theorem (see, e.g., Liptser and Shiryaev [1977] 2001, chp. III, Theorem 3.6; or Revuz and Yor 1999, chp. II, Theorem 3.2), we obtain that the value functions in Equations (11) and (12) admit the representations

$$\begin{aligned} U_j^*(s,i) &= G_j(s)\,(1-i) \\ &+ (-1)^j \sup_\tau E_{s,i} \int_0^\tau e^{-rt}\,\Big(\big((r+\lambda)\,K_j + (-1)^j\,\eta_j\big) - \big(\delta_0+\lambda-(-1)^j\varkappa_j\big)\,S_t\Big)\,(1-\Theta_t)\,dt \end{aligned} \tag{17}$$

and

$$\begin{aligned} V_j^*(s,\pi) &= G_j(s)\,(1-\pi) \\ &+ (-1)^j \sup_\zeta E_{s,\pi} \int_0^\zeta e^{-rt}\,\Big(\big((r+\lambda)\,K_j + (-1)^j\,\eta_j\big) - \big(\delta_0+\lambda-(-1)^j\varkappa_j\big)\,S_t\Big)\,(1-\Pi_t)\,dt \end{aligned} \tag{18}$$

for $(s,i) \in (0,\infty) \times \{0,1\}$ and $(s,\pi) \in (0,\infty) \times [0,1]$, respectively. Thus, it is seen from the structure of the integrands in Equations (17) and (18) that the optimal stopping times τ_j^* and ζ_j^*, for $j = 1,2$, are infinite whenever $K_1 \le \eta_1/(r+\lambda)$ or $\delta_0 + \lambda \le \varkappa_2$ holds, respectively. Moreover, it follows from the expressions in Equations (17) and (18) that it is not optimal to exercise the options when $(\delta_0+\lambda+\varkappa_1)S_t - ((r+\lambda)K_1 - \eta_1) > 0$ under $K_1 > \eta_1/(r+\lambda)$ as well as when $(r+\lambda)K_2 + \eta_2 - (\delta_0 + \lambda - \varkappa_2)S_t > 0$ under $\delta_0 + \lambda > \varkappa_2$, for any $t \ge 0$, respectively. We also observe from the structure of the gain functions and integrands in the reward functionals of Equations (11) and (17) that $U_j^*(s,1) = 0$ holds, for all $s > 0$.

2.3. Structure of the Optimal Stopping Times

By means of the results of general theory of optimal stopping (see, e.g., Peskir and Shiryaev 2006, chp. I, sct. 2), it follows from the structure of the rewards in Equations (11) and (12) that the optimal stopping times in these problems are given by

$$\tau_j^* = \inf\{t \geq 0 \,|\, U_j^*(S_t, \Theta_t) = G_j(S_t)\,(1 - \Theta_t)\} \tag{19}$$

and

$$\zeta_j^* = \inf\{t \geq 0 \,|\, V_j^*(S_t, \Pi_t) = G_j(S_t)\,(1 - \Pi_t)\} \tag{20}$$

for every $j = 1, 2$. We further assume that the optimal stopping times in the problems of Equations (11) and (12) are of the form

$$\tau_1^* = \inf\{t \geq 0 \,|\, S_t \leq a^*(\Theta_t)\} \quad \text{and} \quad \tau_2^* = \inf\{t \geq 0 \,|\, S_t \geq b^*(\Theta_t)\} \tag{21}$$

for some numbers $0 < a^*(i) < ((r+\lambda)K_1 - \eta_1)/(\delta_0 + \lambda + \varkappa_1)$ when $K_1 > \eta_1/(r+\lambda)$, and $b^*(i) > ((r+\lambda)K_2 + \eta_2)/(\delta_0 + \lambda - \varkappa_2)$ when $\delta_0 + \lambda > \varkappa_2$, for $i = 0, 1$, as well as

$$\zeta_1^* = \inf\{t \geq 0 \,|\, S_t \leq g^*(\Pi_t)\} \quad \text{and} \quad \zeta_2^* = \inf\{t \geq 0 \,|\, S_t \geq h^*(\Pi_t)\} \tag{22}$$

for some numbers $0 < g^*(\pi) < ((r+\lambda)K_1 - \eta_1)/(\delta_0 + \lambda + \varkappa_1)$ when $K_1 > \eta_1/(r+\lambda)$, and $h^*(\pi) > ((r+\lambda)K_2 + \eta_2)/(\delta_0 + \lambda - \varkappa_2)$ when $\delta_0 + \lambda > \varkappa_2$, for $\pi \in [0, 1]$, to be determined, respectively.

2.4. The Free-Boundary Problems

It can be shown by means of Itô's formula that the infinitesimal operator $\mathbb{L}_{(S,\Theta)}$ of the process (S, Θ) acting on a function $F(s, i)$ from the class $C^{2,0}$ on $(0, \infty) \times \{0, 1\}$ has the form

$$(\mathbb{L}_{(S,\Theta)} F)(s, i) = \big(r - \delta_0 - (\delta_1 - \delta_0)\, i\,\big)\, s\; \partial_s F(s, i) + \frac{\sigma^2 s^2}{2}\, \partial_{ss} F(s, i) - \lambda\, F(s, i)\,(1 - i) \tag{23}$$

for all $(s, i) \in (0, \infty) \times \{0, 1\}$. In order to find analytic expressions for the unknown value functions $U_i^*(s, i)$, for $i = 0, 1$, from Equation (12) and the unknown boundaries $a^*(i)$ and $b^*(i)$, for $i = 0, 1$, from Equation (21), let us use the results of general theory of optimal stopping problems for Markov processes (see, e.g., Shiryaev 1978, chp. III, sct. 8; and Peskir and Shiryaev 2006, chp. IV, sct. 8) and references therein). We can therefore reduce the optimal stopping problems of Equation (11) to the equivalent free-boundary problem:

$$(\mathbb{L}_{(S,\Theta)} U_j - rU_j)(s, i) = -(\eta_j + \varkappa_j s)\,(1 - i) \quad \text{for } s > a(i) \text{ or } s < b(i) \text{ and } j = 1, 2 \tag{24}$$

$$U_1(s, i)\big|_{s=a(i)+} = (K_1 - a(i))\,(1 - i), \quad U_2(s, i)\big|_{s=b-} = (b(i) - K_2)\,(1 - i) \tag{25}$$

$$\partial_s U_1(s, i)\big|_{s=a(i)+} = -(1 - i), \quad \partial_s U_2(s, i)\big|_{s=b(i)-} = 1 - i \tag{26}$$

$$U_1(s, i) = (K_1 - s)\,(1 - i) \text{ for } s < a(i), \quad U_2(s, i) = (s - K_2)\,(1 - i) \text{ for } s > b(i) \tag{27}$$

$$U_1(s, i) > (K_1 - s)\,(1 - i) \text{ for } s > a(i), \quad U_2(s, i) > (s - K_2)\,(1 - i) \text{ for } s < b(i) \tag{28}$$

$$(\mathbb{L}_{(S,\Theta)} U_j - rU_j)(s, i) < -(\eta_j + \varkappa_j s)\,(1 - i) \quad \text{for } s < a(i) \text{ or } s > b(i) \text{ and } j = 1, 2 \tag{29}$$

for some $a(i) > 0$ and $b(i) > 0$, for $i = 0, 1$, to be determined. Observe that the superharmonic characterisation of the value function (see Dynkin 1963; Shiryaev 1978, chp. III; and Peskir and Shiryaev 2006, chp. IV, sct. 9) implies that $U_j^*(s, i)$, for $i = 0, 1$ and $j = 1, 2$, are the smallest functions satisfying Equations (24)–(25) and Equations (27)–(28) with the boundaries $a^*(i)$ and $b^*(i)$, for $i = 0, 1$, respectively.

By means of standard arguments it can be shown that the infinitesimal operator $\mathbb{L}_{(S,\Pi)}$ of the process (S,Π) acting on a function $F(s,\pi)$ from the class $C^{2,2}$ on $(0,\infty)\times(0,1)$ has the form

$$(\mathbb{L}_{(S,\Pi)}F)(s,\pi) = \big(r-\delta_0-(\delta_1-\delta_0)\,\pi\big)\,s\;\partial_s F(s,\pi) + \lambda\,(1-\pi)\,\partial_\pi F(s,\pi) \tag{30}$$
$$+\frac{\sigma^2 s^2}{2}\,\partial_{ss}F(s,\pi) - (\delta_1-\delta_0)\,s\;\pi(1-\pi)\,\partial_{s\pi}F(s,\pi) + \frac{1}{2}\left(\frac{\delta_1-\delta_0}{\sigma}\right)^2 \pi^2(1-\pi)^2\,\partial_{\pi\pi}F(s,\pi)$$

for all $(s,\pi)\in(0,\infty)\times(0,1)$. In order to find analytic expressions for the unknown value functions $V_i^*(s,\pi)$, for $i=1,2$, from Equation (12) and the unknown boundaries $g^*(\pi)$ and $h^*(\pi)$, for $\pi\in[0,1]$, from Equation (22), we also use the results of general theory of optimal stopping problems for Markov processes and reduce the optimal stopping problems of Equation (12) to the equivalent free-boundary problem:

$$(\mathbb{L}_{(S,\Pi)}V_j - rV_j)(s,\pi) = -(\eta_j+\varkappa_j s)\,(1-\pi) \quad \text{for } s>g(\pi) \text{ or } s<h(\pi) \text{ and } j=1,2 \tag{31}$$
$$V_1(s,\pi)\big|_{s=g(\pi)+} = (K_1-g(\pi))\,(1-\pi), \quad V_2(s,\pi)\big|_{s=h(\pi)-} = (h(\pi)-K_2)\,(1-\pi) \tag{32}$$
$$\partial_s V_1(s,\pi)\big|_{s=g(\pi)+} = -(1-\pi), \quad \partial_s V_2(s,\pi)\big|_{s=h(\pi)-} = 1-\pi \tag{33}$$
$$V_1(s,\pi) = (K_1-s)\,(1-\pi) \text{ for } s<g(\pi), \quad V_2(s,\pi) = (s-K_2)\,(1-\pi) \text{ for } s>h(\pi) \tag{34}$$
$$V_1(s,\pi) > (K_1-s)\,(1-\pi) \text{ for } s>g(\pi), \quad V_2(s,\pi) > (s-K_2)\,(1-\pi) \text{ for } s<h(\pi) \tag{35}$$
$$(\mathbb{L}_{(S,\Pi)}V_j - rV_j)(s,\pi) < -(\eta_j+\varkappa_j s)\,(1-\pi) \quad \text{for } s<g(\pi) \text{ or } s>h(\pi) \text{ and } j=1,2 \tag{36}$$

for some $g(\pi)>0$ and $h(\pi)>0$, for $\pi\in[0,1]$, to be determined. Observe that the superharmonic characterisation of the value function implies that $V_j^*(s,\pi)$, for $j=1,2$, are the smallest functions satisfying Equations (31)–(32) and Equations (34)–(35) with the boundaries $g^*(\pi)$ and $h^*(\pi)$, for $\pi\in[0,1]$, respectively.

3. Solutions to the Free-Boundary Problems

In this section, we obtain solutions to the free-boundary problems of Equations (24)–(29) and Equations (31)–(36) and derive explicit expressions for the optimal stopping boundaries in Equations (21) and (22).

3.1. The Case of Full Information

It is shown that the second-order (inhomogeneous) ordinary differential equations in Equation (24) have the general solutions

$$U_j(s,0) = C_{j,+}(0)\,s^{\gamma_+} + C_{j,-}(0)\,s^{\gamma_-} + \frac{\eta_j}{r+\lambda} + \frac{\varkappa_j s}{\delta_0+\lambda} \tag{37}$$

where $C_{j,\pm}(0)$, $j=1,2$, are some arbitrary constants, and $\gamma_\pm$ are given by

$$\gamma_\pm = \frac{1}{2} - \frac{r-\delta_0}{\sigma^2} \pm \sqrt{\left(\frac{1}{2}-\frac{r-\delta_0}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} \tag{38}$$

so that $\gamma_- < 0 < 1 < \gamma_+$ holds. Note that we should have $C_{1,+}(0) = C_{2,-}(0) = 0$ in Equation (37), since otherwise $U_1(s,0)\to\pm\infty$ as $s\uparrow\infty$ and $U_2(s,0)\to\pm\infty$ as $s\downarrow 0$, which must be excluded, by virtue of the fact that the functions $U_j^*(s,0)$, for $j=1,2$, in Equation (11) are bounded. Hence, by applying the boundary conditions of Equation (32) to the functions $U_j(s,0)$, for $j=1,2$, from Equation (37), we obtain the equalities

$$C_{1,-}(0)\,a^{\gamma_-}(0) + \frac{\eta_1}{r+\lambda} + \frac{\varkappa_1 a(0)}{\delta_0+\lambda} = K_1 - a(0) \quad \text{and} \quad C_{1,-}(0)\,\gamma_-\,a^{\gamma_-}(0) + \frac{\varkappa_1 a(0)}{\delta_0+\lambda} = -a(0) \tag{39}$$

and

$$C_{2,+}(0)\, b^{\gamma_+}(0) + \frac{\eta_2}{r+\lambda} + \frac{\varkappa_2 b(0)}{\delta_0+\lambda} = b(0) - K_2 \quad \text{and} \quad C_{2,+}(0)\, \gamma_+\, b^{\gamma_+}(0) + \frac{\varkappa_2 b(0)}{\delta_0+\lambda} = b(0). \tag{40}$$

By solving the systems in Equations (39) and (40) with respect to $a(0)$ and $b(0)$, we obtain that the candidate value functions $U_1(s,0;a(0))$ and $U_2(s,0;b(0))$ admit the representations

$$U_1(s,0;a(0)) = \left(K_1 - a(0) - \frac{\eta_1}{r+\lambda} - \frac{\varkappa_1 a(0)}{\delta_0+\lambda}\right)\left(\frac{s}{a(0)}\right)^{\gamma_-} + \frac{\eta_1}{r+\lambda} + \frac{\varkappa_1 s}{\delta_0+\lambda} \tag{41}$$

for $s > a(0)$, and

$$U_2(s,0;b(0)) = \left(b(0) - K_2 - \frac{\eta_2}{r+\lambda} - \frac{\varkappa_2 b(0)}{\delta_0+\lambda}\right)\left(\frac{s}{b(0)}\right)^{\gamma_-} + \frac{\eta_2}{r+\lambda} + \frac{\varkappa_2 s}{\delta_0+\lambda} \tag{42}$$

for $s < b(0)$. Thus, by applying the conditions of Equation (33) to the functions in Equations (41) and (42), we obtain the expressions

$$a^*(0) = \frac{\gamma_-}{\gamma_- - 1}\,\frac{\delta_0+\lambda}{\delta_0+\lambda+\varkappa_1}\left(K_1 - \frac{\eta_1}{r+\lambda}\right) \tag{43}$$

whenever $K_1 > \eta_1/(r+\lambda)$, and

$$b^*(0) = \frac{\gamma_+}{\gamma_+ - 1}\,\frac{\delta_0+\lambda}{\delta_0+\lambda-\varkappa_2}\left(K_2 + \frac{\eta_2}{r+\lambda}\right) \tag{44}$$

whenever $\delta_0 + \lambda > \varkappa_2$. Observe that the inequalities in Equation (29) take the form

$$a(0) < \overline{a}(0) \equiv \frac{r+\lambda}{\delta_0+\lambda+\varkappa_1}\left(K_1 - \frac{\eta_1}{r+\lambda}\right) \quad \text{and} \quad b(0) > \underline{b}(0) \equiv \frac{r+\lambda}{\delta_0+\lambda-\varkappa_2}\left(K_2 + \frac{\eta_2}{r+\lambda}\right), \tag{45}$$

and it is shown by means of straightforward calculations that the conditions $a^*(0) < \overline{a}(0)$ and $b^*(0) > \underline{b}(0)$ are satisfied whenever $K_1 > \eta_1/(r+\lambda)$ and $\delta_0 + \lambda > \varkappa_2$ hold, respectively.

3.2. The Case of Partial Information

Taking into account the structure of the reward functionals in Equations (11) and (12), let us now look for a solution of the free-boundary problems of Equations (31)–(36) in the form $V_j(s,\pi) = W_j(s)(1-\pi)$, for $j = 1,2$, with $g(\pi) \equiv g$ and $h(\pi) \equiv h$, where the unknown functions $W_j(s)$, for $j = 1,2$, and the boundaries satisfy the free-boundary problems

$$(\mathbb{L}_{(S,0)} W_j - (r+\lambda) W_j)(s) = -(\eta_j + \varkappa_j s) \quad \text{for } s > g \text{ or } s < h \text{ and } j = 1,2 \tag{46}$$

$$W_1(s)\big|_{s=g+} = K_1 - g, \quad W_2(s)\big|_{s=h-} = h - K_2 \tag{47}$$

$$W_1'(s)\big|_{s=g+} = -1, \quad W_2'(s)\big|_{s=h-} = 1 \tag{48}$$

$$W_1(s) = K_1 - s \ \text{ for } \ s < g, \quad W_2(s) = s - K_2 \ \text{ for } \ s > h \tag{49}$$

$$W_1(s) > K_1 - s \ \text{ for } \ s > g, \quad W_2(s) > s - K_2 \ \text{ for } \ s < h \tag{50}$$

$$(\mathbb{L}_{(S,0)} W_j - (r+\lambda) W_j)(s) < -(\eta_j + \varkappa_j s) \quad \text{for } \ s < g \ \text{ or } \ s > h \ \text{ and } \ j = 1,2 \tag{51}$$

for some $g > 0$ and $h > 0$ to be determined. It is shown that the second-order (inhomogeneous) ordinary differential equations in Equation (46) have the general solutions

$$W_j(s) = D_{j,+}\, s^{\gamma_+} + D_{j,-}\, s^{\gamma_-} + \frac{\eta_j}{r+\lambda} + \frac{\varkappa_j s}{\delta_0+\lambda} \tag{52}$$

where $D_{j,\pm}$, for $j = 1, 2$, are arbitrary constants, and $\gamma_- < 0 < 1 < \gamma_+$ are given by Equation (38). Note that we should have $D_{1,+} = D_{2,-} = 0$ in Equation (52), since otherwise $W_1(s) \to \pm\infty$ as $s \uparrow \infty$ and $W_2(s) \to \pm\infty$ as $s \downarrow 0$, which must be excluded, by virtue of the fact that the functions $V_j^*(s, \pi) = W_j^*(s)(1 - \pi)$, for $j = 1, 2$, in Equation (12) are bounded. Hence, by applying the boundary conditions of Equation (32) to the functions $V_j(s, \pi) = W_j(s)(1 - \pi)$ with $W_j(s)$, for $j = 1, 2$, from Equation (52), we obtain the equalities

$$D_{1,-}\, g^{\gamma_-} + \frac{\eta_1}{r + \lambda} + \frac{\varkappa_1 g}{\delta_0 + \lambda} = K_1 - g \quad \text{and} \quad D_{1,-}\, \gamma_-\, g^{\gamma_-} + \frac{\varkappa_1 g}{\delta_0 + \lambda} = -g \tag{53}$$

and

$$D_{2,+}\, h^{\gamma_+} + \frac{\eta_2}{r + \lambda} + \frac{\varkappa_2 h}{\delta_0 + \lambda} = h - K_2 \quad \text{and} \quad D_{2,+}\, \gamma_+\, h^{\gamma_+} + \frac{\varkappa_2 h}{\delta_0 + \lambda} = h. \tag{54}$$

By solving the systems in Equations (53) and (54) with respect to g and h, the candidate value functions admit the representations $V_1(s, \pi; g(\pi)) = W_1(s; g)(1 - \pi)$ and $V_2(s, \pi; h(\pi)) = W_2(s; h)(1 - \pi)$ with

$$W_1(s; g) = \left(K_1 - g - \frac{\eta_1}{r + \lambda} - \frac{\varkappa_1 g}{\delta_0 + \lambda}\right)\left(\frac{s}{g}\right)^{\gamma_-} + \frac{\eta_1}{r + \lambda} + \frac{\varkappa_1 s}{\delta_0 + \lambda} \tag{55}$$

for $s > g$, and

$$W_2(s; h) = \left(h - K_2 - \frac{\eta_2}{r + \lambda} - \frac{\varkappa_2 h}{\delta_0 + \lambda}\right)\left(\frac{s}{h}\right)^{\gamma_-} + \frac{\eta_2}{r + \lambda} + \frac{\varkappa_2 s}{\delta_0 + \lambda} \tag{56}$$

for $s < h$. Thus, by applying the conditions of Equation (33) to the functions in Equations (55) and (56), we obtain the expressions

$$g^*(\pi) \equiv g^* = \frac{\gamma_-}{\gamma_- - 1}\, \frac{\delta_0 + \lambda}{\delta_0 + \lambda + \varkappa_1}\left(K_1 - \frac{\eta_1}{r + \lambda}\right) \tag{57}$$

whenever $K_1 > \eta_1/(r + \lambda)$, and

$$h^*(\pi) \equiv h^* = \frac{\gamma_+}{\gamma_+ - 1}\, \frac{\delta_0 + \lambda}{\delta_0 + \lambda - \varkappa_2}\left(K_2 + \frac{\eta_2}{r + \lambda}\right) \tag{58}$$

whenever $\delta_0 + \lambda > \varkappa_2$. Observe that the inequalities in Equation (51) take the form

$$g < \overline{g} \equiv \frac{r + \lambda}{\delta_0 + \lambda + \varkappa_1}\left(K_1 - \frac{\eta_1}{r + \lambda}\right) \quad \text{and} \quad h > \underline{h} \equiv \frac{r + \lambda}{\delta_0 + \lambda - \varkappa_2}\left(K_2 + \frac{\eta_2}{r + \lambda}\right), \tag{59}$$

and it is shown by means of straightforward calculations that the conditions $g^* < \overline{g}$ and $h^* > \underline{h}$ are satisfied, whenever $K_1 > \eta_1/(r + \lambda)$ and $\delta_0 + \lambda > \varkappa_2$ hold, respectively.

4. Main Results and Proofs

In this section, based on the expressions computed above, we formulate and prove the main results of the paper in models with full and partial information.

Proposition 1. *Let the processes S be given by Equations (1) and (2), with some $r > 0$, $\delta_i > 0$, for $i = 0, 1$, and $\sigma > 0$ fixed, and Θ is the Markov chain defined above. The value functions $U_j^*(s, 0)$, for $j = 1, 2$, of* *the perpetual American dividend-paying put and call options pricing problems of Equation (11) under full information thus admit the representations*

$$U_1^*(s,0) = \begin{cases} U_1(s,0;a^*(0)), & \text{if} \quad s > a^*(0) \\ K_1 - a^*(0), & \text{if} \quad 0 < s \le a^*(0) \end{cases} \tag{60}$$

and

$$U_2^*(s,0) = \begin{cases} U_2(s,0;b^*(0)), & \text{if} \quad 0 < s < b^*(0) \\ b^*(0) - K_2, & \text{if} \quad s \ge b^*(0) \end{cases}, \tag{61}$$

and the optimal exercise times τ_j^*, for $j = 1,2$, have the form of Equation (21). Here, the functions $U_1(s,0;a^*(0))$ and $U_2(s,0;b^*(0))$ take the expressions of Equations (41) and (42), while the optimal exercise boundaries $a^*(0)$ and $b^*(0)$ are given by Equations (43) and (44), whenever $K_1 > \eta_1/(r+\lambda)$ and $\delta_0 + \lambda > \varkappa_2$ hold, respectively. The optimal exercise times τ_j^*, for $j = 1,2$, are infinite, whenever $K_1 \le \eta_1/(r+\lambda)$ or $\delta_0 + \lambda \le \varkappa_2$ hold, respectively. We also have $U_j^*(s,1) \equiv 0$, for $j = 1,2$, as well as $a^*(1) = \infty$ and $b^*(1) = 0$.

Since both assertions formulated above are proved using similar arguments, we only give a proof for the case of optimal stopping problem related to the perpetual American dividend-paying call option.

Proof. In order to verify the assertion stated above, it remains for us to show that the function defined in Equation (61) coincides with the value function in Equation (11) and that the stopping time τ_2^* in Equation (21) is optimal with the boundary $b^*(0)$ specified above. For this purpose, let us denote by $U_2(s,0)$ the right-hand side of the expression in Equation (61) associated with $b^*(0)$, and $U_2(s,1) \equiv 0$. Therefore, by means of straightforward calculations from the previous section, it is shown that the function $U_2(s,i)$ solves the system of Equation (24) with Equations (27)–(29) and satisfies the conditions of Equations (25) and (26). Recall that the function $U_2(s,i)$ is $C^{2,0}$ in $(s,i) \in (0,\infty) \times \{0,1\}$ such that $s \neq b^*(0)$. Hence, by applying the change-of-variable formula from (Peskir 2007, Theorem 3.1) to the process $e^{-rt}U_2(S_t,\Theta_t)$ (see also Peskir and Shiryaev 2006, chp. II, sct. 3.5 for a summary of the related results and further references), we obtain

$$e^{-rt}\, U_2(S_t,\Theta_t) = U_2(s,i) + \int_0^t e^{-ru}\, (\mathbb{L}_{(S,\Theta)}U_2 - rU_2)(S_u,\Theta_u)\, I(S_u \neq b^*(\Theta_u))\, du \, + M_t \tag{62}$$

where the process $M = (M_t)_{t\ge0}$ defined by

$$\begin{aligned} M_t = & \int_0^t e^{-ru}\, \partial_s U_2(S_u,\Theta_u)\, I(S_u \neq b^*(\Theta_u))\, \sigma\, S_u\, dB_u \\ & + \int_0^t e^{-ru}\, \big(U_2(S_u,1) - U_2(S_u,0)\big)\, \big(d\Theta_u - \lambda\,(1-\Theta_u)\, du\big) \end{aligned} \tag{63}$$

is a local martingale with respect to the probability measure $P_{s,i}$. Note that, since the time spent by the process (S,Θ) at the boundary surface $\{(s,i) \in (0,\infty) \times \{0,1\} \,|\, s = b^*(0)\}$ is of Lebesgue measure zero, the indicator in the formula in Equation (62) can be set equal to one.

By using straightforward calculations and the arguments from the previous section, it is verified that $(\mathbb{L}_{(S,\Theta)}U_2 - rU_2)(s,i) \le 0$ holds, for all $(s,i) \in (0,\infty) \times \{0,1\}$ such that $s \neq b^*(0)$. Moreover, it is shown by means of standard arguments that the properties in Equations (28) and (29) also hold, which together with the conditions of Equations (25)–(27) imply that the inequality $U_2(s,i) \ge (s-K_2)^+(1-i)$ is satisfied, for all $(s,i) \in (0,\infty) \times \{0,1\}$. Let $(\tau_k)_{k\in\mathbb{N}}$ be the localising sequence of stopping times for the process M from Equation (63) such that $\tau_k = \inf\{t \geq 0 \,|\, |M_t| \geq k\}$, for each $k \in \mathbb{N}$. It therefore follows from the expression in Equation (62) that the inequalities

$$\begin{aligned} & e^{-r(\tau\wedge\tau_k)}\,(S_{\tau\wedge\tau_k} - K_2)^+\,(1-\Theta_{\tau\wedge\tau_k}) + \int_0^{\tau\wedge\tau_k} e^{-ru}\,(\eta_2 + \varkappa_2\,S_u)\,(1-\Theta_u)\,du \\ & \leq e^{-r(\tau\wedge\tau_k)}\,U_2(S_{\tau\wedge\tau_k},\Theta_{\tau\wedge\tau_k}) + \int_0^{\tau\wedge\tau_k} e^{-ru}\,(\eta_2 + \varkappa_2\,S_u)\,(1-\Theta_u)\,du \;\leq U_2(s,i) + M_{\tau\wedge\tau_k} \end{aligned} \tag{64}$$

hold, for any stopping time τ of the process (S,Θ) and each $k \in \mathbb{N}$ fixed. Taking the expectation with respect to $P_{s,i}$ in Equation (64), by means of Doob's optional sampling theorem, we obtain

$$\begin{aligned} & E_{s,i}\Big[e^{-r(\tau\wedge\tau_k)}\,(S_{\tau\wedge\tau_k} - K_2)^+\,(1-\Theta_{\tau\wedge\tau_k}) + \int_0^{\tau\wedge\tau_k} e^{-ru}\,(\eta_2 + \varkappa_2\,S_u)\,(1-\Theta_u)\,du\Big] \\ & \leq E_{s,i}\Big[e^{-r(\tau\wedge\tau_n)}\,U_2(S_{\tau\wedge\tau_k},\Theta_{\tau\wedge\tau_k}) + \int_0^{\tau\wedge\tau_k} e^{-ru}\,(\eta_2 + \varkappa_2\,S_u)\,(1-\Theta_u)\,du\Big] \\ & \leq U_2(s,i) + E_{s,i}\big[M_{\tau\wedge\tau_k}\big] = U_2(s,i) \end{aligned} \tag{65}$$

for all $(s,i) \in (0,\infty) \times \{0,1\}$ and each $k \in \mathbb{N}$. Hence, letting k go to infinity and using Fatou's lemma, we obtain that the inequalities

$$\begin{aligned} & E_{s,i}\Big[e^{-r\tau}\,(S_\tau - K_2)^+\,(1-\Theta_\tau) + \int_0^{\tau} e^{-ru}\,(\eta_2 + \varkappa_2\,S_u)\,(1-\Theta_u)\,du\Big] \\ & \leq E_{s,i}\Big[e^{-r\tau}\,U_2(S_\tau,\Theta_\tau) + \int_0^{\tau} e^{-ru}\,(\eta_2 + \varkappa_2\,S_u)\,(1-\Theta_u)\,du\Big] \leq U_2(s,i) \end{aligned} \tag{66}$$

are satisfied for any stopping time τ and all $(s,i) \in (0,\infty) \times \{0,1\}$. Taking in Equation (66) the supremum over all stopping times τ, we may therefore conclude that the inequalities

$$E_{s,i}\Big[e^{-r\tau_2^*}\,(S_{\tau_2^*} - K_2)^+\,(1-\Theta_{\tau_2^*}) + \int_0^{\tau_2^*} e^{-ru}\,(\eta_2 + \varkappa_2\,S_u)\,(1-\Theta_u)\,du\Big] \leq U_2(s,i) \tag{67}$$

hold, for all $(s,i) \in (0,\infty) \times \{0,1\}$. By virtue of the structure of the stopping time in Equation (21), it is readily seen that the equalities in Equation (67) hold with τ_2^* instead of τ when $s \geq b^*(0)$.

It remains for us to show that the equalities are attained in Equation (67) when τ_2^* replaces τ when $0 < s < b^*(0)$, for $i = 0,1$. By virtue of the fact that the function $U_2(s,i;b^*(0))$ satisfies the conditions in Equations (24) and (25), it follows from the expression in Equation (62) that the equalities

$$\begin{aligned} & e^{-r(\tau_2^*\wedge\tau_k)}\,(S_{\tau_2^*\wedge\tau_k} - K_2)^+\,\lambda\,(1-\Theta_{\tau_2^*\wedge\tau_k}) + \int_0^{\tau_2^*\wedge\tau_k} e^{-ru}\,(\eta_2 + \varkappa_2\,S_u)\,(1-\Theta_u)\,du \\ & = e^{-r(\tau_2^*\wedge\tau_k)}\,U_2(S_{\tau_2^*\wedge\tau_n},\Theta_{\tau^*\wedge\tau_n}) + \int_0^{\tau_2^*\wedge\tau_k} e^{-ru}\,(\eta_2 + \varkappa_2\,S_u)\,\lambda\,(1-\Theta_u)\,du \\ & = U_2(s,i) + M_{\tau_2^*\wedge\tau_n} \end{aligned} \tag{68}$$

hold, for all $(s,i) \in (0,b^*(0)) \times \{0,1\}$ and each $k \in \mathbb{N}$. Observe that, by virtue of the arguments from (Shepp and Shiryaev 1993, pp. 635–36), it follows from the structure of the stochastic differential equation in (2) and the expression in (13) that the property

$$E_{s,i}\Big[\sup_{t\geq 0}\Big(e^{-r(\tau_2^*\wedge t)}\,S_{\tau_2^*\wedge t} + \int_0^{\tau_2^*\wedge t} e^{-ru}\,(\eta_2 + \varkappa_2\,S_u)\,du\Big)\Big] < \infty \tag{69}$$

holds, for all $(s,i) \in (0,\infty) \times \{0,1\}$, and the variable $e^{-r\tau_2^*}(S_{\tau_2^*} - K_2)^+(1 - \Theta_{\tau_2^*})$ is bounded on the event $\{\tau_2^* = \infty\}$. Hence, letting k go to infinity and using the conditions of Equation (25), we can apply the Lebesgue bounded convergence theorem to the expression in Equation (68) to obtain the equality

$$E_{s,i}\left[e^{-r\tau_2^*}\,(S_{\tau_2^*} - K_2)^+\,(1 - \Theta_{\tau_2^*}) + \int_0^{\tau_2^*} e^{-ru}\,(\eta_2 + \varkappa_2\,S_u)\,(1 - \Theta_u)\,du\right] = U_2(s,i) \tag{70}$$

for all $(s,i) \in (0,\infty) \times \{0,1\}$, which together with the inequalities in Equation (67) directly implies the desired assertion. □

Proposition 2. *Let the processes S and Π be given as the strong solutions of the stochastic differential equations in Equations (3) and (4), with some $r > 0$, $\delta_i > 0$, for $i = 0,1$, and $\sigma > 0$ fixed. The value functions $V_j^*(s,\pi)$, for $j = 1,2$, of the perpetual American dividend-paying put and call options pricing problems of Equation (12) under partial information therefore admit the representations*

$$V_1^*(s,\pi) = \begin{cases} W_1(s;g^*)\,(1-\pi), & \text{if} \quad s > g^* \\ (K_1 - g^*)\,(1-\pi), & \text{if} \quad 0 < s \le g^* \end{cases} \tag{71}$$

and

$$V_2^*(s,\pi) = \begin{cases} W_2(s;h^*)\,(1-\pi), & \text{if} \quad 0 < s < h^* \\ (h^* - K_2)\,(1-\pi), & \text{if} \quad s \ge h^* \end{cases}, \tag{72}$$

and the optimal exercise times ζ_j^*, for $j = 1,2$, have the form of Equation (22). Here, the functions $W_1(s;g^*)$ and $W_2(s;h^*)$ take the expressions of Equations (55) and (56), while the optimal exercise boundaries g^* and h^* are given by Equations (57) and (58), whenever $K_1 > \eta_1/(r+\lambda)$ and $\delta_0 + \lambda > \varkappa_2$ hold, respectively. The optimal exercise times ζ_j^*, for $j = 1,2$, are infinite, whenever $K_1 \le \eta_1/(r+\lambda)$ or $\delta_0 + \lambda \le \varkappa_2$ hold, respectively.

Since both assertions formulated above are proved using similar arguments, we only give a proof for the case of optimal stopping problem related to the perpetual American dividend-paying call option.

Proof. In order to verify the assertion stated above, it remains for us to show that the function defined in Equation (72) coincides with the value function in Equation (12) and that the stopping time ζ_2^* in Equation (22) is optimal with the boundary h^* specified above. For this purpose, let us denote by $V_2(s,\pi)$ the right-hand side of the expression in Equation (72) associated with h^*. Therefore, by means of straightforward calculations from the previous section, it is shown that the function $V_2(s,\pi)$ solves the system of Equation (31) with Equations (34)–(36) and satisfies the conditions of Equations (32) and (33). Recall that the function $V_2(s,\pi)$ is $C^{2,2}$ in $(s,\pi) \in (0,\infty) \times [0,1]$ such that $s \neq h^*$. Hence, by applying the change-of-variable formula from (Peskir 2007, Theorem 3.1) to the process $e^{-rt}V_2(S_t,\Pi_t)$, we obtain

$$e^{-rt}\,V_2(S_t,\Pi_t) = V_2(s,\pi) + \int_0^t e^{-ru}\,(\mathbb{L}_{(S,\Pi)}V_2 - rV_2)(S_u,\Pi_u)\,I(S_u \neq h^*)\,du \; + \overline{M}_t \tag{73}$$

where the process $\overline{M} = (\overline{M}_t)_{t \ge 0}$ defined by

$$\begin{aligned} \overline{M}_t = & \int_0^t e^{-ru}\,\partial_s V_2(S_u,\Pi_u)\,I(S_u \neq h^*)\,\sigma\,S_u\,d\overline{B}_u \\ & - \int_0^t e^{-ru}\,\partial_\pi V_2(S_u,\Pi_u)\,I(S_u \neq h^*)\,\frac{\delta_1 - \delta_0}{\sigma}\,\Pi_u(1-\Pi_u)\,d\overline{B}_u \end{aligned} \tag{74}$$

is a continuous local martingale with respect to the probability measure $P_{s,\pi}$. Note that, since the time spent by the process (S, Π) at the boundary surface $\{(s, \pi) \in (0, \infty) \times [0, 1] \,|\, s = h^*\}$ is of Lebesgue measure zero, the indicator in the formula in Equation (73) can be set equal to one.

By using straightforward calculations and the arguments from the previous section, it is verified that $(\mathbb{L}_{(S,\Pi)} V_2 - rV_2)(s, \pi) \leq 0$ holds for all $(s, \pi) \in (0, \infty) \times [0, 1]$ such that $s \neq h^*$. Moreover, it is shown by means of standard arguments that the properties in Equations (35)–(36) also hold, which together with the conditions of Equations (32)–(34) imply that the inequality $V_2(s, \pi) \geq (s - K_2)^+(1 - \pi)$ is satisfied, for all $(s, \pi) \in (0, \infty) \times [0, 1]$. Let $(\zeta_k)_{k \in \mathbb{N}}$ be the localising sequence of stopping times for the process $\overline{M}$ from Equation (74) such that $\zeta_k = \inf\{t \geq 0 \,|\, |\overline{M}_t| \geq k\}$, for each $k \in \mathbb{N}$. It therefore follows from the expression in Equation (73) that the inequalities

$$\begin{aligned} & e^{-r(\zeta \wedge \zeta_k)} \, (S_{\zeta \wedge \zeta_k} - K_2)^+ \, (1 - \Pi_{\zeta \wedge \zeta_k}) + \int_0^{\zeta \wedge \zeta_k} e^{-ru} \, (\eta_2 + \varkappa_2 \, S_u) \, (1 - \Pi_u) \, du \\ & \leq e^{-r(\zeta \wedge \zeta_k)} \, V_2(S_{\zeta \wedge \zeta_k}, \Pi_{\zeta \wedge \zeta_k}) + \int_0^{\zeta \wedge \zeta_k} e^{-ru} \, (\eta_2 + \varkappa_2 \, S_u) \, (1 - \Pi_u) \, du \; \leq V_2(s, \pi) + \overline{M}_{\zeta \wedge \zeta_k} \end{aligned} \tag{75}$$

hold, for any stopping time ζ of the process (S, Π) and each $k \in \mathbb{N}$ fixed. Taking the expectation with respect to $P_{s,\pi}$ in Equation (75), by means of Doob's optional sampling theorem, we obtain

$$\begin{aligned} & E_{s,\pi}\Big[e^{-r(\zeta \wedge \zeta_k)} \, (S_{\zeta \wedge \zeta_k} - K_2)^+ \, (1 - \Pi_{\zeta \wedge \zeta_k}) + \int_0^{\zeta \wedge \zeta_k} e^{-ru} \, (\eta_2 + \varkappa_2 \, S_u) \, (1 - \Pi_u) \, du \Big] \\ & \leq E_{s,\pi}\Big[e^{-r(\zeta \wedge \zeta_k)} \, V_2(S_{\zeta \wedge \zeta_k}, \Pi_{\zeta \wedge \zeta_k}) + \int_0^{\zeta \wedge \zeta_k} e^{-ru} \, (\eta_2 + \varkappa_2 \, S_u) \, (1 - \Pi_u) \, du \Big] \\ & \leq V_2(s, \pi) + E_{s,\pi}\big[\overline{M}_{\zeta \wedge \zeta_k}\big] = V_2(s, \pi) \end{aligned} \tag{76}$$

for all $(s, \pi) \in (0, \infty) \times [0, 1]$ and each $k \in \mathbb{N}$. Hence, letting k go to infinity and using Fatou's lemma, we obtain that the inequalities

$$\begin{aligned} & E_{s,\pi}\Big[e^{-r\zeta} \, (S_\zeta - K_2)^+ \, (1 - \Pi_\zeta) + \int_0^\zeta e^{-ru} \, (\eta_2 + \varkappa_2 \, S_u) \, (1 - \Pi_u) \, du \Big] \\ & \leq E_{s,\pi}\Big[e^{-r\tau} \, V_2(S_\zeta, \Pi_\zeta) + \int_0^\zeta e^{-ru} \, (\eta_2 + \varkappa_2 \, S_u) \, (1 - \Pi_u) \, du \Big] \leq V_2(s, \pi) \end{aligned} \tag{77}$$

are satisfied for any stopping time ζ and all $(s, \pi) \in (0, \infty) \times [0, 1]$. Taking in Equation (77) the supremum over all stopping times ζ, we may therefore conclude that the inequalities

$$E_{s,\pi}\Big[e^{-r\zeta_2^*} \, (S_{\zeta_2^*} - K_2)^+ \, (1 - \Pi_{\zeta_2^*}) + \int_0^{\zeta_2^*} e^{-ru} \, (\eta_2 + \varkappa_2 \, S_u) \, (1 - \Pi_u) \, du \Big] \leq V_2(s, \pi) \tag{78}$$

hold, for all $(s, \pi) \in (0, \infty) \times [0, 1]$. By virtue of the structure of the stopping time in Equation (22), it is readily seen that the equalities in Equation (78) hold with ζ_2^* instead of ζ when $s \geq h^*$.

It remains for us to show that the equalities are attained in Equation (78) when ζ_2^* replaces ζ when $0 < s < h^*$, for all $\pi \in [0, 1]$. By virtue of the fact that the function $V_2(s, \pi; h^*) \equiv W_2(s; h^*)(1 - \pi)$ satisfies the conditions in Equations (31) and (32), it follows from the expression in Equation (73) that the equalities

$$\begin{aligned} & e^{-r(\zeta_2^* \wedge \zeta_k)} \, (S_{\zeta_2^* \wedge \zeta_k} - K_2)^+ \, \lambda \, (1 - \Pi_{\zeta_2^* \wedge \zeta_k}) + \int_0^{\zeta_2^* \wedge \zeta_k} e^{-ru} \, (\eta_2 + \varkappa_2 \, S_u) \, (1 - \Pi_u) \, du \\ & = e^{-r(\zeta_2^* \wedge \zeta_k)} \, V_2(S_{\zeta_2^* \wedge \zeta_k}, \Pi_{\zeta^* \wedge \zeta_k}) + \int_0^{\zeta_2^* \wedge \zeta_k} e^{-ru} \, (\eta_2 + \varkappa_2 \, S_u) \, (1 - \Pi_u) \, du \\ & = V_2(s, \pi) + \overline{M}_{\zeta_2^* \wedge \zeta_k} \end{aligned} \tag{79}$$

hold, for all $(s, \pi) \in (0, \infty) \times [0, 1]$ and each $k \in \mathbb{N}$. Observe that, by virtue of the arguments from (Shepp and Shiryaev 1993, pp. 635–36), it follows from the structure of the stochastic differential equation in (2) and the expression in (14) that the property

$$E_{s,\pi}\left[\sup_{t\geq 0}\left(e^{-r(\zeta_2^*\wedge t)}\, S_{\zeta_2^*\wedge t} + \int_0^{\zeta_2^*\wedge t} e^{-ru}\,(\eta_2 + \varkappa_2\, S_u)\, du\right)\right] < \infty \tag{80}$$

holds, for all $(s, \pi) \in (0, h^*) \times [0, 1]$, and the variable $e^{-r\zeta_2^*}(S_{\zeta_2^*} - K_2)^+(1 - \Pi_{\zeta_2^*})$ is bounded on the event $\{\zeta_2^* = \infty\}$. Hence, letting k go to infinity and using the conditions of Equation (32), we can apply the Lebesgue bounded convergence theorem to the expression in Equation (79) to obtain the equality

$$E_{s,\pi}\left[e^{-r\zeta_2^*}\,(S_{\zeta_2^*} - K_2)^+\,(1 - \Pi_{\zeta_2^*}) + \int_0^{\zeta_2^*} e^{-ru}\,(\eta_2 + \varkappa_2\, S_u)\,(1 - \Pi_u)\, du\right] = V_2(s, \pi) \tag{81}$$

for all $(s, \pi) \in (0, \infty) \times [0, 1]$, which together with the inequalities in Equation (78) directly implies the desired assertion. □

Remark 1. *Observe that the solutions in Equations (71) and (72) of the optimal stopping problems in Equation (12) can be represented in a product form with the factor* $(1 - \pi)$ *when the gain and the integrand in Equation (12) contain the factors* $(1 - \Pi_\zeta)$ *and* $(1 - \Pi_t)$*, respectively, because of the special form* $\lambda(1 - \Pi_t)$ *of the local drift rate of the filtering estimation process* Π *in Equation (4).*

Author Contributions: P.V.G. has formulated the mathematical problems considered in the paper. P.V.G. and H.A.M. have obtained the solutions to these problems and prepared the text of the article jointly.

Acknowledgments: The authors are grateful to four anonymous referees for their helpful suggestions, which significantly improved the paper. The paper was essentially written during the time when the second author was visiting the Department of Mathematics at the London School of Economics and Political Science, and she is grateful for the hospitality. This research was supported by a Small Grant from the Suntory and Toyota International Centres for Economics and Related Disciplines (STICERD) at the London School of Economics and Political Science.

Conflicts of Interest: The authors declare no conflict of interest.

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