

[N. H. Bingham](#) and A. J. Ostaszewski

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## Variants on the Berz sublinearity theorem

N. H. BINGHAM AND A. J. OSTASZEWSKI

*To Roy O. Davies on his 90th birthday.*

**Abstract.** We consider variants on the classical Berz sublinearity theorem, using only DC, the Axiom of Dependent Choices, rather than AC, the Axiom of Choice, which Berz used. We consider thinned versions, in which conditions are imposed on only part of the domain of the function—results of quantifier-weakening type. There are connections with classical results on subadditivity. We close with a discussion of the extensive related literature.

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**Keywords.** Dependent Choices, Homomorphism extension, Subadditivity, Sublinearity, Steinhaus–Weil property, Thinning, Quantifier weakening.

### 1. Introduction: sublinearity

We are concerned here with two questions. The first is to prove, as directly as possible, a linearity result via an appropriate group-homomorphism analogue of the classical Hahn–Banach Extension Theorem *HBE* [5]—see [18] for a survey. Much of the *HBE* literature most naturally elects as its context *real* Riesz spaces (ordered linear spaces equipped with semigroup action, see Sect. 4.8), where some naive analogues can fail—see [19]. These do not cover our test-case of the additive reals  $\mathbb{R}$ , with focus on the fact (e.g. [14]) that for  $\mathbb{A} \subseteq \mathbb{R}$  a dense subgroup, if  $f : \mathbb{A} \rightarrow \mathbb{R}$  is additive (i.e. a partial homomorphism) and locally bounded (see Theorem R in Sect. 2), then it is linear:  $f(a) := ca$  for some  $c \in \mathbb{R}$  and all  $a \in \mathbb{A}$ . Can this result be deduced by starting with some natural, continuous, subadditive majorant  $S : \mathbb{R} \rightarrow \mathbb{R}$  (so that, equivalently,  $\bar{S}|_{\mathbb{A}} \leq f \leq S|_{\mathbb{A}}$  for  $\bar{S}(\cdot) = -S(-\cdot)$ , which is super-additive) and then invoking an (interpolating) additive extension  $F$  majorized by  $S$ ? For then  $F$ , automatically being continuous, is linear, because its restriction to the rationals  $F|_{\mathbb{Q}}$  is so (as in Th. 1 below). Assuming additionally positive-homogeneity, *HBE* yields an  $F$ , but this strategy relies very heavily on powerful selection axioms

(formally a weakened version of the Prime Ideal Theorem, itself a weakening of the Axiom of Choice, AC, see Sect. 4.7). The alternative is to apply either semigroup results in [29, 43, 48], or the recent group-theoretic result in [3], but all these again rely on AC (see Sect. 4.7 again). We give an answer in Theorem 3 that relies on the much weaker axiom of Dependent Choices, DC (see Sect. 4.7 once more). We stress that, throughout the paper, *all our results need only DC*.

The group analogue (for  $\mathbb{R}$ ) of *sublinearity* used by [9] (cf. [43]) requires subadditivity as in Banach’s result [5, §2.2 Th. 1], but restricts Banach’s positive-homogeneity condition to just  $\mathbb{N}$ -homogeneity:

$$S(nx) = nS(x) \quad (x \in \mathbb{R}, n = 0, 1, 2, \dots)$$

(with the universal quantifier  $\forall$  on  $x$  and  $n$  understood here, as is usual in mathematical logic). From here onwards we take this to be our definition of sublinearity. This is of course equivalent to ‘positive-rational-homogeneity’. Berz proves and uses a Hahn–Banach theorem in the context of  $\mathbb{R}$  as a vector space over  $\mathbb{Q}$  (for which see also [50, §10.1]) to show that if  $S : \mathbb{R} \rightarrow \mathbb{R}$  is measurable and sublinear, then  $S|_{\mathbb{R}_+}$  and  $S|_{\mathbb{R}_-}$  are both linear; for generalizations to Baire (i.e. having the *Baire* property) and universally measurable functions in contexts including Banach spaces, again using only DC, see [13]. Berz’s motivation was questions of *normability* in topological spaces [9]. The key result here is Kolmogorov’s theorem [47]: normability is equivalent to the origin having a bounded convex neighbourhood ([61, Th. 1.39 and p. 400]).

Our second, linked, question asks whether the universal quantifier ( $x \in \mathbb{R}$ ) above can be *weakened* to range over an additive subgroup. Since  $S = \mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$  is subadditive and  $\mathbb{N}$ -homogeneous on  $\mathbb{Q}$ , but not linear on  $\mathbb{R}_{\pm}$ , the *quantifier weakening* must be accompanied by an appropriate side-condition. We give in Theorem 5 a necessary and sufficient condition (referring also to a thinned-out domain), by extending the standard asymptotic analysis—as in [34] (see Theorem HP below)—of the ratio  $S(t)/t$  near 0 and at infinity; this, indeed, permits thinning-out the universal quantifier of  $\mathbb{N}$ -homogeneity to a dense additive subgroup  $\mathbb{A}$ .

We come at these questions here employing ideas on quantifier weakening previously applied in [14] to additivity issues in classical regular variation, and in [13] to Jensen-style convexity in Banach spaces. We borrow from [14] two key tools: Theorem 0 below on continuity (exploiting an idea of Goldie), and Theorem 0<sup>+</sup> on linear (upper) bounding (exploiting early use by Kingman of the Baire Category Theorem—see [11]), the latter delayed till Sect. 3, when we have the preparatory results needed.

**Theorem 0.** *For subadditive  $S : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  with  $S(0+) = S(0) = 0$  :  $S$  is continuous at 0 iff  $S(z_n) \rightarrow 0$ , for some sequence  $z_n \uparrow 0$ , and then  $S$  is continuous everywhere, if finite-valued.*

In Sect. 2 below we discuss subadditivity, sublinearity and theorems of Berz type, proving Theorems 1–3, Th. BM (for Baire/measurable) and Th. HP (for Hille–Phillips). The work of Hille and Phillips is a major ingredient in the Kingman subadditive ergodic theorem (Sect. 4.9) of probability theory. In Sect. 3 we give stronger versions of Berz’s theorem by thinning the domain of definition, under appropriate side-conditions, results of quantifier-weakening type (Theorems 4 and 5). We close in Sect. 4 with a discussion of the extensive background literature.

## 2. Subadditivity, sublinearity and theorems of Berz type

We first justify our preferred use of local boundedness. Here and below we write  $B_\delta(x) := (x - \delta, x + \delta)$  for the open  $\delta$ -ball around  $x$ .

**Theorem R.** (cf. [50, Th. 16.22]) *For  $\mathbb{A} \subseteq \mathbb{R}$  a subgroup and  $S|_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{R}$  subadditive: if  $S$  is bounded above on some interval, say by  $K$  on  $B_\delta(a) \cap \mathbb{A}$ , then for any  $b \in \mathbb{A}$*

$$S(b + a) - K \leq S(x) \leq S(b - a) + K \quad (x \in B_\delta(b) \cap \mathbb{A}).$$

*In particular, it is locally bounded on  $\mathbb{A}$ : so here, local boundedness from above is equivalent to local boundedness.*

*Proof.* Mutatis mutandis, this is [14, Prop. 5(i)], as the proof there relies only on group structure.  $\square$

The proof in [50] is more involved, and not immediately adaptable to the subgroup setting that we need. The proof offered here we learned from the Referee of [14] (see the Acknowledgements) and in view of that and of some related helpful correspondence with Professor van Rooij we gladly label this Theorem R. As it pinpoints the group dependence, we thank the present Referee for urging us to make explicit reference to local upper boundedness as an alternative assumption (calling to mind the Darboux–Ostrowski-type assumption—see [12]).

We may now begin with a sharpened form of the Berz theorem, with a proof that seems new. Below we write  $\mathbb{R}_+ := [0, \infty)$ ,  $\mathbb{R}_- := (-\infty, 0]$ , and  $\mathbb{A}_\pm := \mathbb{A} \cap \mathbb{R}_\pm$ .

**Theorem 1.** (cf. [14]) *For  $S : \mathbb{R} \rightarrow \mathbb{R}$  a sublinear function (i.e. subadditive, with  $S(nx) = nS(x)$  for  $x \in \mathbb{R}$  and  $n = 0, 1, 2, \dots$ ), if  $S$  is locally bounded, then both  $S|_{\mathbb{R}_+}$  and  $S|_{\mathbb{R}_-}$  are linear.*

*Proof.* For  $M$  a bound on  $S$  in  $B_\delta(0) = (-\delta, \delta)$ ,

$$|S(x)| = |S(kx)/k| \leq M/k,$$

for  $k \in \mathbb{N}$ ,  $x \in B_{\delta/k}(0)$ ; so  $S$  is continuous at 0, and so everywhere uniformly, since  $|S(x+h)| - S(x) \leq \max\{|S(h)|, |S(-h)|\}$ , [33, Th.7.8.2] cf. [14, Proof of Th. 0]. But  $S(q) = qS(1)$  for rational  $q > 0$ , so by continuity  $S(x) = xS(1)$  for  $x \in \mathbb{R}_+$ ; likewise  $S(x) = |x|S(-1)$  for  $x \in \mathbb{R}_-$ .  $\square$

*Remark.* The argument can be repeated for  $S : X \rightarrow \mathbb{R}$  with  $X$  a normed vector space; then  $S(x) = \|x\|S(u_x)$  for  $u_x$  the unit vector on the ray:  $\{\lambda x : \lambda \geq 0\}$ . Here  $|S(u_x)| \leq M/\delta$  for all  $x \neq 0$ .

This gives us a corollary

**Theorem BM.** ([9, 13], cf. [14]) *For  $S : \mathbb{R} \rightarrow \mathbb{R}$  a sublinear function, if  $S$  is Baire/measurable, then both  $S|_{\mathbb{R}_+}$  and  $S|_{\mathbb{R}_-}$  are linear.*

*Proof.* For  $S$  Baire/measurable,  $S$  is bounded above on a non-negligible set and so, being subadditive, is bounded above on some interval (by the Steinhaus–Weil Theorem, [13, 56]), and so, being subadditive, is locally bounded.  $\square$

The following is a slightly sharper form of results in [14] with a simpler proof (the subgroup here is initially arbitrary). This extension theorem may be interpreted in Hahn–Banach style as involving a subadditive function  $S$  which, relative to a subgroup  $\mathbb{A}$ , majorizes an additive function  $G$  that happens to agree with the restriction  $S|_{\mathbb{A}}$ .

**Theorem 2.** *If  $S : \mathbb{R} \rightarrow \mathbb{R}$  is a subadditive locally bounded function and  $\mathbb{A}$  any non-trivial additive subgroup such that  $S|_{\mathbb{A}}$  is additive, then  $S|_{\mathbb{A}}$  is linear.*

*In particular, for  $\mathbb{A}$  dense, any additive function  $G$  on  $\mathbb{A}$  has at most one continuous subadditive extension  $S : \mathbb{R} \rightarrow \mathbb{R}$ .*

For the proof, we will need the following theorem; for completeness, we show how to make the simple modification needed for the result given in [34, Th. 7.6.1]. (We replace their additional blanket condition of measurability of  $S$  by local boundedness, and give more of the details, as they are needed later.)

**Theorem HP.** *For  $S : \mathbb{R} \rightarrow \mathbb{R}$  a locally bounded subadditive function*

$$\beta = \beta_S := \inf_{t>a} \frac{S(t)}{t} = \lim_{t \rightarrow \infty} \frac{S(t)}{t} < \infty \quad (a > 0),$$

*so  $\beta$  does not depend on the choice of  $a > 0$ . In particular,*

$$\beta_S := \inf_{t>0} \frac{S(t)}{t} \in \mathbb{R}.$$

*Proof.* Following [34, Th.7.5.1], for  $a > 0$  and  $ma \leq t < (m+1)a$  with  $m = 2, 3, \dots$ , we note two inequalities, valid according as  $S(a) \geq 0$  or  $S(a) < 0$ :

$$\frac{S(t)}{t} \leq \frac{mS(a) + S(t - ma)}{t} \leq \frac{S(a)}{a} + \frac{K}{a}, \text{ if } S(a) \geq 0, \quad \frac{S(a)}{2a} + \frac{K}{a}, \text{ if } S(a) < 0, \tag{\dagger}$$

for  $K := \sup |S([0, 2a])|$ ; indeed

$$\frac{1}{2a} \leq \frac{1}{a} - \frac{1}{t} \leq \frac{m}{t} \leq \frac{1}{a}.$$

Also  $S(t)/t$  itself is bounded on  $[a, 2a]$ , so  $\beta = \beta(a) < \infty$  is well-defined.

Suppose first that  $\beta > -\infty$ , and let  $\varepsilon > 0$ . As  $\inf_{t>0} S(t)/t < \beta + \varepsilon$ , choose and fix  $b \geq a$  with  $S(b)/b \leq (\beta + \varepsilon)$ . For any  $t \geq 2b$ , let  $n = n(t) \in \mathbb{N}$  satisfy  $(n + 1)b \leq t < (n + 2)b$ ; then  $b < t - nb < 2b$  and

$$1 - \frac{2b}{t} < \frac{nb}{t} \leq 1 - \frac{b}{t}.$$

This time with  $K = \sup |S([0, 2b])|$  a bound on  $S$  as above, since  $S(t) = S(nb + t - nb) \leq S(nb) + S(t - nb)$ ,

$$\beta \leq \frac{S(t)}{t} \leq \frac{nb S(b)}{t} + \frac{S(t - nb)}{t} \leq \frac{nb}{t}(\beta + \varepsilon) + \frac{K}{t} \leq (\beta + \varepsilon) + \varepsilon,$$

for  $t > \max\{nb, K/\varepsilon\}$ . So  $\lim_{t \rightarrow \infty} S(t)/t = \beta$ .

The case  $\beta = -\infty$  would be similar albeit simpler. In fact it does not arise. Indeed, writing  $T(t) = S(-t)$ , which is subadditive and locally bounded, since also  $\beta_T < \infty$ , we have

$$\beta_S + \beta_T = \lim_{t \rightarrow \infty} \left[ \frac{S(t)}{t} + \frac{T(t)}{t} \right] \geq 0,$$

as  $0 \leq S(0) \leq S(t) + T(t)$ . So  $\beta_S > -\infty$ , since  $\beta_S \geq -\beta_T > -\infty$ . □

*Remark.* In fact

$$\begin{aligned} -\beta_T &= -\lim_{t \rightarrow \infty} \frac{T(t)}{t} = -\inf_{t>0} \frac{T(t)}{t} = \sup_{t>0} \frac{S(-t)}{(-t)} \\ &= \sup_{z<0} \frac{S(z)}{z} = \lim_{z \rightarrow -\infty} \frac{S(z)}{z}. \end{aligned}$$

*Proof of Theorem 2.* Put  $G := S|_{\mathbb{A}}$ , and let  $\beta_S$  denote the unique  $\beta$  of Th. HP. For any  $a \in \mathbb{A} \cap (0, \infty)$ , we have, by Th. HP, that

$$\beta_S = \lim_{n \rightarrow \infty} \frac{S(na)}{na} = \frac{G(a)}{a}.$$

Now for all  $a \in \mathbb{A}$ , as  $G(-a) = -G(a)$  and  $G(0) = 0$  (by additivity),  $G(a) = \beta_S a$ . In particular, for  $S$  continuous and  $\mathbb{A}$  dense,  $S(t) = \beta_S t$  for all  $t \in \mathbb{R}$ . □

The next extension theorem employs majorization and minorization on a subspace. The assumption of subgroup divisibility—gives (the more convenient)  $\mathbb{Q}_{+-}$  homogeneity from  $\mathbb{N}$ -homogeneity, but otherwise  $a/k \in \mathbb{A}$  ( $a \in \mathbb{A}$ ,  $k \in \mathbb{N}$ )—gives (the more convenient)  $\mathbb{Q}_{+-}$  homogeneity from  $\mathbb{N}$ -homogeneity, but otherwise is innocuous (as any (infinite) subgroup may be extended to a divisible one without change of cardinality—as with the rationals from the integers).

**Theorem 3.** For  $\mathbb{A}$  a non-trivial, divisible (so dense) subgroup of  $\mathbb{R}$  and a locally bounded sublinear (in particular, additive)  $S : \mathbb{A} \rightarrow \mathbb{R}$ ,

$$S_{\mathbb{A}}^{+}(x) := \lim_{\delta \rightarrow 0} \sup\{S(t) : t \in B_{\delta}(x) \cap \mathbb{A}\},$$

$$S_{\mathbb{A}}^{-}(x) := \lim_{\delta \rightarrow 0} \inf\{S(t) : t \in B_{\delta}(x) \cap \mathbb{A}\},$$

define locally bounded, subadditive, indeed sublinear, functions  $S_{\mathbb{A}}^{\pm} : \mathbb{R} \rightarrow \mathbb{R}$  with

$$S_{\mathbb{A}}^{-}(x) \leq S_{\mathbb{A}}^{+}(x) \quad (x \in \mathbb{R}) \quad \text{and} \quad S_{\mathbb{A}}^{-}(a) \leq S(a) \leq S_{\mathbb{A}}^{+}(a) \quad (a \in \mathbb{A}).$$

Hence

- (i)  $S_{\mathbb{A}}^{+}|_{\mathbb{R}_{\pm}} = S_{\mathbb{A}}^{-}|_{\mathbb{R}_{\pm}}$ ;
- (ii)  $S_{\mathbb{A}}^{\pm}|_{\mathbb{R}_{\pm}}, S|_{\mathbb{A}_{\pm}}$  are linear;
- (iii)  $S_{\mathbb{A}}^{\pm}|_{\mathbb{A}} = S$ .
- (iv) In particular, for  $S$  additive,  $S_{\mathbb{A}}^{+} = S_{\mathbb{A}}^{-}$  and is linear, as is also  $S = S_{\mathbb{A}}^{+}|_{\mathbb{A}}$ .

*Proof.* To lighten the notation, we write  $S^{\pm}$  for  $S_{\mathbb{A}}^{\pm}$ . Local boundedness of  $S^{\pm}$  follows immediately from local boundedness of  $S$ . Subadditivity is routine, and follows much as in [34, §7.8]. As regards sublinearity of  $S^{\pm}$ , note that if  $a_n \rightarrow x$  for  $a_n \in \mathbb{A}$  with  $\lim_{n \rightarrow \infty} S(a_n) = S^{\pm}(x)$ , then, as  $ka_n \in \mathbb{A}$  for  $k \in \mathbb{N}$ , by sublinearity of  $S$

$$kS^{+}(x) = \lim_{n \rightarrow \infty} kS(a_n) = \lim_{n \rightarrow \infty} S(ka_n) \leq S^{+}(kx),$$

$$kS^{-}(x) = \lim_{n \rightarrow \infty} kS(a_n) = \lim_{n \rightarrow \infty} S(ka_n) \geq S^{-}(kx).$$

Similarly, for  $k \in \mathbb{N}$ , if  $a_n \rightarrow kx$  for  $a_n \in \mathbb{A}$  with  $\lim_{n \rightarrow \infty} S(a_n) = S^{\pm}(kx)$ , then  $a_n/k \rightarrow x$  with  $a_n/k \in \mathbb{A}$ , and so again by sublinearity of  $S$ ,

$$S^{+}(kx)/k = \lim_{n \rightarrow \infty} S(a_n/k) \leq S^{+}(x),$$

$$S^{-}(kx)/k = \lim_{n \rightarrow \infty} S(a_n/k) \geq S^{-}(x).$$

So  $kS^{\pm}(x) = S^{\pm}(kx)$ . By Theorem 1 the four functions  $S_{\mathbb{A}}^{\pm}|_{\mathbb{R}_{\pm}}$  are linear, and so by dominance the two functions  $S|_{\mathbb{A}_{\pm}}$  are continuous at 0 and so continuous everywhere (as in Th. 1). So if  $a_n \rightarrow a$  with  $a, a_n \in \mathbb{A}$ , then  $\lim_{n \rightarrow \infty} S(a_n) = S(a) = S^{\pm}(a)$ , proving (iii). So  $S^{\pm}|_{\mathbb{A}_{\pm}} = S|_{\mathbb{A}_{\pm}}$ ; this implies  $S|_{\mathbb{A}_{+}}$  is linear and also that  $S^{+}|_{\mathbb{R}_{+}} = S^{-}|_{\mathbb{R}_{+}}$ , since  $\mathbb{A}$  is dense, proving (i) and (ii) on  $\mathbb{R}_{+}$  and  $\mathbb{A}_{+}$ ; similarly on  $\mathbb{R}_{-}$  and  $\mathbb{A}_{-}$ . For additive  $S$  this means that  $S^{+} = S^{-}$  is linear, as is  $S$ , proving (iv).  $\square$

*Remark.* The conclusions (i)-(iv) of Theorem 3 continue to hold for (just) a dense subgroup  $\mathbb{A}$  and a locally bounded, sublinear  $S : \mathbb{A} \rightarrow \mathbb{R}$ , by similar reasoning as follows. As in Th. 1,  $S$  is continuous on  $\mathbb{A}$ , and, since in particular  $|S(a) - S(a')| \leq \max\{|S(a - a')|, |S(a' - a)|\}$  for  $a, a' \in \mathbb{A}$ , this gives both  $S_{\mathbb{A}}^{+}(x) = S_{\mathbb{A}}^{-}(x) (= \bar{S}_{\mathbb{A}}(\bar{x}))$  say for any  $x \in \mathbb{R}$  and the equality  $\bar{S}_{\mathbb{A}}(a) = S(a)$  for  $a \in \mathbb{A}$ . This equality implies directly (from the same properties of  $S$ ) that  $\bar{S}_{\mathbb{A}}(x)$  is sublinear and locally bounded, so linear on  $\mathbb{R}_{\pm}$  and continuous, by

Th. 1. For a more detailed exposition, see the extended arXiv version of the paper.

Actually  $S_{\mathbb{A}}^+$  ( $S_{\mathbb{A}}^-$ ) is upper (lower) semicontinuous, hence Baire cf. [17, Prop. 3]. Of course  $S^\pm(kx) \leq kS^\pm(x)$ , so (in view of the first displayed inequality above, etc.) a less symmetric proof would have fewer steps. Inducing functions (such as  $S^\pm$  from  $S$ , above) is a method followed variously in e.g. [49, Th. 1], [13, Th. 5].

### 3. Thinning: a stronger Berz theorem

We now give a stronger version of the Berz theorem by weakening a condition of Heiberg–Seneta type by thinning, as in [14], and requiring the homogeneity assumption to hold on only a dense additive subgroup  $\mathbb{A}$  of  $\mathbb{R}$ ; all in all, with rather less than sublinearity, we improve on Theorem BM. This comes at the price of assuming more about  $S$ . To motivate the next definition, note that for locally bounded subadditive  $S$ , the inequalities ( $\dagger$ ) of Sect. 2 imply that for any  $a > 0$

$$\gamma(a) := \sup_{t>a} \frac{S(t)}{t} < \infty,$$

as  $|S(t)/t|$  is bounded on  $[a, 2a]$ . As  $\gamma(a)$  is decreasing for  $a > 0$ , we have

$$-\infty < \beta_S \leq \limsup_{t \downarrow 0} \frac{S(t)}{t} \leq \infty.$$

We note that, with  $T$  as in Theorem HP,  $\alpha_S := \sup_{z<0} S(z)/z = -\beta_T$  is finite (by the remark above), so the definition below fills the gap for  $\sup_{t>0} S(t)/t$ , by asking apparently a little less.

**Definition.** Say that  $S : \mathbb{R} \rightarrow \mathbb{R}$  satisfies the *strong Heiberg–Seneta (SHS)* condition if

$$\gamma = \gamma_S^+ := \limsup_{t \downarrow 0} \frac{S(t)}{t} < \infty. \tag{SHS}$$

See Sect. 4.4 for the origin of this term. For  $S$  subadditive, we will see in Proposition 2 that this implies its dual:

$$-\infty < \gamma_S^- := \liminf_{t \uparrow 0} \frac{S(t)}{t} \leq \gamma_S^+.$$

Proposition 1, to which we now turn, associates to each subadditive function  $S$  a sublinear function  $S^*$  dominating  $S$ , here and below to be called the (upper) *sublinear envelope* of  $S$ . (Albeit multiplicatively, [43] studies the lower envelope dominated by  $S$ , using instead  $S(nx)/n$ —also noted in [29], cf. [3] and [50, 16.2.9]—an approach followed in [31] employing the decreasing sequence  $S(2^n x)/2^n$ .) However, some assumption on  $S$  is needed to ensure that  $S^*$  is



finite-valued: recall that the subadditive function  $S = \mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$  is  $\mathbb{N}$ -homogeneous on  $\mathbb{Q}$ , yet  $S^* = (+\infty) \cdot \mathbf{1}_{\mathbb{R} \setminus \mathbb{Q}}$ .

**Proposition 1.** *For  $S : \mathbb{R} \rightarrow \mathbb{R}$  locally bounded and subadditive with  $S(0) = 0$ , the function defined by*

$$S^*(x) := \limsup_{n \rightarrow \infty} nS(x/n) \quad (x \in \mathbb{R})$$

*is subadditive and sublinear and dominates  $S$ . If further  $S$  satisfies (SHS), then, for  $t \geq 0$ ,*

$$\beta_S t \leq S(t) \leq S^*(t) \leq \gamma_S^+ t.$$

*In particular,  $S(0+) = S^*(0+) = 0$  and  $S^*$  is locally bounded; furthermore,  $\gamma_S^+ > -\infty$  and*

$$\sup_{t>0} \frac{S(t)}{t} \leq \gamma_S^+.$$

*Proof.* By subadditivity of  $S$ , for any  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$

$$S(x) = S(nx/n) \leq nS(x/n) \leq S^*(x).$$

Evidently  $S^*$  is subadditive (cf. [34, 7.2.2, 7.2.3]). Moreover, as  $S(0) = 0$ ,  $S^*$  is  $\mathbb{Q}_+$ -homogeneous, since for fixed  $k \in \mathbb{N}$

$$\begin{aligned} kS^*(x) &= \limsup_{m \rightarrow \infty} k \cdot mS(kx/km) \\ &\leq \limsup_{n \rightarrow \infty} nS(kx/n) \quad (\text{via specialization: } n = km) \\ &= S^*(kx) \leq kS^*(x), \end{aligned}$$

the latter by subadditivity. Combining,

$$S^*(kx) = kS^*(x).$$

Suppose now that (SHS) holds. Let  $\varepsilon > 0$ . Then there is  $\delta > 0$  with

$$S(x)/x \leq \gamma_S^+ + \varepsilon \quad (0 < x < \delta).$$

Fix  $t > 0$ . Then for integer  $n > t/\delta$

$$\frac{S(t/n)}{t/n} \leq \gamma_S^+ + \varepsilon : \quad nS(t/n) \leq (\gamma_S^+ + \varepsilon)t,$$

and so taking limsup as  $n \rightarrow \infty$

$$S^*(t) \leq (\gamma_S^+ + \varepsilon)t,$$

for  $t \geq 0$ , as  $S^*(0) = 0$  (since  $S(0) = 0$ ). Taking limits as  $\varepsilon \downarrow 0$  yields

$$S^*(t) \leq \gamma_S^+ t.$$

Furthermore, for  $t \geq 0$ ,

$$\beta_S t \leq S(t) \leq S^*(t).$$

Finally, by Th. R,  $S^*$  is locally bounded, since  $S^*$  is locally bounded for  $t > 0$ .  $\square$

In view of the linear bounding of  $S^*$  (and hence of  $S$ ) just proved from *SHS*, we proceed to a *weaker property* of  $S$  in which the domain of the limsup operation is *thinned-out*. This will nevertheless also yield linear bounding of  $S$  (from above), hence finiteness of  $\gamma_S^+$ , and in turn the bounding of  $S^*$ . We need a definition and a theorem from [14].

**Definition.** [13,14,17] Say that  $\Sigma \subseteq \mathbb{R}$  is *locally Steinhaus–Weil (SW)*, or has the *SW property locally*, if for  $x, y \in \Sigma$  and, for all  $\delta > 0$  sufficiently small, the sets

$$\Sigma_z^\delta := \Sigma \cap B_\delta(z),$$

for  $z = x, y$ , have the *interior-point property*, that  $\Sigma_x^\delta \pm \Sigma_y^\delta$  has  $x \pm y$  in its interior. (Here again  $B_\delta(x)$  is the open ball about  $x$  of radius  $\delta$ .) See [15, §6.9] or [17, §7] for conditions under which this property is implied by the interior-point property of the sets  $\Sigma_x^\delta - \Sigma_x^\delta$  (cf. [7]); for a rich list of examples, see Sect. 4.5. An obvious example is an open set  $\Sigma$ .

We now cite from [14] the following result.

**Theorem 0<sup>+</sup>.** *Let  $\Sigma \subseteq [0, \infty)$  be locally SW accumulating at 0. Suppose that  $S : \mathbb{R} \rightarrow \mathbb{R}$  is subadditive with  $S(0) = 0$  and:  $S|\Sigma$  is linearly bounded above by  $G(x) := cx$ , i.e.  $S(\sigma) \leq c\sigma$  for some  $c$  and all  $\sigma \in \Sigma$ , so that in particular,*

$$\limsup_{\sigma \downarrow 0, \sigma \in \Sigma} S(\sigma) \leq 0.$$

*Then  $S(x) \leq cx$  for all  $x > 0$ , so*

$$\limsup_{x \downarrow 0} S(x) \leq 0,$$

*and so  $S(0+) = 0$ .*

*In particular, if furthermore there exists a sequence  $\{z_n\}_{n \in \mathbb{N}}$  with  $z_n \uparrow 0$  and  $S(z_n) \rightarrow 0$ , then  $S$  is continuous at 0 and so everywhere.*

**Definition.** Say that  $S : \mathbb{R} \rightarrow \mathbb{R}$  satisfies the *weak Heiberg–Seneta (WHS)* condition if for some  $\Sigma \subseteq (0, \infty)$ , a locally SW set accumulating at 0,

$$\gamma_S^\Sigma := \limsup_{t \downarrow 0, t \in \Sigma} \frac{S(t)}{t} < \infty.$$

**Corollary.** *For  $S : \mathbb{R} \rightarrow \mathbb{R}$  locally bounded and subadditive with  $S(0) = 0$ , if  $S$  satisfies WHS, then  $S$  is linearly bounded by  $\gamma_S^\Sigma t$  for  $t \geq 0$ , and so satisfies SHS with  $\gamma_S^+ \leq \gamma_S^\Sigma$ .*

*Proof.* Write  $\gamma = \gamma_S^\Sigma$ . Let  $\varepsilon > 0$ . Then there is  $\delta > 0$  with

$$S(t) \leq (\gamma + \varepsilon)t \quad (t \in \Sigma \cap (0, \delta)).$$

So  $S(t) \leq (\gamma + \varepsilon)t$  for all  $t > 0$ , by Th. 0<sup>+</sup> applied to  $c = \gamma + \varepsilon$ . Taking limits as  $\varepsilon \downarrow 0$  yields  $S(t) \leq \gamma t$  for all  $t > 0$  and so

$$\gamma_S^+ = \limsup_{t \downarrow 0} \frac{S(t)}{t} \leq \gamma_S^\Sigma < \infty.$$

So  $S$  satisfies the *SHS*. □

We now derive in Theorem 5 below a form of Berz’s Theorem, in which the weak Heiberg–Seneta condition on  $S$  permits a thinned-out assumption of homogeneity; the argument is based on the following result, a corollary of Theorem 1 and Prop 1.

**Theorem 4.** *For  $S : \mathbb{R} \rightarrow \mathbb{R}$  locally bounded and subadditive with  $S(0) = 0$ , if  $S$  satisfies *WHS*, and  $S^*(t_n) \rightarrow 0$  for some sequence  $t_n \uparrow 0$ , then  $S$  and its sublinear envelope  $S^*$  are continuous, and further, by sublinearity, both  $S^*|_{\mathbb{R}_+}$  and  $S^*|_{\mathbb{R}_-}$  are linear.*

*Proof.* By the Corollary we may assume that *SHS* holds. By Prop. 1,  $S^*(0+) = 0$ , so  $S^*$  is continuous by Theorem 0, and now linearity on half-lines follows by Theorem 1, as  $S^*$  is sublinear (and locally bounded at 0, so everywhere—see Theorem R). In fact, it follows directly, since, by  $\mathbb{N}$ -homogeneity and continuity,  $S^*(x) = xS^*(1)$  and  $S^*(-x) = xS^*(-1)$  for  $x > 0$ , as  $\mathbb{Q}$  is dense and  $S^*(\pm q) = qS^*(\pm 1)$  for  $q \in \mathbb{Q}_+$ .

Now for  $x \geq 0$ , by subadditivity  $-S(x) \leq S(-x)$  (as  $S(0) = 0$ ) and so

$$-S^*(x) \leq -S(x) \leq S(-x) \leq S^*(-x) = xS^*(-1).$$

So  $S(0-) = 0$ , as  $S^*$  is continuous; so  $S$  is continuous. □

**Proposition 2.** *For  $S : \mathbb{R} \rightarrow \mathbb{R}$  locally bounded and subadditive with  $S(0) = 0$ : if  $S$  satisfies *WHS*, then, for  $t \geq 0$ ,*

$$S^*(-t)/(-t) = \gamma_S^- \leq \alpha_S \leq \beta_S \leq \gamma_S^+ = S^*(t)/t.$$

*In particular,*

$$-\infty < \gamma_S^- \leq \gamma_S^+ < \infty.$$

*Proof.* Just as in Theorem 4 we may assume that *SHS* holds. By Prop. 1 and Th. 1, write  $\gamma^\pm := S^*(\pm t)/(\pm t)$  for  $t > 0$ . From Prop. 1  $\beta_S \leq \gamma_S^+$ ; as  $S \leq S^*$ , for  $t > 0$ ,  $\gamma^+ = S^*(t)/t \geq S(t)/t$ , so  $\gamma_S^+ \leq \gamma^+$ . For the reverse inequality, take any  $\varepsilon > 0$  and choose  $\delta > 0$  such that  $S(t)/t \leq \gamma_S^+ + \varepsilon$  for all  $0 < t < \delta$ . As  $S^*(1) = \gamma^+$ , there exists  $m > 1/\delta$  with  $\gamma^+ - \varepsilon \leq mS(1/m)$ . Taking  $t = 1/m < \delta$  yields

$$\gamma^+ - \varepsilon \leq \gamma_S^+ + \varepsilon : \quad \gamma^+ \leq \gamma_S^+ + 2\varepsilon.$$

Taking limits as  $\varepsilon \downarrow 0$  yields  $\gamma^+ \leq \gamma_S^+$ . Combining,  $\gamma^+ = \gamma_S^+$ .

Now recall from Theorem HP above that  $\alpha_S \leq \beta_S$  (via  $\alpha_S = -\beta_T$ ). We obtain  $\gamma^- \leq \alpha_S$  from

$$\lim_{t \rightarrow -\infty} \frac{S^*(t)}{t} \leq \lim_{t \rightarrow -\infty} \frac{S(t)}{t} = \alpha_S,$$

again as  $S \leq S^*$  ( $t$  being negative here).

Put  $T^*(t) := S^*(-t) = \gamma^-(-t)$  for  $t > 0$ . Then, by linearity of  $S^*$  for  $t < 0$ ,

$$\gamma^- = \liminf_{z \uparrow 0} \frac{S^*(z)}{z};$$

so by the definition of  $\gamma_S^-$ ,

$$\gamma_S^- = \liminf_{z \uparrow 0} \frac{S(z)}{z} \geq \liminf_{z \uparrow 0} \frac{S^*(z)}{z} = \gamma^-,$$

as  $S \leq S^*$  ( $z$  here being negative). So

$$\gamma_S^- \geq \gamma^-.$$

We now show that  $\gamma_S^- \leq \gamma^-$ . This runs analogously to the plus version. Let  $\varepsilon > 0$ . Choose  $\delta > 0$  with  $\gamma_S^- - \varepsilon \leq S(t)/t$  for  $t \in (-\delta, 0)$ . As  $-\gamma^- = S^*(-1)$ , pick  $m$  with  $m > 1/\delta$  and  $-\gamma^- - \varepsilon \leq mS(-1/m)$ . Then taking  $t = -1/m$  gives

$$\gamma_S^- - \varepsilon \leq -mS(-1/m) \leq \gamma^- + \varepsilon: \quad \gamma_S^- \leq \gamma^- + 2\varepsilon.$$

Taking limits as  $\varepsilon \downarrow 0$  yields  $\gamma_S^- \leq \gamma^-$ . Combining,  $\gamma_S^- = \gamma^-$ . □

*Remark.* The burden of proof falls on showing that  $\gamma_S^\pm = \gamma^\pm$ ; of course, for  $t > 0$ ,  $S(t) + S(-t) = \gamma^+t - \gamma^-t \geq 0$  yields directly that  $\gamma^- \leq \gamma^+$ .

**Theorem 5.** (*Quantifier-weakened Berz Theorem*) For  $S : \mathbb{R} \rightarrow \mathbb{R}$  locally bounded and subadditive (in particular for  $S$  Baire/measurable and subadditive) with  $S(0) = 0$ , if

- (i)  $S$  satisfies WHS and,
- (ii)  $\mathbb{A}$  is a (dense) divisible additive subgroup of  $\mathbb{R}$  with  $S|_{\mathbb{A}}$   $\mathbb{N}$ -homogeneous – then both  $S|_{\mathbb{R}_+}$  and  $S|_{\mathbb{R}_-}$  are linear: for  $t \geq 0$  :

$$S(t) = \beta_S t, \text{ and } S(-t) = -\alpha_S t.$$

In particular, for  $S$  additive

$$S(t) = \beta_S t \quad (t \in \mathbb{R}).$$

*Proof.* By the Corollary we may assume that SHS holds. Consider any  $a \in \mathbb{A}$ . Then  $S^*(a) = \limsup_{n \rightarrow \infty} nS(a/n) = S(a)$  by  $\mathbb{N}$ -homogeneity of  $\mathbb{A}$ , and further, by sublinearity of  $S^*$  (Prop. 1),  $S^*(a/n) = S^*(a)/n = S(a)/n \rightarrow 0$ , taking limits through  $n \in \mathbb{N}$ . Taking  $a < 0$  gives, via Theorem 4, that  $S^*$  and so  $S$  is continuous on  $\mathbb{A}$ . Now  $S = S^*$ , by continuity and density of  $\mathbb{A}$ , as  $S^*|_{\mathbb{A}} = S|_{\mathbb{A}}$ . So  $S|_{\mathbb{R}_+}$  and  $S|_{\mathbb{R}_-}$  are linear, again by Theorem 4. The first

formulas come from Th. HP, and the final one, in the additive case, from  $S(-t) = -S(t)$  (and then  $\alpha_S = \beta_S$ ).  $\square$

*Remark.* The assumption of divisibility placed on  $\mathbb{A}$  is innocuous: it may be omitted by the following reasoning. By the Corollary, we may assume that  $S$  satisfies  $SHS$ , being in fact linearly bounded for  $t > 0$ . So, by Theorem  $0^+$ ,  $S(0+ = 0)$ . By the Remarks after Theorem 3,  $S|_{\mathbb{A}_-}$  is linear. So by density of  $\mathbb{A}$  there is a sequence in  $\mathbb{A}_- : a_n \uparrow 0$  say, with  $S(a_n) \rightarrow 0$ . So, by Theorem 0,  $S$  is continuous. So by the density of  $\mathbb{A}$ ,  $S|_{\mathbb{R}_\pm}$  is linear.

In the result above, the particular case of  $S$  additive includes [14, Th. 1 and Th. 1’].

## 4. Complements

### 4.1. Approximate homomorphisms

There are results in which one has a property, such as additivity, which holds only *approximately*, and then deduces that, under suitable restrictions, it holds *exactly*. For example, in Badora’s almost-everywhere version of the Hahn–Banach theorem [4], if the relevant differences are *bounded*, as in [2], then they *vanish*. That is, the relevant differences are either identically plus infinity or identically zero. This is a *dichotomy*, reminiscent of those that occur in probability theory in connection with 0-1 laws (for example, *Belyaev’s dichotomy* [8]; [53, 5.3.10]).

### 4.2. Popa (circle) group subadditivity

We recall from [14] that the Popa circle operation on  $\mathbb{R}$ , introduced in [60] (cf. [39]), given by

$$a \circ b = a + b\eta(a), \text{ for } \eta(t) := 1 + \rho t \text{ with } \rho \geq 0,$$

turns  $\mathbb{G}_+ := \{x \in \mathbb{R} : 1 + \rho x > 0\}$  into a group with  $\mathbb{R}_+$  as a subsemigroup. The latter induces an order on  $\mathbb{G}_+$  which agrees with the usual order (cf. e.g. [30]). So a function  $f : (\mathbb{R}_+, \times) \rightarrow (\mathbb{G}_+, \circ)$  satisfying

$$f(xy) \leq f(x) \circ f(y)$$

may be viewed as subadditive in the group context. This abstract viewpoint encompasses both the current context of subadditivity (for  $\rho = 0$ ), and a further significant one arising in the theory of regular variation (the ‘Goldie Functional Inequality’, for  $\rho = 1$  — cf. [36]); for the latter see [13]. We hope to return to these matters elsewhere—cf. [14, §7].

### 4.3. Restricted domain

There are results when, as in Sect. 3 on quantifier weakening, a property such as additivity or subadditivity holds off some exceptional set (say, almost everywhere), and the conclusion is also similarly restricted. This goes back to work of Hyers and Ulam [1, 20]. See also de Bruijn [25], Ger [32, 33].

### 4.4. Origin of the Heiberg–Seneta condition

This condition, introduced in regular variation (see [10, Th. 3.2.5], prompting its recent study in [14]), as applied to a subadditive function  $S : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ , took the form

$$\limsup_{t \downarrow 0} S(t) \leq 0. \quad (HS)$$

For  $\mathbb{A} \subseteq \mathbb{R}$  a dense subgroup, the assumption that  $S|_{\mathbb{A}}$  is linear together with (HS) guaranteed not only that  $S$  is finite-valued with  $S(0+) = 0$ , but that in fact  $S$  is linear, as in Th. 5, which relates directly to [10, Th. 3.2.5].

### 4.5. Examples of families of locally Steinhaus–Weil sets

The sets listed below are typically, though not always, members of a topology on an underlying set.

- (o)  $\Sigma$  a usual (Euclidean) open set in  $\mathbb{R}$  (and in  $\mathbb{R}^n$ )—this is the ‘trivial’ example;
- (i)  $\Sigma$  density-open subset of  $\mathbb{R}$  (similarly in  $\mathbb{R}^n$ ) (by Steinhaus’s Theorem—see e.g. [10, Th. 1.1.1], [17], [56, Ch. 8]);
- (ii)  $\Sigma$  Baire, locally non-meagre at all points  $x \in \Sigma$  (by the Piccard-Pettis Theorem—as in [10, Th. 1.1.2], [17], [56, Ch. 8]—such sets can be ‘thinned out’, i.e. extracted as subsets of a second-category set, using separability or by reference to the Banach Category Theorem [56, Ch.16]);
- (iii)  $\Sigma$  the Cantor ‘excluded middle-thirds’ subset of  $[0, 1]$  (since  $\Sigma + \Sigma = [0, 2]$ );
- (iv)  $\Sigma$  universally measurable and open in the *ideal* topology ([15, 51]) generated by omitting Haar null sets (by the Christensen–Solecki Interior-points Theorem of [21, 22] and [64]);
- (v)  $\Sigma$  a Borel subset of a Polish abelian group and open in the ideal topology generated by omitting *Haar meagre* sets in the sense of Darji [24] (by Jabłońska’s generalization of the Piccard Theorem, [35, Th.2], cf. [37], and since the Haar-meagre sets form a  $\sigma$ -ideal [24, Th. 2.9]); for details see [17].

If  $\Sigma$  is *Baire* (has the Baire property) and is locally non-meagre, then it is co-meagre (since its quasi interior is everywhere dense).

**Caveats.**

1. Care is needed in identifying locally SW sets: Matoůsková and Zelený [54] show that in any non-locally compact abelian Polish group there are closed *non-Haar null* sets  $A, B$  such that  $A + B$  has empty interior. Recently, Jabłońska [38] has shown that likewise in any *non-locally compact* abelian Polish group there are closed *non-Haar meager* sets  $A, B$  such that  $A + B$  has empty interior.
2. For an example on  $\mathbb{R}$  of a compact subset  $S$  such that  $S - S$  contains an interval, but  $S + S$  has measure zero and so does not, see [23] and the recent [6].
3. Here we were concerned with subsets  $\Sigma \subseteq \mathbb{R}$  where such ‘anomalies’ are assumed not to occur.

**4.6. Baire/measurable  $S$  and  $S^*$** 

Of course if  $S$  is Baire/measurable, then so is  $S^*$ , as the limsup is sequential. Also for  $\mathbb{A}$  a *countable* subgroup, the upper and lower limit functions  $S_{\mathbb{A}}^{\pm}$  derived from a subadditive function  $S$  are Baire/measurable, as the image  $S(\mathbb{A})$  is countable.

**4.7. The Hahn–Banach theorem: variants**

There are various theorems of Hahn–Banach type. Text-book accounts, as in e.g. Rudin [61, § 3.2, 3.3, 3.4], [30], deal with dominated extension theorems (without any assumed continuity on the partial function  $f$  nor on the dominating function  $p$ , HB below), separation theorems for convex sets, and continuous extension theorems. Variations include the assumption that the dominating function  $p$  is continuous, e.g. [26] (implying continuity of the minorant partial function); another variation—from [28], call this ‘HB-lite’ for our needs in Sect. 4.8 below—assumes for given  $p$  merely the existence of some linear functional dominated by  $p$ . (Here, if the variant axiom is satisfied for all  $p$  continuous, then HB follows for all continuous  $p$  [28, §4]). For a most insightful survey of very many variations in earlier literature see [18]. The context also varies, correspondingly, from vector spaces, to topological vector spaces and beyond, so to F-spaces (i.e. topological vector spaces with topology generated by some complete translation-invariant metric, [42]) and Banach spaces. One needs to distinguish between the variants, including the category of space over which the assertions range, when discussing their axiomatic status. Kalton proved ([27, 41], cf. [42, Ch. 4]) that an F-space in which the continuous extension theorem (in which  $f$  is continuous) holds is necessarily locally convex, a result that is false without metrizability; it is not known whether completeness is necessary.

The dominated extension theorem HB (i.e. without any continuity) is equivalent to a weakened form of PIT, the Prime Ideal Theorem, namely the existence of a non-trivial finitely-additive probability measure (as opposed to a two-valued measure implicit in PIT) on any non-trivial Boolean algebra ([40, 52, 55, 66])—MB (for ‘measure-Boolean’) in the terminology of [55].

For the relative strengths of HB and the Axiom of Choice AC, see [57, 58]; [59] provides a model of set theory in which the Axiom of Dependent Choices DC holds but HB fails. Moreover, HB for separable normed spaces is not provable from DC [26, Cor. 4]. On the other hand, any separable normed space satisfies the version of HB in which the dominating function  $p$  is continuous; indeed the partial function  $f$  may first be explicitly extended to the linear span of the union of its domain with the dense countable set—as in the original Banach proof [5] by inductive assignment of function-values using the least possible function-value at each stage (as in [26, Lemma 9])—and then to the rest of space, essentially as in Theorem 3, using the continuity conferred by  $p$  and our sequential analysis. (Compare [28] for various completeness and compactness notions here.) Further to [26], we raise, and leave open here, the question as to whether the separable case of Badora’s result in [3] can be proved with only DC rather than AC, and the role that completeness (sequential or otherwise) may play here [42].

For more on axiomatics (with references), see [18, §12, 20], Appendix 1 of the fuller arXiv version of [13] and also [16].

#### 4.8. The Hahn–Banach Theorem: group analogues

The group analogue of the ‘HB-lite’ property of Sect. 4.7 (mutatis mutandis, with ‘additive’ replacing ‘linear’ etc.) delineates a class of groups providing the context for Badora’s ‘general’ Hahn–Banach extension theorem for groups [3, Th. 1], and includes amenable groups; the class is characterized in [3, Th. 3] by the group analogue of HB with a side-condition on  $p$ . The more special Hahn–Banach-type extension property for the case of a group  $G$  of linear operators  $g : V \rightarrow V$  on a real vector space  $V$  is concerned with a  $p$ -dominated  $G$ -invariant extension of a  $G$ -invariant partial linear operator  $f$  (defined on a  $G$ -invariant subspace  $W$ ) satisfying  $f(w) \leq p(w)$  for  $w \in W$ , where  $p$  is a subadditive and positive-homogeneous functional  $p : V \rightarrow \mathbb{R}$  with  $p(g(v)) \leq p(v)$ . This as a property of  $G$  turns out to be equivalent to  $G$  being *amenable* (Silverman [62, 63])—see [46] for a clear albeit early approach. See also [66, Th. 12.11].

#### 4.9. Kingman’s Subadditive Ergodic Theorem

Detailed study of subadditivity is partially motivated by links with the Kingman subadditive ergodic theorem, which has been very widely used in probability theory. For background and details, see e.g. [44, 45], Steele [65].



## Postscript.

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## References

- [1] Badora, R.: On some generalized invariant means and their application to the stability of the Hyers–Ulam type. *Ann. Polon. Math.* LVII I, 147–159 (1993)
- [2] Badora, R.: On approximate ring homomorphisms. *J. Math. Anal. Appl.* **276**, 589–597 (2002)
- [3] Badora, R.: On the Hahn–Banach theorem for groups. *Arch. Math. (Basel)* **86**, 517–528 (2006)
- [4] Badora, R.: The Hahn–Banach theorem almost everywhere. *Aequ. Math.* **90**(1), 173–179 (2016)
- [5] Banach, S.: *Théorie des opérations linéaires*. Monografie Mat. **1**, Warszawa, 1932; Chelsea Publishing Co., New York, 1955; in *Oeuvres*, II, PWN-1979 (see <http://matwbn-old.icm.edu.pl/>), Éditions Jacques Gabay, Sceaux, 1993; *Theory of linear operations*. Translated from the French by F. Jellet. North-Holland Mathematical Library **38** (1987)
- [6] Bartoszewicz, A., Filipczak, M.: Remarks on sets with small differences and large sums. *J. Math. Anal. Appl.* **456**, 245–250 (2017)
- [7] Bartoszewicz, A., Filipczak, M., Natkaniec, T.: On Smítal properties. *Topol. Appl.* **158**, 2066–2075 (2011)
- [8] Belyaev, Yu.K.: Continuity and Hölder’s conditions for sample functions of stationary Gaussian processes. In: *Proceedings of 4th Berkeley Symposium on Mathematical Statistics and Probability*, vol. II, pp. 23–33. University of California Press (1961)
- [9] Berz, E.: Sublinear functions on  $\mathbb{R}$ . *Aequ. Math.* **12**, 200–206 (1975)
- [10] Bingham, N.H., Goldie, C.M., Teugels, J.L.: *Regular Variation*, 2nd edn. Cambridge University Press, Cambridge (1989). (1st ed. 1987)
- [11] Bingham, N.H., Ostaszewski, A.J.: Kingman, category and combinatorics. In: Sir John Kingman Festschrift, Bingham, N.H., Goldie, C.M. (eds) *Probability and Mathematical Genetics*, London Mathematical Society. *Lecture Notes in Mathematics*, vol. 378, pp. 135–168. Cambridge University Press (2010)
- [12] Bingham, N.H., Ostaszewski, A.J.: Dichotomy and infinite combinatorics: the theorems of Steinhaus and Ostrowski. *Math. Proc. Camb. Philos. Soc.* **150**, 1–22 (2011)
- [13] Bingham, N.H., Ostaszewski, A.J.: Category-measure duality: convexity, mid-point convexity and Berz sublinearity. *Aequ. Math.* **91.5**, 801–836 (2017) (fuller version: [arXiv:1607.05750](https://arxiv.org/abs/1607.05750))
- [14] Bingham, N.H., Ostaszewski, A.J.: Additivity, subadditivity and linearity: automatic continuity and quantifier weakening. *Indag. Math.* **29**, 687–713 (2018). [arXiv:1405.3948v3](https://arxiv.org/abs/1405.3948v3)
- [15] Bingham, N.H., Ostaszewski, A.J.: Beyond Lebesgue and Baire IV: density topologies and a converse Steinhaus–Weil theorem. *Topol. Appl.* **239**, 274–292 (2018). Extended version: [arXiv:1607.00031v2](https://arxiv.org/abs/1607.00031v2)

- [16] Bingham, N.H., Ostaszewski, A.J.: Set theory and the analyst. *Eur. J. Math.* (to appear) [arXiv:1801.099149](https://arxiv.org/abs/1801.099149)
- [17] Bingham, N.H., Ostaszewski, A.J.: The Steinhaus–Weil property: its converse, subcontinuity and Solecki amenability. [arXiv:1607.00049v3](https://arxiv.org/abs/1607.00049v3)
- [18] Buskes, G.: The Hahn–Banach theorem surveyed. *Dissertationes Math. (Rozprawy Mat.)* **327**, 1–49 (1993)
- [19] Buskes, G., van Rooij, A.: Hahn–Banach for Riesz homomorphisms. *Indag. Math.* **51**, 25–34 (1989)
- [20] Cabello Sanchez, F., Castillo, J.M.F.: Banach space techniques underpinning a theory for nearly additive mappings. *Dissertationes Math. (Rozprawy Mat.)* **404**, 1–73 (2002)
- [21] Christensen, J.P.R.: On sets of Haar measure zero in abelian Polish groups. In: *Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972)*. *Israel J. Math.* **13**, 255–260 (1973)
- [22] Christensen, J.P.R.: *Topology and Borel Structure. Descriptive Topology and Set Theory with Applications to Functional Analysis and Measure Theory*. North-Holland Mathematics Studies, vol. 10. North-Holland, Amsterdam (1974)
- [23] Crnjac, M., Guljaš, B., Miller, H.I.: On some questions of Ger, Grubb and Kraljević. *Acta Math. Hungar.* **57**, 253–257 (1991)
- [24] Darji, U.B.: On Haar meager sets. *Topol. Appl.* **160**, 2396–2400 (2013)
- [25] de Bruijn, N.G.: On almost additive functions. *Colloq. Math.* **XV**, 59–63 (1966)
- [26] Dodu, J., Morillon, M.: The Hahn–Banach property and the axiom of choice. *Math. Log. Q.* **45**(3), 299–314 (1999)
- [27] Drewnowski, L.: The weak basis theorem fails in non-locally convex F-spaces. *Can. J. Math.* **29**, 1069–1071 (1977)
- [28] Fossy, J., Morillon, M.: The Baire category property and some notions of compactness. *J. Lond. Math. Soc.* **57**(2), 1–19 (1998)
- [29] Fuchssteiner, B.: On exposed semigroup homomorphisms. *Semigroup Forum* **13**, 189–204 (1976/77)
- [30] Fuchssteiner, B., Lusky, W.: *Convex Cones*. North-Holland Mathematical Studies, vol. 56. North-Holland, Amsterdam (1981)
- [31] Gajda, Z., Kominek, Z.: On separation theorems for subadditive and superadditive functionals. *Stud. Math.* **100**, 25–38 (1991)
- [32] Ger, R.: On some functional equations with a restricted domain, I. *Fund. Math.* **89**, 131–149 (1975)
- [33] Ger, R.: On some functional equations with a restricted domain, II. *Fund. Math.* **98**, 249–272 (1978)
- [34] Hille, E., Phillips, R.S.: *Functional Analysis and Semi-groups*. Colloquium Publications, vol. 31, 3rd edn. American Mathematical Society, Providence (1974). (1st edn, 1957)
- [35] Jabłońska, E.: Some analogies between Haar meager sets and Haar null sets in abelian Polish groups. *J. Math. Anal. Appl.* **421**, 1479–1486 (2015)
- [36] Jabłońska, E.: On solutions of a composite type functional inequality. *Math. Ineq. Apps.* **18**, 207–215 (2015)
- [37] Jabłońska, E.: A theorem of Piccard’s type and its applications to polynomial functions and convex functions of higher orders. *Topol. Appl.* **209**, 46–55 (2016)
- [38] Jabłońska, E.: A theorem of Piccard’s type in abelian Polish groups. *Anal. Math.* **42**, 159–164 (2016)
- [39] Javor, P.: On the general solution of the functional equation  $f(x + yf(x)) = f(x)f(y)$ . *Aequ. Math.* **1**, 235–238 (1968)
- [40] Jech, T.: *The Axiom of Choice. Studies in Logic and the Foundations of Mathematics*, vol. 75. North-Holland, Amsterdam (1973)
- [41] Kalton, N.J.: Basic sequences in F-spaces and their applications. *Proc. Edinburgh Math. Soc. (2)* **19**(2), 151–167 (1974/75)
- [42] Kalton, N.J., Peck, N.T., Roberts, J.W.: *An F-Space Sampler*. London Mathematical Society. Lecture Notes Series, vol. 89. Cambridge University Press, Cambridge (1984)

- [43] Kaufman, R.: Extension of functionals and inequalities on an abelian semi-group. *Proc. Am. Math. Soc.* **17**, 83–85 (1966)
- [44] Kingman, J.F.C.: Subadditive ergodic theory. *Ann. Probab.* **1**, 883–909 (1973)
- [45] Kingman, J.F.C.: Subadditive processes. *Ecole d’été de Probabilités de Saint Fleur V. Lect. Notes Math.* **539**, 167–223 (1976)
- [46] Klee, V.: Invariant extension of linear functionals. *Pac. J. Math.* **4**, 37–46 (1954)
- [47] Kolmogoroff, A.N.: Zur Normierbarkeit eines allgemeinen topologischen linearen Raumes. *Stud. Math.* **5**, 29–33 (1934)
- [48] Kranz, P.: Additive functionals on abelian semigroups. *Comment. Math. (Prace Mat.)* **16**, 239–246 (1972)
- [49] Kuczma, M.: Almost convex functions. *Coll. Math.* **XXI**, 279–284 (1070)
- [50] Kuczma, M.: *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy’s Equation and Jensen’s Inequality*, 2nd edn. Birkhäuser, Basel (2009). (1st ed. PWN, Warszawa, 1985)
- [51] Lukeš, J., Malý, J., Zajíček, L.: *Fine Topology Methods in Real Analysis and Potential Theory. Lecture Notes in Mathematics*, vol. 1189. Springer, Berlin (1986)
- [52] Luxemburg, W.A.J.: Reduced powers of the real number system and equivalents of the Hahn–Banach extension theorem. In: *Applications of Model Theory to Algebra, Analysis, and Probability (International Symposium, Pasadena, CA, 1967)*, pp. 123–137. Holt, Rinehart and Winston (1969)
- [53] Marcus, M.B., Rosen, J.: *Gaussian Processes, Markov Processes and Local Times*. Cambridge University Press, Cambridge (2006)
- [54] Matošková, E., Zelený, M.: A note on intersections of non-Haar null sets. *Colloq. Math.* **96**, 1–4 (2003)
- [55] Mycielski, J., Tomkowicz, G.: Shadows of the axiom of choice in the universe  $L(\mathbb{R})$ . *Arch. Math. Logic.* **57**, 607–616 (2018)
- [56] Oxtoby, J.C.: *Measure and Category*. Graduate Texts in Mathematics, 2nd edn. Springer, Berlin (1980)
- [57] Pincus, D.: Independence of the prime ideal theorem from the Hahn–Banach theorem. *Bull. Am. Math. Soc.* **78**, 766–770 (1972)
- [58] Pincus, D.: The strength of the Hahn-Banach theorem. *Victoria Symposium on Non-standard Analysis (University of Victoria, Victoria, BC, 1972)*, *Lecture Notes in Mathematics*, vol. 369, pp. 203–248. Springer, Berlin (1974)
- [59] Pincus, D., Solovay, R.: Definability of measures and ultrafilters. *J. Symb. Logic* **42**(2), 179–190 (1977)
- [60] Popa, C.G.: Sur l’équation fonctionnelle  $f[x + yf(x)] = f(x)f(y)$ . *Ann. Polon. Math.* **17**, 193–198 (1965)
- [61] Rudin, W.: *Functional Analysis*, 2nd edn. McGraw-Hill, New York (1991)
- [62] Silverman, R.J.: Invariant linear functions. *Trans. Am. Math. Soc.* **81**, 411–424 (1956)
- [63] Silverman, R.J.: Means on semigroups and the Hahn–Banach extension property. *Trans. Am. Math. Soc.* **83**, 222–237 (1956)
- [64] Solecki, S.: Amenability, free subgroups, and Haar null sets in non-locally compact groups. *Proc. Lond. Math. Soc.* **93**(3), 693–722 (2006)
- [65] Steele, J.M.: Kingman’s subadditive ergodic theorem. *Ann. de l’Institut Henri Poincaré* **25**, 93–98 (1989)
- [66] Tomkowicz, G., Wagon, S.: *The Banach-Tarski Paradox*. Cambridge University Press, Cambridge (2016). (1st edn. 1985)

N. H. Bingham  
Mathematics Department  
Imperial College  
London SW7 2AZ  
UK  
e-mail: n.bingham@ic.ac.uk

Variants on the Berz sublinearity theorem

A. J. Ostaszewski  
Mathematics Department  
London School of Economics  
Houghton Street  
London WC2A 2AE  
UK  
e-mail: A.J.Ostaszewski@lse.ac.uk

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