Vít Jelínek and Stanislav Kučera

On the structure of matrices avoiding interval-minor patterns

Article (Accepted version) (Refereed)

Original citation:
Jelínek, Vít and Kučera, Stanislav (2018) *On the structure of matrices avoiding interval-minor patterns*. Advances in Applied Mathematics, 101. pp. 70-99. ISSN 0196-8858
DOI: 10.1016/j.aam.2018.07.005

On the structure of matrices avoiding interval-minor patterns

Vít Jelínek[a,*], Stanislav Kučera[b]

[a] *Computer Science Institute, Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, Prague, Czech Republic*
[b] *Department of Mathematics, London School of Economics, Houghton Street, London, WC2A 2AE, United Kingdom*

Abstract

We study the structure of 01-matrices avoiding a pattern P as an interval minor. We focus on critical P-avoiders, i.e., on the P-avoiding matrices in which changing a 0-entry to a 1-entry always creates a copy of P as an interval minor.

Let Q be the 3×3 permutation matrix corresponding to the permutation 231. As our main result, we show that for every pattern P that has no rotated copy of Q as interval minor, there is a constant c_P such that any row and any column in any critical P-avoiding matrix can be partitioned into at most c_P intervals, each consisting entirely of 0-entries or entirely of 1-entries. In contrast, for any pattern P that contains a rotated copy of Q, we construct critical P-avoiding matrices of arbitrary size $n \times n$ having a row with $\Omega(n)$ alternating intervals of 0-entries and 1-entries.

Keywords: interval minor, 01–matrix, pattern avoidance

1. Introduction

A *binary matrix* is a matrix with entries equal to 0 or 1. All matrices considered in this paper are binary. The study of extremal problems of binary matrices has been initiated by the papers of Bienstock and Győri [1] and of Füredi [7]. Since these early works, most of the research in this area has focused on the concept of forbidden submatrices: a matrix M is said to contain a pattern P as a submatrix if we can transform M into P by deleting some rows and columns, and by changing 1-entries into 0-entries. This notion of submatrix is a matrix analogue of the notion of subgraph in graph theory.

The main problem in the study of pattern-avoiding matrices is to determine the extremal function $\mathrm{ex}(n;P)$, defined as the largest number of 1-entries in an $n \times n$ binary matrix avoiding the pattern P as submatrix. This is an analogue

*Corresponding author
Email addresses: `jelinek@iuuk.mff.cuni.cz` (Vít Jelínek), `s.kucera@lse.ac.uk` (Stanislav Kučera)

Preprint submitted to Elsevier

of the classical Turán-type problem of finding a largest number of edges in an n-vertex graph avoiding a given subgraph. Despite the analogy, the function $\mathrm{ex}(n; P)$ may exhibit an asymptotic behaviour not encountered in Turán theory. For instance, for the pattern[1] $P = \begin{pmatrix} \bullet & & \bullet & \\ & \bullet & & \bullet \end{pmatrix}$ Füredi and Hajnal [8] proved that $\mathrm{ex}(n; P) = \Theta(n\alpha(n))$, where $\alpha(n)$ is the inverse of the Ackermann function.

The asymptotic behaviour of $\mathrm{ex}(n; P)$ for general P is still not well understood. Füredi and Hajnal [8] posed the problem of characterising the *linear* patterns, i.e., the patterns P satisfying $\mathrm{ex}(n; P) = O(n)$. Marcus and Tardos [15] proved that $\mathrm{ex}(n; P) = O(n)$ whenever P is a *permutation matrix*, i.e., P has exactly one 1-entry in each row and each column. This result, combined with previous work of Klazar [12], has confirmed the long-standing Stanley–Wilf conjecture. However, the problem of characterising linear patterns is still open despite a number of further partial results [3, 6, 9, 11, 17, 19].

Fox [5] has introduced a different notion of containment among binary matrices, based on the concept of interval minors. Informally, a matrix M contains a pattern P as an interval minor if we can transform M into P by contracting adjacent rows or columns and changing 1-entries into 0-entries; see Section 2 for the precise definition. In this paper, we mostly deal with containment and avoidance of interval minors rather than submatrices. Therefore, the phrases *M avoids P* or *M contains P* always refer to avoidance or containment of interval minors, and the term *P-avoider* always refers to a matrix that avoids P as interval minor.

In analogy with $\mathrm{ex}(n; P)$, it is natural to consider the corresponding extremal function $\mathrm{ex}_{\preccurlyeq}(n; P)$ as the largest number of 1-entries in an $n \times n$ matrix that avoids P as an interval minor. If M contains P as a submatrix, it also contains it as an interval minor, and therefore $\mathrm{ex}_{\preccurlyeq}(n; P) \leq \mathrm{ex}(n; P)$. Moreover, it can be easily seen that for a permutation matrix P the two notions of containment are equivalent, and hence $\mathrm{ex}_{\preccurlyeq}(n; P) = \mathrm{ex}(n; P)$.

Fox [5] used interval minors as a key tool in his construction of permutation patterns with exponential Stanley–Wilf limits. In view of the results of Cibulka [2], this is equivalent to constructing a permutation matrix P for which the limit of the ratio $\mathrm{ex}(n; P)/n$ (which is equal to $\mathrm{ex}_{\preccurlyeq}(n; P)/n$) is exponential in the size of P.

Even before the work of Fox, interval minors have been implicitly used by Guillemot and Marx [10], who proved that a permutation matrix M which avoids as interval minor a fixed complete square pattern (i.e., a square pattern with all entries equal to 1) admits a type of recursive decomposition of bounded complexity. This result can be viewed as an analogue of the grid theorem from graph theory [18], which states that graphs avoiding a large square grid as a minor have bounded tree-width. Guillemot and Marx used their result on forbidden interval minors to design a linear-time algorithm for testing the containment of a fixed pattern in a permutation.

[1] We use the convention of representing 1-entries in binary matrices by dots and 0-entries by blanks.

Subsequent research into interval-minor avoidance has focused on avoiders of a complete matrix. In particular, Mohar et al. [16] obtained exact values for the extremal function for matrices simultaneously avoiding a complete pattern of size $2 \times \ell$ and its transpose, and they obtained bounds for patterns of size $3 \times \ell$. Their results were further generalised by Mao et al. [14] to a multidimensional setting.

While the functions $\mathrm{ex}(n; P)$ exhibit diverse forms of asymptotic behaviour, the function $\mathrm{ex}_{\preccurlyeq}(n; P)$ is linear for every nontrivial pattern P. This is a consequence of the Marcus–Tardos theorem and the fact that any binary matrix is an interval minor of a permutation matrix; see Fox [5]. Therefore, in the interval-minor avoidance setting, it is not as natural to classify patterns by the growth of $\mathrm{ex}_{\preccurlyeq}(n; P)$ alone as in the submatrix avoidance setting.

In our paper, we instead classify the patterns P based on the structure of the P-avoiders. We introduce the notion of *line complexity* of a binary matrix M, as the largest number of maximal runs of consecutive 0-entries in a single row or a single column of M. We focus on the *critical* P-avoiders, which are the matrices that avoid P as interval minor, but lose this property when any 0-entry is changed into a 1-entry.

Our main result is a sharp dichotomy for line complexity of critical P-avoiders. Let $Q_1, \dots, Q_4$ be defined as follows:

$$Q_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}, \; Q_2 = \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}, \; Q_3 = \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix} \text{ and } Q_4 = \begin{pmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix}.$$

We show that if a pattern P avoids the four patterns Q_i as interval minors (or equivalently, as submatrices), then the line-complexity of every critical P-avoider is bounded by a constant c_P depending only on P. On the other hand, if P contains at least one of the Q_i, then there are critical P-avoiders of size $n \times n$ with line complexity $\Omega(n)$, for any n.

After properly introducing our terminology and proving several simple basic facts in Section 2, we devote Section 3 to the statement and proof of our main result. In Section 4, we discuss the possibility of extending our approach to general minor-closed matrix classes, and present several open problems.

2. Preliminaries

Basic notation. For integers m and n, we let $[m, n]$ denote the set $\{m, m + 1, \dots, n\}$. We will also use the notation $[m, n)$ for the set $[m, n - 1]$, $(m, n]$ for the set $[m+1, n]$, and $[n]$ for $[1, n]$. We will avoid using (m, n) for $[m+1, n-1]$, however; instead, we will use the notation (m, n) to denote ordered pairs of integers.

We write $\{0, 1\}^{m \times n}$ for the set of binary matrices with m rows and n columns. We will always assume that rows of matrices are numbered top-to-bottom starting with 1, that is, the first row is the topmost.

For a matrix $M \in \{0, 1\}^{m \times n}$, we let $M(i, j)$ denote the value of the entry in row i and column j of M. We say that the pair (i, j) is a *1-entry* of M if $M(i, j) = 1$, otherwise it is a *0-entry*. The set of 1-entries of a matrix $M \in$

$\{0,1\}^{m\times n}$ is called the *support* of M, denoted by $\mathrm{supp}(M)$; formally, $\mathrm{supp}(M) = \{(i,j) \in [m] \times [n];\ M(i,j) = 1\}$.

We say that a binary matrix M' *dominates* a binary matrix M, if the two matrices have the same number of rows and the same number of columns, and moreover, $\mathrm{supp}(M) \subseteq \mathrm{supp}(M')$. In other words, M can be obtained from M' by changing some 1-entries into 0-entries.

For a matrix $M \in \{0,1\}^{m\times n}$ and for a set of row-indices $R \subseteq [m]$ and column-indices $C \subseteq [n]$, we let $M[R \times C]$ denote the submatrix of M induced by the rows in R and columns in C. More formally, if $R = \{r_1 < r_2 < \cdots < r_k\}$ and $C = \{c_1 < c_2 < \cdots < c_\ell\}$, then $M[R \times C]$ is a matrix $P \in \{0,1\}^{k\times\ell}$ such that $P(i,j) = M(r_i, c_j)$ for every $(i,j) \in [k] \times [\ell]$.

A *line* in a matrix M is either a row or a column of M. We view a line as a special case of a submatrix. For instance, the i-th row of a matrix $M \in \{0,1\}^{m\times n}$ is the submatrix $M[\{i\} \times [n]]$. A *horizontal interval* is a submatrix formed by consecutive entries belonging to a single row, i.e., a submatrix of the form $M[\{i\} \times [j_1, j_2]\}$ where i is a row index and j_1, j_2 are column indices. Vertical intervals are defined analogously.

We say that a submatrix of M is *empty* if it does not contain any 1-entries.

For a matrix $M \in \{0,1\}^{m\times n}$ and an entry $e \in [m] \times [n]$, we let $M\Delta e$ denote the matrix obtained from M by changing the value of the entry e from 0 to 1 or from 1 to 0.

Interval minors. A *row contraction* in a matrix $M \in \{0,1\}^{m\times n}$ is an operation that replaces a pair of adjacent rows r and $r+1$ by a single row, so that the new row contains a 1-entry in a column j if and only if at least one of the two original rows contained a 1-entry in column j. Formally, the row contraction transforms M into a matrix $M' \in \{0,1\}^{(m-1)\times n}$ whose entries are defined by

$$M'(i,j) = \begin{cases} M(i,j) \text{ if } i < r, \\ \max\{M(r,j), M(r+1,j)\} \text{ if } i = r, \\ M(i+1,j) \text{ if } i > r. \end{cases}$$

A column contraction is defined analogously.

We say that a matrix $P \in \{0,1\}^{k\times\ell}$ is an *interval minor* of a matrix $M \in \{0,1\}^{m\times n}$, denoted $P \preccurlyeq M$, if we can transform M by a sequence of row contractions and column contractions to a matrix $P' \in \{0,1\}^{k\times\ell}$ that dominates P. When P is an interval minor of M, we also say that M *contains* P, otherwise we say that M *avoids* P, or M is P-avoiding.

There are several alternative ways to define interval minors. One possible approach uses the concept of matrix partition. For $P \in \{0,1\}^{k\times\ell}$ and $M \in \{0,1\}^{m\times n}$, a *partition of M containing P* is the sequence of row indices $r_0, r_1, \ldots, r_k$ and column indices $c_0, c_1, \ldots, c_\ell$ with $0 \le r_0 < r_1 < \cdots < r_k \le m$ and $0 \le c_0 < c_1 < \cdots < c_\ell \le n$, such that for every 1-entry (i,j) of P, the submatrix $M[(r_{i-1}, r_i] \times (c_{j-1}, c_j]]$ has at least one 1-entry. See Figure 1.

An *embedding* of a matrix $P \in \{0,1\}^{k\times\ell}$ into a matrix $M \in \{0,1\}^{m\times n}$ is a function $\phi\colon [k] \times [\ell] \to [m] \times [n]$ with the following properties:

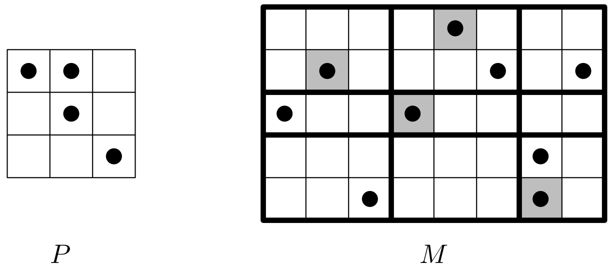

Figure 1: A pattern P and a matrix M that contains P. The thick lines indicate a partition of M containing P, and the shaded 1-entries form an image of P.

- If $e = (i, j)$ is a 1-entry of P, then $\phi(e)$ is a 1-entry of M.
- Let $e_1 = (i_1, j_1)$ and $e_2 = (i_2, j_2)$ be two entries of P, and suppose that $\phi(e_1) = (i_1^*, j_1^*)$ and $\phi(e_2) = (i_2^*, j_2^*)$. If $i_1 < i_2$ then $i_1^* < i_2^*$, and if $j_1 < j_2$ then $j_1^* < j_2^*$.

Notice that in an embedding ϕ of P into M, two entries of P belonging to the same row may be mapped to different rows of M, and similarly for columns.

In practice, it is often inconvenient and unnecessary to specify completely an embedding of P into M. In particular, it is usually unnecessary to specify the image of all the 0-entries in P. This motivates the notion of partial embedding, which we now formalise. Consider again binary matrices $P \in \{0,1\}^{k\times\ell}$ and $M \in \{0,1\}^{m\times n}$. Let S be a nonempty subset of $[k] \times [\ell]$. We say that a function $\psi \colon S \to [m] \times [n]$ is a *partial embedding* of P into M if the following holds:

- If $e = (i, j)$ is a 1-entry of P, then e is in S and $\psi(e)$ is a 1-entry of M.
- An entry $e = (i, j) \in S$ is mapped by ψ to an entry $\psi(e) = (i^*, j^*)$ of M satisfying the following inequalities: $i \le i^*$, $j \le j^*$, $k - i \le m - i^*$ and $\ell - j \le n - j^*$. Informally, the entry $\psi(e)$ is at least as far from the top, left, bottom and right edge of the corresponding matrix as the entry e.
- Let $e_1 = (i_1, j_1)$ and $e_2 = (i_2, j_2)$ be two entries in S, with $\psi(e_1) = (i_1^*, j_1^*)$ and $\psi(e_2) = (i_2^*, j_2^*)$. If $i_1 < i_2$ then $i_2 - i_1 \le i_2^* - i_1^*$, and if $j_1 < j_2$ then $j_2 - j_1 \le j_2^* - j_1^*$.

For a partial embedding ψ of a pattern P into a matrix M, the *image of* P (with respect to ψ) is the set of entries $\{\psi(e);\ e \in \mathrm{supp}(P)\}$ in the matrix M. Note that all the entries in the image of P are 1-entries.

Lemma 2.1. *For matrices $P \in \{0,1\}^{k\times\ell}$ and $M \in \{0,1\}^{m\times n}$ the following properties are equivalent.*

1. *P is an interval minor of M.*
2. *M has a partition containing P.*

3. P has an embedding into M.

4. P has a partial embedding into M.

Proof. We will prove the implications $2 \implies 1 \implies 3 \implies 4 \implies 2$.

To see that 2 implies 1, suppose M has a partition containing P, determined by row indices $r_0, r_1, \dots, r_k$ and column indices $c_0, c_1, \dots, c_\ell$, where we may assume that $r_0 = c_0 = 0$, $r_k = m$ and $c_\ell = n$. We may then contract the rows from each interval of the form $(r_{i-1}, r_i]$ into a single row, and contract the columns from each interval $(c_{i-1}, c_i]$ to a single column, to obtain a matrix $P' \in \{0,1\}^{k\times\ell}$ that dominates P.

To see that 1 implies 3, suppose that P is an interval minor of M. This means that there is a sequence of matrices $M_0, M_1, M_2, \dots, M_s$ with $s = (m-k)+(n-\ell)$, where $M_0 \in \{0,1\}^{k\times\ell}$ is a matrix that dominates P, and for each $i \in [s]$, the matrix M_{i-1} can be obtained from M_i by contracting a pair of adjacent rows or columns. We can then easily observe that for every $i = 0, 1, \dots, s$ there is an embedding ϕ_i of P into M_i. Indeed, reasoning by induction, the embedding ϕ_0 is the identity map, and for a given $i \in [s]$, if there is an embedding ϕ_{i-1} of P into M_{i-1}, then an embedding ϕ_i can be obtained by an obvious modification of ϕ_{i-1}.

Clearly, 3 implies 4, since every embedding is also a partial embedding.

To show that 4 implies 2, assume that $\psi\colon S \to [m]\times[n]$ is a partial embedding of P into M. We will define a sequence of row indices $0 \le r_0 < r_1 < \cdots < r_k \le m$ with these two properties:

- For each entry $e \in S$ that belongs to row i of P, the entry $\psi(e)$ belongs to a row i^* of M for some $i^* \in (r_{i-1}, r_i]$.
- If S contains at least one entry from row i in P, then S contains an entry e in row i such that $\psi(e)$ is in row r_i of M.

We define the numbers r_i inductively, starting with $r_0 = 0$. Suppose that $r_0, \dots, r_{i-1}$ have been defined, for some $i \ge 1$. If S contains no entry from row i of P, define $r_i = r_{i-1}+1$. On the other hand, if S contains an entry from row i, we let r_i be the largest row index of M such that ψ maps an entry from row i of P to an entry in row r_i of M. Notice that any entry $e \in S$ that does not belong to the first i rows of P must be mapped by ψ to an entry strictly below row r_i of M, otherwise ψ would not satisfy the properties of a partial embedding.

In an analogous way, we also define a sequence of column indices $0 \le c_0 < c_1 < \cdots < c_\ell \le n$. These sequences will satisfy that for every $e = (i,j) \in S$ we have $\psi(e) \in (r_{i-1}, r_i] \times (c_{j-1}, c_j]$. Since ψ is a partial embedding, S contains all the 1-entries of P, and ψ maps these 1-entries to 1-entries of M. In particular, the sequences $(r_i)_{i=0}^k$ and $(c_j)_{j=0}^\ell$ form a partition of M containing P. $\square$

Minor-closed classes. For a matrix P, we let $Av_{\preccurlyeq}(P)$ denote the set of all binary matrices that do not contain P as an interval minor. We call the matrices in $Av_{\preccurlyeq}(P)$ the *avoiders* of P, or *P-avoiders.*

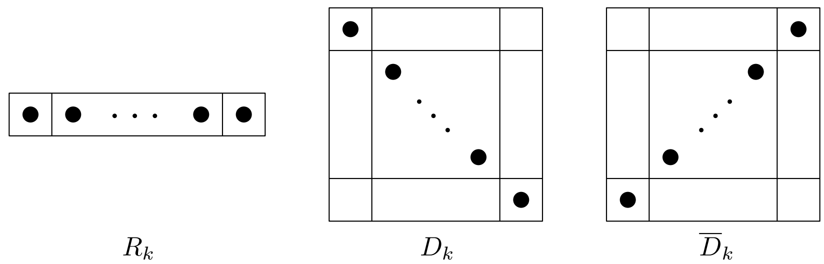

Figure 2: The patterns R_k, D_k and $\overline{D}_k$.

More generally, if $\mathcal{F}$ is a set of matrices, we let $Av_{\preccurlyeq}(\mathcal{F})$ denote the set of binary matrices that avoid all elements of $\mathcal{F}$ as interval minors.

We call a set $\mathcal{C}$ of binary matrices a *minor-closed class* (or just *class*, for short) if for every matrix $M \in \mathcal{C}$, all the interval minors of M are in $\mathcal{C}$ as well. Clearly, $Av_{\preccurlyeq}(\mathcal{F})$ is a class, and for every class $\mathcal{C}$ there is a (possibly infinite) set $\mathcal{F}$ such that $\mathcal{C} = Av_{\preccurlyeq}(\mathcal{F})$. A *principal class* is a class of matrices determined by a single forbidden pattern, i.e., a class of the form $Av_{\preccurlyeq}(P)$ for a matrix P.

For a class $\mathcal{C}$ of matrices, we say that a matrix $M \in \mathcal{C}$ is *critical for* $\mathcal{C}$ if the change of any 0-entry of M to a 1-entry creates a matrix that does not belong to $\mathcal{C}$. In other words, $M \in \mathcal{C}$ is critical for $\mathcal{C}$ if it is not dominated by any other matrix in $\mathcal{C}$. For a pattern P, we let $Av_{crit}(P)$ be the set of critical matrices for $Av_{\preccurlyeq}(P)$, and similarly for a set of patterns $\mathcal{F}$, $Av_{crit}(\mathcal{F})$ is the set of all critical matrices for $Av_{\preccurlyeq}(\mathcal{F})$.

2.1. Simple examples of P-avoiders

We conclude this section by presenting several examples of avoiders of certain simple patterns. These examples will play a role in Section 3, in the proof of our main result. We begin with a very simple example, which we present without proof.

Observation 2.2. *Let R_k be the matrix with 1 row and k columns, whose every entry is a 1-entry (see Figure 2). A matrix $M \in \{0,1\}^{m\times n}$ avoids R_k if and only if M has at most $k-1$ nonempty columns. Consequently, M is a critical R_k-avoider if and only if* $\operatorname{supp}(M)$ *is a union of* $\min\{k-1, n\}$ *columns.*

Next, we will consider the diagonal patterns $D_k \in \{0,1\}^{k\times k}$, defined by $\operatorname{supp}(D_k) = \{(i,i);\ i \in [k]\}$, and their mirror image $\overline{D}_k \in \{0,1\}^{k\times k}$, defined by $\operatorname{supp}(\overline{D}_k) = \{(i, k-i+1);\ i \in [k]\}$ (see again Figure 2). To describe the avoiders of these patterns, we first introduce some terminology.

Let e and e' be two entries of a matrix M. An *increasing walk* from e to e' in M is a set of entries $W = \{e_i = (r_i, c_i);\ i = 0, \dots, t\}$ such that $e_0 = e$, $e_t = e'$, and for every $i \in [t]$ we have either $r_i = r_{i-1}$ and $c_i = c_{i-1} + 1$ (that is, e_i is to the right of e_{i-1}), or $r_i = r_{i-1} - 1$ and $c_i = c_{i-1}$ (that is, e_i is above e_{i-1}). A *decreasing walk* is defined analogously, except now e_i is either to the right or below e_{i-1}.

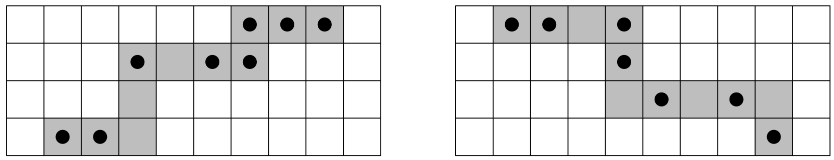

Figure 3: An increasing matrix (left) and a decreasing matrix (right). The shaded entries form an increasing and a decreasing walk in the respective matrices.

We say a matrix M is an *increasing matrix* if $\mathrm{supp}(M)$ is a subset of an increasing walk. A *decreasing matrix* is defined analogously. See Figure 3.

Proposition 2.3. *A matrix $M \in \{0,1\}^{m\times n}$ avoids the pattern D_k if and only if M contains $k-1$ increasing walks $W_1, \dots, W_{k-1}$ from $(m,1)$ to $(1,n)$ such that*

$$\mathrm{supp}(M) \subseteq W_1 \cup W_2 \cup \cdots \cup W_{k-1}.$$

Proof. Clearly, if M contains D_k, then M has k 1-entries no two of which can belong to a single increasing walk, and therefore $\mathrm{supp}(M)$ cannot be covered by $k-1$ increasing walks.

Suppose now that M avoids D_k. Consider a partial order $\triangleleft$ on the set $\mathrm{supp}(M)$, defined as $(i,j) \triangleleft (i',j') \iff i < i'$ and $j < j'$. Since M avoids D_k, this order has no chain of length k. By the classical Dilworth theorem [4], $\mathrm{supp}(M)$ is a union of $k-1$ antichains of $\triangleleft$. We may easily observe that each antichain of $\triangleleft$ is contained in an increasing walk. □

Proposition 2.3 shows, in particular, that a matrix M avoids the pattern $D_2 = (\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix})$ if and only if M is an increasing matrix. By symmetry, M avoids $\overline{D}_2$ if and only if it is a decreasing matrix.

Another direct consequence of the proposition is the following corollary, describing the structure of critical D_k-avoiders.

Corollary 2.4. *A critical D_k-avoiding matrix M contains $k-1$ increasing walks $W_1, \dots, W_{k-1}$ from $(m,1)$ to $(1,n)$ such that* $\mathrm{supp}(M) = W_1 \cup W_2 \cup \cdots \cup W_{k-1}$.

Note that Corollary 2.4 only gives a necessary condition for a matrix to be a critical D_k-avoider, therefore it is not a characterisation of critical D_k-avoiders. With only a little bit of extra effort, we could state and prove such a characterisation, but we omit doing so, as we do not need it for our purposes.

A simple but useful observation is that adding an empty row or column to the boundary of a pattern affects the P-avoiders in a predictable way. We state it here without proof.

Observation 2.5. *Let $P \in \{0,1\}^{k\times\ell}$ be a pattern, and let $P' \in \{0,1\}^{k\times(\ell+1)}$ be the pattern obtained by appending an empty column to P; in other words, we have $P'[[k]\times[\ell]] = P$, and the last column of P' is empty. Then a matrix $M \in \{0,1\}^{m\times n}$ avoids P' if and only if the matrix obtained by removing the last column from M avoids P. Consequently, M is in $\mathrm{Av}_{crit}(P')$ if and only if all the* *entries in the last column of M are 1-entries, and the preceding columns form a matrix from $Av_{crit}(P)$. Analogous properties hold for a pattern P'' obtained by prepending an empty column in front of all the columns of P, and also for rows instead of columns.*

3. Line complexity

In the previous section, we have seen several examples of matrices avoiding a fixed pattern as an interval minor. At a glance, it is clear that these matrices are highly structured. We would now like to make the notion of 'highly structured matrices' rigorous, and generalize it to other forbidden patterns.

We will focus on the local structure of matrices, i.e., the structure observed by looking at a single row or column. For a forbidden pattern P with at least two rows and two columns, any binary vector can appear as a row or column of a P-avoiding matrix; indeed, if P has at least two rows, then any matrix with a single row is P-avoiding.

However, the situation changes when we restrict our attention to critical P-avoiders. In the examples of critical P-avoiders we saw in Subsection 2.1, the 1-entries in each row or column were clustered into a bounded number of intervals. In particular, for these patterns P, the number of distinct vectors of a given length n that may appear as rows or columns of a critical P-avoider is at most polynomial in n.

In this section, we study this phenomenon in detail. We show that it generalizes to many other forbidden patterns P, but not all of them. As our main result, we will present a complete characterisation of the patterns P exhibiting this phenomenon.

Let us begin by formalising our main concepts.

A *horizontal 0-run* in a matrix M is a maximal sequence of consecutive 0-entries in a single row. More formally, a horizontal interval $M[\{r\} \times [c_1, c_2]]$ is a *horizontal 0-run* if all its entries are 0-entries, $c_1 = 1$ or $M(r, c_1 - 1) = 1$, and $c_2 = n$ or $M(r, c_2 + 1) = 1$. Symmetrically, a vertical interval is a *vertical 0-run* if it is a maximal vertical interval that only contains 0-entries. In the same manner, we define a (horizontal or vertical) *1-run* to be a maximal interval of consecutive 1-entries in a single line of M.

Note that each line in a matrix M can be uniquely decomposed into an alternating sequence of 0-runs and 1-runs.

Let M be a binary matrix. The *complexity* of a line of M is the number of 0-runs contained in this line. The *row-complexity* of M is the maximum complexity of a row of M, i.e., the least number k such that each row has complexity at most k. Similarly, the *column-complexity* of M is the maximum complexity of a column of M.

For a class of matrices $\mathcal{C}$, we define its *row-complexity*, denoted $r(\mathcal{C})$, as the supremum of the row-complexities of the critical matrices in $\mathcal{C}$. We say that $\mathcal{C}$ is *row-bounded* if $r(\mathcal{C})$ is finite, and *row-unbounded* otherwise. Symmetrically, we define the *column-complexity* $c(\mathcal{C})$ of $\mathcal{C}$ and the property of being *column-bounded* and *column-unbounded*. We say that a class $\mathcal{C}$ is *bounded* if it is both row-bounded and column-bounded; otherwise, it is *unbounded*.

We stress that when defining the row-complexity and column-complexity of a class of matrices, we only take into account the matrices that are critical for the class.

We are now ready to state our main result.

Theorem 3.1. *Let P be a pattern. The class $Av_{\preccurlyeq}(P)$ is row-bounded if and only if P does not contain any of Q_1, Q_2, Q_3, Q_4 as an interval minor, where*

$$Q_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix},\ Q_2 = \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix},\ Q_3 = \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix} \text{ and } Q_4 = \begin{pmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix}.$$

Before we prove Theorem 3.1, we point out two of its direct consequences.

Corollary 3.2. *For a pattern P, these statements are equivalent:*

- *$Av_{\preccurlyeq}(P)$ is row-bounded.*
- *$Av_{\preccurlyeq}(P)$ is column-bounded.*
- *$Av_{\preccurlyeq}(P)$ is bounded.*

Corollary 3.3. *Let $\mathcal{C} = Av_{\preccurlyeq}(P)$ and $\mathcal{C}' = Av_{\preccurlyeq}(P')$ be principal classes, and suppose that $\mathcal{C} \subseteq \mathcal{C}'$ (or equivalently, $P \preccurlyeq P'$). If $\mathcal{C}'$ is bounded, then $\mathcal{C}$ is bounded as well.*

Although each of these two corollaries is stating a seemingly basic property of the boundedness notion, we are not able to prove either of them without first proving Theorem 3.1. We also remark that neither of the two corollaries can be generalized to non-principal classes of matrices, as we will see in Section 4.

Let us say that a pattern P is *row-bounding* if $Av_{\preccurlyeq}(P)$ is row-bounded, otherwise P is *non-row-bounding*. Similarly, P is *bounding* if $Av_{\preccurlyeq}(P)$ is bounded and *non-bounding* otherwise.

Let $\mathcal{Q}$ be the set of patterns $\{Q_1, Q_2, Q_3, Q_4\}$. Theorem 3.1 states that a pattern P is row-bounding if and only if P is in $Av_{\preccurlyeq}(\mathcal{Q})$. To prove this, we will proceed in several steps. We first show, in Subsection 3.1, that if P contains a pattern from $\mathcal{Q}$, then P is not row-bounding. This is the easier part of the proof, though by no means trivial. Next, in Subsection 3.2, we show that every pattern in $Av_{\preccurlyeq}(\mathcal{Q})$ is row-bounding. This part is more technical, and requires a characterisation the structure of the patterns in $Av_{\preccurlyeq}(\mathcal{Q})$.

3.1. Non-row-bounding patterns

Our goal in this subsection is to show that any pattern P that contains one of the matrices from $\mathcal{Q}$ is not row-bounding. Let us therefore fix such a pattern P. Without loss of generality, we may assume that $Q_1 \preccurlyeq P$.

Theorem 3.4. *For every matrix P such that $Q_1 \preccurlyeq P$, the class $Av_{\preccurlyeq}(P)$ is row-unbounded.*

Proof. Refer to Figure 4. Let $P \in \{0,1\}^{k\times\ell}$ be a pattern containing Q_1 as an interval minor. In particular, there are row indices $1 \le r_1 < r_2 < r_3 \le k$ and column indices $1 \le c_1 < c_2 < c_3 \le \ell$ such that $P(r_1, c_2) = P(r_2, c_1) = P(r_3, c_3) = 1$.

For an arbitrary integer p, we will show how to construct a matrix in $Av_{crit}(P)$ of row-complexity at least p. We first describe a matrix $M \in \{0,1\}^{m\times n}$ with $m = r_1 + p(r_3 - r_1) + (k - r_3)$ and $n = (c_1 - 1) + p(c_3 - c_1 + 1) + (\ell - c_3)$.

In the matrix M, the leftmost $c_1 - 1$ columns, the rightmost $\ell - c_3$ columns, the topmost $r_1 - 1$ rows and the bottommost $k - r_3$ rows have all entries equal to 1. We call these entries the *frame* of M.

In the r_1-th row of M, there are p 0-entries appearing in columns $c_2 + i(c_3 - c_1 + 1)$ for $i = 0, \dots, p-1$, and the remaining entries in row r_1 are 1-entries.

The remaining entries of M, that is, the entries in rows $r_1+1, \dots, m-(k-r_3)$ and columns $c_1, \dots, n - (\ell - c_3)$, form a submatrix with $p(r_3 - r_1)$ rows and $p(c_3 - c_1 + 1)$ columns. We partition these entries into rectangular blocks, each block with $r_3 - r_1$ rows and $c_3 - c_1 + 1$ columns. For $i, j \in \{0, \dots, p-1\}$, let $B_{i,j}$ be such a block, with top-left corner in row $r_1 + 1 + i(r_3 - r_1)$ and column $c_1 + j(c_3 - c_1 + 1)$. The entries in $B_{i,j}$ are all equal to 1 if $i + j = p - 1$, otherwise they are all equal to 0.

We claim that the matrix M avoids P. To see this, assume there is an embedding ϕ of P into M, and consider where ϕ maps the three 1-entries $e_1 = (r_1, c_2)$, $e_2 = (r_2, c_1)$, and $e_3 = (r_3, c_3)$. Note that none of these three entries can be mapped into the frame of M, and moreover, neither e_2 nor e_3 can be mapped to the r_1-th row of M. In particular, $\phi(e_3)$ is inside a block $B_{i,j}$ for some $i + j = p - 1$. Since $\phi(e_2)$ is to the top-left of $\phi(e_3)$, it must belong to the same block $B_{i,j}$. It follows that $\phi(e_2)$ is in the leftmost column of $B_{i,j}$, which is the column $c_1 + j(c_3 - c_1 + 1)$, and $\phi(e_3)$ in its rightmost column, i.e., the column $c_3 + j(c_3 - c_1 + 1)$. Therefore, $\phi(e_1)$ is in column $c_2 + j(c_3 - c_1 + 1)$; however, all the entries in this column where ϕ could map e_1 are 0-entries. Therefore M is in $Av_{\preccurlyeq}(P)$.

The matrix M is not necessarily a critical P-avoider. However, we can transform it into a critical P-avoider by greedily changing 0-entries to 1-entries as long as the resulting matrix stays in $Av_{\preccurlyeq}(P)$. By this process, we obtain a matrix $M' \in Av_{crit}(P)$ that dominates M. We claim that the r_1-th row of M' is the same as the r_1-th row of M. This is because changing any 0-entry in the r_1-th row of M to a 1-entry produces a matrix containing the complete pattern $1^{k\times\ell}$ as a submatrix, and in particular also containing P as a minor.

We conclude that the matrix $M' \in Av_{crit}(P)$ has row-complexity at least p, showing that $Av_{\preccurlyeq}(P)$ is indeed row-unbounded. □

3.2. Row-bounding patterns

We now prove the second implication of Theorem 3.1, that is, we show that any pattern P avoiding the four patterns in $\mathcal{Q}$ is row-bounding (and therefore, by symmetry, also column-bounding). We first prove a result describing the structure of the patterns $P \in Av_{\preccurlyeq}(\mathcal{Q})$.

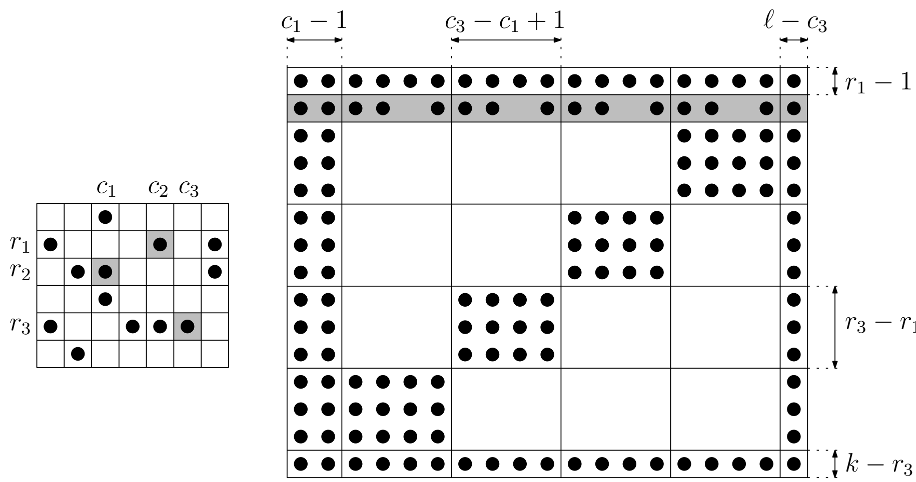

Figure 4: Illustration of the proof of Theorem 3.4. Left: a pattern P with a shaded image of Q_1. Right: a P-avoider with a shaded row of complexity $p = 4$.

We say that a matrix M can be *covered by k lines* if there is a set of lines $\ell_1, \dots, \ell_k$ such that each 1-entry of M belongs to some ℓ_i. The following fact is a version of the classical Kőnig–Egerváry theorem. We present it here without proof; a proof can be found, e.g., in Kung [13].

Fact 3.5 (Kőnig–Egerváry theorem)**.** *A matrix M cannot be covered by k lines if and only if M contains a set of $k+1$ 1-entries, no two of which are in the same row or column.*

Proposition 3.6. *If a pattern P belongs to $Av_{\preccurlyeq}(\mathcal{Q})$, then*

1. *P avoids the pattern $D_2 = (^{\bullet}{}_{\bullet})$, or*
2. *P avoids the pattern $\overline{D}_2 = ({}_{\bullet}{}^{\bullet})$, or*
3. *P can be covered by three lines.*

Proof. Assume P cannot be covered by three lines. By Fact 3.5, P contains four 1-entries $e_1 = (r_1, c_1)$, $e_2 = (r_2, c_2)$, $e_3 = (r_3, c_3)$ and $e_4 = (r_4, c_4)$, no two of which are in the same row or column. We may assume that $r_1 < r_2 < r_3 < r_4$. Moreover, since P does not contain any pattern from $\mathcal{Q}$, we see that any three entries among e_1, e_2, e_3, e_4 must form an image of D_3 or of $\overline{D}_3$. Consequently, the four entries e_i form an image of D_4 or of $\overline{D}_4$, i.e., we must have either $c_1 < c_2 < c_3 < c_4$ or $c_1 > c_2 > c_3 > c_4$. Suppose that $c_1 < c_2 < c_3 < c_4$ holds, the other case being symmetric.

We will now show that P avoids the pattern $\overline{D}_2$. Note first that the submatrix $P[[r_3] \times [c_3]]$ avoids $\overline{D}_2$, since an image of $\overline{D}_2$ there would form an image of Q_1 with e_4. Therefore, by Proposition 2.3, all the 1-entries in $P[[r_3] \times [c_3]]$

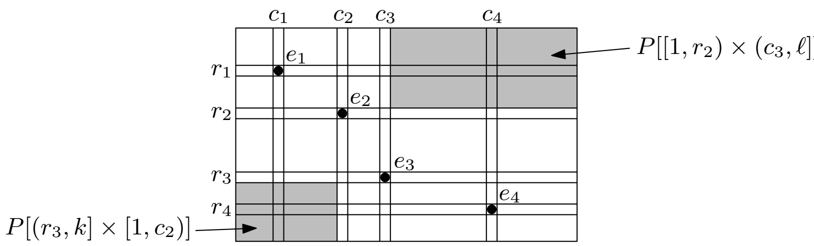

Figure 5: Illustration of the proof of Proposition 3.6.

belong to a single decreasing walk from $(1,1)$ to e_3. Symmetrically, all 1-entries in the submatrix $P[[r_2, k] \times [c_2, \ell]]$ belong to a decreasing walk from e_2 to (k, ℓ).

Moreover, there can be no 1-entry in $P[(r_3, k] \times [1, c_2)]$ or in $P[[1, r_2) \times (c_3, \ell]]$, since such a 1-entry would form a forbidden pattern with e_2 and e_3. We conclude that all the 1-entries of P belong to a single decreasing walk and therefore P avoids $\overline{D}_2$. □

We note that Proposition 3.6 is not an equivalent characterisation of patterns from $Av_{\preccurlyeq}(\mathcal{Q})$, since a matrix covered by three lines may contain a pattern from $\mathcal{Q}$. Later, in Lemma 3.17, we will give a more precise description of the avoiders of $\mathcal{Q}$ that cannot be covered by two lines.

Relative row-boundedness. Before we prove that each pattern P in the set $Av_{\preccurlyeq}(\mathcal{Q})$ is row-bounding, we need some technical preparation. First of all, we shall need a more refined notion of row-boundedness, which considers individual 1-entries of the pattern P separately.

Let P be a pattern, let e be a 1-entry of P, let M be a P-avoiding matrix, and let f be a 0-entry of M. Recall that $M\Delta f$ is the matrix obtained from M by changing the entry f from 0 to 1. We say that the entry f of M is *critical for e (with respect to P)* if there is an embedding of P into $M\Delta f$ that maps e to f. Moreover, if z is a 0-run in M, we say that z is *critical for e* if at least one 0-entry in z is critical for e.

Note that a P-avoiding matrix is critical for $Av_{\preccurlyeq}(P)$ if and only if each 0-entry of M is critical for at least one 1-entry of P.

Let e be a 1-entry of a pattern P. Let M be a matrix avoiding P. The *complexity of a row r of M relative to e* is the number of 0-runs in row r that are critical for e. The *row-complexity of M relative to e* is the maximum complexity of a row of M relative to e, and the *row-complexity of $Av_{\preccurlyeq}(P)$ relative to e*, denoted $r(Av_{\preccurlyeq}(P), e)$, is the supremum of the row-complexities of the matrices in $Av_{\preccurlyeq}(P)$ relative to e. When $r(Av_{\preccurlyeq}(P), e)$ is finite, we say that $Av_{\preccurlyeq}(P)$ is *row-bounded relative to e* and *e is row-bounding*, otherwise $Av_{\preccurlyeq}(P)$ is *row-unbounded relative to e*.

Notice that in the definition of $r(Av_{\preccurlyeq}(P), e)$, we are taking supremum over all the matrices in $Av_{\preccurlyeq}(P)$, not just the critical ones. This makes the definition more convenient to work with, but it does not make any substantial difference.

In fact, for a pattern P with a row-bounding 1-entry e, the row-complexity relative to e in $Av_{\preccurlyeq}(P)$ is maximized by a critical P-avoider. To see this, suppose that M is a P-avoiding matrix, M^+ is any critical P-avoiding matrix that dominates M, and f is a 0-entry of M that is critical for e; then f is necessarily also a 0-entry in M^+, and is still critical for e in M^+. Therefore, the row-complexity of M^+ relative to e is at least as large as the row-complexity of M relative to e.

Observe that the following inequalities hold for any pattern P:

$$\max_{e\in\operatorname{supp}(P)} r\left(Av_{\preccurlyeq}(P), e\right) \leq r\left(Av_{\preccurlyeq}(P)\right) \leq \sum_{e\in\operatorname{supp}(P)} r\left(Av_{\preccurlyeq}(P), e\right).$$

In particular, a pattern P is row-bounding if and only if each 1-entry of P is row-bounding.

Lemma 3.7. *Let P be a pattern, and let M be a P-avoiding matrix. Let z be a horizontal 0-run of M, and let $f \in z$ be a 0-entry in this 0-run. Assume that there is an embedding ϕ of P into $M\Delta f$. Then P has a 1-entry e mapped by ϕ to f, and moreover, every entry of P in the same column as e is mapped by ϕ to a column containing an entry from z.*

Proof. Clearly, ϕ must map a 1-entry of P to the entry f, otherwise ϕ would also be an embedding of P into M and M would not be P-avoiding.

Suppose now that $z = \{r\} \times [c_1, c_2]$ for a row r and columns $c_1 \leq c_2$. Let e' be an entry of P in the same column as e. Suppose that ϕ maps e' to an entry in column c, with $c \notin [c_1, c_2]$. Assume that $c < c_1$, the case $c > c_2$ being analogous. Then we may modify ϕ to map e to the 1-entry $(r, c_1 - 1)$ instead of f, obtaining an embedding of P into M, which is a contradiction. ☐

Criteria for relative row-boundedness. Let us first point out a trivial but useful fact: if $\overline{P} \in \{0,1\}^{k\times\ell}$ is a pattern obtained from a pattern P by reversing the order of rows (i.e., turning P upside down) then a 1-entry $e = (i, j)$ of P is row-bounding if and only if the corresponding 1-entry $\overline{e} = (k - i + 1, j)$ of $\overline{P}$ is row-bounding. Analogous properties hold for reversing the order of columns or 180-degree rotation. Similarly, operations that map rows to columns, such as transposition or 90-degree rotation, will map row-bounding 1-entries to column-bounding ones and vice versa.

We will now state several general criteria for row-boundedness of 1-entries, which we will later use to show that any $\mathcal{Q}$-avoiding pattern is row-bounding.

Lemma 3.8. *If $P \in \{0,1\}^{k\times\ell}$ is a pattern with a row $r \in [k]$ and a column $c \in [\ell]$ such that* $\operatorname{supp}(P) \subseteq (\{r\} \times [\ell]) \cup ([m] \times [c, \ell])$*, then every 1-entry of P in the interval $\{r\} \times [c]$ is row-bounding (see Figure 6).*

Proof. Let $e = (r, j)$ be a 1-entry of P with $j \leq c$. Let $M \in \{0,1\}^{m\times n}$ be a P-avoider, let $f = (r', c')$ be a 0-entry of M critical for e, and let z be the horizontal 0-run containing f.

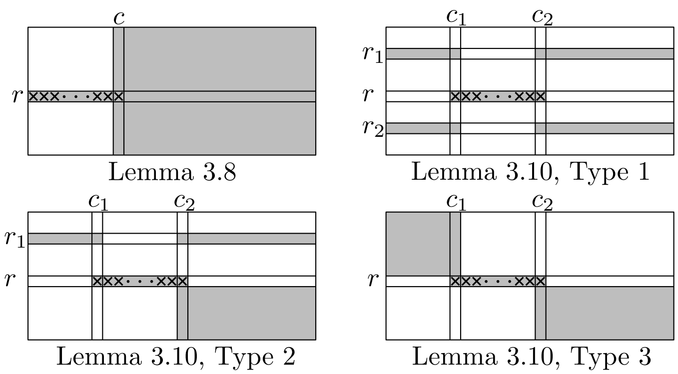

Figure 6: Illustration of Lemma 3.8 and Lemma 3.10. The shaded areas are the possible locations of 1-entries. The 1-entries in the cells marked by crosses are row-bounding.

We claim that in the row r' of M, there are fewer than j 1-entries to the left of f. Suppose this is not the case, i.e., row r' contains j distinct 1-entries $f'_1, f'_2, \dots, f'_j$, numbered left to right, all of them to the left of f.

Let ϕ be an embedding of P into $M\Delta f$ which maps e to f. Recall from Lemma 3.7 that all the entries in column j of P are mapped to columns intersecting z. In particular, all the entries from column j are mapped to the right of f'_j.

We define a partial embedding ψ of P into M, as follows. Firstly, ψ maps the entries $(r,1),(r,2),\dots,(r,j)$ of P to the 1-entries $f'_1, f'_2, \dots, f'_j$ of M. Next, ψ maps each 1-entry of P that is not among $(r,1),(r,2),\dots,(r,j)$ to the same entry as ϕ. We easily see that ψ is a partial embedding of P into M, a contradiction.

Therefore, there are fewer than j 1-entries in row r' to the right of f, and hence row r has at most j 0-runs critical for e. Consequently, $r\left(Av_{\preccurlyeq}\left(P\right), e\right) \le j$ and e is row-bounding. □

The assumptions of Lemma 3.8 are satisfied when c is the leftmost nonempty column of a pattern P and r is an arbitrary row. We state this important special case as a separate corollary.

Corollary 3.9. *Any 1-entry in the leftmost nonempty column of a pattern P is row-bounding.*

Lemma 3.10. *Let $P \in \{0,1\}^{k\times\ell}$ be a pattern with a row r, and two distinct columns $c_1 < c_2$, such that all the 1-entries of P in row r belong to the interval $\{r\} \times [c_1, c_2]$. Moreover, if c is a column index with $c_1 < c < c_2$, then P has no 1-entry in column c except possibly for the entry (r,c). Suppose furthermore that P satisfies one of the following three conditions (see Figure 6):*

Type 1: All the 1-entries of P above row r are in a single row $r_1 < r$, and all the 1-entries below row r are in a single row $r_2 > r$.

Type 2: All the 1-entries of P above row r are in a single row $r_1 < r$, and all the 1-entries below row r are in the submatrix $P[(r,k] \times [c_2,\ell]]$.

Type 3: All the 1-entries of P above row r are in the submatrix $P[[1, r) \times [c_1]]$, and all the 1-entries below row r are in the submatrix $P[(r, k] \times [c_2, \ell]]$.

Then every 1-entry in the interval $\{r\} \times [c_1, c_2]$ is row-bounding.

Proof. Let $P \in \{0,1\}^{k \times \ell}$ be a pattern satisfying the assumptions, and let $d = c_2 - c_1 + 1$. We will show that for each 1-entry $e \in \{r\} \times [c_1, c_2]$ of P and every P-avoiding matrix $M \in \{0,1\}^{m \times n}$, there are at most d 0-runs critical for e in each row of M.

For contradiction, assume that M has a row r' with at least $d+1$ 0-runs critical for e. Let f and f' be the leftmost and the rightmost 0-entries critical for e in row r'. By assumption, M has at least d 1-entries in row r' between f and f'. Let $f_1, f_2, \ldots, f_d$ be d such 1-entries, numbered left to right.

Let ϕ be an embedding of P into $M \Delta f$ which maps e to f, and let ϕ' be an embedding of P into $M \Delta f'$ which maps e to f'. Let us describe a partial embedding ψ of P into M. Firstly, ψ maps the entries $(r, c_1), (r, c_1 + 1), \ldots, (r, c_2)$ to the entries $f_1, f_2, \ldots, f_d$ in row r' of M. Next, ψ maps each 1-entry in $M[[m] \times [c_1]]$ except (r, c_1) to the same entry as ϕ, and ψ maps the 1-entries in $M[[m] \times [c_2, n]]$ except (r, c_2) to the same entry as ϕ'. We easily check that this makes ψ a partial embedding of P into M: note that from Lemma 3.7, it follows that ϕ maps all the entries in column c_1 of P to entries strictly to the left of f_1, and ϕ' maps entries in column c_2 to entries strictly to the right of f_d.

This is impossible, since M is P-avoiding. Therefore, every row of a P-avoiding matrix has at most d 0-runs critical for e, and e is row-bounding. □

Lemma 3.11. *Let $P \in \{0,1\}^{k \times \ell}$ be a pattern with two rows $r_1 \le r_2$ and a column c, such that for every $r \in [r_1, r_2]$, P has no 1-entry in row r except possibly for the entry (r, c). Suppose moreover, that P satisfies one of the following conditions (see Figure 7):*

Type 1: All the 1-entries of P above row r_1 are in column c or in the row $r_1 - 1$, and all the 1-entries below row r_2 are in column c or in the row $r_2 + 1$.

Type 2: All the 1-entries of P above row r_1 are in column c or in the row $r_1 - 1$, and all the 1-entries below row r_2 are in the submatrix $P[(r_2, k] \times [c, \ell]]$.

Type 3: All the 1-entries of P above row r_1 are in the submatrix $P[[1, r_1) \times [c]]$, and all the 1-entries below row r_2 are in the submatrix $P[(r_2, k] \times [c, \ell]]$.

Then every 1-entry in the interval $[r_1, r_2] \times \{c\}$ is row-bounding.

Proof. Let P be a pattern satisfying the assumptions of the lemma, and let $e = (r, c)$ be its 1-entry, with $r \in [r_1, r_2]$. Let $M \in \{0,1\}^{m \times n}$ be a P-avoider. We claim that every row of M has at most one 0-run critical for e. For contradiction, suppose that row i of M has two 0-runs z_L and z_R critical for e, where z_L is to the left of z_R. Let $f_L \in z_L$ and $f_R \in z_R$ be two 0-entries critical for e in the two 0-runs.

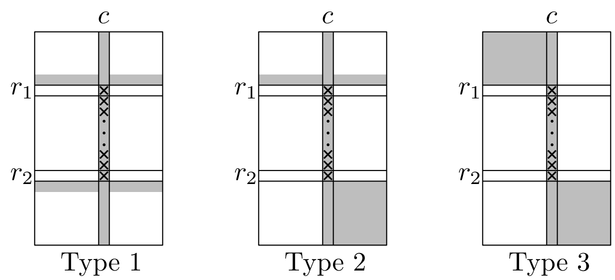

Figure 7: Illustration of Lemma 3.11. The shaded areas correspond to possible locations of 1-entries. The 1-entries in cells marked by crosses are row-bounding.

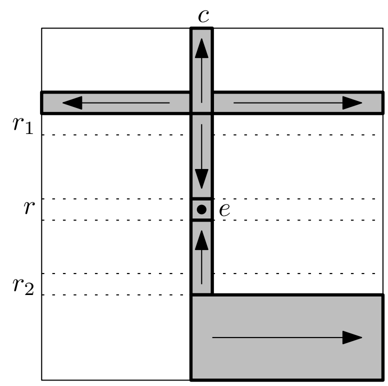

Figure 8: The construction of the embedding ψ in the proof of Lemma 3.11, illustrated on a matrix of Type 2. The shaded areas are possible locations of 1-entries. For a 1-entry e' in the rectangle marked by a left arrow, we put $\psi(e') = \phi_L(e')$, in the rectangles marked by a right arrow, we put $\psi(e') = \phi_R(e')$, in those marked by an up arrow, $\psi(e')$ is the higher of $\phi_L(e')$ and $\phi_R(e')$, and in those marked by a down arrow, $\psi(e')$ is the lower of the two options.

Let ϕ_L be an embedding of P into $M\Delta f_L$ with $\phi_L(e) = f_L$, and ϕ_R be an embedding mapping P into $M\Delta f_R$ with $\phi_R(e) = f_R$. Let $C_L \subseteq [n]$ be the set of columns of M that intersect the 0-run z_L, that is, we have $z_L = \{i\} \times C_L$. Similarly, let C_R be the set of columns intersecting z_R. Note that by Lemma 3.7, ϕ_L maps all the entries in column c of P to entries in the set of columns C_L, and similarly, ϕ_R maps the entries of column c to entries in columns C_R. We will now describe a partial embedding ψ of P into M; see Figure 8.

Since f_L and f_R are in distinct 0-runs, M has a 1-entry f that lies in row i between f_L and f_R. We put $\psi(e) = f$. For any other 1-entry $e' \in \text{supp}(P) \setminus \{e\}$, we will define $\psi(e')$ to be equal to either $\phi_L(e')$ or $\phi_R(e')$, by the following rules.

For a 1-entry e' which is strictly to the left of column c, we let $\psi(e') = \phi_L(e')$ and for a 1-entry e' strictly to the right of column c, we let $\psi(e) = \phi_R(e')$.

It remains to deal with the 1-entries in column c. For a 1-entry e' in $[r_1, r) \times \{c\}$, we choose $\psi(e')$ to be the lower of the two entries $\phi_L(e')$ and $\phi_R(e')$, i.e., we choose the entry that has larger row-index. If $\phi_L(e')$ and $\phi_R(e')$ are in the same row, we choose $\psi(e')$ arbitrarily from the two options.

For a 1-entry e' in $[1, r_1) \times \{c\}$, we distinguish two possibilities. If P is of Type 1 or Type 2, that is, all 1-entries above row r_1 are in column c or row $r_1 - 1$, we choose $\psi(e')$ to be the higher of the two entries $\phi_L(e')$ and $\phi_R(e')$. If, on the other hand, P is of Type 3, so all 1-entries above row r_1 are in columns $1, \dots, c$, we put $\psi(e') = \phi_L(e')$.

We proceed symmetrically for 1-entries below row r. For a 1-entry $e' \in (r, r_2] \times \{c\}$, we choose $\psi(e')$ to be the higher of the two entries $\phi_L(e')$ and $\phi_R(e')$, breaking ties arbitrarily. For a 1-entry $e' \in (r_2, k] \times \{c\}$, if P is of Type 1, we let $\psi(e')$ be the lower of $\phi_L(e')$ and $\phi_R(e')$, and if P is of Type 2 or 3, we put $\psi(e') = \phi_R(e')$.

Let us verify that the mapping ψ is a partial embedding of P into M. For an entry e' of P or of M, let $\mathrm{ri}(e')$ and $\mathrm{ci}(e')$ denote respectively the row index and column index of e'. Choose a pair of 1-entries $e_1 = (i_1, j_1)$ and $e_2 = (i_2, j_2)$ of P, and consider their respective images $f_1 = \psi(e_1)$ and $f_2 = \psi(e_2)$.

Suppose that $i_1 < i_2$; we then need to show that $\mathrm{ri}(f_2) - \mathrm{ri}(f_1) \geq i_2 - i_1$. We may assume here, without loss of generality, that P has no 1-entries in the rows $[i_1 + 1, i_2)$. In particular, we have either $i_2 \leq r$ or $i_1 \geq r$. By the definition of ψ, we see that

$$\begin{aligned}\mathrm{ri}(f_2) - \mathrm{ri}(f_1) &\geq \min\{\mathrm{ri}(\phi_L(e_2)) - \mathrm{ri}(\phi_L(e_1)), \mathrm{ri}(\phi_R(e_2)) - \mathrm{ri}(\phi_R(e_1))\} \\ &\geq i_2 - i_1,\end{aligned}$$

as claimed, where the second inequality follows from the fact that ϕ_L and ϕ_R are both embeddings.

By similar reasoning, we verify that for a pair of 1-entries $e_1 = (i_1, j_1)$ and $e_2 = (i_2, j_2)$ of P with $j_1 < j_2$, we always have $\mathrm{ci}(\psi(e_2)) - \mathrm{ci}(\psi(e_1)) \geq j_2 - j_1$ (recall that ϕ_L maps the entries from column c to the set of columns C_L, while ϕ_R maps them to the set C_R, which is strictly to the right of C_L).

We may also easily see that for any 1-entry e' of P, the distance of its image $\psi(e')$ from the left, right, top and bottom edge of M, is at least as large as the distance of e from the corresponding edge of P. We conclude that ψ is a partial embedding of P into M, which contradicts the assumption that M is P-avoiding. This contradiction shows that the entry $e = (r, c)$ is row-bounding. □

Lemma 3.12. *Let $P \in \{0,1\}^{k \times \ell}$ be a pattern with two rows $r_1 < r_2$ and two columns $c_1 < c_2$ of one of the following two types (see Figure 9):*

Type 1: $\mathrm{supp}(P) \subseteq ([r_1, r_2] \times \{c_1\}) \cup \big(\{r_1, r_2\} \times ([c_1] \cup \{c_2\})\big)$.

Type 2: $\mathrm{supp}(P) \subseteq ([r_1, r_2] \times \{c_1\}) \cup \big(\{r_2\} \times ([c_1] \cup \{c_2\})\big) \cup ([r_1] \times \{c_2\})$.

If $e = (r_1, c_1)$ is a 1-entry of P, then it is row-bounding.

Proof. Suppose that $P \in \{0,1\}^{k \times \ell}$ satisfies the assumptions of the lemma, and that the entry $e = (r_1, c_1)$ is a 1-entry. Let e' be the entry (r_2, c_2) of P. Notice that if e' is a 0-entry, we can deduce that e is row-bounding by Lemma 3.8 (for

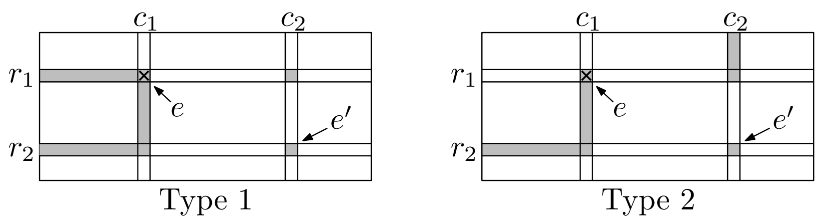

Figure 9: Illustration of the proof of Lemma 3.12. As before, the shaded areas correspond to possible locations of 1-entries, and the 1-entry e, marked by a cross, is row-bounding.

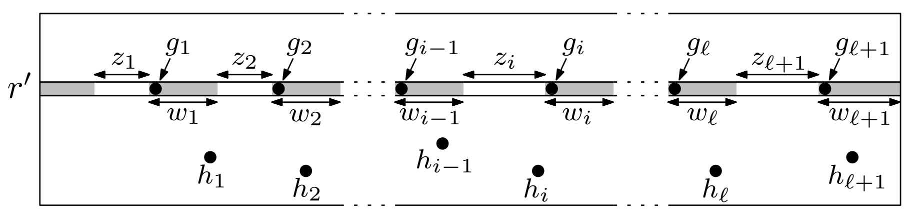

Figure 10: Illustration of the proof of Lemma 3.12: the structure of a P-avoiding matrix with many intervals critical for e in row r'.

Type 1, mirrored by the vertical axis) or by Type 3 of Lemma 3.10 (for Type 2, again mirrored by the vertical axis). Assume therefore that e' is a 1-entry of P.

Let $M \in \{0,1\}^{m\times n}$ be a P-avoider. We will show that every row of M has at most $\ell(\ell+1)$ 0-runs critical for e. Suppose that a row r' of M has more than $\ell(\ell+1)$ 0-runs critical for e. Among these 0-runs, we select a subsequence $z_1, z_2, \dots, z_{\ell+1}$ numbered left to right, with the property that for each $i \in [\ell]$, M has at least ℓ 1-entries in row r' between z_i and z_{i+1}, and M also has at least ℓ 1-entries in row r' to the right of $z_{\ell+1}$.

For each $i \in [\ell+1]$, let f_i be a 0-entry in z_i critical for e, and let ϕ_i be an embedding of P into $M\Delta f_i$ that maps e to f_i. For $i \in [\ell]$, let w_i be the interval of entries that lie between z_i and z_{i+1} in row r' of M, and let $w_{\ell+1}$ be the interval of entries in row r' to the right of $z_{\ell+1}$. Recall that each w_i contains at least ℓ 1-entries. Let g_i be the leftmost entry in w_i, which is necessarily a 1-entry, because z_i is a maximal interval of 0-entries. Finally, let $h_i = (p_i, q_i)$ be the 1-entry $\phi_i(e')$ (recall that $e' = (r_2, c_2)$ is a 1-entry of P).

Let us define a partial embedding ψ of P into M. We let ψ map the entry (r_1, c_2) to the 1-entry $g_{\ell+1}$, and if P is of Type 2, then for every 1-entry e'' in the interval $[1, r_1) \times \{c_2\}$, we define $\psi(e'') = \phi_{\ell+1}(e'')$. Note that all the entries we mapped so far are to the right of $f_{\ell+1}$.

To define ψ for the remaining 1-entries of P, we will distinguish several situations, depending on the positions of the entries $h_i = (p_i, q_i)$.

If, for some $i \in [\ell]$, the entry h_i is to the right of the rightmost column of w_i, we put $\psi(e) = g_i$, and for every 1-entry e'' of P for which ψ has not yet been defined, we put $\psi(e'') = \phi_i(e'')$. To see that the mapping ψ is a partial embedding of P into M, it is enough to observe that all the 1-entries in column c_2 of P are mapped by ψ to entries strictly to the right of w_i, while by Lemma 3.7, all the 1-entries in column c_1 are mapped to the columns intersecting the interval z_i, except for the entry e, which is mapped to g_i. There are therefore at least $\ell - 1$ columns which separate the image of any entry from column c_1 from the image of any entry from column c_2. With this in mind, it is easy to check that ψ is indeed a partial embedding.

Suppose that the situation from the previous paragraph does not occur, that is, for every $i \in [\ell]$, the entry h_i is not to the right of the rightmost column intersecting w_i. Since h_i must by construction be to the right of the column containing f_i, we know that the column q_i containing h_i intersects either z_i or w_i. In particular, we have $q_1 < q_2 < \cdots < q_{\ell+1}$.

Assume now, that for some $i \in [\ell]$, the inequality $p_i \leq p_{i+1}$ holds. We now complete the mapping ψ as follows: we put $\psi(e) = g_i$, $\psi(e') = h_{i+1}$, and for all the 1-entries e'' of P not yet mapped (i.e., the 1-entries in columns $1, \dots, c_1$ except e), we put $\psi(e'') = \phi_i(e'')$. The mapping ϕ is again a partial embedding of P into M.

It remains to deal with the situation when we have $p_1 > p_2 > \cdots > p_\ell > p_{\ell+1}$, which means that the 1-entries $h_1, h_2, \dots, h_{\ell+1}$ form an image of the diagonal pattern $\overline{D}_{\ell+1}$. We complete the mapping ψ as follows: a 1-entry of the form (r_1, j) for $j \leq c_1$ is mapped to the entry g_j, a 1-entry of the form (r_2, j) for any $j \in [\ell]$ is mapped to h_j, and any 1-entry $e'' \in [r_1 + 1, r_2) \times \{c_1\}$ is mapped to $\phi_\ell(e'')$. Note that for $j < c_1$, the mapping ψ maps the 1-entries in column j to 1-entries in columns intersecting $z_j \cup w_j$, and for $j = c_1$, the 1-entries in column j get mapped to columns intersecting $z_j \cup w_j \cup z_\ell$.

In all cases, we found a partial embedding ψ of P into M, which is a contradiction. Therefore, each row of M has at most $\ell(\ell+1)$ 0-runs critical for e, and e is row-bounding. □

Row-boundedness of specific patterns. We now have enough technical tools to establish that any pattern P from $Av_{\preccurlyeq}(\mathcal{Q})$ is row-bounding. Recall from Proposition 3.6 that any $P \in Av_{\preccurlyeq}(\mathcal{Q})$ avoids D_2 or $\overline{D}_2$ or can be covered by three lines.

We will first look at patterns that can be covered by fewer than three lines, and show that they are all row-bounding.

Lemma 3.13. *A pattern P that has at most two nonempty columns or at most one nonempty row is row-bounding.*

Proof. It follows from Lemma 3.8 and trivial symmetries that every 1-entry of P is row-bounding, hence P is row-bounding. □

Lemma 3.14. *If $P \in \{0,1\}^{k \times \ell}$ is a pattern with two nonempty rows, then P is row-bounding.*

Proof. We will show that for every 1-entry e of P, we have $r\left(Av_{\preccurlyeq}(P), e\right) \leq \ell^2$.

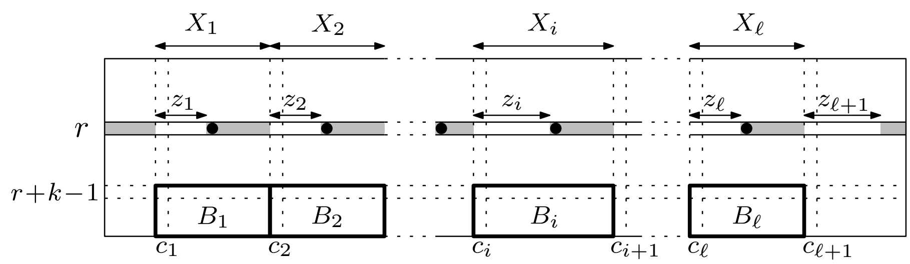

Figure 11: The matrix M considered in the proof of Lemma 3.14.

In view of Observation 2.5, we may assume that only the first row and the last row of P are nonempty. Let e be a 1-entry of P, and suppose without loss of generality that e is in the first row, i.e., $e = (1, c)$ for some c.

Given a matrix $M \in Av_{\preccurlyeq}(P)$, consider an arbitrary row r of M. For contradiction, suppose that the row r has $\ell^2 + 1$ distinct 0-runs $z_1, \ldots, z_{\ell^2+1}$ critical for e, numbered left to right. Let c_i denote the leftmost column intersecting z_i, and for $i \leq \ell^2$, let X_i denote the set of column indices $[c_i, c_{i+1})$. Observe that for every $i \leq \ell^2$, M has at least one 1-entry in the interval $\{r\} \times X_i$.

Let B_i be the submatrix $M[[r + k - 1, m] \times X_i]$ of M (see Figure 11). Note that if there are at least ℓ distinct values of i for which B_i contains at least one 1-entry, then the matrix M contains the pattern P.

Suppose therefore that B_i is empty for each i up to at most $\ell - 1$ exceptions. In particular, there is an index $j \in [\ell^2]$ such that the ℓ consecutive submatrices $B_j, B_{j+1}, \ldots, B_{j+\ell-1}$ are all empty.

Recall that $e = (1, c)$ is a 1-entry of P, and that all the 1-entries of P are in rows 1 and k. Let c' be a column index such that $e' = (k, c')$ is a 1-entry of P, and $|c - c'|$ is as small as possible. Suppose without loss of generality that $c \leq c'$ and let $d := c' - c$.

Let f be a 0-entry in z_j critical for e, and let ϕ be an embedding of P into $M \Delta f$ that maps e to f. Note that by Lemma 3.7, ϕ maps the entries in column c of P to entries in columns intersecting z_j, and in particular, the entry (k, c) is mapped inside B_j. Since B_j is empty, (k, c) is a 0-entry and in particular, c' is greater than c.

It follows that the 1-entry $e' = (k, c')$ is mapped strictly to the right of the column containing f, and since $B_j, \ldots, B_{j+\ell-1}$ are all empty, e' must be mapped to the right of the columns in the set $X_{j+\ell-1}$.

We now define a partial embedding ψ of P into M as follows: the $d+1$ entries in $P[\{1\} \times [c, c']]$ get mapped into $M[\{r\} \times (X_j \cup X_{j+1} \cup \cdots \cup X_{j+d})]$ by ψ (recall that $\{r\} \times X_i$ contains at least one 1-entry for each i). The remaining 1-entries of P are mapped by ψ in the same way as by ϕ. Then ψ is a partial embedding of P into M, a contradiction. $\square$

Lemma 3.15. *A pattern P that can be covered by one row and one column is row-bounding.*

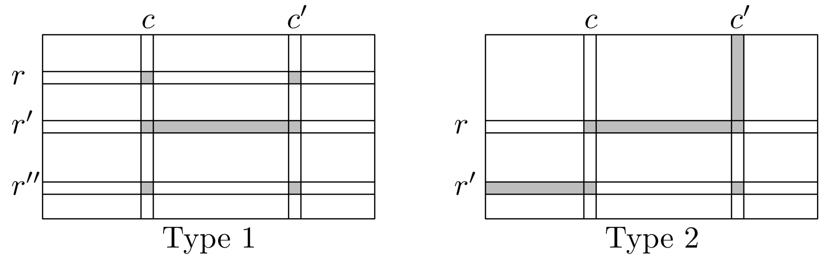

Figure 12: The two types of $\mathcal{Q}$-avoiders considered in Lemma 3.17. The shaded areas are the possible positions of 1-entries.

Proof. Suppose that $P \in \{0,1\}^{k\times\ell}$ is covered by row r and column c. By Lemma 3.8, all the 1-entries in $P[\{r\}\times[c]]$ are row-bounding, and by symmetry, the 1-entries in $P[\{r\}\times[c,\ell]]$ are row-bounding as well. By Lemma 3.11, the 1-entries in $P[[1,r)\times\{c\}]$ and $P[(r,k]\times\{c\}]$ are also row-bounding. □

Lemmas 3.13, 3.14 and 3.15 imply that any pattern that can be covered by two lines is row-bounding. We now proceed with the remaining cases of Proposition 3.6.

Lemma 3.16. *A pattern $P \in \{0,1\}^{k\times\ell}$ that avoids D_2 or $\overline{D}_2$ is row-bounding.*

Proof. Suppose that P avoids $\overline{D}_2$, the other case being symmetric. From Proposition 2.3, we know that P is a decreasing pattern. Every 1-entry of P is row-bounding either by Lemma 3.10 (Type 3), or by Lemma 3.11 (Type 3), and therefore P is row-bounding. □

What follows is the last and the most difficult case of our analysis, which deals with patterns that are not increasing or decreasing and cannot be covered by two lines.

Lemma 3.17. *Let $P \in Av_{\preccurlyeq}(\mathcal{Q})$ be a pattern that contains both D_2 and $\overline{D}_2$, and that cannot be covered by two lines. Then P can be transformed by a rotation or a reflection to a pattern P_0 of one of these two types (see Figure 12).*

Type 1: P_0 has three rows $r < r' < r''$ and two columns $c < c'$ with

$$\mathrm{supp}(P_0) \subseteq \big(\{r'\}\times[c,c']\big) \cup \{(r,c),(r'',c),(r,c'),(r'',c')\}.$$

Type 2: P_0 has two rows $r < r'$ and two columns $c < c'$ with

$$\mathrm{supp}(P_0) \subseteq \big(\{r\}\times[c,c']\big) \cup \big(\{r'\}\times[c]\big) \cup \big([r]\times\{c'\}\big) \cup \{(r',c')\}.$$

Proof. Let $P \in \{0,1\}^{k\times\ell}$ be a pattern satisfying the assumptions of the lemma. Since P cannot be covered by two lines, by Fact 3.5, P contains three 1-entries $e_1 = (r_1,c_1)$, $e_2 = (r_2,c_2)$ and $e_3 = (r_3,c_3)$, with $r_1 < r_2 < r_3$, and such that the columns c_1, c_2, c_3 are all distinct. Since P avoids the patterns from $\mathcal{Q}$, we must have either $c_1 < c_2 < c_3$ or $c_1 > c_2 > c_3$. Without loss of generality, assume $c_1 < c_2 < c_3$.

By Proposition 3.6, P can be covered by three lines. Suppose first that the three lines that cover P are the rows r_1, r_2 and r_3. Suppose moreover, that the three 1-entries e_1, e_2, e_3 were chosen in such a way that c_1 is as large as possible, while c_2 and c_3 are as small as possible, i.e., the choice minimizes the value of $-c_1 + c_2 + c_3$; see Figure 13 (left). In particular, row r_1 of P has no 1-entry in any of the columns $[c_1 + 1, c_2)$, otherwise we could choose a larger value of c_1. Similarly, row r_2 has no 1-entry in columns $[c_1 + 1, c_2)$ and row r_3 has no 1-entry in columns $[c_2 + 1, c_3)$.

Moreover, since P avoids the four patterns from the set $\mathcal{Q}$, row r_1 has no 1-entry in columns $[c_2 + 1, c_3)$ or $(c_3, \ell]$, row r_2 has no 1-entry in columns $[1, c_1)$ or $(c_3, \ell]$, and row r_3 has no 1-entry in columns $[1, c_1)$ or $[c_1 + 1, c_2)$.

Therefore, apart from the three 1-entries e_i, a 1-entry of P can appear in one of the three intervals $\alpha = \{r_1\} \times [1, c_1)$, $\beta = \{r_2\} \times (c_2, c_3]$ and $\gamma = \{r_3\} \times (c_3, \ell]$, or be equal to one of the five entries $a = (r_2, c_1)$, $b = (r_3, c_1)$, $c = (r_1, c_2)$, $d = (r_3, c_2)$ or $e = (r_1, c_3)$; see Figure 13 (left). Note that a and c cannot be simultaneously equal to 1, otherwise they would form a forbidden pattern with e_3, and similarly, if β contains a 1-entry then $d = 0$, if α contains a 1-entry then $b = 0$, and if γ contains a 1-entry then $e = 0$.

Since P contains a copy of $\overline{D}_2$, at least one of b and e must be a 1-entry. Let us go through the cases that may occur.

Case I: $b = 1$. If $b = 1$ then α is empty. We have two subcases:

Ia: β contains a 1-entry. Then $c = 0$ and $d = 0$. If γ is empty, then P is a Type 1 matrix, with $c = c_1$, $c' = c_3$, and $(r, r', r'') = (r_1, r_2, r_3)$. If γ is nonempty, then $e = 0$, and P is a mirror image of a Type 2 matrix, with $(r, r') = (r_2, r_3)$ and $(c, c') = (c_3, c_1)$.

Ib: β is empty. If γ is nonempty, then $e = 0$ and since at most one of a and c is nonempty, rotating P counterclockwise by 90 degrees yields a Type 2 matrix. If γ is empty, then either $a = 0$ and P is the transpose of a Type 1 matrix, or $a = 1$, and therefore $c = 0$, and at least one of d and e is a 0-entry, resulting in a Type 1 matrix or a rotated Type 2 matrix.

Case II: $b = 0$. If $b = 0$, then $e = 1$, otherwise P would avoid $\overline{D}_2$. Consequently, γ is empty. If β were empty as well, then P would be symmetric to a matrix from case Ib by a 180-degree rotation. We may therefore assume that β is nonempty, and hence $d = 0$. At most one of a and c can be a 1-entry, and in either case we get an upside-down copy of a Type 2 matrix.

This completes the analysis of matrices that can be covered by 3 rows. Suppose now that P can be covered by two rows and one column. As each of the three entries e_1, e_2 and e_3 must be covered by a distinct line, there are three possibilities: either P is covered by rows r_1 and r_2 and column c_3; or P is covered by rows r_1 and r_3 and column c_2; or P is covered by rows r_2 and r_3 and

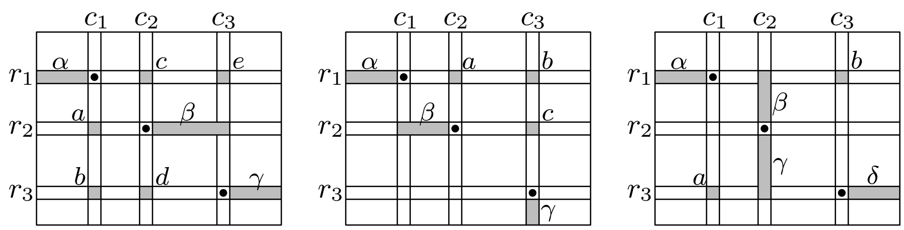

Figure 13: $\mathcal{Q}$-avoiders covered by rows r_1, r_2 and r_3 (left), by rows r_1, r_2 and column c_3 (center), and by rows r_1, r_3 and column c_2 (right). The shaded entries are potential 1-entries, the dots represent the three 1-entries e_1, e_2 and e_3.

column c_1. The last possibility is symmetric to the first one, so we only consider the first two.

Suppose P is covered by rows r_1 and r_2 and column c_3. Choose c_1 and c_2 to be as large as possible, and r_3 to be as small as possible, i.e., the choice maximizes $c_1 + c_2 - r_3$. Together with the absence of patterns from $\mathcal{Q}$, this means that apart from the 1-entries e_1, e_2 and e_3, all the remaining 1-entries must be inside the intervals α, β and γ or at the positions a, b or c depicted in Figure 13 (center). Moreover, if $a = 1$ then β is empty. Therefore, P is an upside-down copy of a matrix of Type 2, with the role of column c played by c_1 if $a = 0$ or by c_2 if $a = 1$.

Let us now suppose that P is covered by rows r_1 and r_3 and column c_2. See Figure 13 (right). Suppose c_1 is largest possible and c_3 smallest possible, i.e., $c_1 - c_3$ is maximized. We make no assumptions about r_2, to keep the configuration symmetric. All the 1-entries are in the intervals α, β, γ and δ or at the positions a and b depicted in the figure. Since P contains $\overline{D}_2$, at least one of a and b is a 1-entry. Suppose without loss of generality that $a = 1$. Then α is empty. If δ is nonempty, then $b = 0$, and P is a Type 2 matrix rotated 90 degrees clockwise. Otherwise δ is empty and P is a rotated Type 1 matrix.

The cases when P can be covered by three columns, or by two columns and a row, are symmetric to the cases handled so far by a 90-degree rotation. □

We now have all the ingredients to complete the proof of our main result.

Theorem 3.18. *Every pattern* $P \in Av_{\preccurlyeq}(\mathcal{Q})$ *is row-bounding.*

Proof. Choose a $P \in Av_{\preccurlyeq}(\mathcal{Q})$. By Proposition 3.6, either P can be covered by three lines, or it avoids D_2, or it avoids $\overline{D}_2$. If P avoids one of the two patterns of size 2, then it is row-bounding by Lemma 3.16. If it can be covered by two lines, it is row-bounding by Lemmas 3.13, 3.14 and 3.15. Finally, if P contains both D_2 and $\overline{D}_2$ and cannot be covered by two lines, Lemma 3.17 shows that, up to symmetry, P corresponds to a matrix of Type 1 or Type 2. We therefore need to argue that the matrices of these two types, as well as their transposes, are row-bounding. See Figure 14.

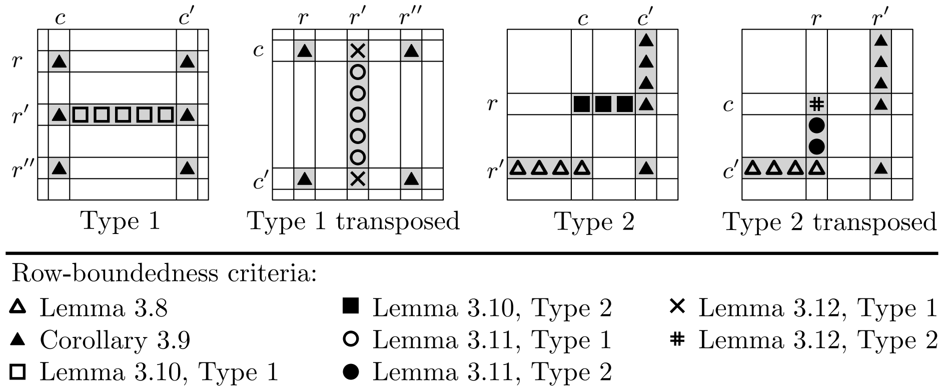

Figure 14: Illustration of the proof of Theorem 3.18. The symbols indicate the criteria used to prove row-boundedness of the 1-entries in the two types of patterns of Lemma 3.17.

If P is of Type 1, its 1-entries in column c or in column c' are row-bounding by Corollary 3.9, and those in row r' are row-bounding by Lemma 3.10, Type 1.

If P is the transpose of a Type 1 matrix, then its 1-entries in columns r and r'' are row-bounding by Corollary 3.9, and those in column r' by Lemmas 3.11 and 3.12.

If P is of Type 2, the 1-entries in row r' and in column c' are row-bounding by Lemma 3.8 and Corollary 3.9, and those in row r are row-bounding by Lemma 3.10.

Finally, if P is the transpose of a Type 2 matrix, the 1-entries in column r' and in row c' are row-bounding by Lemma 3.8 and Corollary 3.9, and the remaining 1-entries are covered by Lemmas 3.11 and 3.12. □

Theorems 3.4 and 3.18 together imply Theorem 3.1.

4. Further directions and open problems

Boundedness of non-principal classes. So far, we have only considered principal classes of matrices, i.e., the classes determined by a single forbidden pattern. It is natural to ask to what extent our results generalize to arbitrary minor-closed classes of matrices, or at least to classes determined by a finite number of forbidden patterns.

All our row-boundedness results for principal classes are based on the study of row-bounding 1-entries in a pattern P. This approach extends straightforwardly to the setting of multiple forbidden patterns. In particular, for a set $\mathcal{F}$ of patterns, a pattern $P \in \mathcal{F}$ and a 1-entry e of P, we say that e is *row-bounding in* $Av_{\preccurlyeq}(\mathcal{F})$ if each row of a matrix $M \in Av_{\preccurlyeq}(\mathcal{F})$ has only a bounded number of 0-runs critical for e with respect to P. Note that if $\mathcal{F}$ is finite, then $Av_{\preccurlyeq}(\mathcal{F})$ is row-bounded if and only if each 1-entry of each pattern $P \in \mathcal{F}$ is row-bounding in $Av_{\preccurlyeq}(\mathcal{F})$.

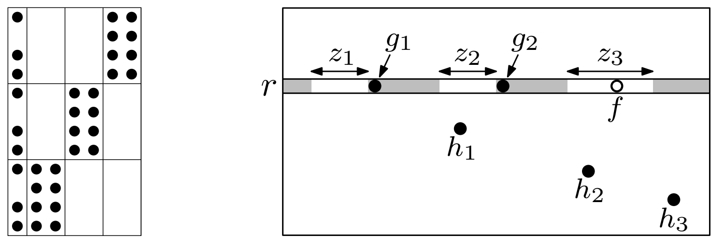

Figure 15: Left: illustration that $Av_{\preccurlyeq}(\mathcal{F})$ has unbounded column-complexity relative to the entry $e = (2,1)$ of P. Right: illustration of the proof that $Av_{\preccurlyeq}(\mathcal{F})$ is row-bounded.

Note also that, by definition, if e is a 1-entry of P that is row-bounding in $Av_{\preccurlyeq}(P)$, then for every set of patterns $\mathcal{F}$ that contains P, the entry e is also row-bounding in $Av_{\preccurlyeq}(\mathcal{F})$, since $Av_{\preccurlyeq}(\mathcal{F})$ is a subclass of $Av_{\preccurlyeq}(P)$. Therefore, all the criteria for row-bounding entries that we derived in Subsection 3.2 are applicable to non-principal classes as well.

We have seen in Corollary 3.2 that a principal class is row-bounded if and only if it is column-bounded. Our next example shows that this property does not generalize to non-principal classes.

Proposition 4.1. *For the set of patterns $\mathcal{F} = \{D_4, P\}$ with*

$$P = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & \bullet & \\ & & \bullet \end{pmatrix} \text{ and } D_4 = \begin{pmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{pmatrix},$$

the class $Av_{\preccurlyeq}(\mathcal{F})$ is row-bounded but not column-bounded.

Proof. To prove that $Av_{\preccurlyeq}(\mathcal{F})$ is not column-bounded, we apply the transpose of the construction of Theorem 3.4, and observe that the constructed matrix avoids D_4 (see Figure 15 (left)).

To prove that the class $Av_{\preccurlyeq}(\mathcal{F})$ is row-bounded, observe first that all the 1-entries in D_4 are row-bounding by Lemma 3.10, the leftmost and the rightmost 1-entry of P are row-bounding by Corollary 3.9, and the 1-entry $(3,2)$ of P is row-bounding by Lemma 3.11. It thus remains to show that the entry $e = (1,2)$ of P is row-bounding in $Av_{\preccurlyeq}(\mathcal{F})$.

We will show that each matrix $M \in Av_{\preccurlyeq}(\mathcal{F})$ has at most two 0-runs critical for e in any given row. Refer to Figure 15 (right). For contradiction, suppose that there are three 0-runs $z_1 < z_2 < z_3$ in a row r of M. Let g_1 be a 1-entry of M that lies in row r between z_1 and z_2, let g_2 be a 1-entry of M in row r between z_2 and z_3, and let f be a 0-entry in z_3 critical for the entry e. Let ϕ be an embedding of P into $M\Delta f$ with $\phi(e) = f$.

Consider the three 1-entries $h_1 = \phi(2,1)$, $h_2 = \phi(3,2)$ and $h_3 = \phi(4,3)$. If h_1 is in a column strictly to the right of g_1, then g_1 forms an image of D_4 with the three h_is, a contradiction. If, on the other hand, h_1 is not to the right of g_1, then h_1 is strictly to the left of g_2, and g_2 forms an image of P with the three h_is (recall that h_3 is to the right of $f = \phi(1,2)$, and therefore also to the right of g_2). This shows that $Av_{\preccurlyeq}(\mathcal{F})$ is row-bounded. □

Recall from Corollary 3.3, that any principal subclass of a bounded principal class is again bounded. The example of Proposition 4.1 shows that this result does not generalize to non-principal classes: indeed, the class $Av_{\preccurlyeq}(D_4)$ is bounded by Theorem 3.1 (or by Corollary 2.4), while its subclass $Av_{\preccurlyeq}(\mathcal{F})$ is not bounded.

On the positive side, it is not hard to show that row-boundedness (and therefore also boundedness) is closed under union and intersection of classes.

Proposition 4.2. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are row-bounded classes of matrices, then the classes $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2$ are row-bounded as well.*

Proof. Let K_i be the row-complexity of the class $\mathcal{C}_i$, for $i \in \{1, 2\}$. Since every matrix that is critical for $\mathcal{C}_1 \cup \mathcal{C}_2$ is also critical for $\mathcal{C}_1$ or for $\mathcal{C}_2$, we observe that $\mathcal{C}_1 \cup \mathcal{C}_2$ has row-complexity at most $\max\{K_1, K_2\}$. In particular, $\mathcal{C}_1 \cup \mathcal{C}_2$ is row-bounded.

Let us argue that $\mathcal{C}_1 \cap \mathcal{C}_2$ is row-bounded as well. We claim that $\mathcal{C}_1 \cap \mathcal{C}_2$ has row-complexity at most $K := K_1 + K_2$. For contradiction, suppose that there is a matrix M critical for $\mathcal{C}_1 \cap \mathcal{C}_2$ with row-complexity at least $K + 1$.

Let r be a row of M with maximum complexity, let $z_1, z_2, \ldots, z_{K+1}$ be a sequence of 0-runs in this row, and let f_i be a 0-entry in z_i. By criticality of M, we know that for each $i \in [K + 1]$, the matrix $M\Delta f_i$ does not belong to $\mathcal{C}_1$ or does not belong to $\mathcal{C}_2$.

In particular, there are either at least $K_1 + 1$ values of i for which $M\Delta f_i$ is not in $\mathcal{C}_1$, or at least $K_2 + 1$ values of i for which $M\Delta f_i$ is not in $\mathcal{C}_2$. Suppose without loss of generality that the former situation occurs. Let M^+ be a critical matrix for the class $\mathcal{C}_1$ that dominates the matrix M. If f_i is a 0-entry of M such $M\Delta f_i$ is not in $\mathcal{C}_1$, then f_i is also a 0-entry of M^+. It follows that M^+ has at least $K_1 + 1$ 0-runs in row r, which is impossible, since K_1 is the row-complexity of $\mathcal{C}_1$. □

In contrast with Proposition 4.2, an intersection of two unbounded classes is not necessarily unbounded, as we will now show. Consider the two patterns $Q_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$ and $Q_2 = \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}$, and recall from Theorem 3.1 that both $Av_{\preccurlyeq}(Q_1)$ and $Av_{\preccurlyeq}(Q_2)$ are unbounded classes.

Proposition 4.3. *The class $Av_{\preccurlyeq}(\{Q_1, Q_2\}) = Av_{\preccurlyeq}(Q_1) \cap Av_{\preccurlyeq}(Q_2)$ is bounded.*

Proof. Let us first show that every 1-entry of the two patterns Q_1 and Q_2 is row-bounding for $\mathcal{C} := Av_{\preccurlyeq}(\{Q_1, Q_2\})$. For a 1-entry that belongs to the first or the last column of either pattern, this follows from Corollary 3.9.

Consider the 1-entry $e = (1, 2)$ of the pattern Q_1. We claim that each row in a matrix $M \in \mathcal{C}$ has at most two 0-runs critical for e. Suppose that a matrix $M \in \mathcal{C}$ has a row r with three 0-runs $z_1 < z_2 < z_3$ critical for e. Let f_i be a 0-entry in z_i critical for e, and let g_i be a 1-entry in row r between z_i and z_{i+1}, for $i \in \{1, 2\}$.

For $i \in \{1, 2, 3\}$, let ϕ_i be an embedding of Q_1 into $M\Delta f_i$ that maps e to f_i. Consider the three 1-entries $h_1 = \phi_1(2, 1)$, $h_2 = \phi_1(3, 3)$, and $h_3 = \phi_3(3, 3)$. Let p_i and q_i be the row and the column containing h_i.

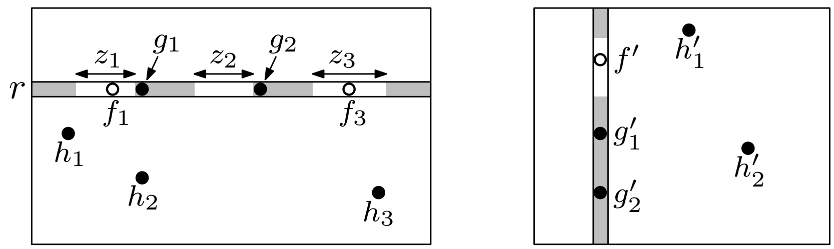

Figure 16: Illustration of the row-boundedness (left) and column-boundedness (right) of $Av_{\preccurlyeq}(\{Q_1, Q_2\})$.

Note that g_1 cannot be to the left of column q_2, since then h_1, g_1 and h_2 would form an image of Q_1. It follows that g_1 is in the column q_2 or to the right of it, and consequently, we have $q_2 < q_3$. Moreover, if $p_3 > p_1$, then h_1, g_2 and h_3 form an image of Q_1, so p_3 is no larger than p_1 and hence $p_3 < p_2$. But then h_2, g_2 and h_3 form an image of Q_2, a contradiction.

By symmetry, the 1-entry $(1, 2)$ of Q_2 is row-bounding as well, and therefore $\mathcal{C}$ is row-bounded.

Let us now argue that $\mathcal{C}$ is column-bounded. It is enough to show that the 1-entry $e' = (2, 1)$ of Q_1 is column-bounding for $\mathcal{C}$, the rest follows from symmetry and from Corollary 3.9. Suppose that a matrix $M \in \mathcal{C}$ has a column c with three 0-runs critical for e'. In particular, column c contains a 0-entry f' critical for e' such that below f', there are at least two 1-entries g'_1 and g'_2 in column c of M. Suppose that g'_1 is above g'_2.

Let ϕ be an embedding of Q_1 into $M \Delta f'$ with $\phi(e') = f'$. Define $h'_1 = \phi(1, 2)$ and $h'_2 = \phi(3, 3)$. Let r' be the row containing h'_2. If g'_1 is above row r', then g'_1, h'_1 and h'_2 form a copy of Q_1, and if g'_1 is not above row r', then g'_2 is below row r' and g'_2, h'_1 and h'_2 form a copy of Q_2, a contradiction. □

Open problems. A natural question arising from our results is to extend the dichotomy of Theorem 3.1 to non-principal classes of matrices.

Problem 4.4. For which sets $\mathcal{F}$ of patterns is the class $Av_{\preccurlyeq}(\mathcal{F})$ row-bounded? Can we characterize such sets $\mathcal{F}$, at least when $\mathcal{F}$ is finite?

The notion of complexity we used in this paper is quite crude, in the sense that it only takes into account single lines of the corresponding matrix. It is reasonable to expect that matrices from a class of unbounded complexity possess nontrivial properties that could be revealed by a more refined approach.

Problem 4.5. Is there a refinement of our complexity notion that would provide nontrivial insight into the structure of critical matrices in unbounded classes?

Throughout the paper, we focused on distinguishing bounded classes from unbounded ones. We made no attempts to obtain tight estimates for the actual value of the complexity of a bounded class. This might be a line of research worth pursuing.

Problem 4.6. What is the highest possible value of $r\left(Av_{\preccurlyeq}\left(P\right)\right)$, over all row-bounding patterns P of a given size $k \times \ell$? For which pattern is this maximum attained?

By Observation 2.5, if P^+ is a pattern obtained by adding an empty row or column to the boundary of a pattern P, then $Av_{\preccurlyeq}\left(P\right)$ has the same complexity as $Av_{\preccurlyeq}\left(P^+\right)$, and the avoiders of P^+ can be easily described in terms of the avoiders of P.

It is, however, more challenging to deal with a pattern P^+ obtained by inserting an empty line into the interior of a pattern P. Theorem 3.1 implies that P is bounding if and only if P^+ is bounding, but we are not aware of any direct proof of this.

Problem 4.7. Let P^+ be a pattern obtained from a pattern P by inserting a new empty row or column to an arbitrary position inside P. Can we bound $r\left(Av_{\preccurlyeq}\left(P^+\right)\right)$ in terms of $r\left(Av_{\preccurlyeq}\left(P\right)\right)$? Can we describe the avoiders of P^+ in terms of the avoiders of P? If $\mathcal{F}$ is a set of patterns and $\mathcal{F}^+$ a set of patterns obtained by inserting empty rows and columns to the patterns in $\mathcal{F}$, is it true that $Av_{\preccurlyeq}\left(\mathcal{F}^+\right)$ is bounded if and only if $Av_{\preccurlyeq}\left(\mathcal{F}\right)$ is?

Acknowledgements

This research was supported by project 16-01602Y of the Czech Science Foundation (GAČR), and by the Neuron Foundation for the Support of Science.

References

[1] Dan Bienstock and Ervin Győri. An extremal problem on sparse 0–1 matrices. *SIAM Journal on Discrete Mathematics*, 4(1):17–27, 1991.

[2] Josef Cibulka. On constants in the Füredi–Hajnal and the Stanley–Wilf conjecture. *Journal of Combinatorial Theory, Series A*, 116(2):290–302, 2009.

[3] P. A. CrowdMath. Bounds on parameters of minimally nonlinear patterns. *Electronic Journal of Combinatorics*, 25(1):#P1.5, 2018.

[4] Robert P. Dilworth. A decomposition theorem for partially ordered sets. *Annals of Mathematics*, 51(1):161–166, 1950.

[5] Jacob Fox. Stanley-Wilf limits are typically exponential. *arXiv:1310.8378 (to appear in Advances in Mathematics)*, 2013.

[6] Radoslav Fulek. Linear bound on extremal functions of some forbidden patterns in 0–1 matrices. *Discrete Mathematics*, 309(6):1736–1739, 2009.

[7] Zoltán Füredi. The maximum number of unit distances in a convex n-gon. *Journal of Combinatorial Theory, Series A*, 55(2):316–320, 1990.

[8] Zoltán Füredi and Péter Hajnal. Davenport–Schinzel theory of matrices. *Discrete Mathematics*, 103(3):233–251, 1992.

[9] Jesse T. Geneson. Extremal functions of forbidden double permutation matrices. *Journal of Combinatorial Theory, Series A*, 116(7):1235–1244, 2009.

[10] Sylvain Guillemot and Dániel Marx. Finding small patterns in permutations in linear time. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 82–101. ACM, New York, 2014.

[11] Balázs Keszegh. On linear forbidden submatrices. *Journal of Combinatorial Theory, Series A*, 116(1):232–241, 2009.

[12] Martin Klazar. The Füredi–Hajnal conjecture implies the Stanley–Wilf conjecture. In Daniel Krob, Alexander A. Mikhalev, and Alexander V. Mikhalev, editors, *Formal Power Series and Algebraic Combinatorics*, pages 250–255. Springer Berlin Heidelberg, 2000.

[13] Joseph P. S. Kung. Jacobi's identity and the König–Egerváry theorem. *Discrete Mathematics*, 49(1):75–77, 1984.

[14] Yaping Mao, Hongjian Lai, Zhao Wang, and Zhiwei Guo. Interval minors of complete multipartite graphs. *arXiv:1508.01263*, 2015.

[15] Adam Marcus and Gábor Tardos. Excluded permutation matrices and the Stanley–Wilf conjecture. *Journal of Combinatorial Theory, Series A*, 107(1):153–160, 2004.

[16] Bojan Mohar, Arash Rafiey, Behruz Tayfeh-Rezaie, and Hehui Wu. Interval minors of complete bipartite graphs. *Journal of Graph Theory*, 2015.

[17] Seth Pettie. Degrees of nonlinearity in forbidden 0-1 matrix problems. *Discrete Mathematics*, 311(21):2396–2410, 2011.

[18] Neil Robertson and P. D. Seymour. Graph minors. V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(1):92–114, 1986.

[19] Gábor Tardos. On 0–1 matrices and small excluded submatrices. *Journal of Combinatorial Theory, Series A*, 111(2):266–288, 2005.