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Optimizing Periodic Patrols against Short Attacks on the Line and Other Networks.

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Abstract

On a given network, a Patroller and Attacker play the following win-lose game: The Patroller adopts a periodic walk on the network while the Attacker chooses a node and two consecutive periods (to attack there). The Patroller wins if he successfully intercepts the attack, that is, if he occupies the attacked node in one of the two periods of the attack. We solve this game in mixed strategies for line graphs, the first class of graphs to be solved for the periodic patrolling game. We also solve the game for arbitrary graphs when the period is even.

Keywords: Game Theory, Networks, Search/Surveillance

1 Introduction

The periodic patrolling game was introduced in Alpern et al. (2011) to model the defense of the nodes of a network from attack by an antagonistic opponent. This is a discrete game model in which the network is modeled as a graph, the Patroller chooses a walk on the graph with a given period and the Attacker picks a node and a discrete time interval of fixed duration m for his attack. The Patroller wins the game if he is present at the attacked node during the time interval in which it is attacked, in which case we say that he intercepts the attack. Otherwise the Attacker wins. Compared with other patrolling models in the literature, for example Chung et al. (2011), the patrolling game model represents only an idealization of the patrolling problem. However it is the only model in which the Patroller and Attacker are treated symmetrically, rather than the more usual Stackelberg approach where the Patroller picks his strategy first.

This paper considers the periodic patrolling game on general graphs and then in more detail on the class of line graphs L_n consisting of n nodes $1, 2, \ldots, n$ with consecutive numbers considered to be adjacent. The case of a unit attack duration $m = 1$ is covered by the field of geometric games as defined by Ruckle (1983), so we here consider the next smallest duration $m = 2$, which is the only case thus far susceptible to analysis. We note that the easier version of non-periodic patrolling games is able to handle line graphs for larger values of m, as recently solved by Papadaki et al. (2016). It is likely that the techniques introduced here will be extended to larger attack durations in the future, but clearly additional ideas will be required.

In the case of the line graph, our discrete model could be applied for example to the problem of patrolling, possibly with a sniffer dog, a bank of linearly arranged airport security scanners, or a mountainous border with a discrete set of passes that can be crossed. In such cases, the "nodes" can be attacked at any time, around the clock, so the period T is likely to be the number of nodes that can be patrolled in a day. Other possibilities for defining T might be the attention span of the sniffer dog or the optimal time between refueling by a mobile vehicle, robot or UAV. In the last case, it is likely that the refuelling station might not be public knowledge, so we do not need to assume the Attacker can make use of this.

The paper is organized as follows. In Section 2, we review the related literature, then in Section 3 we formally define the game. In Section 4 we discuss some results for general graphs, showing how the game can be solved using notions from fractional graph theory if the patrol period is even. We then give a complete solution to the game played on a line graph in Section 5. In Section 6 we consider an extension of the game to the case of multiple patrollers, and show how our results on the line may be extended to this setting. Finally, we conclude in Section 7.

2 Literature Review

The problem of patrolling a border or channel against attack or infiltration goes back to the classical work of Morse and Kimball (1951). Since then many attempts have been made to improve the theory and practice of patrolling. Washburn (1982) considers an infiltrator who wants to maximize the probability of getting across a line in a channel. The case where the channel is blocked by fixed barriers has been consider by Baston and Bostock (1987) and the case when the barriers are moving has been analyzed by Washburn (2010). The case of a thick infiltrator has been considered by Baston and Kikuta (2009). If there are many infiltrators and they arrive in a Poisson manner, the analysis is given by Szechtman et al. (2008). Multiple infiltrators are also considered by Zoroa et al. (2012) where the infiltration is through a circular rather than a linear boundary. Multiple patrollers, when only some portions of the boundary need to be protected, are considered by Collins et al. (2013), who show how the problem can be divided up. Papadaki et al. (2016) consider the discrete border patrol problem, where the infiltration can only be accomplished at certain points of the border (perhaps mountain passes). When patrollers are restricted to periodic patrols, as here, the analysis of the continuous problem (with elements such as turning radius included) has been analyzed by Chung et al. (2011).

The more general problem of patrolling an arbitrary network against attacks at its nodes has been modeled as a game by Alpern et al. (2011), including a definition of the periodic patrolling game which we adopt here. Lin et al. (2013) developed more general approximate methods which cover such extensions as varying values for attacks at different nodes. Their methods, extended in Lin et al. (2014) to imperfect detection, can solve large scale problems. In the computer science literature, patrolling games with mobile robots and a Stackelberg model have been developed by Basilico et al. (2009, 2012). Multi vehicle patrolling problems have been solved by Hochbaum et al. (2014).

Infiltration games without mobile patrollers are analyzed in Garnaev et al. (1997), Alpern (1992), Baston and Garnaev (1996) and Baston and Kikuta (2004, 2009).

3 The Periodic Patrolling Game

In this section we formally define the patrolling game. There are three parameters: a graph $Q = Q(N, E)$ (where N is the set of nodes and E is the set of edges of Q), a period T, and an attack duration m (which we will take as 2 in this paper). The Attacker chooses a node i of Q to attack and a time interval of m consecutive periods in which to attack it. These m periods can be considered as an arc of the time circle $\mathcal{T} = \{1, 2, \dots, T, T+1 = 1\}$, on which arithmetic is carried out modulo T. So in the periodic game with $T = 24$ and $m = 5$, for example, a valid Attacker strategy would be the "overnight" attack, with attack interval $J = \{22, 23, 24, 1, 2\}$. Note that if Q has n nodes, then the number of possible attacks is given by nT, and the mixed attack strategy which chooses among them equiprobably will be called the *uniform attack strategy.* To foil the attack, the Patroller walks along the graph in an attempt to intercept it, that is, to be at the attacked node at some time during the attack interval. More precisely, a patrol is a walk w on Q with period T, that is, $w : \{1, 2, \ldots\} \to N$ with $w(t)$ and $w(t + 1)$ the same or adjacent nodes and $w(t + T) = w(T)$ for all t. A patrol w intercepts an attack at node i during attack interval J if $i \in w(J)$ or equivalently if $w(t) = i$ for some time t in the attack interval J. In such a case we say that the Patroller wins, and the payoff is 1; otherwise we say the Attacker wins, and the payoff is 0. Thus the payoff of the game corresponding to mixed strategies is the probability that the Patroller intercepts the attack. The value V of the game is the expected payoff (interception probability) with optimal play on both sides.

We note that in Alpern et al. (2011), this game is called the periodic patrolling game (one of two forms of the game considered there) and the value is denoted V^p. We assume throughout that the period is at least 2 and that the graph Q has at least $n = 2$ nodes.

4 General Graphs

In this section we obtain some bounds on the value V of the patrolling game on a general graph. The tools comprise the well known covering and independence numbers and a decomposition result taken from Alpern et al. (2011).

4.1 Covering and independence numbers $\mathcal{I}$ and $\mathcal{C}$.

We recall some elementary definitions about a graph Q. A set of nodes is called *independent* if no two of them are adjacent. The maximum cardinality of an independent set is called the independence number $\mathcal{I}$. Similarly a set of edges is called a *covering set* if every node of the graph is incident to one of these edges. The minimum cardinality of a covering set is called the covering number $\mathcal{C}$ of the graph. If I is a set of nodes and C is a covering set of edges there is, by the definition of covering set, a function $f : I \to C$ such that the edge $f(i)$ is incident to node i for all $i \in I$. If I is an independent set then, by the definition of independent set, the function f is injective, so that $|I| \leq |C|$ and hence $\mathcal{I} \leq \mathcal{C}$.

Suppose the Attacker attacks in some fixed time interval $\{t, t+1\}$ at a node chosen equiprobably from a set of $\mathcal{I}$ independent nodes. We call this an *independent attack strategy.* If a patrol intercepts one of these attacks at node $i \in \mathcal{I}$ at time t, he cannot intercept another at time $t + 1$, since none of the other attacks are at a node adjacent to i. Hence the probability of intercepting an attack cannot exceed $1/\mathcal{I}$ and therefore $V \leq 1/\mathcal{I}$. Next suppose T is even. In this case the Patroller fixes a covering set of $\mathcal{C}$ edges, picks a single edge amongst these randomly, and on that edge goes back and forth in an oscillation of length T. We call this Patroller mixed strategy an *unbiased covering strategy*, or, if the covering set consists of only an edge, an *unbiased oscillation.* Every node is visited by one of these patrols in every pair of consecutive time periods, and hence every attack of duration $m = 2$ is intercepted by at least one of these $\mathcal{C}$ patrols. Therefore the Patroller wins with probability at least $1/\mathcal{C}$. Hence we have shown the following.

Lemma 1 *The value of the Patrolling Game on any graph Q satisfies*

$$V \leq 1/\mathcal{I}, \textit{ and futhermore} \tag{1}$$
$$1/\mathcal{C} \leq V \leq 1/\mathcal{I}, \textit{ if } T \textit{ is even.} \tag{2}$$

A graph is called bipartite if its nodes can be partitioned into two sets such that no two nodes within the same set are adjacent. For bipartite graphs, we can say more.

Proposition 2 *Let Q be a bipartite graph. Then $\mathcal{C} = \mathcal{I}$ and the value V satisfies*

$$V = \frac{1}{\mathcal{C}} = \frac{1}{\mathcal{I}}, \textit{ if } T \textit{ is even, and} \tag{3}$$
$$\left(\frac{2T-1}{2T}\right)\frac{1}{\mathcal{C}} = \left(\frac{2T-1}{2T}\right)\frac{1}{\mathcal{I}} \leq V \leq \frac{1}{\mathcal{I}} \textit{ if } T \textit{ is odd.} \tag{4}$$

Proof. The first result (3) follows immediately from (2) and the well known fact (Konig's Theorem) that $\mathcal{C} = \mathcal{I}$ for bipartite graphs. The upper bound of (4) follows from (1). For the lower bound let $\{e_k\}_{k=1}^{\mathcal{C}}$ be a covering set of $\mathcal{C}$ edges, and let w_k denote the randomized walk of period T which oscillates on e_k except that it stays at a randomly chosen node of e_k for two consecutive times, also randomly chosen. We call this strategy of the Patroller a *biased covering strategy.* For example if $T = 7$ and the endpoints of e_k are a and b, the repeated sequence might be *ababbab*. Consider the Patroller strategy that chooses one of the randomized walks w_k equiprobably. If one of the nodes of e_k is attacked then the attack is detected if the Patroller chooses w_k (which happens with probabiliy $1/\mathcal{C}$) and the Patroller does not happen to choose to repeat this node for two consecutive periods that coincide with the time of attack (this happens with probability $1 - 1/(2T)$. So the total probabilty the attack is detected is $(1/\mathcal{C})\,(1 - 1/\,(2T))$, giving the lower bound for the value in (4). ■

We now give an example based on Lemma 1 and Proposition 2 for the line graph L_7 with nodes $\{1, \ldots, 7\}$ and edges $(i, i+1)$ $i = 1, \ldots, 6$. Since L_n is bipartite we can use the result in (3). We demonstrate the result for even period $T = 12$ (any even period would suffice but we pick 12 to be able to compare it with a later example in Section 5.6). A minimum covering set is $\{(1, 2), (3, 4), (5, 6), (6, 7)\}$ and thus $\mathcal{C} = 4$. An unbiased covering strategy for the Patroller consists of picking an edge at random from a minimum covering set (with probability 1/4) and performing an oscillation on that edge with period $T = 12$. Since $T = 12$ is even the oscillations performed on the chosen edges are unbiased (nodes are visited equally often). This is shown in Figure 1. This Patroller strategy intercepts attacks at nodes $1 - 5, 7$ with probability 1/4 and at node 6 with probability 1/2. Thus, the Patroller at worst can guarantee interception probability of at least 1/4. The Attacker would use the independent attack strategy and attack equiprobably on the independent set $\{1, 3, 5, 7\}$, which clearly guarantees him interception probability of at most 1/4. This gives the value of the game $V = 1/C = 1/4$.

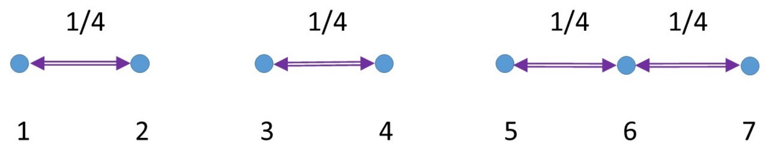

Figure 1: Unbiased covering strategy for the Patroller to oscillate on edges $\{(1, 2), (3, 4), (5, 6), (6, 7)\}$ of this minimum covering set.

The following gives an alternative upper bound to $1/\mathcal{I}$ on V based on the *uniform attack strategy*, which chooses equiprobably among the nT possible attacks (pure strategies). The reason that there are nT pure strategies is because in a game with period T, there are T periods that the attacker can start the attack: $1, 2, \ldots, T, T + 1 = 1$, at each node. The new upper bound is sometimes but not always better (lower) than $1/\mathcal{I}$.

Proposition 3 *By adopting the uniform strategy on a graph Q, the Patroller ensures the value of the periodic patrol game is bounded above by $2/n$. If T is odd and Q is bipartite, then the upper bound can be strengthened to $(2T - 1)/(nT)$.*

Proof. Suppose the Attacker adopts the uniform strategy on a graph Q, and let w be any Patroller pure strategy. If $w(t) = i$ and $w(t + 1) = j \neq i$ then in these two periods the Patroller can intercept at most four pure Attacker strategies, namely $[i, (t - 1, t)]$, $[i, (t, t + 1)]$ and $[j, (t, t + 1))]$, $[j, (t + 1, t + 2))]$, so 2 in each period and $2T$ in all. If $i = j$ then only the three attacks $[i,(t-1,t)]$, $[i,(t,t+1)]$ and $[i,(t+1,t+2)]$ can be intercepted. Since there are nT possible attacks, we have $V \le (2T)/(nT) = 2/n$.

If T is odd and Q is bipartite then $w(t) = w(t+1)$ for some t, so at most $2T-1$ attacks can be intercepted. Hence $V \le \frac{2T-1}{nT}$. ■

Note that it follows from the proof of Proposition 3 that against the uniform attack strategy, the interception probability will be strictly less than $2/n$ for any Patroller walk which repeats a node. This observation can be used to show that in some cases oscillations on an edge cannot be optimal. Consider the triangle graph shown in Figure 2, with $T=3$. If the Patroller adopts a random cyclic patrol, he intercepts any attack with probability $2/3$. Similarly, Proposition 3 shows that the uniform attack strategy is intercepted by any walk with probability not exceeding $2/3$, and so $V = 2/3$. On the other hand, if the Patroller uses oscillations on edges (or any walks other than the cycles), then he has repeated vertices and by the above remark cannot achieve interception probability $2/3$. So this example shows that in general, the Patroller cannot restrict to walks restricted to individual edges.

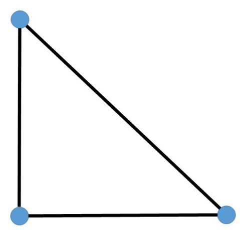

Figure 2: The triangle graph

The following situation will be important in analyzing the patrolling game on the n node line graph L_n with n even. For example, consider the edge covering of L_4 consisting of the edges $(1,2)$ and $(3,4)$ with $\mathcal{C} = 2 = n/2$. The covering edges are disjoint, unlike the graph of Figure 1.

Corollary 4 *Suppose T is odd, n is even and let Q be a bipartite graph with $\mathcal{C} = n/2$. Then $V = (2T-1)\,/\,(nT)$.*

Proof. Since $\mathcal{C} = n/2$, we have from (4) that

$$V \ge \frac{(2T-1)}{2\mathcal{C}T} = \frac{2\,(2T-1)}{2nT} = \frac{2T-1}{nT}$$

The result follows since for odd T we have from Proposition 3 that $V \le (2T-1)\,/\,(nT)$. ■

4.2 Even periods T

When the period T is even, we can solve the patrolling game on any graph $Q = Q(N, E)$ (where N is the set of nodes and E is the set of edges of Q) by extending the notions of covering and independence numbers to fractional forms. A more explicit solution for even T will be obtained later for line graphs.

Let $\mu : E \to [0,1]$ assign *edge weights* $\mu(e)$ to every edge e so that the total weight $\hat{\mu} = \sum_{e \in E} \mu(e)$ is minimized subject to the condition that for every node $i \in N$ the weights $\mu(e)$ of the edges e incident to i sum to at least 1. Such a μ is called an *optimal edge weighting* and $\hat{\mu}$ is called the *fractional covering number*.

Similarly let $\nu : N \to [0,1]$ assign node weights $\nu(i)$ to every node i so that the total weight $\hat{\nu} = \sum_i \nu(i)$ is maximized subject to the condition that sum of the weights $\nu(i)$ of the two endpoints i of every edge e is at most 1. Such a ν is called an *optimal node weighting* and $\hat{\nu}$ is called the *fractional independence number*. It is well known (see Scheinerman and Ullman, 2011) that $\hat{\mu} = \hat{\nu}$, a result that follows from either duality theory or the minimax theorem applied to the game where the maximizer picks an edge, the minimizer picks a node and the payoff is 1 if the node is incident to the edge and 0 otherwise. Note that, since the number of strategies in this game is polynomial in the number of nodes of the graph, an optimal edge weighting, an optimal node weighting and $\hat{\mu} = \hat{\nu}$ can be found efficiently.

Theorem 5 *If T is even, then the value of the patrolling game is given by $V = 1/\hat{\mu} = 1/\hat{\nu}$. An optimal strategy for the Patroller is to oscillate on edge e with probability $\mu(e)/\hat{\mu}$, where μ is any optimal edge weighting. An optimal strategy for Attacker to fix any interval $\{t, t+1\}$ and attack at node i with probabiltiy $\nu(i)/\hat{\nu}$, where ν is an optimal node weighting.*

Proof. Suppose the Patroller chooses the stated mixed strategy and the attack is at node i, in any time interval. The Patroller will intercept the attack if he has chosen to oscillate on an interval incident to i, which has probability at least $1/\hat{\mu}$ because the numerater is the sum of weights on edges incident to i. Similarly, suppose the Attacker adopts the stated mixed strategy. Let i and j be the nodes occupied by the Patroller at the attack times t and $t+1$. If $i \neq j$, and $e = \{i, j\}$ is the edge determined by $i \neq j$ then the probability of intercepting the attack is given by $\nu(i)/\hat{\nu} + \nu(j)/\hat{\nu} = (\nu(i) + \nu(j))/\hat{\nu} \le 1/\hat{\nu}$. If $i = j$ the same inequality holds. ■

Note that if we restrict the weights $\mu(e)$ and $\nu(i)$ to being 0 or 1 we get the usual covering number $\hat{\mu} = \mathcal{C}$ and independence number $\hat{\nu} = \mathcal{I}$. Thus, from linear programming theory and duality we have: $\mathcal{I} \le \hat{\nu} = \hat{\mu} \le \mathcal{C}$.

We consider, as an example, the graph depicted in Figure 3. It is not bipartite, so the covering number and independence number are not equal. The covering number is 3, and an

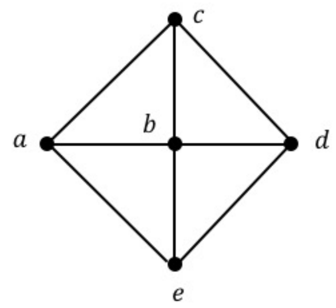

Figure 3: A non-bipartite graph

optimal covering is $\{ab, ac, de\}$ (where, for example ab denotes the edge with endpoints a and b). The independence number of the graph is 2, and a maximum cardinality independent set is $\{a, d\}$.

One optimal edge weighting is $\mu(ae) = 1, \mu(bc) = \mu(cd) = \mu(db) = 1/2$ and an optimal node weighting is given by $\nu(a) = \nu(b) = \nu(c) = \nu(d) = \nu(e) = 1/2$. Hence $\hat{\mu} = \hat{\nu} = 5/2$. This translates to an optimal Patroller strategy that oscillates on ae with probability $\mu(ae)/\hat{\mu} = 2/5$, and oscillates on bc, cd or bd each with probability $\mu(bc)/\hat{\mu} = 1/5$. And it translates to an optimal Attacker strategy of attacking at node i with probability $\nu(i)/\hat{\nu} = 1/5$, which is equivalent to the uniform Attacker strategy. We have $V = 1/\hat{\mu} = 1/\hat{\nu} = 1/(5/2) = 2/5$.

4.3 Patroller decomposition

As observed earlier in Alpern *et al.* (2011) the Patroller has the option of decomposing the given graph Q into subgraphs Q_1 and Q_2 and randomly choosing whether to play an optimal patrolling strategy on Q_1 or on Q_2. Specifically, suppose we write the node set N of Q as the (not necessarily disjoint) union $N_1 \cup N_2$, and define Q_i to be the graph with nodes N_i and edges between nodes that are adjacent in Q. Let V_i denote the value of the patrolling game on Q_i (with the same parameters as on Q). If the Patroller optimally patrols on Q_i with probability p_i, then any attack on a node in Q_i will be intercepted with probability at least $p_i V_i$. If the Patroller equalizes these two probabilities ($p_1 V_1 = p_2 V_2$) by choosing $p_1 = V_2/(V_1 + V_2)$, then he wins with probability at least

$$p_2 V_2 = p_1 V_1 = \frac{V_1 V_2}{V_1 + V_2}, \text{ and hence we have}$$

$$V \geq \frac{V_1 V_2}{V_1 + V_2}. \tag{5}$$

The right-hand side of (5) represents the highest interception probability that the Patroller can obtain by restricting patrols to one of the two subgraphs Q_1 or Q_2. So if strict inequality holds in (5) then it is suboptimal for the Patroller to decompose Q in this way. If (5) holds with equality, we say that the patrolling game on Q with period T is *decomposable.* Note that if the game for Q, T is decomposable this means that removing edges (or barring the Patroller from using them) connecting nodes in Q_1 to nodes in Q_2 does not lower the value of the game.

This derivation is simpler than that given in Alpern et al. (2011). We will use this method to solve one of the cases for the line graph in Section 5.5.

Consider the example in Figure 3. Take $N_1 = \{a, e\}$ and $N_2 = \{b, c, d\}$. We have $V_1 = 1$ (an oscillation intercepts any attack at a or e) and $V_2 = 2/3$, as shown in the analysis of the triangle graph in Figure 2. Using the decomposition result (5), we have

$$\begin{aligned} V &\geq \frac{V_1\ V_2}{V_1 + V_2} = \frac{2/3}{1+2/3} = \frac{2}{5} \text{ and Proposition 3 gives} \\ V &\leq \frac{2}{n} = \frac{2}{5}, \text{ so } V = 2/5, \end{aligned}$$

as shown earlier in the analysis of Figure 3, using different methods.

5 The Line Graph

We now concentrate our attention on the line graph L_n with node set $N = \{1, 2, \ldots, n\}$ and edges between consecutive numbers. This graph is bipartite, with the two node sets made up of the odd numbers and the even numbers. As mentioned in Proposition 2, this implies that $\mathcal{I} = \mathcal{C}$, and we may take the odd numbered nodes as a maximum independent set, giving

$$\mathcal{I} = \mathcal{C} = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even, and} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases} \tag{6}$$

The solution of the periodic patrolling game on the line breaks up into five cases, as outlined in Table 1. For the Attacker the strategies are simpler and have been defined earlier. However, for the Patroller the strategies are more complicated and specific details for some of them can be found at the corresponding propositions.

Case	Description	Value	Patroller strategy	Attacker strategy
1	T, n even Proposition 6	$\frac{2}{n}$	unbiased covering strategy Lemma 1	independent Lemma 1
2	T even, n odd Propostion 6	$\frac{2}{n+1}$	unbiased covering strategy Lemma 1	independent Lemma 1
3	T odd, n even Proposition 6	$\frac{2T-1}{nT}$	biased covering strategy Proposition 2	uniform Proposition 3
4	T, n odd, $n \geq 2T+1$ Propositions 9, 11	$\frac{2T-1}{nT}$	mixture of p-biased oscillations (Prop 9) or decomposed (Prop 11)	uniform Prop 7
5	T, n odd, $n \leq 2T-1$ Proposition 10	$\frac{2}{n+1}$	mixture of p-biased oscillations Proposition 10	independent Prop 7

Table 1: Solution of Patrolling Game on L_n, period T.

We give in Figure 4 a partition of (n, T) into the five cases of Table 1. The pattern is quite complicated.

T\n	2	3	4	5	6	7	8	9	10	11	12	13
2	1	2	1	2	1	2	1	2	1	2	1	2
3	3	5	3	5	3	4	3	4	3	4	3	4
4	1	2	1	2	1	2	1	2	1	2	1	2
5	3	5	3	5	3	5	3	5	3	4	3	4
6	1	2	1	2	1	2	1	2	1	2	1	2
7	3	5	3	5	3	5	3	5	3	5	3	5
8	1	2	1	2	1	2	1	2	1	2	1	2
9	3	5	3	5	3	5	3	5	3	5	3	5
10	1	2	1	2	1	2	1	2	1	2	1	2

Figure 4: Cases from Table 1 for pairs of (n, T).

5.1 Cases 1 to 3 (one of T or n is even)

If either T or n is even, there are three different forms for the value, but all follow easily from previous results.

Proposition 6 *For L_n, if T is even, then*

$$V = \frac{1}{\mathcal{C}} = \begin{cases} \frac{2}{n} & \text{if } n \text{ is even,} \\ \frac{2}{n+1} & \text{if } n \text{ is odd.} \end{cases} \tag{7}$$

If T is odd and n is even we have

$$V = \frac{2T-1}{nT}. \tag{8}$$

Proof.

First suppose that T is even. In this case, the result (7) easily follows from Proposition 2 and (6), since L_n is bipartite.

For T odd and n even, there is an edge covering of L_n with $\mathcal{C} = n/2$ disjoint edges of the form $\{2i-1, 2i\}$, $i = 1, \ldots, n/2$. Thus the result follows from Corollary 4. ■

Thus the only remaining cases (4 and 5) are when T and n are both odd. These are the complicated cases.

5.2 Comparison of uniform and independent attack strategies

For the remaining cases when T and n are both odd, we must compare the effectiveness of two different strategies for the Attacker: the uniform strategy, mentioned above, chooses equiprobably among all the nT possible pure stategies (at all n nodes at all T starting times); the independent strategy starts at time, say, 1 and chooses equiprobably among the $\mathcal{I}$ independent nodes. That is, the independent strategy chooses among $\mathcal{I}$ *simultaneous* attacks. We have already obtained two different upper bounds on V for these cases: $(2T-1)/(nT)$ from Proposition 3, for the uniform attack strategy; and $2/(n+1)$ from (4), for the independent strategy (since $\mathcal{I} = \frac{n+1}{2}$).

Note that the upper bound $(2T-1)/(nT)$ is smaller than the upper bound $2/(n+1)$ for $n \geq 2T+1$ and the reverse holds for $n \leq 2T-1$. Since the Attacker can choose the attack (uniform or independent) which gives the smaller upper bound on the value, we can summarize his options as follows.

Proposition 7 *Suppose T and n are both odd, and $Q = L_n$. Then*

$$V \leq \min\left(\frac{2T-1}{nT}, \frac{2}{n+1}\right) = \begin{cases} \frac{2}{n+1} & \text{if } n \leq 2T-1, \\ \frac{2T-1}{nT} & \text{if } n \geq 2T+1. \end{cases}$$

We now analyze these two cases in sections 5.3 and 5.4 respectively. For the Patroller strategies we shall use oscillations which are similar to the walks w_k which appeared in the proof of Proposition 2.

5.3 Case 4 (T, n odd, $n \geq 2T+1$)

To deal with the case of $n \geq 2T+1$ and noting that the oddness of T requires a stunted type of oscillation, we define *p-biased oscillations* as follows.

Definition 8 *For $p \in [0,1]$, a* **right p-biased oscillation** $\overrightarrow{b}_p(i)$ *(for $i = 1, \ldots, n-1$) is a T-periodic walk between i and $i+1$ where i and $i+1$ alternate except that with probability p, at a random time, the right-hand node $i+1$ is repeated (if $T = 2q+1$, it is at node $i+1$ for $q+1$ periods and at i for q periods); with probability $1-p$, at a random time, the left-hand node is repeated. For convenience, we define a* **left p-biased oscillation** $\overleftarrow{b}_p(i)$ *as* $\overrightarrow{b}_{1-p}(i)$*. If $p = 1/2$, we will refer to a right (or left) p-biased oscillation as an* **unbiased oscillation**.

For the following result note that for larger n the uniform attack strategy is better for the Attacker than the independent attack strategy.

Proposition 9 *For L_n, assume that both T and n are odd and that $2T \leq n-1$. Then $V = \frac{2T-1}{nT}$. The uniform attack strategy is optimal for the Attacker and a probabilistic choice of biased oscillations is optimal for the Patroller.*

The reader is invited to read the example in Table 2 and commentary to obtain some intuition for the proof.

Proof. From Proposition 7 we know that $V \leq \frac{2T-1}{nT}$, so it is enough to demonstrate a Patroller strategy which intercepts an attack at any node i with probability at least $\frac{2T-1}{nT}$.

For $j = 1, \ldots, (n+1)/2$, let A_j be the set of edges of the form $(2i-1, 2i)$ for $i < j$. For example, A_1 is empty and $A_3 = \{(1,2),(3,4)\}$. Also let B_j be the set of edges of the form $(2i, 2i+1)$ for $i \geq j$, so $B_1 = \{(2,3),(4,5),\ldots,(n-1,n)\}$ and $B_3 = \{(6,7),(8,9),\ldots,(n-1,n)\}$. Finally let $D_j = A_j \cup B_j$.

For example when $n = 7$ we have $A_2 = \{(1,2)\}$, $B_2 = \{(4,5),(6,7)\}$ and $D_2 = \{(1,2),(4,5),(6,7)\}$, as shown by the three arrows (for edges) on the second line from the top in Table 2. The arrows are oriented left for edges in A_2 and right for those in B_2 to indicate the Patroller's use of left or right biased oscillations on these edges in his optimal strategy.

There are $(n-1)/2$ edges in D_j, and each node in the line graph except one is incident to some edge in D_j, for each j.

Consider the following Patroller strategy. First some j is chosen uniformly at random, $j = 1, \ldots, (n+1)/2$ and an edge $(i, i+1)$ in D_j is chosen uniformly at random. If $(i, i+1)$ is contained in A_j then the Patroller performs a left p-biased oscillation $\overleftarrow{b}_p(i)$. If $(i, i+1)$ is in B_j then the Patroller performs a right p-biased oscillation $\overrightarrow{b}_p(i)$. This probability p will be determined later.

If a node is either on the left of an edge in some A_j that is being patrolled or if it is on the right of an edge in some B_j that is being patrolled, then an attack at that node is intercepted with probability:

$$p \cdot 1 + (1-p) \cdot (T-1)/T = (T+p-1)/T. \tag{9}$$

If a node is either on the *right* of an edge in some A_j that is being patrolled or if it is on the *left* of an edge in some B_j that is being patrolled, then an attack at that node is intercepted with probability:

$$p \cdot (T-1)/T + (1-p) \cdot 1 = (T-p)/T. \tag{10}$$

We first calculate the probability p_{2i} that an attack at an even numbered node $2i$ is intercepted, $i = 1, \ldots, (n-1)/2$. Observe that for every one of the $(n+1)/2$ values of j, the node $2i$ is either on the right of an edge in A_j or on the left of an edge in B_j, so

$$p_{2i} = \left(\frac{1}{(n-1)/2}\right)\left(\frac{T-p}{T}\right) = \frac{2(T-p)}{(n-1)T}. \tag{11}$$

For an odd numbered node $2i-1, i = 1, \ldots, (n+1)/2$, we observe that there are $(n-1)/2$ values of j such that the node $2i-1$ is either on the left of an edge in A_j or on the right of an edge in B_j. There is one value of j such that node $2i-1$ is not incident to any edge in A_j or B_j. So the probability p_{2i-1} that an attack at node $2i-1$ is intercepted is

$$p_{2i-1} = \left(\frac{1}{(n-1)/2}\right)\left(\frac{(n-1)/2}{(n+1)/2}\right)\left(\frac{T+p-1}{T}\right) = \frac{2(T+p-1)}{(n+1)T}. \tag{12}$$

Since $2T \leq n-1 \leq n+1$, we may choose $p = (2T+n-1)/(2n)$ so that the probabilities p_{2i} and p_{2i-1} are equal, and substituting this value of p into (11) or (12), we obtain the bound

$$V \geq \frac{2(T-(2T+n-1)/2n)}{(n-1)T} = \frac{(2T-1)}{nT}.$$

Combining this with our lower bound, this establishes the proposition. ■

We illustrate the Patroller's optimal strategy, taking L_7 as an example, with $T = 3$ in Table 2. The four choices of $D_1, \ldots, D_4$ correspond to the four rows in Table 2. The left pointing arrows correspond to the edges in the A_j and the right pointing arrows correspond to the edges in the B_j. Nodes which are incident to one of the edges in D_j, are indicated by a solid disk, those which are not, by an outlined disk.

The Patroller picks one of the rows of the table at random, and then one of the arrows in that row at random, corresponding to an edge $(i, i+1)$. Equivalently, he picks one of the 12 arrows at random. Then he performs a left or right p-biased oscillation, depending on the

1		2		3		4		5		6		7	
○		•	$\Longrightarrow$	•		•	$\Longrightarrow$	•		•	$\Longrightarrow$	•	D_1
•	$\Longleftarrow$	•		○		•	$\Longrightarrow$	•		•	$\Longrightarrow$	•	D_2
•	$\Longleftarrow$	•		•	$\Longleftarrow$	•		○		•	$\Longrightarrow$	•	D_3
•	$\Longleftarrow$	•		•	$\Longleftarrow$	•		•	$\Longleftarrow$	•		○	D_4

Table 2: Optimal strategy for L_7 with $T = 3$.

direction of the arrow, where $p = (2T + n - 1)/(2n) = 12/14 = 6/7$. If a node has three arrows pointing toward it (odd nodes), then an attack at that node is intercepted with probability $(3/12)(p + (1-p)(T-1)/T) = (1/4)(6/7 + (1/7)(2/3)) = 5/21$. If, on the other hand, a node has four arrows pointing away from it (even nodes), then an attack at that node is intercepted with probability $(4/12)((1-p) + p(T-1)/T) = (1/3)(1/7 + (6/7)(2/3)) = 5/21$. So the value is $5/21 = (2T-1)/(nT)$.

5.4 Case 5 (T, n odd, $n \leq 2T - 1$)

We now consider the remaining open case of n and T odd and $n \leq 2T - 1$.

Proposition 10 *For L_n, assume that both T and n are odd and that $n \leq 2T - 1$. Then $V = \frac{2}{n+1}$. The independent strategy is optimal for the Attacker and a probabilistic choice of biased oscillations is optimal for the Patroller.*

Proof. It follows from Proposition 7 that $V \leq 2/(n+1)$. To prove the reverse bound on the value, we simply use the Patroller strategy described in the proof of Proposition 9, but this time taking $p = 1$ in Equations (12) and (11) to obtain

$$p_{2i-1} = \frac{2(T+p-1)}{(n+1)T} = \frac{2}{n+1} \text{ and } p_{2i} = \frac{2(T-1)}{(n-1)T} \geq \frac{2}{n+1},$$

where the last inequality follows directly from $n \leq 2T - 1$. Thus, we have $V \geq 2/(n+1)$. ■

5.5 Patroller decomposition strategies

We may now also give an alternative optimal strategy for the Patroller in case 4, using a decomposition of the line graph.

Proposition 11 *For L_n, if T and n are odd and $n > 2T - 1$ then $V = \frac{2T-1}{nT}$. The uniform strategy is optimal for the Attacker. For the Patroller there is an optimal strategy which* *decomposes the graph* $Q = L_n$ *into a left graph* $\mathcal{L} = L_{n_\mathcal{L}}$ *with the odd number* $n_\mathcal{L} = 2T - 1$ *of nodes* $\{1, 2, \ldots, 2T - 1\}$ *and a right graph* $\mathcal{R} = L_{n_\mathcal{R}}$ *with the remaining even number* $n_\mathcal{R} = n - (2T - 1)$ *of nodes* $\{2T, 2T + 1, \ldots, n\}$.

Proof. The adoption of the uniform attacker strategy guarantees that $V \geq \frac{2T-1}{nT}$ by Proposition 3. The left graph $\mathcal{L}$ satisfies the hypotheses of Proposition 10, because $n_\mathcal{L} \leq 2T-1$ (equality holds) and T and $n_\mathcal{L}$ are odd. Hence Proposition 10 gives

$$V(\mathcal{L}) = \frac{2}{n_\mathcal{L} + 1} = \frac{2}{2T}.$$

The subgraph $\mathcal{R}$ has an even number of nodes $n_\mathcal{R}$, so it satisfies the hypothesis of Proposition 6, hence equation (8) gives

$$V(\mathcal{R}) = \frac{2T - 1}{n_\mathcal{R} T} = \frac{2T - 1}{(n - (2T - 1))\,T}.$$

It follows from the decomposition estimate (5) that

$$V = V(L_n) \geq \frac{V(\mathcal{L})\ V(\mathcal{R})}{V(\mathcal{L}) + V(\mathcal{R})} = \frac{2T - 1}{nT}.$$

■

As an example, consider again the case $T = 3$ and $n = 7 > 2T - 1 = 5$, as considered in Section 5.3. As we know, $V_7 = 5/21$. We decompose L_7 into $\mathcal{L} = L_5$ and $\mathcal{R} = L_2$. On L_5, the optimal Patroller strategy is given by Proposition 10. On L_2 the optimal Patroller strategy is an unbiased oscillation on the single edge $(6, 7)$.

According to Section 4.3, the probabilities p_5 and p_2 of patrolling on L_5 and L_2 should satisfy $p_5 V_5 = p_2 V_2$. Since $V_5 = 2/(5 + 1) = 1/3$ and $V_2 = (2 \cdot 3 - 1)/(2 \cdot 3) = 5/6$, we have $p_5 = 5/7$ and $p_2 = 2/7$.

We may represent this strategy by the diagram in Table 3, where L_7 is decomposed into $\mathcal{L} = L_5$ (on the left) and $\mathcal{R} = L_2$ (on the right). The Patroller first chooses L_5 with probability $p_5 = 5/7$ and L_2 with probability $2/7$. If he chooses L_2 then he performs an unbiased oscillation (indicated by the double-ended arrow) on edge $(6, 7)$. If he chooses L_5 then he chooses one of the single-ended arrows at random and performs a left or right biased p-oscillation, depending on the direction of the arrow, with $p = 1$.

We can now determine for which values of T and n the line graph L_n is decomposable (equality in (5)), in the sense that the Patroller can restrict his patrols to one of two disjoint subgraphs without loss of optimality.

Proposition 12 *The patrolling game on the line is decomposable unless* T *and* n *are odd and*

1		2		3		4		5	6		7
$\circ$		$\bullet$	$\Longrightarrow$	$\bullet$		$\bullet$	$\Longrightarrow$	$\bullet$	$\bullet$	$\Longleftrightarrow$	$\bullet$
$\bullet$	$\Longleftarrow$	$\bullet$		$\circ$		$\bullet$	$\Longrightarrow$	$\bullet$	$\bullet$	$\Longleftrightarrow$	$\bullet$
$\bullet$	$\Longleftarrow$	$\bullet$		$\bullet$	$\Longleftarrow$	$\bullet$		$\circ$	$\bullet$	$\Longleftrightarrow$	$\bullet$

Table 3: Decomposed strategy for L_7 with $T = 3$.

$n \leq 2T - 1$ *(case 5).*

Proof. First we show that for cases 1 through 4 in Table 1, the patrolling game is decomposable (by the Patroller). In cases 1, 2 and 3, the Patroller uses what we call covering strategies, in that his pure patrols are on edges forming a minimum covering set. For $n \geq 4$, such as set can include the edge $(1, 2)$ and $(3, 4)$ and in particular the Patroller can avoid using the edge $(2, 3)$. It follows that he is decomposing L_n into $\mathcal{L} = L_2$ and $\mathcal{R} = L_{n-2}$ with disjoint nodes sets $\{1, 2\}$ and $\{3, \ldots, n\}$. (If T is even and $n = 3$, then instead of using the covering strategy involving edges $(1, 2)$ and $(2, 3)$, the Patroller decomposes the game by equiprobably oscillating on edge $(1, 2)$ and remaining stationary on node 3 to obtain an interception probability of $1/2 = V(L_3)$.) For case 4, the optimal Patroller strategy given in Proposition 9 does not decompose the game. However an optimal strategy which does decompose the game is given in Proposition 10, where L_n, n odd, is decomposed into $\mathcal{L} = L_{2T-1}$ and $\mathcal{R} = L_{n-(2T-1)}$. This is a strategy where the Patroller never traverses the edge $(2T - 1, 2T)$.

So assume that T and n are odd and $n \leq 2T - 1$ (case 5). So any decomposition of L_n is into an even node line graph L_{2j}, $j > 0$ and an odd one L_{n-2j}. The assumptions on T and n are covered by Proposition 9, so we have

$$V(L_n) = \frac{2}{n+1}.$$

Since $2j$ is even, it follows from (8) in Proposition 5, that

$$V(L_{2j}) = \frac{2T-1}{2jT}.$$

Since $n - 2j$ is odd and $n - 2j < n \leq 2T - 1$, it follows from Proposition 9 that

$$V(L_{n-2j}) = \frac{2}{(n-2j)+1}.$$

The best the Patroller can do by such a decomposition (see Section 3.2) is to obtain an interception probability of

$$\frac{V\left(L_{2j}\right) * V\left(L_{n-2j}\right)}{V\left(L_{2j}\right) + V\left(L_{n-2j}\right)}.$$

The difference between the unrestricted value and the restricted one is given above is

$$\begin{aligned} V\left(L_n\right) - \frac{V\left(L_{2j}\right) * V\left(L_{n-2j}\right)}{V\left(L_{2j}\right) + V\left(L_{n-2j}\right)} &= \frac{2}{n+1} - \frac{\left(\frac{2T-1}{2jT}\right) * \left(\frac{2}{(n-2j)+1}\right)}{\left(\frac{2T-1}{2jT}\right) + \left(\frac{2}{(n-2j)+1}\right)} \\ &= \frac{4j}{(n+1)\left(2j + (2T-1)(1+n)\right)} > 0. \end{aligned}$$

■

5.6 Connections to non-periodic game

Compared with games with simply a fixed time horizon T, the problem with *period* T is more difficult for the Patroller, as he has the additional requirement that he has to end at the same node as he started. However as the period gets large, this restriction is less oppressive to the Patroller, because the amount of time he must use to get back to his start is the fixed diameter of the graph. In this subsection we check that the limiting value of $V(T, n)$ for the game with period T approaches the value $V(n)$ found for the patrolling game on the graph L_n without periodic patrols. For $m = 2$ the values found in Papadaki et al. (2016) are simply $V(n) = 1/\lceil n/2 \rceil$, that is, $2/n$ for even n and $2/(n+1)$ for odd n. If we look at the values $V(T, n)$ for periodic patrols found for the five cases, looking back at Table 1, for cases 1, 2, 3 and 5 (case 4 does not hold as T goes to infinity), we obtain the same limiting value

$$\lim_{T \to \infty} V(T, n) = V(n) = 1/\lceil n/2 \rceil, \text{ for all } n.$$

Of course this is not an easy way of establishing the nonperiodic result, as the periodic case dealt with here is more complicated.

The solution of the non-periodic game on L_n as given in Papadaki et al. (2016) involves periodic patrols of different periods $T_1, \ldots, T_k$. Setting T^* to be the least common multiple of $\{T_1, T_2, \ldots, T_k\}$, we see that the solution has period T^*. If we were seeking a solution to the periodic game with set period T^* the same solution would be valid.

Let us consider the example with $n = 7$, $m = 2$ (in the non-periodic game there is no given T). The solution given there is as follows: with probability $1/8$ adopt unbiased oscillations on edges $(1, 2)$ and $(6, 7)$ and with probability $6/8$ adopt a tour of L_n of period $2(n-1) = 12$ that goes back and forth between the end nodes. This is illustrated in Figure 5.

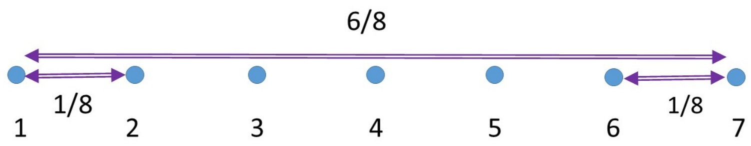

Figure 5: Patroller oscillates between end nodes with probability $6/8$ and on edges $(1,2)$ and $(6,7)$ each with probability $1/8$ in L_7.

It is easy to check that the probability that the tour of L_n (of period 12) intercepts attacks at nodes $1, 2, \ldots, 7$ is given respectively by $2/12, 4/12, 4/12, 4/12, 4/12, 4/12, 2/12$. The 12-cycle can be written, starting at say node 3, as $3^*, 4^*, 5^*, 6^*, 7, 6, 5, 4, 3, 2, 1^*, 2^*, \ldots,$ where * indicates going to the right. Note that an attack at node 4 starting at time t will be intercepted if the Patroller following this cycle is at one of the four steps $5, 4$ or $3^*, 4^*$ out of the twelve steps in the cycle, that is, with probability $4/12$. The other probabilities are calculated in a similar manner.

We now calculate the probability that the mixed strategy stated above intercepts an attack at each node. For node 1 such an attack is intercepted with probability $2/12$ by the big oscillation and with probability 1 by the oscillation on edge $(1,2)$. Hence, the total interception probability is given by $(6/8)(2/12) + (1/8)(1) = 1/4$. Similarly we calculate the probability at nodes 2 and 3 as $3/8$ and $1/4$; by symmetry the interception probability for node 4 is the same as node 3 and the interception probabilities for nodes $5, 6, 7$ are the same as nodes $3, 2, 1$ respectively. So the overall interception probabilities for nodes $\{1, 2, ..., 7\}$ are given by $\{1/4, 3/8, 1/4, 1/4, 1/4, 3/8, 1/4\}$. The minimum is $1/4$, which is also the value of $m/(n+m-1) = 1/4$, given by Papadaki et al. (2016). Note that the Attacker can achieve a successful attack with probability $1/4$ by attacking equiprobably simultaneously at the nodes of the independent set $\{1, 3, 5, 7\}$.

To compare the above analysis with the periodic game of this paper, observe that the three oscillations used in the optimal mixed strategy above have periods $T_1 = T_2 = 2$ and $T_3 = 12$, with least common multiple of $T^* = 12$. So this also gives a solution to the periodic game with $n = 7$ and $T = T^* = 12$. Since T^* is even and n is odd our formula given in Proposition 6, case 2, is $2/(n+1) = 1/4$. The two analyses agree on the value. Note however, that the patrolling strategy given above differs from that given by our analysis of the periodic game with $T = 12$ and $n = 7$ given in Section 4.1, Figure 1. Note also that for both patrolling strategies the nodes which are unfavourable to attack are the penultimate nodes 2 and 6. This shows that the Patroller strategies that we give in our analysis are not uniquely optimal. While this gives an alternative method of analyzing the periodic game $T = 12$, $n = 7$, it does not solve it in general. For example it would not solve the game for, say, $T = 11$.

6 Multiple Patrollers on the Line

We now consider a generalization of the game, where there are k Patrollers. The Attacker's strategy set is the same, but his opponent chooses k periodic walks on L_n, corresponding to k patrols. The attack is intercepted and the payoff is 1 if any of the Patrollers intercept the attack.

Let $V^{(k)}$ denote the value of the game when there are k Patrollers, and write $V_n^{(k)}$ for the value of the k Patroller game on L_n. Suppose in the single Patroller game the Patroller plays first as in the k game but then picks a Patroller randomly. Thus he wins with probability at least $V^{(k)}/k$, and hence

$$V^{(k)} \leq kV. \tag{13}$$

That is, k Patrollers can intercept an attack with probability at most k times the probability that a single Patroller can intercept an attack.

The estimate holds with equality if and only if the k Patrollers can jointly patrol in such a way that each one is following an optimal strategy for $k = 1$ and furthermore no possible attack is simultaneously intercepted by more than one of the Patrollers.

If we assume $k \leq n/2$ then it is easy to adapt our optimal strategies described in the sections above for $k = 1$ to the more general game where $k > 1$. As an example, take case 4, with $n = 7$, $T = 3$ and $k = 3$. An optimal Patroller strategy for $k = 1$ is depicted in Table 2: recall that the Patroller chooses one of the 12 arrows at random and performs a left or right p-biased oscillation, depending on the direction of the arrow, where $p = 6/7$.

An optimal strategy for $k = 3$ simply chooses a row at random and assigns one of the 3 Patrollers to each arrow. This clearly implies that $V_n^{(k)} = 3V_n$. For $k = 2$ the Patroller chooses a row at random and randomly assigns the 2 Patrollers to 2 of the 3 arrows. Note that this extension to $k > 1$ Patrollers works for any $k \leq 3$ but not for $k > 3$. For example this particular argument does not work for $k = 4$. Note also that the alternative decomposed strategy for case 4, described in Section 5.5 cannot be extended to $k > 1$ Patrollers in the same way.

Similarly, for the other cases, as long as $k \leq n/2$, the Patroller's strategy for $k = 1$ can be extended to $k > 1$. We omit the details, as the extensions are straightforward. Hence we have the following theorem.

Theorem 13 *For $k \leq n/2$, the value $V_n^{(k)}$ of the k Patroller game on the line graph L_n satisfies $V_n^{(k)} = kV_n$.*

It is natural to question whether, for $k > n/2$, the value of the game is $\min\{kV_n, 1\}$. Indeed, for T even, it is easy to see that this is true, since for $k > n/2$, the Patroller can win the game with probability 1 by oscillating on k covering edges.

But for T odd, it is not true. Consider the same example of $n = 7$ and $T = 3$ but this time with $k = 4$ Patrollers. In this case, the bound (13) gives $V_7^{(4)} \leq 4V_7 = 20/21$. In fact we will show that $V_7^{(4)} = 19/21$. Suppose the Attacker employs the uniform strategy, so that he chooses uniformly between all 21 possible attacks. We will show that the Patrollers can intercept no more than 19 of the attacks, so that $V_7^{(4)} \leq 19/21$.

Suppose, for a contradiction, that the Patrollers were able to intercept 20 of the attacks. Then there must be 6 nodes at which all 3 attacks are intercepted, because if there were only 5 such nodes, then the total possible number of attacks that were intercepted would be at most $5 \cdot 3 + 2 \cdot 2 = 19$. This means that each of these 6 nodes must be visited by some Patroller in at least 2 of the 3 time periods, which accounts for $6 \cdot 2 = 12$ pairs of nodes and time periods. But since there are only 4 Patrollers, they cannot occupy more than $4 \cdot 3 = 12$ pairs of nodes and time periods in total. So none of the 3 attacks at the 7th node can be intercepted, so that the total number of attacks intercepted is only 18, a contradiction. Hence, $V_7^{(4)} \leq 19/21$.

To see that the value $V_7^{(4)}$ is in fact exactly equal to 19/21, we present a strategy for the Patrollers that mixes between 21 pure strategies. The strategies are indexed by pairs $(i, j)_t$, where (i, j) ranges over the values $\{(1, 3), (2, 3), (2, 4), (1, 7), (5, 7), (5, 6), (4, 6)\}$ and t takes the values $1, 2$ or 3. Strategy $(i, j)_t$ has the property that all 21 possible attacks are intercepted except the two attacks that take place at nodes i and j at times $(t, t + 1)$ (where addition here is modulo 3). If such patrols exist, then it is easy to check that any given attack is intercepted by exactly 19 of the 21 patrols. This means that the mixed strategy that chooses one of these 21 pure strategies uniformly at random guarantees an expected interception probability of at least 19/21 against any given attack, so that $V \geq 19/21$.

So we just need to show that such patrols do indeed exist. Table 4 illustrates seven of the pure strategies $(i, j)_t$. The nodes are listed in the first row and time in the second column and the entries in the table correspond to the four Patrollers.

This accounts for 7 of the 21 pure strategies $(i, j)_t$, and the other 14 can be obtained from these ones by a translation through time. For example, the strategy $(5, 6)_2$ can be obtained by translating $(5, 6)_1$ in time by $+1$.

	Time	1	2	3	4	5	6	7
Strategy $(1,3)_1$	1		1		2		3	4
	2		1		2	3		4
	3	1		2		3	4	
Strategy $(2,3)_1$	1	1			2	3	4	
	2	1			2	3		4
	3		1	2			3	4
Strategy $(2,4)_1$	1	1		2		3	4	
	2	1		2		3		4
	3		1		2		3	4
Strategy $(1,7)_1$	1		1	2	3		4	
	2		1	2		3	4	
	3	1			2	3		4
Strategy $(5,7)_1$	1	1	2		3		4	
	2	1		2	3		4	
	3		1	2		3		4
Strategy $(5,6)_1$	1		1	2	3			4
	2	1		2	3			4
	3	1	2			3	4	
Strategy $(4,6)_1$	1		1	2		3		4
	2	1		2		3		4
	3	1	2		3		4	

Table 4: Part of the optimal Patroller strategy for L_7 with $T = 3$ and $k = 4$.

It is clear that for $T = 3$ and $n = 7$, the value of the game is equal to 1 for $k \geq 5$, since the addition of one more suitably placed Patroller in, say strategy $(2,3)_1$, results in a pure strategy for the Patrollers that guarantees an interception probability of 1.

7 Conclusions

This paper has begun the study of periodic patrols on the line, by giving a complete solution to the case of short attack duration $m = 2$. One reason that the case $m = 2$ is susceptible to our analysis is that, at least for even T, the covering number can be identified with the minimum number of patrols that are required to intercept any attack. This is not true for large m. The periodic patrolling game is much more difficult to solve than the unrestricted version of the game (where patrols are not required to have a given period). The latter can be solved for line graphs of arbitrary size and arbitrary attack duration, as long as the time horizon is sufficiently large, as shown in Papadaki et al. (2016).

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References

[1] Alpern, S., Morton, A., & Papadaki, K. (2011). Patrolling games. Operations Research, 59(5), 1246–1257.

[2] Alpern, S. (1992). Infiltration games on arbitrary graphs. Journal of Mathematical Analysis and Applications, 163(1), 286–288.

[3] Basilico, N., Gatti, N., & Amigoni, F. (2009). A formal framework for mobile robot patrolling in arbitrary environments with adversaries. arXiv preprint arXiv:0912.3275.

[4] Basilico, N., Gatti, N., Amigoni, F. (2012). Patrolling security games: Definition and algorithms for solving large instances with single Patroller and single intruder. Artificial Intelligence, 184, 78–123.

[5] Baston, V. J., & Bostock, F. A. (1987). A continuous game of ambush. Naval Research Logistics, 34(5), 645–654.

[6] Baston, V. J., & Garnaev A. Y. (1996). A fast infiltration game on n arcs. Naval Research Logistics, 43(4), 481–490.

[7] Baston, V., & Kikuta, K. (2004). An ambush game with an unknown number of infiltrators. Operations Research 52(4), 597–605.

[8] Baston, V., & Kikuta, K. (2009). Technical Note - An Ambush Game with a Fat Infiltrator. Operations Research 57(2), 514–519.

[9] Baykal-Gürsoy, M., Duan, Z., Poor, H. V., & Garnaev, A. (2014). Infrastructure security games. European Journal of Operational Research, 239(2), 469–478.

[10] Chung, H., Polak, E., Royset, J. O., & Sastry, S. (2011). On the optimal detection of an underwater intruder in a channel using unmanned underwater vehicles. Naval Research Logistics, 58(8), 804–820.

[11] Collins, A., Czyzowicz, J., Gasieniec, L., Kosowski, A., Kranakis, E., Krizanc, D., & Morales Ponce, O. (2013). Optimal patrolling of fragmented boundaries. ACM Proceedings of the twenty-fifth annual ACM symposium on Parallelism in algorithms and architectures, 241–250.

[12] Fokkink, R., & Lindelauf R. (2013). The Application of Search Games to Counter Terrorism Studies. Handbook of Computational Approaches to Counterterrorism, 543–557, Springer New York.

[13] Gal, S. (1979). Search games with mobile and immobile hider. SIAM Journal of Control and Optimization, 17, 99–122.

[14] Gal, S. (2000). On the optimality of a simple strategy for searching graphs. International Journal of Game Theory, 6(29), 533–542.

[15] Garnaev, A., Garnaeva, G., & Goutal, P. (1997). On the infiltration game. International Journal of Game Theory, 26(2), 15–221.

[16] Hochbaum, D. S., Lyu, C., & Ordóñez, F. (2014). Security routing games with multivehicle Chinese postman problem. Networks, 64(3), 181–191.

[17] Lin, K. Y., Atkinson, M. P., Chung, T. H., & Glazebrook, K. D. (2013). A graph patrol problem with random attack times. Operations Research, 61(3), 694–710.

[18] Lin, K. Y., Atkinson, M. P., & Glazebrook, K. D. (2014). Optimal patrol to uncover threats in time when detection is imperfect. Naval Research Logistics, 61(8), 557–576.

[19] Morse, P. M., & Kimball, G. E. (1951). Methods of Operations Research. MIT Press and Wiley, New York.

[20] Papadaki, K., Alpern, S., Lidbetter, T., & Morton, A. (2016). Patrolling a Border. Operations Research, 64(6), 1256–1269.

[21] Ruckle, W. (1983). Geometric Games and Their Applications. Pitman, Boston.

[22] Scheinerman ER, Ullman DH (2011). *Fractional graph theory: a rational approach to the theory of graphs*. Courier Corporation.

[23] Szechtman, R., Kress, M., Lin, K., & Cfir, D. (2008). Models of sensor operations for border surveillance. Naval Research Logistics, 55(1), 27–41.

[24] Washburn, A. R. (1982). On patrolling a channel. Naval Research Logistics, 29(4), 609–615.

[25] Washburn, A. (2010). Barrier games. Military Operations Research, 15(3), 31–41.

[26] Zoroa, N., Fernández-Sáez, M. J., & Zoroa, P. (2012). Patrolling a perimeter. European Journal of Operational Research, 222(3), 571–582.