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Game, set, and graph

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Game, Set, and Graph

In the twentieth century the Theory of Games was transformed. It began as an amusing pastime, and ended as a major branch of mathematical research and a key paradigm of economic theory. Here it will be argued that the transformation was the result of the work of mathematicians, such as Ernst Zermelo, John von Neumann and Dénes Kőnig, who also contributed to two other areas of mathematics that were emerging at the same time: the Theory of Sets and the Theory of Graphs.

Keywords: game theory, set theory, graph theory.

Introduction

The theme of this paper is the interaction between three areas of mathematical research, Game Theory, Set Theory and Graph Theory. All three flourished in the twentieth century. Although their origins varied considerably, the three disciplines were forged by mathematicians whose interests spanned several areas of mathematics, and we shall explain how these links were central to the development of the disciplines and led to significant advances.

Paris 1900

The second International Congress of Mathematicians (ICM) took place in Paris in August 1900. It is now mainly remembered for the talk delivered by David Hilbert, which began with a rhetorical question: 'Who among us would not be glad to lift the veil behind which the future lies hidden and to cast a glance at the next advances of our science...' (Hilbert 1901; English translation by Winston Newson 1902). Hilbert went on to discuss 23 problems, covering many fields of mathematics, the solution of which, he believed, would lead to significant advances in the subject. Work on the problems did indeed have a profound influence on the development of mathematics in the twentieth century (Gray 2000).

Hilbert's first problem concerned the foundations of mathematics. In the latter part of the nineteenth century Cantor and Dedekind had given formal definitions of $\mathbb{R}$, the system of *real numbers*, also known as the *continuum*. Cantor had also given a good definition of the *cardinal number* of a set, which was particularly significant because it clarified the notion of infinity. He had shown that some infinite subsets of $\mathbb{R}$, such as the set $\mathbb{Q}$ of *rational numbers*, had a smaller cardinal number than $\mathbb{R}$ itself, and he had conjectured that no other kinds of infinity could occur among the subsets of $\mathbb{R}$. This was to become known as the *continuum hypothesis* (CH).

Hilbert remarked that 'it would follow at once that the continuum has the next cardinal number beyond that of the countable set', and he speculated about how this result might be proved. He referred to another statement of Cantor, concerning the ordering of the continuum. The standard ordering of the real numbers has the property that a subset which has a lower bound need not have a least member: for example, the set of strictly positive real numbers has the lower bound 0, but it has no least member. On the other hand, the set of strictly positive *integers* does have a least member, 1, and this set is said to be *well-ordered*, because the property holds quite generally. 'The question now arises' said Hilbert, 'whether the totality of all numbers [the continuum] may not be arranged in another manner' so that the well-ordering property holds for it too. Cantor believed that this was so, but he had not been able to prove it. This question was to lead, indirectly, to a significant contribution to the Theory of Games.

Heidelberg 1904

The next International Congress of Mathematicians took place in Heidelberg in 1904. Work on several of Hilbert's problems was in progress, and a Hungarian mathematician, Gyula (Julius) Kőnig startled the congress by announcing that the continuum cannot be well-ordered (Ebbinghaus 2007). Furthermore, the Continuum Hypothesis is false! This announcement created a sensation, although several of those present were unconvinced – and, unfortunately, Kőnig was wrong.

The flaw in Kőnig's argument was quickly spotted: he had relied on a theorem of Bernstein which specifically did not hold under the conditions needed. Luckily, some ephemeral evidence of how the error was spotted has survived (Ebbinghaus 2007). One of those present in Heidelberg was a young German mathematician, Ernst Zermelo, who had been working in Göttingen where Hilbert was a professor, He had already published a paper on cardinal numbers, and had doubtless worked on the continuum hypothesis. A few weeks after the ICM he sent a postcard to his friend Max Dehn, explaining that he had been unable to check Bernstein's paper at the time, but, on his return to

Göttingen, he had found a copy and verified that Kőnig was mistaken. He foresaw that 'some fine polemic will still unfold in the [Mathematische] Annalen'. In fact, Zermelo succeeded in proving the antithesis of Kőnig's result, which was equally startling: *every* set can be well-ordered (Zermelo 1904; English translation in van Heijenoort 1967).

As he had expected, Zermelo's proof of the well-ordering theorem did not put an end to controversy in the mathematical community. One reason was that he relied on a property of sets that he called the *Auswahlaxiom*; also known as the Axiom of Choice (AC). The AC asserts that given any collection of non-empty sets, it is possible to construct a set that contains one member of each of them. Some mathematicians regarded this as 'obvious', while others were sceptical because it seemed to have consequences that were highly counter-intuitive. The publication of some correspondence in the Bulletin of the French Mathematical Society put the debate into sharp focus (Hadamard 1905).

The work of Zermelo

One result of the controversy was that Zermelo himself set out to examine the foundations of set theory. Cantor had expressed his arguments in the prevailing literary style, and the subject was still only in its infancy. So Zermelo hoped to establish a more mathematical treatment, as (for example) Peano had done with his axioms for the system of natural numbers. In 1908 he published two important papers. First, he gave a new proof of his well-ordering theorem, in which he tried to avoid reliance on the Axiom of Choice (Zermelo 1908a). He also produced the first attempt to formulate explicitly a collection of axioms on which set theory could be based (Zermelo 1908b). This work was prompted partly by the ongoing controversy about the status of the AC and CH, but also by the fact that several difficulties in the naïve approach to the subject had been discovered. These difficulties led to self-contradictory statements about such things as the 'set of all sets which are not members of themselves'. They were euphemistically classified as 'paradoxes' or 'antinomies', and Zermelo proposed to avoid them by imposing some restrictions on the notions of 'set' and 'membership'.

Thus it was that the foundations of set theory became very controversial in the first decade of the twentieth century. Zermelo's work was at the heart of the controversy, as he himself had foreseen in his postcard to Dehn back in 1904. It is possible that he found the continuing excitement rather tiresome, because, when he was invited to address the fifth ICM in Cambridge (England) in 1912, he chose a relatively safe subject for his talk. It was 'Über eine Anwendung der Mengenlehre auf die Theorie Schachspeils' ('On an Application of Set Theory to the Theory of the Game of Chess').

The talk was published in German in the Proceedings of the ICM (Zermelo 1913). Much later, when the subject that we now know as the Theory of Games had been established, it was often referred to in historical accounts of that subject. In 2001 Ulrich Schwalbe and Paul Walker published an English translation, with a careful analysis of what Zermelo had actually written and the various interpretations that had been suggested (Schwalbe and Walker 2001). Some of the earlier interpretations were flawed, and we shall not expand on them here, focussing instead on the part played by Zermelo's paper in later developments.

It must be stressed that Zermelo's main aim was to express some questions in the formal language of his *Mengenlehre*, and to apply some basic ideas from the newly-formulated theory of infinite sets. Although this approach to writing a mathematical paper is now almost universal, at that time it was quite unusual: sets, functions, relations, and the associated terminology, were not the common currency of mathematical discourse. Zermelo did not discuss chess in the form that the game is usually played; specifically, he did not consider the rules which require the game to be ended as a draw in certain circumstances. He set out to clarify the notion of a *winning position*, by which he meant a position in which one player (usually White) can force a win in a finite number of moves, whatever the other player (Black) does. This is the situation studied in 'Chess Problems', where the solver is given an instruction such as 'White to move and mate in three'. Later in this paper we shall describe what Zermelo did in a way that illuminates its relationship with subsequent work; but, given the misleading accounts of his paper that have appeared, it is as well to begin by stating what he did not do.

First, Zermelo did not attempt to determine the strategy which White should employ in order to win. He was concerned only with the definition of a winning position, not the method by which the win could be achieved. Consequently, he was not responsible for what is now often referred to by game-theorists as 'Zermelo's Algorithm' or 'backwards induction'. Furthermore, his arguments about an upper bound for the number of moves required to force a win were incomplete, as we shall explain when we return to this point.

It must also be noted that Zermelo did not refer to the extensive literature on games of all kinds which was already in existence. In the 17th century the analysis of games of chance had led to the establishment of probability theory as a branch of mathematics. As we shall see, probability was to become a basic tool in the analysis of games of strategy, but Zermelo himself made no use of it. The notion of a winning position had also been recognised by several authors, including Charles Babbage (1864), who discussed the design of an automaton to play the game he called Tit-tat-to (Noughts-and-crosses). More relevant to Zermelo's purpose, but also not mentioned by him, were the papers of Bouton (1901) and Moore (1910) on the game of Nim. In such simple games it is possible to define precisely what is meant by a winning position, and to specify a winning strategy—a stark contrast with the situation in Chess.

The Kőnigs and the Theory of Graphs

Gyula Kőnig had to admit that his claims about well-ordering and the CH were wrong, but he continued to work on the foundations of mathematics. Today, he is best known for his elegant proof (Kőnig 1906) of a fundamental theorem due to Cantor, Schröder and Bernstein. In modern terminology, this says that if there are injections $S \to T$ and $T \to S$ then there is a bijection $S \leftrightarrow T$. Equivalently, let $\#S$ denote the cardinal number of the set *S*; then the statements $\#S \leq \#T$ and $\#T \leq \#S$ imply that $\#S = \#T$. Kőnig's proof of this fact is significant because it uses the methods of what we now call the Theory of Graphs.

Formally, a *graph* is an abstract object defined by a set of vertices and a set of edges. The fact that the vertices and edges can be represented pictorially by points and lines often helps us to understand arguments about graphs, and that is why Kőnig's proof of the Cantor-Schröder-Bernstein theorem is often quoted. In Figure 1 the members of the (possibly infinite) sets S and T are represented by points.

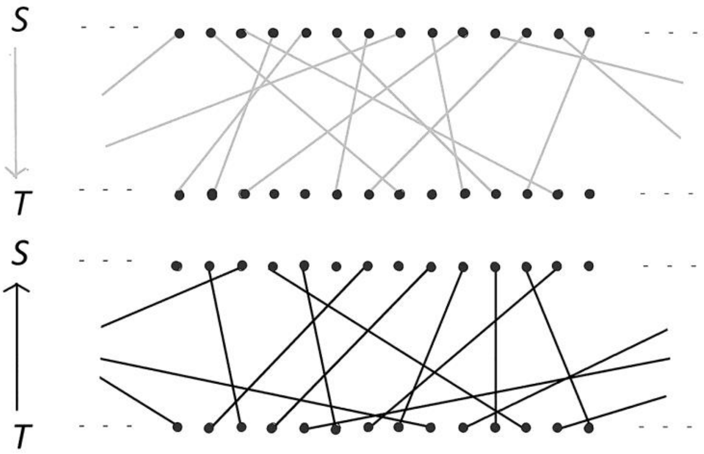

Figure 1. Injections $S \to T$ and $T \to S$.

In the top diagram, an injection $S \to T$ is represented by drawing a grey line from each point in S to the corresponding point in T. Note that every point in S is at the end of a grey line, while each point in T may be at the end of either one grey line or no grey line. Similarly, an injection $T \to S$ can be represented by drawing a black line from each point in T to the corresponding point in S. When we merge the two diagrams (Figure 2) every point now belongs to a unique path, comprising grey lines and black lines alternately.

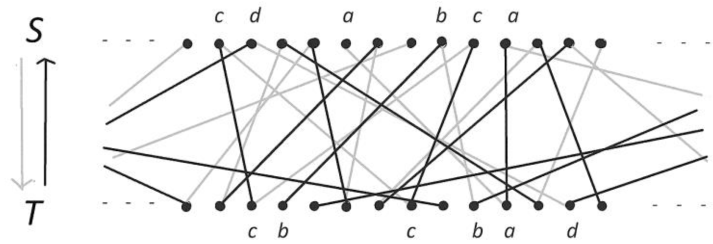

Figure 2. Merging the two injections.

These paths can be of four kinds only:

(1) semi-infinite with an end in S (for example, the points labelled *a*);

(2) semi-infinite with an end in T (for example, the points labelled *b*);

(3) finite, with no ends (for example, the points labelled *c*):

(4) doubly-infinite (for example the points labelled *d*).

In each case it is easy to construct a bijection between the S-points and the T-points in the path. For example, in case (1) the grey lines define a bijection. Combining these bijections we obtain a bijection $S \leftrightarrow T$.

Significant results about graphs had been published in the nineteenth century, and many of the original papers have been described and translated into English by Biggs, Lloyd and Wilson (1976). However, when Gyula Kőnig wrote his paper in 1906, the Theory of Graphs was not an established field of mathematical research. The fact that it eventually became so was largely the result of the work of Gyula's son, Dénes.

Dénes Kőnig was born in Budapest on 21 September 1884. His mathematical abilities were soon apparent, and he published his first paper in 1899. He attended the University of Budapest, and then moved to Göttingen, where he encountered graph-theoretical notions in various contexts. For example, he would have heard about the paper of Julius Petersen (1891), in which graph terminology was applied to problems of invariant theory, following Gordan and Hilbert. In this paper Petersen also proved a fundamental theorem on the factorization of graphs, which was a precursor of Kőnig's later results on this subject. Dénes attended Hermann Minkowski's course on topology, and he may have been present on the famous occasion when the lecturer attempted (unsuccessfully) to give an impromptu solution of the Four Colour Problem. On his return to Budapest his interest was maintained, and in 1911 he wrote two papers on the topology of graphs and surfaces (Kőnig 1911).

Dénes Kőnig's interest in graphs was clearly reinforced by his father's work on Set Theory, especially the proof of the Cantor-Schröder-Bernstein Theorem. Shortly after Gyula's death, Dénes completed and published a book that his father had written on the foundations of Set Theory and the philosophical background to it (Konig 1914). In that year he also gave an address at a conference on mathematical philosophy in Paris, in which he discussed some problems about correspondences between infinite sets. The outbreak of the Great War delayed publication of the conference proceedings, and the talk was not published until many years later, in French (Kőnig 1923). However, in 1916 a version appeared in both Hungarian and German (Kőnig 1916). This paper contained an important result about finite graphs, obtained as a special case of a more general problem about infinite graphs.

In his 1916 paper Kőnig considered what we now refer to as *edge-colourings* of a graph. Here colours are assigned to the edges in such a way that no two edges ending at the same vertex have the same colour. An obvious lower bound on the number of colours required is given by D, the largest degree of a vertex, where the *degree* is the number of edges which have that vertex as an end. Petersen (1898) had given an example of a graph in which all vertices have degree three, but there is no edge-colouring with three colours. However, Kőnig was able to prove that the lower bound *D* can always be achieved if the graph has a very simple property.

A graph is *bipartite* if its vertex-set can be split into two parts in such a way that every edge has an end in both parts. Kőnig proved that, in a finite bipartite graph with maximum degree *D*, an edge-colouring with *D* colours is always possible. His proof used the principle of induction, and it was based on a construction reminiscent of his father's proof of the Cantor-Schröder-Bernstein theorem. The key idea was to consider an *alternating path*, consisting of edges that have two colours alternately, and to switch the colours on that path. This simple idea has many applications, and it became known as the *Hungarian method*. In fact, it was suggested by a technique used by A.B. Kempe in his early work on the map-colouring problem, as Kőnig himself wrote later (Kőnig 1936, 172).

Equilibrium in games of strategy

Dénes Kőnig remained in Budapest throughout the Great War of 1914-18 and its aftermath, teaching at the Technische Hochschule. This was a period of great uncertainty and unrest in Hungary, but Dénes continued his mathematical work, and he remained in contact with foreign mathematicians. For example, in the Foreword to his father's book (Kőnig 1914) he thanked Felix Hausdorff for his help, and 1915 he wrote to Georg Frobenius about a graph-theoretical proof of a theorem. (The latter contact had unpleasant consequences, which Dénes later explained, at some length, in his book (Kőnig 1936, note on page 240)).

In Budapest, a new star was rising. John von Neumann was born into a wealthy family of bankers in 1903 and was enrolled as a student of mathematics at the University of Budapest in 1921. At the same time, he studied chemistry in Berlin and Zurich, returning to Budapest only to take his examinations in mathematics. He was soon recognised as a gifted mathematician, and in the 1920s he published several papers on the foundations of set theory, in which (among other things) he modified and extended the axiom-system of Zermelo (von Neumann 1923, 1925). He also contributed to several other branches of mathematics, including quantum theory, functional analysis, and game theory.

In game theory, he proved a fundamental theorem about games with two players, X and Y, whose interests are directly opposed. A simple model of this situation assumes that each player has a finite number of *strategies* available, and when X uses strategy r and Y uses strategy s there is a number $m(r, s)$ representing the *payoff* to X. The corresponding payoff to Y is $-m(r, s)$. This is what we now call a *finite*, *two-person*, *zero-sum* game, and it is the setting for the remarkable *minimax theorem* first proved by von Neumann in 1926.

In fact, the underlying idea was noticed in the early years of the eighteenth century and was referred to in an appendix to the 1713 edition of Pierre de Montmort's *Essai d'analyse sur les jeux de hazard* (Kuhn 1968). The subject was Le Her, a card game which can be represented as a two-person zero-sum game with two strategies for each player. There was much debate about how the players should behave when the game is played repeatedly, because one player could gain an advantage if it became clear that the other player was always using one of the two available strategies. An English aristocrat named Waldegrave suggested that each player should adopt a *mixed strategy*, alternating between the two strategies randomly, but according to a rule that determined their relative frequency. For example, if one strategy was to be played in 70% of the games and the other in 30%, the player could have a bag containing 7 red balls and 3 blue balls, and draw one to determine the strategy for each game. Waldegrave was able to show that, in the game of Le Her, a mixed strategy was preferable to either of the pure strategies.

Waldegrave's suggestion was almost forgotten, but not quite. Over a century later it was mentioned in Isaac Todhunter's massive *History of the Mathematical Theory of Probability* (1865). However, Todhunter's account (on pages 106-110 of his book), was mainly concerned with details of the game, and he did not highlight the crucial idea of a mixed strategy. There would be another long wait before the idea finally emerged in a recognizably modern form.

Émile Borel was one of the leading French mathematicians of the first half of the 20th century, and he was among those who had expressed doubts about Zermelo's use of the Axiom of Choice (Hadamard 1905). He had interests in many areas of mathematics, but his major focus was the theory of probability and its applications. He may well have read Montmort's *Essai* or Todhunter's *History*, but we have no evidence of this. We do know that in the 1920s he began to develop a theory of strategic games, in which the concept of a mixed strategy was central. His papers on this subject (Borel 1921, 1924, 1927) developed the theory in the case where both players have the same set of strategies, so that the payoffs satisfy the condition $m(s,r) = -m(r,s)$. In modern terms, this means that the matrix of payoffs is skew-symmetric, and Borel was able to use the properties of such matrices to show that, when the number of strategies is small, the optimal mixed strategies for the two players have a remarkable property.

To explain the general problem, suppose that X and Y both adopt mixed strategies, with respective frequencies given by the probability vectors $\boldsymbol{p} = (p_r)$ and $\boldsymbol{q} = (q_s)$. Then the pair of strategies (r,s) will occur with probability $p_r q_s$, and the expectation of the payoff to X will be

$$E(\boldsymbol{p},\boldsymbol{q}) = \sum_{r,s} m(r,s)\, p_r q_s.$$

How should X choose $\boldsymbol{p}$ in order to guarantee the best possible payoff, given that Y may choose any mixed strategy $\boldsymbol{q}$? For any given choice of $\boldsymbol{p}$ there is a minimum value $X(\boldsymbol{p})$ of $E(\boldsymbol{p},\boldsymbol{q})$, taken over all possible choices of $\boldsymbol{q}$, and X is guaranteed to gain this amount whatever Y does. So the best strategy for X is to choose $\boldsymbol{p}$ so that $X(\boldsymbol{p})$ is maximised. In other words, X should maximise the minimum expected payoff.

Now consider the problem from Y's point of view. Since we are considering a zero-sum game, the payoffs to Y are the negatives of those to X. In simple terms, this means that Y loses what X gains. So Y should choose $\boldsymbol{q}$ so that the maximum loss, the maximum value $Y(\boldsymbol{q})$ of $E(\boldsymbol{p},\boldsymbol{q})$, taken over all possible choices of $\boldsymbol{p}$, is minimised. The minimax theorem of von Neumann asserts that, when X and Y act in this way, the outcome is the same. The max-min strategy for X and the min-max strategy for Y are both optimal from their respective viewpoints, and both are satisfied that they can do no better, even though they act independently. The implication is that there are competitive situations in which an *equilibrium* exists. Of course, life (although finite) is not a two-person zero-sum game, so we must be careful not to place too much faith in this beautiful mathematical theorem.

Borel had verified that an equilibrium exists in a few special cases, as mentioned above, but he had not found a general proof. Indeed, he was uncertain as to whether a proof was possible. The first general proof was presented by von Neumann in a talk to the Göttingen Mathematical Society on 7 December 1926. The origins of von Neumann's interest in this problem are unknown. He was certainly familiar with Zermelo's work on set theory and his paper on chess, but that work did not involve any probabilistic notions. It is possible that he had read Montmort or Todhunter, but it is more likely that he learned of Borel's work on mixed strategies during one of his visits to Göttingen. He certainly knew of Borel, since (we are told) he had mastered the eminent Frenchman's *Théorie des Fonctions* at the age of 12 (Baumol and Goldberg 1968, 294). Von Neumann was also in contact with Dénes Kőnig in Budapest (see below), so that could equally well have been his source.

Von Neumann sent Borel a brief outline of his proof, which Borel communicated for publication in the *Comptes Rendus* (von Neumann 1928a). In this paper von Neumann acknowledged Borel's contributions and stated that he had been working independently on a more general version of the same problem. He said that he had obtained an affirmative answer to the main question, and indeed his proof of the minimax theorem was soon published in full (von Neumann 1928b). Borel and von Neumann appear to have developed a mutual coolness about their respective contributions, as described by Leonard (1992) for example. Borel failed to mention von Neumann's proof in some of his later publications, while von Neumann (1953) was dismissive of Borel's papers.

The infinity lemma and its application to chess

In the 1920s, while John von Neumann was travelling around Europe, Dénes Kőnig remained in Budapest. He would surely have known that the proofs of some of the results of his 1916 paper on edge-colourings did not extend to infinite graphs. For example, he had proved that if a finite bipartite graph has maximum degree D, then it has an edge-colouring with D colours. But the induction argument he had used for the finite case did not extend to the infinite case. Eventually, with the aid of a young Hungarian mathematician, Istvan Valko, a new approach was discovered, and was published in a joint paper (Kőnig and Valko 1926). Here a simple but powerful new idea was applied to several problems in the theory of infinite sets. Many details of this work have been described by Franchella (1997).

Soon afterwards there appeared a paper (Kőnig 1927) in which the fundamental idea was presented as a result about infinite graphs. We now know it as the *Infinity Lemma*. In 1936 it would feature in Kőnig's famous book on the Theory of Graphs, about which we shall have more to say later. The presentation in the book is similar to that in the 1927 paper, but rather more discursive, and we shall base our story upon it. In the book the Infinity Lemma is stated in the following way (Kőnig 1936, 81).

> Let $V_1, V_2, V_3, \ . \ . \ .$ be a countably infinite sequence of finite, non-empty, pairwise disjoint sets, and consider their union as the vertex-set of a graph. If the graph has the property that, for $i = 1, 2, 3, \ . \ . \ . ,$ every vertex in V_{i+1} is joined by an edge to some vertex in V_i, then there is an infinite path $x_1 x_2 x_3 \ . \ . \ . ,$ with x_i in V_i.

This apparently simple result does indeed require proof, and Kőnig's 1927 proof is still the canonical one that appears in recent texts on the Theory of Graphs, for example (Diestel 1997, 190). If just one of the sets V_i is infinite, the conclusion is false, as we shall see below (Figure 4).

The Infinity Lemma turns out to be useful in many cases where results about finite sets have to be extended to the infinite case. It can also be applied to a problem discussed by Zermelo in his 1913 paper on chess. Zermelo had described the game of chess along the following lines (but not exactly in these terms). Define a *position* to be an allocation of pieces to the squares of the board which can occur in a legal game of chess. For technical reasons it is necessary to add information about such things as whose turn to move it is, and the situation regarding the promotion of pawns and castling. These additions do not affect the fact that the set of all positions is finite. A *W-move* is an ordered pair (f, g) of positions where f is a position in which it is White's turn, and g is the result of one legal move by White. A *B-move* is defined similarly. Given any set S of positions with Black to move, let S^* be the set of all positions y such that (x, y) is a B-move for some x in S.

Zermelo defined f to be a *winning position for White* if there is a sequence of non-empty sets $S_1, S_2, S_3, \ldots$ satisfying three conditions. In the terminology defined above (which is a paraphrase of Kőnig's version of Zermelo's original) they are as follows (Figure 3).

> (1) S_1 comprises a single element g and (f, g) is a W-move.
>
> (2) For $i = 1, 2, \ldots,$ and each position y in S_i^*, there is a W-move (y, z) with z in S_{i+1}.
>
> (3) The sequence is finite and results in a win for White: that is, there is an integer N such that S_N contains only positions in which White has won by checkmate.

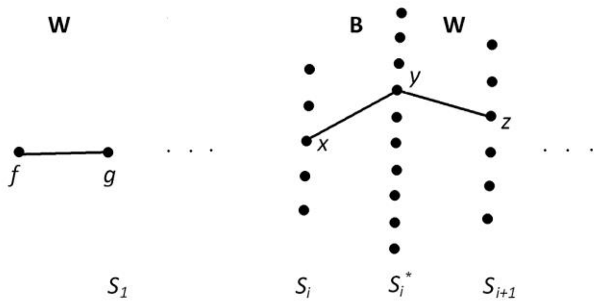

Figure 3. Illustrating Zermelo's definition.

In Condition 1, the assumption that the set S_1 contains only one position, so that White's opening move (f, g) is the unique 'key' to winning (as is usually the case in chess problems) is not strictly necessary. Condition 2 says that whatever Black does, White has a reply that will eventually result in a win. Condition 3 says that Black cannot prolong the game indefinitely, a necessary condition since Zermelo's framework allowed infinite sequences of moves. Note that this condition implies that S_N^* is empty, but not conversely, because of the possibility of stalemate.

Zermelo's objective was to formulate a precise definition. He did not develop any theory based on it, and he did not suggest rules for constructing a winning strategy. However, regarding Condition 3, he did claim that if such a number N exists, then it can be chosen to be not greater than N_0, the total number of possible positions. His argument was that if more than N_0 moves have been made, then some position p must have been repeated. Since White has a strategy which wins after the last occurrence of p, White could have won after the first occurrence of p by using that strategy. This argument assumes that when a position is repeated for a second time, Black will play in the same way as before, which is clearly not the case. Kőnig (1927, and 1936, 116) credits John von Neumann for drawing his attention to the relevance of the Infinity Lemma, so the two men must have been in contact around 1926, when von Neumann was working on the minimax theorem.

The application of the Infinity Lemma to Zermelo's formulation of chess is straightforward. Take the sets V to be the sets S and S^*, linked by edges as in Figure 3; by definition, these sets are finite. If all these sets were non-empty, the Infinity Lemma would guarantee an infinite path, and Black could prolong the game indefinitely. If that is not so, one of the sets must be empty, and there is a number N, as required. Kőnig communicated this result to Zermelo, who responded with a revised proof of his claim that $N \leq N_0$, and this was published as an Addendum to the 1927 paper.

In Kőnig's more general formulation of a game, the set of all possible positions is not assumed to be finite. But for each position he allows only finitely many possible moves, and he gives an example to show why this is the critical condition (Kőnig 1936, 115). His example is illustrated in Figure 4, using our terminology. Suppose that, after White's opening move (f, g), Black has available a countably infinite set of moves (g, h_j), $j = 1, 2, 3, \ldots$. Suppose also that the rules of the game allow the set S_2 to be defined so that, for each position h_j, there is a W-move (h_j, s_{2j}), where s_{21} is a 'checkmate' position and s_{2j} $(j > 1)$ admits only one possible B-move $(s_{2j}, s^*_{2,j-1})$. The sets $S_i = \{s_{i1}, s_{i2}, \ldots\}$ and $S^*_i = \{s^*_{i1}, s^*_{i2}, \ldots\}$ for $i > 2$ are defined similarly. The W-moves are of the form $(s^*_{ij},\ s_{i+1,j})$, the B-moves are of the form $(s_{ij},\ s^*_{i,j-1})$, and the positions $s_{21}, s_{31}, \ldots$ are checkmates by White.

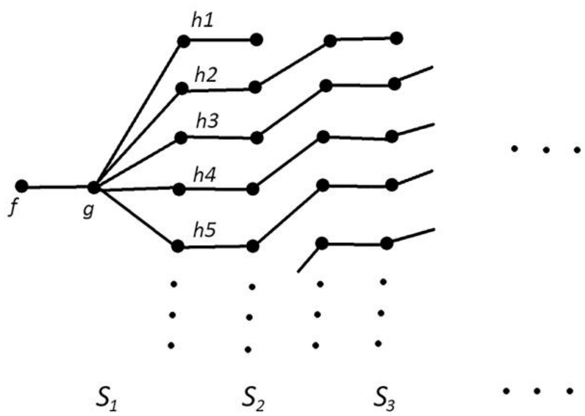

Figure 4. Kőnig's example: Black has infinitely many choices for the first move.

In this game White is certain to win after a finite number of moves, but there is no upper bound on the number of moves required. If Black chooses h_j on the first move, then the game will end after *j* moves by White.

Conclusion

Ernst Zermelo's work on the foundations of Set Theory led, almost incidentally, to his attempt to formulate a mathematical basis for the game of chess, and he later returned to this topic in the light of the Infinity Lemma. The Kőnigs, Gyula amd Dénes, developed graph-theoretical ideas in order to prove results about infinite sets, and Dénes applied these ideas to Game Theory. John von Neumann expanded Zermelo's foundation for Set Theory and, for reasons unknown to us, went on to prove the Minimax Theorem for two-person zero-sum games. These links benefitted the development of the three disciplines, and extended the scope of Mathematics generally. In the first decades of the twentieth century there were many opportunities for the exchange of information by word-of-mouth, and some of the contacts that facilitated this process have left no trace.

The work covered in this article was the first slim volume in an ever-increasing library of mathematical books. Kőnig's book (1936) was the first in which the Theory of Graphs was treated as a subject in its own right. It contains several sections on games of various kinds, and the theory of infinite sets is prominent, including the Infinity Lemma and its applications, along the lines described above. There are now many books on the Theory of Graphs, and the Lemma has found many new applications. For example, it is frequently employed as a 'compactness argument' in the branch of combinatorics that we call Ramsey Theory (Diestel 1997). John von Neumann's proof of the Minimax Theorem for two-person zero-sum games was to have a profound effect on the study of games of strategy. In the 1930s von Neumann began to collaborate with the economist Oskar Morgenstern, and the result was their famous book, *Theory of Games and Economic Behavior* (von Neumann and Morgenstern 1944). The central idea of equilibrium has since been extended to a much wider class of games, and it is a key paradigm of modern economic theory.

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