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Article (Accepted version)
(Refereed)

Original citation:
DOI: 10.1016/j.aam.2018.07.005
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Available in LSE Research Online: August 2018

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On the structure of matrices avoiding interval-minor patterns

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Abstract

We study the structure of 01-matrices avoiding a pattern $P$ as an interval minor. We focus on critical $P$-avoiders, i.e., on the $P$-avoiding matrices in which changing a 0-entry to a 1-entry always creates a copy of $P$ as an interval minor.

Let $Q$ be the $3 \times 3$ permutation matrix corresponding to the permutation $231$.

As our main result, we show that for every pattern $P$ that has no rotated copy of $Q$ as interval minor, there is a constant $c_P$ such that any row and any column in any critical $P$-avoiding matrix can be partitioned into at most $c_P$ intervals, each consisting entirely of 0-entries or entirely of 1-entries. In contrast, for any pattern $P$ that contains a rotated copy of $Q$, we construct critical $P$-avoiding matrices of arbitrary size $n \times n$ having a row with $\Omega(n)$ alternating intervals of 0-entries and 1-entries.

Keywords: interval minor, 01–matrix, pattern avoidance

1. Introduction

A binary matrix is a matrix with entries equal to 0 or 1. All matrices considered in this paper are binary. The study of extremal problems of binary matrices has been initiated by the papers of Bienstock and Győri [1] and of Füredi [7]. Since these early works, most of the research in this area has focused on the concept of forbidden submatrices: a matrix $M$ is said to contain a pattern $P$ as a submatrix if we can transform $M$ into $P$ by deleting some rows and columns, and by changing 1-entries into 0-entries. This notion of submatrix is a matrix analogue of the notion of subgraph in graph theory.

The main problem in the study of pattern-avoiding matrices is to determine the extremal function $\text{ex}(n; P)$, defined as the largest number of 1-entries in an $n \times n$ binary matrix avoiding the pattern $P$ as submatrix. This is an analogue

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Preprint submitted to Elsevier
of the classical Turán-type problem of finding a largest number of edges in an \(n\)-vertex graph avoiding a given subgraph. Despite the analogy, the function \(\text{ex}(n; P)\) may exhibit an asymptotic behaviour not encountered in Turán theory. For instance, for the pattern\(^1\) \(P = (\bullet\bullet\bullet\bullet)\) Füredi and Hajnal [8] proved that \(\text{ex}(n; P) = \Theta(n\alpha(n))\), where \(\alpha(n)\) is the inverse of the Ackermann function.

The asymptotic behaviour of \(\text{ex}(n; P)\) for general \(P\) is still not well understood. Füredi and Hajnal [8] posed the problem of characterising the linear patterns, i.e., the patterns \(P\) satisfying \(\text{ex}(n; P) = O(n)\). Marcus and Tardos [15] proved that \(\text{ex}(n; P) = O(n)\) whenever \(P\) is a permutation matrix, i.e., \(P\) has exactly one 1-entry in each row and each column. This result, combined with previous work of Klazar [12], has confirmed the long-standing Stanley–Wilf conjecture. However, the problem of characterising linear patterns is still open despite a number of further partial results [3, 6, 9, 11, 17, 19].

Fox [5] has introduced a different notion of containment among binary matrices, based on the concept of interval minors. Informally, a matrix \(M\) contains a pattern \(P\) as an interval minor if we can transform \(M\) into \(P\) by contracting adjacent rows or columns and changing 1-entries into 0-entries; see Section 2 for the precise definition. In this paper, we mostly deal with containment and avoidance of interval minors rather than submatrices. Therefore, the phrases \(M\) avoids \(P\) or \(M\) contains \(P\) always refer to avoidance or containment of interval minors, and the term \(P\)-avoider always refers to a matrix that avoids \(P\) as interval minor.

In analogy with \(\text{ex}(n; P)\), it is natural to consider the corresponding extremal function \(\text{ex}_<(n; P)\) as the largest number of 1-entries in an \(n \times n\) matrix that avoids \(P\) as an interval minor. If \(M\) contains \(P\) as a submatrix, it also contains it as an interval minor, and therefore \(\text{ex}_<(n; P) \leq \text{ex}(n; P)\). Moreover, it can be easily seen that for a permutation matrix \(P\) the two notions of containment are equivalent, and hence \(\text{ex}_<(n; P) = \text{ex}(n; P)\).

Fox [5] used interval minors as a key tool in his construction of permutation patterns with exponential Stanley–Wilf limits. In view of the results of Cibulka [2], this is equivalent to constructing a permutation matrix \(P\) for which the limit of the ratio \(\text{ex}(n; P)/n\) (which is equal to \(\text{ex}_<(n; P)/n\)) is exponential in the size of \(P\).

Even before the work of Fox, interval minors have been implicitly used by Guillemot and Marx [10], who proved that a permutation matrix \(M\) which avoids as interval minor a fixed complete square pattern (i.e., a square pattern with all entries equal to 1) admits a type of recursive decomposition of bounded complexity. This result can be viewed as an analogue of the grid theorem from graph theory [18], which states that graphs avoiding a large square grid as a minor have bounded tree-width. Guillemot and Marx used their result on forbidden interval minors to design a linear-time algorithm for testing the containment of a fixed pattern in a permutation.

\(^1\)We use the convention of representing 1-entries in binary matrices by dots and 0-entries by blanks.
Subsequent research into interval-minor avoidance has focused on avoiders of a complete matrix. In particular, Mohar et al. [16] obtained exact values for the extremal function for matrices simultaneously avoiding a complete pattern of size $2 \times \ell$ and its transpose, and they obtained bounds for patterns of size $3 \times \ell$. Their results were further generalised by Mao et al. [14] to a multidimensional setting.

While the functions $\text{ex}(n; P)$ exhibit diverse forms of asymptotic behaviour, the function $\text{ex}_\preceq(n; P)$ is linear for every nontrivial pattern $P$. This is a consequence of the Marcus–Tardos theorem and the fact that any binary matrix is an interval minor of a permutation matrix; see Fox [5]. Therefore, in the interval-minor avoidance setting, it is not as natural to classify patterns by the growth of $\text{ex}_\preceq(n; P)$ alone as in the submatrix avoidance setting.

In our paper, we instead classify the patterns $P$ based on the structure of the $P$-avoiders. We introduce the notion of line complexity of a binary matrix $M$, as the largest number of maximal runs of consecutive 0-entries in a single row or a single column of $M$. We focus on the critical $P$-avoiders, which are the matrices that avoid $P$ as interval minor, but lose this property when any 0-entry is changed into a 1-entry.

Our main result is a sharp dichotomy for line complexity of critical $P$-avoiders. Let $Q_1, \ldots, Q_4$ be defined as follows:

\begin{align*}
Q_1 &= \begin{pmatrix} \bullet & \bullet \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \bullet & \bullet & \bullet \end{pmatrix}, \quad Q_3 = \begin{pmatrix} \bullet & \bullet \bullet \end{pmatrix}, \quad \text{and} \quad Q_4 = \begin{pmatrix} \bullet & \bullet & \bullet \end{pmatrix}.
\end{align*}

We show that if a pattern $P$ avoids the four patterns $Q_i$ as interval minors (or equivalently, as submatrices), then the line-complexity of every critical $P$-avoider is bounded by a constant $c_P$ depending only on $P$. On the other hand, if $P$ contains at least one of the $Q_i$, then there are critical $P$-avoiders of size $n \times n$ with line complexity $\Omega(n)$, for any $n$.

After properly introducing our terminology and proving several simple basic facts in Section 2, we devote Section 3 to the statement and proof of our main result. In Section 4, we discuss the possibility of extending our approach to general minor-closed matrix classes, and present several open problems.

2. Preliminaries

Basic notation. For integers $m$ and $n$, we let $[m, n]$ denote the set \{\(m, m + 1, \ldots, n\}\}. We will also use the notation $[m, n]$ for the set $[m, n - 1]$, $(m, n)$ for the set $[m + 1, n]$, and $[n]$ for $[1, n]$. We will avoid using $(m, n)$ for $[m + 1, n - 1]$, however; instead, we will use the notation $(m, n)$ to denote ordered pairs of integers.

We write $\{0, 1\}^{m \times n}$ for the set of binary matrices with $m$ rows and $n$ columns. We will always assume that rows of matrices are numbered top-to-bottom starting with 1, that is, the first row is the topmost.

For a matrix $M \in \{0, 1\}^{m \times n}$, we let $M(i, j)$ denote the value of the entry in row $i$ and column $j$ of $M$. We say that the pair $(i, j)$ is a 1-entry of $M$ if $M(i, j) = 1$, otherwise it is a 0-entry. The set of 1-entries of a matrix $M \in \{0, 1\}^{m \times n}$
\( \{0,1\}^{m \times n} \) is called the \textit{support} of \( M \), denoted by \( \text{supp}(M) \); formally, \( \text{supp}(M) = \{(i,j) \in [m] \times [n]; \ M(i,j) = 1\} \).

We say that a binary matrix \( M' \) \textit{dominates} a binary matrix \( M \), if the two matrices have the same number of rows and the same number of columns, and moreover, \( \text{supp}(M) \subseteq \text{supp}(M') \). In other words, \( M \) can be obtained from \( M' \) by changing some 1-entries into 0-entries.

For a matrix \( M \in \{0,1\}^{m \times n} \) and for a set of row-indices \( R \subseteq [m] \) and column-indices \( C \subseteq [n] \), we let \( M[R \times C] \) denote the submatrix of \( M \) induced by the rows in \( R \) and columns in \( C \). More formally, if \( R = \{r_1 < r_2 < \cdots < r_k\} \) and \( C = \{c_1 < c_2 < \cdots < c_\ell\} \), then \( M[R \times C] \) is a matrix \( P \in \{0,1\}^{k \times \ell} \) such that \( P(i,j) = M(r_i,c_j) \) for every \( (i,j) \in [k] \times [\ell] \).

A \textit{line} in a matrix \( M \) is either a row or a column of \( M \). We view a line as a special case of a submatrix. For instance, the \( i \)-th row of a matrix \( M \in \{0,1\}^{m \times n} \) is the submatrix \( M[[i] \times [n]] \). A \textit{horizontal interval} is a submatrix formed by consecutive entries belonging to a single row, i.e., a submatrix of the form \( M[[i] \times [j_1,j_2]] \) where \( i \) is a row index and \( j_1,j_2 \) are column indices. Vertical intervals are defined analogously.

We say that a submatrix of \( M \) is \textit{empty} if it does not contain any 1-entries.

For a matrix \( M \in \{0,1\}^{m \times n} \) and an entry \( e \in [m] \times [n] \), we let \( M \Delta e \) denote the matrix obtained from \( M \) by changing the value of the entry \( e \) from 0 to 1 or from 1 to 0.

\textit{Interval minors.} A \textit{row contraction} in a matrix \( M \in \{0,1\}^{m \times n} \) is an operation that replaces a pair of adjacent rows \( r \) and \( r+1 \) by a single row, so that the new row contains a 1-entry in a column \( j \) if and only if at least one of the two original rows contained a 1-entry in column \( j \). Formally, the row contraction transforms \( M \) into a matrix \( M' \in \{0,1\}^{(m-1) \times n} \) whose entries are defined by

\[
M'(i,j) = \begin{cases} 
M(i,j) & \text{if } i < r, \\
\max\{M(r,j), M(r+1,j)\} & \text{if } i = r, \\
M(i+1,j) & \text{if } i > r.
\end{cases}
\]

A column contraction is defined analogously.

We say that a matrix \( P \in \{0,1\}^{k \times \ell} \) is an \textit{interval minor} of a matrix \( M \in \{0,1\}^{m \times n} \), denoted \( P \preceq M \), if we can transform \( M \) by a sequence of row contractions and column contractions to a matrix \( P' \in \{0,1\}^{k \times \ell} \) that dominates \( P \). When \( P \) is an interval minor of \( M \), we also say that \( M \) \textit{contains} \( P \), otherwise we say that \( M \) \textit{avoids} \( P \), or \( M \) is \( P \)-avoiding.

There are several alternative ways to define interval minors. One possible approach uses the concept of matrix partition. For \( P \in \{0,1\}^{k \times \ell} \) and \( M \in \{0,1\}^{m \times n} \), a \textit{partition of \( M \) containing \( P \)} is the sequence of row indices \( r_0, r_1, \ldots, r_k \) and column indices \( c_0, c_1, \ldots, c_\ell \) with \( 0 \leq r_0 < r_1 < \cdots < r_k \leq m \) and \( 0 \leq c_0 < c_1 < \cdots < c_\ell \leq n \), such that for every 1-entry \( (i,j) \) of \( P \), the submatrix \( M[[r_{i-1},r_i] \times [c_{j-1},c_j]] \) has at least one 1-entry. See Figure 1.

An \textit{embedding} of a matrix \( P \in \{0,1\}^{k \times \ell} \) into a matrix \( M \in \{0,1\}^{m \times n} \) is a function \( \phi: [k] \times [\ell] \to [m] \times [n] \) with the following properties:
Figure 1: A pattern $P$ and a matrix $M$ that contains $P$. The thick lines indicate a partition of $M$ containing $P$, and the shaded 1-entries form an image of $P$.

- If $e = (i, j)$ is a 1-entry of $P$, then $\phi(e)$ is a 1-entry of $M$.
- Let $e_1 = (i_1, j_1)$ and $e_2 = (i_2, j_2)$ be two entries of $P$, and suppose that $\phi(e_1) = (i_1^*, j_1^*)$ and $\phi(e_2) = (i_2^*, j_2^*)$. If $i_1 < i_2$ then $i_1^* < i_2^*$, and if $j_1 < j_2$ then $j_1^* < j_2^*$.

Notice that in an embedding $\phi$ of $P$ into $M$, two entries of $P$ belonging to the same row may be mapped to different rows of $M$, and similarly for columns.

In practice, it is often inconvenient and unnecessary to specify completely an embedding of $P$ into $M$. In particular, it is usually unnecessary to specify the image of all the 0-entries in $P$. This motivates the notion of partial embedding, which we now formalise. Consider again binary matrices $P \in \{0, 1\}^{k \times \ell}$ and $M \in \{0, 1\}^{m \times n}$. Let $S$ be a nonempty subset of $[k] \times [\ell]$. We say that a function $\psi : S \to [m] \times [n]$ is a partial embedding of $P$ into $M$ if the following holds:

- If $e = (i, j)$ is a 1-entry of $P$, then $e$ is in $S$ and $\psi(e)$ is a 1-entry of $M$.
- An entry $e = (i, j) \in S$ is mapped by $\psi$ to an entry $\psi(e) = (i^*, j^*)$ of $M$ satisfying the following inequalities: $i \leq i^*$, $j \leq j^*$, $k - i \leq m - i^*$ and $\ell - j \leq n - j^*$. Informally, the entry $\psi(e)$ is at least as far from the top, left, bottom and right edge of the corresponding matrix as the entry $e$.
- Let $e_1 = (i_1, j_1)$ and $e_2 = (i_2, j_2)$ be two entries in $S$, with $\psi(e_1) = (i_1^*, j_1^*)$ and $\psi(e_2) = (i_2^*, j_2^*)$. If $i_1 < i_2$ then $i_2 - i_1 \leq i_2^* - i_1^*$, and if $j_1 < j_2$ then $j_2 - j_1 \leq j_2^* - j_1^*$.

For a partial embedding $\psi$ of a pattern $P$ into a matrix $M$, the image of $P$ (with respect to $\psi$) is the set of entries \{\psi(e) : e \in \text{supp}(P)\} in the matrix $M$. Note that all the entries in the image of $P$ are 1-entries.

**Lemma 2.1.** For matrices $P \in \{0, 1\}^{k \times \ell}$ and $M \in \{0, 1\}^{m \times n}$ the following properties are equivalent.

1. $P$ is an interval minor of $M$.
2. $M$ has a partition containing $P$. 

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3. $P$ has an embedding into $M$.

4. $P$ has a partial embedding into $M$.

Proof. We will prove the implications $2 \implies 1 \implies 3 \implies 4 \implies 2$.

To see that 2 implies 1, suppose $M$ has a partition containing $P$, determined by row indices $r_0, r_1, \ldots, r_k$ and column indices $c_0, c_1, \ldots, c_t$, where we may assume that $r_0 = c_0 = 0$, $r_k = m$ and $c_t = n$. We may then contract the rows from each interval of the form $(r_{i-1}, r_i]$ into a single row, and contract the columns from each interval $(c_{i-1}, c_i]$ to a single column, to obtain a matrix $P' \in \{0,1\}^{k \times t}$ that dominates $P$.

To see that 1 implies 3, suppose that $P$ is an interval minor of $M$. This means that there is a sequence of matrices $M_0, M_1, M_2, \ldots, M_s$ with $s = (m-k)+(n-t)$, where $M_0 \in \{0,1\}^{k \times t}$ is a matrix that dominates $P$, and for each $i \in [s]$, the matrix $M_{i-1}$ can be obtained from $M_i$ by contracting a pair of adjacent rows or columns. We can then easily observe that for every $i = 0, 1, \ldots, s$ there is an embedding $\psi_i$ of $P$ into $M_i$. Indeed, reasoning by induction, the embedding $\phi_0$ is the identity map, and for a given $i \in [s]$, if there is an embedding $\phi_{i-1}$ of $P$ into $M_{i-1}$, then an embedding $\phi_i$ can be obtained by an obvious modification of $\phi_{i-1}$.

Clearly, 3 implies 4, since every embedding is also a partial embedding.

To show that 4 implies 2, assume that $\psi : S \to [m] \times [n]$ is a partial embedding of $P$ into $M$. We will define a sequence of row indices $0 \leq r_0 < r_1 < \cdots < r_k \leq m$ with these two properties:

- For each entry $e \in S$ that belongs to row $i$ of $P$, the entry $\psi(e)$ belongs to a row $i^*$ of $M$ for some $i^* \in (r_{i-1}, r_i]$.

- If $S$ contains at least one entry from row $i$ in $P$, then $S$ contains an entry $e$ in row $i$ such that $\psi(e)$ is in row $r_i$ of $M$.

We define the numbers $r_i$ inductively, starting with $r_0 = 0$. Suppose that $r_0, \ldots, r_{i-1}$ have been defined, for some $i \geq 1$. If $S$ contains no entry from row $i$ of $P$, define $r_i = r_{i-1} + 1$. On the other hand, if $S$ contains an entry from row $i$, we let $r_i$ be the largest row index of $M$ such that $\psi$ maps an entry from row $i$ of $P$ to an entry in row $r_i$ of $M$. Notice that any entry $e \in S$ that does not belong to the first $i$ rows of $P$ must be mapped by $\psi$ to an entry strictly below row $r_i$ of $M$, otherwise $\psi$ would not satisfy the properties of a partial embedding.

In an analogous way, we also define a sequence of column indices $0 \leq c_0 < c_1 < \cdots < c_t \leq n$. These sequences will satisfy that for every $e = (i, j) \in S$ we have $\psi(e) \in (r_{i-1}, r_i) \times (c_{j-1}, c_j]$. Since $\psi$ is a partial embedding, $S$ contains all the 1-entries of $P$, and $\psi$ maps these 1-entries to 1-entries of $M$. In particular, the sequences $(r_i)_{i=0}^k$ and $(c_j)_{j=0}^t$ form a partition of $M$ containing $P$. 

Minor-closed classes. For a matrix $P$, we let $Av_\prec(P)$ denote the set of all binary matrices that do not contain $P$ as an interval minor. We call the matrices in $Av_\prec(P)$ the avoiders of $P$, or $P$-avoiders.
More generally, if $F$ is a set of matrices, we let $Av_\prec(F)$ denote the set of binary matrices that avoid all elements of $F$ as interval minors.

We call a set $C$ of binary matrices a \textit{minor-closed class} (or just \textit{class}, for short) if for every matrix $M \in C$, all the interval minors of $M$ are in $C$ as well. Clearly, $Av_\prec(F)$ is a class, and for every class $C$ there is a (possibly infinite) set $F$ such that $C = Av_\prec(F)$. A \textit{principal class} is a class of matrices determined by a single forbidden pattern, i.e., a class of the form $Av_\prec(P)$ for a matrix $P$.

For a class $C$ of matrices, we say that a matrix $M \in C$ is \textit{critical} for $C$ if the change of any 0-entry of $M$ to a 1-entry creates a matrix that does not belong to $C$. In other words, $M \in C$ is critical for $C$ if it is not dominated by any other matrix in $C$. For a pattern $P$, we let $Av_{\prec \text{crit}}(P)$ be the set of critical matrices for $Av_\prec(P)$, and similarly for a set of patterns $F$, $Av_{\prec \text{crit}}(F)$ is the set of all critical matrices for $Av_\prec(F)$.

2.1. Simple examples of $P$-avoiders

We conclude this section by presenting several examples of avoiders of certain simple patterns. These examples will play a role in Section 3, in the proof of our main result. We begin with a very simple example, which we present without proof.

\textbf{Observation 2.2.} Let $R_k$ be the matrix with 1 row and $k$ columns, whose every entry is a 1-entry (see Figure 2). A matrix $M \in \{0, 1\}^{m \times n}$ avoids $R_k$ if and only if $M$ has at most $k - 1$ nonempty columns. Consequently, $M$ is a critical $R_k$-avoider if and only if $\text{supp}(M)$ is a union of $\min\{k - 1, n\}$ columns.

Next, we will consider the diagonal patterns $D_k \in \{0, 1\}^{k \times k}$, defined by $\text{supp}(D_k) = \{(i, i): i \in [k]\}$, and their mirror image $\overline{D}_k \in \{0, 1\}^{k \times k}$, defined by $\text{supp}(\overline{D}_k) = \{(i, k - i + 1): i \in [k]\}$ (see again Figure 2). To describe the avoiders of these patterns, we first introduce some terminology.

Let $e$ and $e'$ be two entries of a matrix $M$. An \textit{increasing walk} from $e$ to $e'$ in $M$ is a set of entries $W = \{e_i = (r_i, c_i): i = 0, \ldots, t\}$ such that $e_0 = e$, $e_t = e'$, and for every $i \in [t]$ we have either $r_i = r_{i-1}$ and $c_i = c_{i-1} + 1$ (that is, $e_i$ is to the right of $e_{i-1}$), or $r_i = r_{i-1} - 1$ and $c_i = c_{i-1}$ (that is, $e_i$ is above $e_{i-1}$). A \textit{decreasing walk} is defined analogously, except now $e_i$ is either to the right or below $e_{i-1}$.
Figure 3: An increasing matrix (left) and a decreasing matrix (right). The shaded entries form an increasing and a decreasing walk in the respective matrices.

We say a matrix $M$ is an increasing matrix if $\text{supp}(M)$ is a subset of an increasing walk. A decreasing matrix is defined analogously. See Figure 3.

**Proposition 2.3.** A matrix $M \in \{0, 1\}^{m \times n}$ avoids the pattern $D_k$ if and only if $M$ contains $k - 1$ increasing walks $W_1, \ldots, W_{k-1}$ from $(m, 1)$ to $(1, n)$ such that $\text{supp}(M) \subseteq W_1 \cup W_2 \cup \cdots \cup W_{k-1}$.

**Proof.** Clearly, if $M$ contains $D_k$, then $M$ has $k$ 1-entries no two of which can belong to a single increasing walk, and therefore $\text{supp}(M)$ cannot be covered by $k - 1$ increasing walks.

Suppose now that $M$ avoids $D_k$. Consider a partial order $\triangleleft$ on the set $\text{supp}(M)$, defined as \((i, j) \triangleleft (i', j') \iff i < i' \text{ and } j < j'\). Since $M$ avoids $D_k$, this order has no chain of length $k$. By the classical Dilworth theorem [4], $\text{supp}(M)$ is a union of $k - 1$ antichains of $\triangleleft$. We may easily observe that each antichain of $\triangleleft$ is contained in an increasing walk. \qed

Proposition 2.3 shows, in particular, that a matrix $M$ avoids the pattern $D_2 = (\bullet \bullet)$ if and only if $M$ is an increasing matrix. By symmetry, $M$ avoids $D_2$ if and only if it is a decreasing matrix.

Another direct consequence of the proposition is the following corollary, describing the structure of critical $D_k$-avoiders.

**Corollary 2.4.** A critical $D_k$-avoiding matrix $M$ contains $k - 1$ increasing walks $W_1, \ldots, W_{k-1}$ from $(m, 1)$ to $(1, n)$ such that $\text{supp}(M) = W_1 \cup W_2 \cup \cdots \cup W_{k-1}$.

Note that Corollary 2.4 only gives a necessary condition for a matrix to be a critical $D_k$-avoider, therefore it is not a characterisation of critical $D_k$-avoiders. With only a little bit of extra effort, we could state and prove such a characterisation, but we omit doing so, as we do not need it for our purposes.

A simple but useful observation is that adding an empty row or column to the boundary of a pattern affects the $P$-avoiders in a predictable way. We state it here without proof.

**Observation 2.5.** Let $P \in \{0, 1\}^{k \times \ell}$ be a pattern, and let $P' \in \{0, 1\}^{k \times (\ell+1)}$ be the pattern obtained by appending an empty column to $P$; in other words, we have $P'[k] \times [\ell] = P$, and the last column of $P'$ is empty. Then a matrix $M \in \{0, 1\}^{m \times n}$ avoids $P'$ if and only if the matrix obtained by removing the last column from $M$ avoids $P$. Consequently, $M$ is in $\text{Av}_{\text{crit}}(P')$ if and only if all the
entries in the last column of $M$ are 1-entries, and the preceding columns form a matrix from $\text{Av}_{\text{crit}}(P)$. Analogous properties hold for a pattern $P''$ obtained by prepending an empty column in front of all the columns of $P$, and also for rows instead of columns.

3. Line complexity

In the previous section, we have seen several examples of matrices avoiding a fixed pattern as an interval minor. At a glance, it is clear that these matrices are highly structured. We would now like to make the notion of ‘highly structured matrices’ rigorous, and generalize it to other forbidden patterns.

We will focus on the local structure of matrices, i.e., the structure observed by looking at a single row or column. For a forbidden pattern $P$ with at least two rows and two columns, any binary vector can appear as a row or column of a $P$-avoiding matrix; indeed, if $P$ has at least two rows, then any matrix with a single row is $P$-avoiding.

However, the situation changes when we restrict our attention to critical $P$-avoiders. In the examples of critical $P$-avoiders we saw in Subsection 2.1, the 1-entries in each row or column were clustered into a bounded number of intervals. In particular, for these patterns $P$, the number of distinct vectors of a given length $n$ that may appear as rows or columns of a critical $P$-avoider is at most polynomial in $n$.

In this section, we study this phenomenon in detail. We show that it generalizes to many other forbidden patterns $P$, but not all of them. As our main result, we will present a complete characterisation of the patterns $P$ exhibiting this phenomenon.

Let us begin by formalising our main concepts.

A horizontal 0-run in a matrix $M$ is a maximal sequence of consecutive 0-entries in a single row. More formally, a horizontal interval $M[\{r\} \times [c_1, c_2]]$ is a horizontal 0-run if all its entries are 0-entries, $c_1 = 1$ or $M(r, c_1 - 1) = 1$, and $c_2 = n$ or $M(r, c_2 + 1) = 1$. Symmetrically, a vertical interval is a vertical 0-run if it is a maximal vertical interval that only contains 0-entries. In the same manner, we define a (horizontal or vertical) 1-run to be a maximal interval of consecutive 1-entries in a single line of $M$.

Note that each line in a matrix $M$ can be uniquely decomposed into an alternating sequence of 0-runs and 1-runs.

Let $M$ be a binary matrix. The complexity of a line of $M$ is the number of 0-runs contained in this line. The row-complexity of $M$ is the maximum complexity of a row of $M$, i.e., the least number $k$ such that each row has complexity at most $k$. Similarly, the column-complexity of $M$ is the maximum complexity of a column of $M$.

For a class of matrices $\mathcal{C}$, we define its row-complexity, denoted $r(\mathcal{C})$, as the supremum of the row-complexities of the critical matrices in $\mathcal{C}$. We say that $\mathcal{C}$ is row-bounded if $r(\mathcal{C})$ is finite, and row-unbounded otherwise. Symmetrically,
we define the column-complexity $c(C)$ of $C$ and the property of being column-bounded and column-unbounded. We say that a class $C$ is bounded if it is both row-bounded and column-bounded; otherwise, it is unbounded.

We stress that when defining the row-complexity and column-complexity of a class of matrices, we only take into account the matrices that are critical for the class.

We are now ready to state our main result.

**Theorem 3.1.** Let $P$ be a pattern. The class $Av_\ll (P)$ is row-bounded if and only if $P$ does not contain any of $Q_1, Q_2, Q_3, Q_4$ as an interval minor, where

$$Q_1 = \left( \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \right), \quad Q_2 = \left( \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \right), \quad Q_3 = \left( \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \right) \text{ and } Q_4 = \left( \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \right).$$

Before we prove Theorem 3.1, we point out two of its direct consequences.

**Corollary 3.2.** For a pattern $P$, these statements are equivalent:

- $Av_\ll (P)$ is row-bounded.
- $Av_\ll (P)$ is column-bounded.
- $Av_\ll (P)$ is bounded.

**Corollary 3.3.** Let $C = Av_\ll (P)$ and $C' = Av_\ll (P')$ be principal classes, and suppose that $C \subseteq C'$ (or equivalently, $P \ll P'$). If $C'$ is bounded, then $C$ is bounded as well.

Although each of these two corollaries is stating a seemingly basic property of the boundedness notion, we are not able to prove either of them without first proving Theorem 3.1. We also remark that neither of the two corollaries can be generalized to non-principal classes of matrices, as we will see in Section 4.

Let us say that a pattern $P$ is row-bounding if $Av_\ll (P)$ is row-bounded, otherwise $P$ is non-row-bounding. Similarly, $P$ is bounding if $Av_\ll (P)$ is bounded and non-bounding otherwise.

Let $Q$ be the set of patterns $\{Q_1, Q_2, Q_3, Q_4\}$. Theorem 3.1 states that a pattern $P$ is row-bounding if and only if $P$ is in $Av_\ll (Q)$. To prove this, we will proceed in several steps. We first show, in Subsection 3.1, that if $P$ contains a pattern from $Q$, then $P$ is not row-bounding. This is the easier part of the proof, though by no means trivial. Next, in Subsection 3.2, we show that every pattern in $Av_\ll (Q)$ is row-bounding. This part is more technical, and requires a characterisation the structure of the patterns in $Av_\ll (Q)$.

### 3.1. Non-row-bounding patterns

Our goal in this subsection is to show that any pattern $P$ that contains one of the matrices from $Q$ is not row-bounding. Let us therefore fix such a pattern $P$. Without loss of generality, we may assume that $Q_1 \ll P$.

**Theorem 3.4.** For every matrix $P$ such that $Q_1 \ll P$, the class $Av_\ll (P)$ is row-unbounded.
Proof. Refer to Figure 4. Let \( P \in \{0, 1\}^{k \times \ell} \) be a pattern containing \( Q_1 \) as an interval minor. In particular, there are row indices \( 1 \leq r_1 < r_2 < r_3 \leq k \) and column indices \( 1 \leq c_1 < c_2 < c_3 \leq \ell \) such that \( P(r_1, c_2) = P(r_2, c_1) = P(r_3, c_3) = 1 \).

For an arbitrary integer \( p \), we will show how to construct a matrix in \( \text{Av}_{\text{crit}}(P) \) of row-complexity at least \( p \). We first describe a matrix \( M \in \{0, 1\}^{m \times n} \) with \( m = r_1 + p(r_3 - r_1) + (k - r_3) \) and \( n = (c_1 - 1) + p(c_3 - c_1 + 1) + (\ell - c_3) \).

In the matrix \( M \), the leftmost \( c_1 - 1 \) columns, the rightmost \( \ell - c_3 \) columns, the topmost \( r_1 - 1 \) rows and the bottommost \( k - r_3 \) rows have all entries equal to 1. We call these entries the frame of \( M \).

In the \( r_1 \)-th row of \( M \), there are \( p \) 0-entries appearing in columns \( c_2 + i(c_3 - c_1 + 1) \) for \( i = 0, \ldots, p - 1 \), and the remaining entries in row \( r_1 \) are 1-entries.

The remaining entries of \( M \), that is, the entries in rows \( r_1 + 1, \ldots, m - (k - r_3) \) and columns \( c_1, \ldots, n - (\ell - c_3) \), form a submatrix with \( p(r_3 - r_1) \) rows and \( p(c_3 - c_1 + 1) \) columns. We partition these entries into rectangular blocks, each block with \( r_3 - r_1 \) rows and \( c_3 - c_1 + 1 \) columns. For \( i, j \in \{0, \ldots, p - 1\} \), let \( B_{i,j} \) be such a block, with top-left corner in row \( r_1 + 1 + i(r_3 - r_1) \) and column \( c_1 + j(c_3 - c_1 + 1) \). The entries in \( B_{i,j} \) are all equal to 1 if \( i + j = p - 1 \), otherwise they are all equal to 0.

We claim that the matrix \( M \) avoids \( P \). To see this, assume there is an embedding \( \phi \) of \( P \) into \( M \), and consider where \( \phi \) maps the three 1-entries \( e_1 = (r_1, c_2), e_2 = (r_2, c_1), \) and \( e_3 = (r_3, c_3) \). Note that none of these three entries can be mapped into the frame of \( M \), and moreover, neither \( e_2 \) nor \( e_3 \) can be mapped to the \( r_1 \)-th row of \( M \). In particular, \( \phi(e_3) \) is inside a block \( B_{i,j} \) for some \( i + j = p - 1 \). Since \( \phi(e_2) \) is to the top-left of \( \phi(e_3) \), it must belong to the same block \( B_{i,j} \). It follows that \( \phi(e_2) \) is in the leftmost column of \( B_{i,j} \), which is the column \( c_1 + j(c_3 - c_1 + 1) \), and \( \phi(e_3) \) in its rightmost column, i.e., the column \( c_3 + j(c_3 - c_1 + 1) \). Therefore, \( \phi(e_1) \) is in column \( c_2 + j(c_3 - c_1 + 1) \); however, all the entries in this column where \( \phi \) could map \( e_1 \) are 0-entries. Therefore \( M \) is in \( \text{Av}_{\text{crit}}(P) \).

The matrix \( M \) is not necessarily a critical \( P \)-avoider. However, we can transform it into a critical \( P \)-avoider by greedily changing 0-entries to 1-entries as long as the resulting matrix stays in \( \text{Av}_{\text{crit}}(P) \). By this process, we obtain a matrix \( M' \in \text{Av}_{\text{crit}}(P) \) that dominates \( M \). We claim that the \( r_1 \)-th row of \( M' \) is the same as the \( r_1 \)-th row of \( M \). This is because changing any 0-entry in the \( r_1 \)-th row of \( M \) to a 1-entry produces a matrix containing the complete pattern \( 1^{k \times \ell} \) as a submatrix, and in particular also containing \( P \) as a minor.

We conclude that the matrix \( M' \in \text{Av}_{\text{crit}}(P) \) has row-complexity at least \( p \), showing that \( \text{Av}_{\text{crit}}(P) \) is indeed row-unbounded. \( \square \)

3.2. Row-bounding patterns

We now prove the second implication of Theorem 3.1, that is, we show that any pattern \( P \) avoiding the four patterns in \( Q \) is row-bounding (and therefore, by symmetry, also column-bounding). We first prove a result describing the structure of the patterns \( P \in \text{Av}_{\text{crit}}(Q) \).
We say that a matrix $M$ can be covered by $k$ lines if there is a set of lines $\ell_1, \ldots, \ell_k$ such that each 1-entry of $M$ belongs to some $\ell_i$. The following fact is a version of the classical König–Egerváry theorem. We present it here without proof; a proof can be found, e.g., in Kung [13].

**Fact 3.5** (König–Egerváry theorem). *A matrix $M$ cannot be covered by $k$ lines if and only if $M$ contains a set of $k+1$ 1-entries, no two of which are in the same row or column.*

**Proposition 3.6.** If a pattern $P$ belongs to $\text{Av}_{\preceq}(Q)$, then

1. $P$ avoids the pattern $D_2 = (\bullet \bullet)$, or
2. $P$ avoids the pattern $\overline{D}_2 = (\bullet \circ)$, or
3. $P$ can be covered by three lines.

**Proof.** Assume $P$ cannot be covered by three lines. By Fact 3.5, $P$ contains four 1-entries $e_1 = (r_1, c_1), e_2 = (r_2, c_2), e_3 = (r_3, c_3)$ and $e_4 = (r_4, c_4)$, no two of which are in the same row or column. We may assume that $r_1 < r_2 < r_3 < r_4$. Moreover, since $P$ does not contain any pattern from $Q$, we see that any three entries among $e_1, e_2, e_3, e_4$ must form an image of $D_3$ or of $\overline{D}_3$. Consequently, the four entries $e_i$ form an image of $D_4$ or of $\overline{D}_4$, i.e., we must have either $c_1 < c_2 < c_3 < c_4$ or $c_1 > c_2 > c_3 > c_4$. Suppose that $c_1 < c_2 < c_3 < c_4$ holds, the other case being symmetric.

We will now show that $P$ avoids the pattern $\overline{D}_2$. Note first that the submatrix $P[r_3 \times [c_3]]$ avoids $\overline{D}_2$, since an image of $\overline{D}_2$ there would form an image of $Q_1$ with $e_4$. Therefore, by Proposition 2.3, all the 1-entries in $P[r_3 \times [c_3]]$...
belong to a single decreasing walk from $(1, 1)$ to $e_4$. Symmetrically, all 1-entries in the submatrix $P[[r_2, k] \times [c_2, \ell]]$ belong to a decreasing walk from $e_2$ to $(k, \ell)$.

Moreover, there can be no 1-entry in $P[[r_3, k] \times [1, c_2]]$ or in $P[[1, r_2] \times (c_3, \ell)]$, since such a 1-entry would form a forbidden pattern with $e_2$ and $e_3$. We conclude that all the 1-entries of $P$ belong to a single decreasing walk and therefore $P$ avoids $\overline{D}_2$.

We note that Proposition 3.6 is not an equivalent characterisation of patterns from $Av_\leq (Q)$, since a matrix covered by three lines may contain a pattern from $Q$. Later, in Lemma 3.17, we will give a more precise description of the avoiders of $Q$ that cannot be covered by two lines.

**Relative row-boundedness.** Before we prove that each pattern $P$ in the set $Av_\leq (Q)$ is row-bounding, we need some technical preparation. First of all, we shall need a more refined notion of row-boundedness, which considers individual 1-entries of the pattern $P$ separately.

Let $P$ be a pattern, let $e$ be a 1-entry of $P$, let $M$ be a $P$-avoiding matrix, and let $f$ be a 0-entry of $M$. Recall that $M \Delta f$ is the matrix obtained from $M$ by changing the entry $f$ from 0 to 1. We say that the entry $f$ of $M$ is critical for $e$ (with respect to $P$) if there is an embedding of $P$ into $M \Delta f$ that maps $e$ to $f$. Moreover, if $z$ is a 0-run in $M$, we say that $z$ is critical for $e$ if at least one 0-entry in $z$ is critical for $e$.

Note that a $P$-avoiding matrix is critical for $Av_\leq (P)$ if and only if each 0-entry of $M$ is critical for at least one 1-entry of $P$.

Let $e$ be a 1-entry of a pattern $P$. Let $M$ be a matrix avoiding $P$. The complexity of a row $r$ of $M$ relative to $e$ is the number of 0-runs in row $r$ that are critical for $e$. The row-complexity of $M$ relative to $e$ is the maximum complexity of a row of $M$ relative to $e$, and the row-complexity of $Av_\leq (P)$ relative to $e$, denoted $r(Av_\leq (P), e)$, is the supremum of the row-complexities of the matrices in $Av_\leq (P)$ relative to $e$. When $r(Av_\leq (P), e)$ is finite, we say that $Av_\leq (P)$ is row-bounded relative to $e$ and $e$ is row-bounding, otherwise $Av_\leq (P)$ is row-unbounded relative to $e$.

Notice that in the definition of $r(Av_\leq (P), e)$, we are taking supremum over all the matrices in $Av_\leq (P)$, not just the critical ones. This makes the definition more convenient to work with, but it does not make any substantial difference.
In fact, for a pattern $P$ with a row-bounding 1-entry $e$, the row-complexity relative to $e$ in $Av_e(P)$ is maximized by a critical $P$-avoider. To see this, suppose that $M$ is a $P$-avoiding matrix, $M^+$ is any critical $P$-avoiding matrix that dominates $M$, and $f$ is a 0-entry of $M$ that is critical for $e$; then $f$ is necessarily also a 0-entry in $M^+$, and is still critical for $e$ in $M^+$. Therefore, the row-complexity of $M^+$ relative to $e$ is at least as large as the row-complexity of $M$ relative to $e$.

Observe that the following inequalities hold for any pattern $P$:

$$\max_{e \in \text{supp}(P)} r(Av_e(P), e) \leq r(Av_e(P)) \leq \sum_{e \in \text{supp}(P)} r(Av_e(P), e).$$

In particular, a pattern $P$ is row-bounding if and only if each 1-entry of $P$ is row-bounding.

**Lemma 3.7.** Let $P$ be a pattern, and let $M$ be a $P$-avoiding matrix. Let $z$ be a horizontal 0-run of $M$, and let $f \in z$ be a 0-entry in this 0-run. Assume that there is an embedding $\phi$ of $P$ into $M \Delta f$. Then $P$ has a 1-entry $e$ mapped by $\phi$ to $f$, and moreover, every entry of $P$ in the same column as $e$ is mapped by $\phi$ to a column containing an entry from $z$.

**Proof.** Clearly, $\phi$ must map a 1-entry of $P$ to the entry $f$, otherwise $\phi$ would also be an embedding of $P$ into $M$ and $M$ would not be $P$-avoiding.

Suppose now that $z = \{r\} \times [c_1, c_2]$ for a row $r$ and columns $c_1 \leq c_2$. Let $e'$ be an entry of $P$ in the same column as $e$. Suppose that $\phi$ maps $e'$ to an entry in column $c$, with $c \notin [c_1, c_2]$. Assume that $c < c_1$, the case $c > c_2$ being analogous. Then we may modify $\phi$ to map $e$ to the 1-entry $(r, c_1 - 1)$ instead of $f$, obtaining an embedding of $P$ into $M$, which is a contradiction.

**Criteria for relative row-boundedness.** Let us first point out a trivial but useful fact: if $P \in \{0, 1\}^{k \times \ell}$ is a pattern obtained from a pattern $P$ by reversing the order of rows (i.e., turning $P$ upside down) then a 1-entry $e = (i, j)$ of $P$ is row-bounding if and only if the corresponding 1-entry $e' = (k - i + 1, j)$ of $P$ is row-bounding. Analogous properties hold for reversing the order of columns or 180-degree rotation. Similarly, operations that map rows to columns, such as transposition or 90-degree rotation, will map row-bounding 1-entries to column-bounding ones and vice versa.

We will now state several general criteria for row-boundedness of 1-entries, which we will later use to show that any $Q$-avoiding pattern is row-bounding.

**Lemma 3.8.** If $P \in \{0, 1\}^{k \times \ell}$ is a pattern with a row $r \in [k]$ and a column $c \in [\ell]$ such that $\text{supp}(P) \subseteq (\{r\} \times [\ell]) \cup ([m] \times [c, \ell])$, then every 1-entry of $P$ in the interval $\{r\} \times [c]$ is row-bounding (see Figure 6).

**Proof.** Let $e = (r, j)$ be a 1-entry of $P$ with $j \leq c$. Let $M \in \{0, 1\}^{m \times n}$ be a $P$-avoider, let $f = (r', c')$ be a 0-entry of $M$ critical for $e$, and let $z$ be the horizontal 0-run containing $f$. 

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Lemma 3.8

We claim that in the row $r'$ of $M$, there are fewer than $j$ 1-entries to the left of $f$. Suppose this is not the case, i.e., row $r'$ contains $j$ distinct 1-entries $f'_1, f'_2, \ldots, f'_j$, numbered left to right, all of them to the left of $f$.

Let $\phi$ be an embedding of $P$ into $M \Delta f$ which maps $e$ to $f$. Recall from Lemma 3.7 that all the entries in column $j$ of $P$ are mapped to columns intersecting $z$. In particular, all the entries from column $j$ are mapped to the right of $f'_j$.

We define a partial embedding $\psi$ of $P$ into $M$, as follows. Firstly, $\psi$ maps the 1-entries $(r,1), (r,2), \ldots, (r,j)$ of $P$ to the 1-entries $f'_1, f'_2, \ldots, f'_j$ of $M$. Next, $\psi$ maps each 1-entry of $P$ that is not among $(r,1), (r,2), \ldots, (r,j)$ to the same entry as $\phi$. We easily see that $\psi$ is a partial embedding of $P$ into $M$, a contradiction.

Therefore, there are fewer than $j$ 1-entries in row $r'$ to the right of $f$, and hence row $r$ has at most $j$ 0-runs critical for $e$. Consequently, $r \langle Av_\phi (P), e \rangle \leq j$ and $e$ is row-bounding.

We claim that in the row $r'$ of $M$, there are fewer than $j$ 1-entries to the left of $f$. Suppose this is not the case, i.e., row $r'$ contains $j$ distinct 1-entries $f'_1, f'_2, \ldots, f'_j$, numbered left to right, all of them to the left of $f$.

Let $\phi$ be an embedding of $P$ into $M \Delta f$ which maps $e$ to $f$. Recall from Lemma 3.7 that all the entries in column $j$ of $P$ are mapped to columns intersecting $z$. In particular, all the entries from column $j$ are mapped to the right of $f'_j$.

We define a partial embedding $\psi$ of $P$ into $M$, as follows. Firstly, $\psi$ maps the 1-entries $(r,1), (r,2), \ldots, (r,j)$ of $P$ to the 1-entries $f'_1, f'_2, \ldots, f'_j$ of $M$. Next, $\psi$ maps each 1-entry of $P$ that is not among $(r,1), (r,2), \ldots, (r,j)$ to the same entry as $\phi$. We easily see that $\psi$ is a partial embedding of $P$ into $M$, a contradiction.

Therefore, there are fewer than $j$ 1-entries in row $r'$ to the right of $f$, and hence row $r$ has at most $j$ 0-runs critical for $e$. Consequently, $r \langle Av_\phi (P), e \rangle \leq j$ and $e$ is row-bounding.

The assumptions of Lemma 3.8 are satisfied when $c$ is the leftmost nonempty column of a pattern $P$ and $r$ is an arbitrary row. We state this important special case as a separate corollary.

**Corollary 3.9.** Any 1-entry in the leftmost nonempty column of a pattern $P$ is row-bounding.

#### Lemma 3.10

Let $P \in \{0,1\}^{k \times t}$ be a pattern with a row $r$, and two distinct columns $c_1 < c_2$, such that all the 1-entries of $P$ in row $r$ belong to the interval $\{r\} \times [c_1, c_2]$. Moreover, if $c$ is a column index with $c_1 < c < c_2$, then $P$ has no 1-entry in column $c$ except possibly for the entry $(r,c)$. Suppose furthermore that $P$ satisfies one of the following three conditions (see Figure 6):

1. **Type 1:** All the 1-entries of $P$ above row $r$ are in a single row $r_1 < r$, and all the 1-entries below row $r$ are in a single row $r_2 > r$.

2. **Type 2:** All the 1-entries of $P$ above row $r$ are in a single row $r_1 < r$, and all the 1-entries below row $r$ are in the submatrix $P[(r,k) \times [c_2, t]]$.
Type 3: All the 1-entries of $P$ above row $r$ are in the submatrix $P[[1, r) \times [c_1]]$, and all the 1-entries below row $r$ are in the submatrix $P[[r, k) \times [c_2, \ell]]$.

Then every 1-entry in the interval $\{r\} \times [c_1, c_2]$ is row-bounding.

Proof. Let $P \in \{0, 1\}^{k \times \ell}$ be a pattern satisfying the assumptions, and let $d = c_2 - c_1 + 1$. We will show that for each 1-entry $e \in \{r\} \times [c_1, c_2]$ of $P$ and every $P$-avoiding matrix $M \in \{0, 1\}^{m \times n}$, there are at most $d$ 0-runs critical for $e$ in each row of $M$.

For contradiction, assume that $M$ has a row $r'$ with at least $d + 1$ 0-runs critical for $e$. Let $f$ and $f'$ be the leftmost and the rightmost 0-entries critical for $e$ in row $r'$. By assumption, $M$ has at least $d$ 1-entries in row $r'$ between $f$ and $f'$. Let $f_1, f_2, \ldots, f_d$ be $d$ such 1-entries, numbered left to right.

Let $\phi$ be an embedding of $P$ into $M\Delta f$ which maps $e$ to $f$, and let $\phi'$ be an embedding of $P$ into $M\Delta f'$ which maps $e$ to $f'$. Let us describe a partial embedding $\psi$ of $P$ into $M$. Firstly, $\psi$ maps the entries $(r, c_1), (r, c_1 + 1), \ldots, (r, c_2)$ to the entries $f_1, f_2, \ldots, f_d$ in row $r'$ of $M$. Next, $\psi$ maps each 1-entry in $M[[m] \times [c_1]]$ except $(r, c_1)$ to the same entry as $\phi$, and $\psi$ maps each 1-entry in $M[[m] \times [c_2, n]]$ except $(r, c_2)$ to the same entry as $\phi'$. We easily check that this makes $\psi$ a partial embedding of $P$ into $M$: note that from Lemma 3.7, it follows that $\phi$ maps all the entries in column $c_1$ of $P$ to entries strictly to the left of $f_1$, and $\phi'$ maps entries in column $c_2$ to entries strictly to the right of $f_d$.

This is impossible, since $M$ is $P$-avoiding. Therefore, every row of a $P$-avoiding matrix has at most $d$ 0-runs critical for $e$, and $e$ is row-bounding. □

Lemma 3.11. Let $P \in \{0, 1\}^{k \times \ell}$ be a pattern with two rows $r_1 \leq r_2$ and a column $c$, such that for every $r \in [r_1, r_2]$, $P$ has no 1-entry in row $r$ except possibly for the entry $(r, c)$. Suppose moreover, that $P$ satisfies one of the following conditions (see Figure 7):

Type 1: All the 1-entries of $P$ above row $r_1$ are in column $c$ or in the row $r_1 - 1$, and all the 1-entries below row $r_2$ are in column $c$ or in the row $r_2 + 1$.

Type 2: All the 1-entries of $P$ above row $r_1$ are in column $c$ or in the row $r_1 - 1$, and all the 1-entries below row $r_2$ are in the submatrix $P[[r_2, k) \times [c, \ell]]$.

Type 3: All the 1-entries of $P$ above row $r_1$ are in the submatrix $P[[1, r_1) \times [c]]$, and all the 1-entries below row $r_2$ are in the submatrix $P[[r_2, k) \times [c, \ell]]$.

Then every 1-entry in the interval $[r_1, r_2] \times \{c\}$ is row-bounding.

Proof. Let $P$ be a pattern satisfying the assumptions of the lemma, and let $e = (r, c)$ be its 1-entry, with $r \in [r_1, r_2]$. Let $M \in \{0, 1\}^{m \times n}$ be a $P$-avoider. We claim that every row of $M$ has at most one 0-run critical for $e$. For contradiction, suppose that row $i$ of $M$ has two 0-runs $z_L$ and $z_R$ critical for $e$, where $z_L$ is to the left of $z_R$. Let $f_L \in z_L$ and $f_R \in z_R$ be two 0-entries critical for $e$ in the two 0-runs.
Let $\phi_L$ be an embedding of $P$ into $M \Delta f_L$ with $\phi_L(e) = f_L$, and $\phi_R$ be an embedding mapping $P$ into $M \Delta f_R$ with $\phi_R(e) = f_R$. Let $C_L \subseteq [n]$ be the set of columns of $M$ that intersect the 0-run $z_L$, that is, we have $z_L = \{i\} \times C_L$. Similarly, let $C_R$ be the set of columns intersecting $z_R$. Note that by Lemma 3.7, $\phi_L$ maps all the entries in column $c$ of $P$ to entries in the set of columns $C_L$, and similarly, $\phi_R$ maps the entries of column $c$ to entries in columns $C_R$. We will now describe a partial embedding $\psi$ of $P$ into $M$; see Figure 8.

Since $f_L$ and $f_R$ are in distinct 0-runs, $M$ has a 1-entry $f$ that lies in row $i$ between $f_L$ and $f_R$. We put $\psi(e) = f$. For any other 1-entry $e' \in \text{supp}(P) \setminus \{e\}$, we will define $\psi(e')$ to be equal to either $\phi_L(e')$ or $\phi_R(e')$, by the following rules.

For a 1-entry $e'$ which is strictly to the left of the left of column $c$, we let $\psi(e') = \phi_L(e')$ and for a 1-entry $e'$ strictly to the right of column $c$, we let $\psi(e') = \phi_R(e')$.

It remains to deal with the 1-entries in column $c$. For a 1-entry $e'$ in $[r_1, r) \times \{c\}$, we choose $\psi(e')$ to be the lower of the two entries $\phi_L(e')$ and $\phi_R(e')$, i.e., we choose the entry that has larger row-index. If $\phi_L(e')$ and $\phi_R(e')$ are in the
that the entry.

Suppose that \( e \) is a 1-entry. Let \( \psi(e') \) denote respectively the row index and column index of \( e' \). Choose a pair of 1-entries \( e_1 = (i_1, j_1) \) and \( e_2 = (i_2, j_2) \) of \( P \), and consider their respective images \( f_1 = \psi(e_1) \) and \( f_2 = \psi(e_2) \).

We proceed symmetrically for 1-entries below row \( r \). For a 1-entry \( e' \in (r, r_2] \times \{c\} \), we choose \( \psi(e') \) to be the higher of the two entries \( \phi_L(e') \) and \( \phi_R(e') \), breaking ties arbitrarily. For a 1-entry \( e' \in (r, k] \times \{c\} \), if \( P \) is of Type 1, we let \( \psi(e') \) be the lower of \( \phi_L(e') \) and \( \phi_R(e') \), and if \( P \) is of Type 2 or 3, we put \( \psi(e') = \phi_R(e') \).

Let us verify that the mapping \( \psi \) is a partial embedding of \( P \) into \( M \). For an entry \( e' \) of \( P \) or of \( M \), let \( ri(e') \) and \( ci(e') \) denote respectively the row index and column index of \( e' \). Choose a pair of 1-entries \( e_1 = (i_1, j_1) \) and \( e_2 = (i_2, j_2) \) of \( P \), and consider their respective images \( f_1 = \psi(e_1) \) and \( f_2 = \psi(e_2) \).

Suppose that \( i_1 < i_2 \); we then need to show that \( ri(f_2) - ri(f_1) \geq i_2 - i_1 \). We may assume here, without loss of generality, that \( P \) has no 1-entries in the rows \( [i_1 + 1, i_2] \). In particular, we have either \( i_2 \leq r \) or \( i_1 \geq r \). By the definition of \( \psi \), we see that

\[
ri(f_2) - ri(f_1) \geq \min\{ri(\phi_L(e_2)) - ri(\phi_L(e_1)), ri(\phi_R(e_2)) - ri(\phi_R(e_1))\} \geq i_2 - i_1,
\]
as claimed, where the second inequality follows from the fact that \( \phi_L \) and \( \phi_R \) are both embeddings.

By similar reasoning, we verify that for a pair of 1-entries \( e_1 = (i_1, j_1) \) and \( e_2 = (i_2, j_2) \) of \( P \) with \( j_1 < j_2 \), we always have \( ci(\psi(e_2)) - ci(\psi(e_1)) \geq j_2 - j_1 \) (recall that \( \phi_L \) maps the entries from column \( c \) to the set of columns \( C_L \), while \( \phi_R \) maps them to the set \( C_R \), which is strictly to the right of \( C_L \)).

We may also easily see that for any 1-entry \( e' \) of \( P \), the distance of its image \( \psi(e') \) from the left, right, top and bottom edge of \( M \), is at least as large as the distance of \( e \) from the corresponding edge of \( P \). We conclude that \( \psi \) is a partial embedding of \( P \) into \( M \), which contradicts the assumption that \( M \) is \( P \)-avoiding. This contradiction shows that the entry \( e = (r, c) \) is row-bounding. \( \square \)

**Lemma 3.12.** Let \( P \in \{0, 1\}^{k \times \ell} \) be a pattern with two rows \( r_1 < r_2 \) and two columns \( c_1 < c_2 \) of one of the following two types (see Figure 9):

- **Type 1:** supp(\( P \)) \( \subseteq ([r_1, r_2] \times \{c_1\}) \cup ([r_1, r_2] \times ([c_1] \cup \{c_2\})) \).
- **Type 2:** supp(\( P \)) \( \subseteq ([r_1, r_2] \times \{c_1\}) \cup ([r_2] \times ([c_1] \cup \{c_2\})) \cup ([r_1] \times \{c_2\}) \).

If \( e = (r_1, c_1) \) is a 1-entry of \( P \), then it is row-bounding.

**Proof.** Suppose that \( P \in \{0, 1\}^{k \times \ell} \) satisfies the assumptions of the lemma, and that the entry \( e = (r_1, c_1) \) is a 1-entry. Let \( e' \) be the entry \( (r_2, c_2) \) of \( P \). Notice that if \( e' \) is a 0-entry, we can deduce that \( e \) is row-bounding by Lemma 3.8 (for
Type 1, mirrored by the vertical axis) or by Type 3 of Lemma 3.10 (for Type 2, again mirrored by the vertical axis). Assume therefore that $e'$ is a 1-entry of $P$.

Let $M \in \{0,1\}^{m \times n}$ be a $P$-avoider. We will show that every row of $M$ has at most $\ell(\ell + 1)$ 0-runs critical for $e$. Suppose that a row $r'$ of $M$ has more than $\ell(\ell + 1)$ 0-runs critical for $e$. Among these 0-runs, we select a subsequence $z_1, z_2, \ldots, z_{\ell+1}$ numbered left to right, with the property that for each $i \in [\ell]$, $M$ has at least $\ell$ 1-entries in row $r'$ between $z_i$ and $z_{i+1}$, and $M$ also has at least $\ell$ 1-entries in row $r'$ to the right of $z_{\ell+1}$.

For each $i \in [\ell+1]$, let $f_i$ be a 0-entry in $z_i$ critical for $e$, and let $\psi_i$ be an embedding of $P$ into $M \Delta f_i$ that maps $e$ to $f_i$. For $i \in [\ell]$, let $w_i$ be the interval of entries that lie between $z_i$ and $z_{i+1}$ in row $r'$ of $M$, and let $w_{\ell+1}$ be the interval of entries in row $r'$ to the right of $z_{\ell+1}$. Recall that each $w_i$ contains at least $\ell$ 1-entries. Let $g_i$ be the leftmost entry in $w_i$, which is necessarily a 1-entry, because $z_i$ is a maximal interval of 0-entries. Finally, let $h_i = (p_i, q_i)$ be the 1-entry $\psi_i(e')$ (recall that $e' = (r_2, c_2)$ is a 1-entry of $P$).

Let us define a partial embedding $\psi$ of $P$ into $M$. We let $\psi$ map the entry $(r_1, c_2)$ to the 1-entry $g_{\ell+1}$, and if $P$ is of Type 2, then for every 1-entry $e''$ in the interval $[1, r_1] \times \{c_2\}$, we define $\psi(e'') = \phi_{\ell+1}(e'')$. Note that all the entries we mapped so far are to the right of $f_{\ell+1}$.

To define $\psi$ for the remaining 1-entries of $P$, we will distinguish several situations, depending on the positions of the entries $h_i = (p_i, q_i)$.

If, for some $i \in [\ell]$, the entry $h_i$ is to the right of the rightmost column of $w_i$, we put $\psi(e) = g_i$, and for every 1-entry $e''$ of $P$ for which $\psi$ has not yet
been defined, we put \( \psi(e'') = \phi_i(e'') \). To see that the mapping \( \psi \) is a partial embedding of \( P \) into \( M \), it is enough to observe that all the 1-entries in column \( c_2 \) of \( P \) are mapped by \( \psi \) to entries strictly to the right of \( w_i \), while by Lemma 3.7, all the 1-entries in column \( c_1 \) are mapped to the columns intersecting the interval \( z_i \), except for the entry \( e \), which is mapped to \( g_i \). There are therefore at least \( \ell - 1 \) columns which separate the image of any entry from column \( c_1 \) from the image of any entry from column \( c_2 \). With this in mind, it is easy to check that \( \psi \) is indeed a partial embedding.

Suppose that the situation from the previous paragraph does not occur, that is, for every \( i \in [\ell] \), the entry \( h_i \) is not to the right of the rightmost column intersecting \( w_i \). Since \( h_i \) must by construction be to the right of the column containing \( f_i \), we know that the column \( q_i \) containing \( h_i \) intersects either \( z_i \) or \( w_i \). In particular, we have \( q_1 < q_2 < \cdots < q_{\ell+1} \).

Assume now, that for some \( i \in [\ell] \), the inequality \( p_i \leq p_{i+1} \) holds. We now complete the mapping \( \psi \) as follows: we put \( \psi(e) = g_i \), \( \psi(e') = h_{i+1} \), and for all the 1-entries \( e'' \) of \( P \) not yet mapped (i.e., the 1-entries in columns 1, \ldots, \( c_1 \) except \( e \)), we put \( \psi(e'') = \phi_i(e'') \). The mapping \( \phi \) is again a partial embedding of \( P \) into \( M \).

It remains to deal with the situation when we have \( p_1 > p_2 > \cdots > p_k > p_{k+1} \), which means that the 1-entries \( h_1, h_2, \ldots, h_{\ell+1} \) form an image of the diagonal pattern \( \overline{T}_{\ell+1} \). We complete the mapping \( \psi \) as follows: a 1-entry of the form \( (r_1, j) \) for \( j \leq c_1 \) is mapped to the entry \( g_j \), a 1-entry of the form \( (r_2, j) \) for any \( j \in [\ell] \) is mapped to \( h_j \), and any 1-entry \( e'' \in [r_1 + 1, r_2] \times \{c_1\} \) is mapped to \( \phi_i(e'') \). Note that for \( j < c_1 \), the mapping \( \psi \) maps the 1-entries in column \( j \) to 1-entries in columns intersecting \( z_j \cup w_j \), and for \( j = c_1 \), the 1-entries in column \( j \) get mapped to columns intersecting \( z_j \cup w_j \cup z_{\ell} \).

In all cases, we found a partial embedding \( \psi \) of \( P \) into \( M \), which is in contradiction. Therefore, each row of \( M \) has at most \( \ell(\ell + 1) \) 0-runs critical for \( e \), and \( e \) is row-bounding.

**Row-boundedness of specific patterns.** We now have enough technical tools to establish that any pattern \( P \) from \( \text{Av}_{\leq}(Q) \) is row-bounding. Recall from Proposition 3.6 that any \( P \in \text{Av}_{\leq}(Q) \) avoids \( D_2 \) or \( \overline{D}_2 \) or can be covered by three lines.

We will first look at patterns that can be covered by fewer than three lines, and show that they are all row-bounding.

**Lemma 3.13.** A pattern \( P \) that has at most two nonempty columns or at most one nonempty row is row-bounding.

**Proof.** It follows from Lemma 3.8 and trivial symmetries that every 1-entry of \( P \) is row-bounding, hence \( P \) is row-bounding. \( \square \)

**Lemma 3.14.** If \( P \in \{0,1\}^{k \times \ell} \) is a pattern with two nonempty rows, then \( P \) is row-bounding.

**Proof.** We will show that for every 1-entry \( e \) of \( P \), we have \( r(Av_{\leq}(P), e) \leq \ell^2 \).
In view of Observation 2.5, we may assume that only the first row and the last row of $P$ are nonempty. Let $e$ be a 1-entry of $P$, and suppose without loss of generality that $e$ is in the first row, i.e., $e = (1, c)$ for some $c$.

Given a matrix $M \in Av_{\leq} (P)$, consider an arbitrary row $r$ of $M$. For contradiction, suppose that the row $r$ has $\ell^2 + 1$ distinct 0-runs $z_1, \ldots, z_{\ell^2+1}$ critical for $e$, numbered left to right. Let $c_i$ denote the leftmost column intersecting $z_i$, and for $i \leq \ell^2$, let $X_i$ denote the set of column indices $[c_i, c_{i+1})$. Observe that for every $i \leq \ell^2$, $M$ has at least one 1-entry in the interval $\{r\} \times X_i$.

Let $B_i$ be the submatrix $M[r + k - 1, m] \times X_i$ of $M$ (see Figure 11). Note that if there are at least $\ell$ distinct values of $i$ for which $B_i$ contains at least one 1-entry, then the matrix $M$ contains the pattern $P$.

Suppose therefore that $B_i$ is empty for each $i$ up to at most $\ell - 1$ exceptions. In particular, there is an index $j \in [\ell^2]$ such that the $\ell$ consecutive submatrices $B_j, B_{j+1}, \ldots, B_{j+\ell-1}$ are all empty.

Recall that $e = (1, c)$ is a 1-entry of $P$, and that all the 1-entries of $P$ are in rows 1 and $k$. Let $c'$ be a column index such that $e' = (k, c')$ is a 1-entry of $P$, and $|c - c'|$ is as small as possible. Suppose without loss of generality that $c \leq c'$ and let $d := c' - c$.

Let $f$ be a 0-entry in $z_j$ critical for $e$, and let $\phi$ be an embedding of $P$ into $M \Delta f$ that maps $e$ to $f$. Note that by Lemma 3.7, $\phi$ maps the entries in column $c$ of $P$ to entries in columns intersecting $z_j$, and in particular, the entry $(k, c)$ is mapped inside $B_j$. Since $B_j$ is empty, $(k, c)$ is a 0-entry and in particular, $c'$ is greater than $c$.

It follows that the 1-entry $e' = (k, c')$ is mapped strictly to the right of the column containing $f$, and since $B_j, \ldots, B_{j+\ell-1}$ are all empty, $e'$ must be mapped to the right of the columns in the set $X_{j+\ell-1}$.

We now define a partial embedding $\psi$ of $P$ into $M$ as follows: the $d+1$ entries in $P[\{1\} \times [c, c']]$ get mapped into $M[\{r\} \times (X_j \cup X_{j+1} \cup \ldots \cup X_{j+d})]$ by $\psi$ (recall that $\{r\} \times X_i$ contains at least one 1-entry for each $i$). The remaining 1-entries of $P$ are mapped by $\psi$ in the same way as by $\phi$. Then $\psi$ is a partial embedding of $P$ into $M$, a contradiction.

\begin{lemma}
A pattern $P$ that can be covered by one row and one column is row-bounding.
\end{lemma}
Proof. Suppose that \( P \in \{0,1\}^{k \times \ell} \) is covered by row \( r \) and column \( c \). By Lemma 3.8, all the 1-entries in \( P[[r] \times [c]] \) are row-bounding, and by symmetry, the 1-entries in \( P[[r] \times [c, \ell]] \) are row-bounding as well. By Lemma 3.11, the 1-entries in \( P[[1,r] \times \{c\}] \) and \( P[[r,k] \times \{c\}] \) are also row-bounding.

Lemmas 3.13, 3.14 and 3.15 imply that any pattern that can be covered by two lines is row-bounding. We now proceed with the remaining cases of Proposition 3.6.

**Lemma 3.16.** A pattern \( P \in \{0,1\}^{k \times \ell} \) that avoids \( D_2 \) or \( \overline{D_2} \) is row-bounding.

Proof. Suppose that \( P \) avoids \( \overline{D_2} \), the other case being symmetric. From Proposition 2.3, we know that \( P \) is a decreasing pattern. Every 1-entry of \( P \) is row-bounding either by Lemma 3.10 (Type 3), or by Lemma 3.11 (Type 3), and therefore \( P \) is row-bounding.

What follows is the last and the most difficult case of our analysis, which deals with patterns that are not increasing or decreasing and cannot be covered by two lines.

**Lemma 3.17.** Let \( P \in \text{Av}_\prec (Q) \) be a pattern that contains both \( D_2 \) and \( \overline{D_2} \), and that cannot be covered by two lines. Then \( P \) can be transformed by a rotation or a reflection to a pattern \( P_0 \) of one of these two types (see Figure 12).

- **Type 1:** \( P_0 \) has three rows \( r < r' < r'' \) and two columns \( c < c' \) with
  \[
  \text{supp}(P_0) \subseteq \{(r',c),(r',c'),(r,c),(r,c'),(r'',c), (r'',c')\}.
  \]

- **Type 2:** \( P_0 \) has two rows \( r < r' \) and two columns \( c < c' \) with
  \[
  \text{supp}(P_0) \subseteq \{(r,c),(r,c'),(r',c),(r',c')\}.
  \]

Proof. Let \( P \in \{0,1\}^{k \times \ell} \) be a pattern satisfying the assumptions of the lemma. Since \( P \) cannot be covered by two lines, by Fact 3.5, \( P \) contains three 1-entries \( e_1 = (r_1,c_1), e_2 = (r_2,c_2) \) and \( e_3 = (r_3,c_3) \), with \( r_1 < r_2 < r_3 \), and such that...
the columns $c_1, c_2, c_3$ are all distinct. Since $P$ avoids the patterns from $Q$, we must have either $c_1 < c_2 < c_3$ or $c_1 > c_2 > c_3$. Without loss of generality, assume $c_1 < c_2 < c_3$.

By Proposition 3.6, $P$ can be covered by three lines. Suppose first that the three lines that cover $P$ are the rows $r_1$, $r_2$ and $r_3$. Suppose moreover, that the three 1-entries $e_1, e_2, e_3$ were chosen in such a way that $c_1$ is as large as possible, while $c_2$ and $c_3$ are as small as possible, i.e., the choice minimizes the value of $-c_1 + c_2 + c_3$; see Figure 13 (left). In particular, row $r_1$ of $P$ has no 1-entry in any of the columns $[c_1 + 1, c_2)$, otherwise we could choose a larger value of $c_1$. Similarly, row $r_2$ has no 1-entry in columns $[c_1 + 1, c_2)$ and row $r_3$ has no 1-entry in columns $[c_2 + 1, c_3)$.

Moreover, since $P$ avoids the four patterns from the set $Q$, row $r_1$ has no 1-entry in columns $[c_2 + 1, c_3)$ or $(c_3, \ell)$, row $r_2$ has no 1-entry in columns $[1, c_1)$ or $(c_3, \ell)$, and row $r_3$ has no 1-entry in columns $[1, c_1)$ or $[c_1 + 1, c_2)$.

Therefore, apart from the three 1-entries $e_i$, a 1-entry of $P$ can appear in one of the three intervals $\alpha = [r_1] \times [1, c_1)$, $\beta = [r_2] \times (c_2, c_3]$ and $\gamma = [r_3] \times (c_3, \ell]$, or be equal to one of the five entries $a = (r_2, c_1)$, $b = (r_3, c_1)$, $c = (r_1, c_2)$, $d = (r_3, c_2)$ or $e = (r_1, c_3)$; see Figure 13 (left). Note that $a$ and $c$ cannot be simultaneously equal to 1, otherwise they would form a forbidden pattern with $e_3$, and similarly, if $\beta$ contains a 1-entry then $d = 0$, if $\alpha$ contains a 1-entry then $b = 0$, and if $\gamma$ contains a 1-entry then $e = 0$.

Since $P$ contains a copy of $D_2$, at least one of $b$ and $e$ must be a 1-entry. Let us go through the cases that may occur.

**Case I: $b = 1$.** If $b = 1$ then $\alpha$ is empty. We have two subcases:

**Ia: $\beta$ contains a 1-entry.** Then $c = 0$ and $d = 0$. If $\gamma$ is empty, then $P$ is a Type 1 matrix, with $c = e_1$, $e' = e_3$, and $(r, r', r'') = (r_1, r_2, r_3)$. If $\gamma$ is nonempty, then $e = 0$, and $P$ is a mirror image of a Type 2 matrix, with $(r, r') = (r_2, r_3)$ and $(c, e') = (e_3, c_1)$.

**Ib: $\beta$ is empty.** If $\gamma$ is nonempty, then $e = 0$ and since at most one of $a$ and $c$ is nonempty, rotating $P$ counterclockwise by 90 degrees yields a Type 2 matrix. If $\gamma$ is empty, then either $a = 0$ and $P$ is the transpose of a Type 1 matrix, or $a = 1$, and therefore $c = 0$, and at least one of $d$ and $e$ is a 0-entry, resulting in a Type 1 matrix or a rotated Type 2 matrix.

**Case II: $b = 0$.** If $b = 0$, then $e = 1$, otherwise $P$ would avoid $D_2$. Consequently, $\gamma$ is empty. If $\beta$ were empty as well, then $P$ would be symmetric to a matrix from case Ib by a 180-degree rotation. We may therefore assume that $\beta$ is nonempty, and hence $d = 0$. At most one of $a$ and $c$ can be a 1-entry, and in either case we get an upside-down copy of a Type 2 matrix.

This completes the analysis of matrices that can be covered by 3 rows. Suppose now that $P$ can be covered by two rows and one column. As each of the three entries $e_1, e_2$ and $e_3$ must be covered by a distinct line, there are three possibilities: either $P$ is covered by rows $r_1$ and $r_2$ and column $c_3$; or $P$ is covered by rows $r_1$ and $r_3$ and column $c_2$; or $P$ is covered by rows $r_2$ and $r_3$ and
Figure 13: $Q$-avoiders covered by rows $r_1$, $r_2$ and $r_3$ (left), by rows $r_1$, $r_2$ and column $c_3$ (center), and by rows $r_1$, $r_3$ and column $c_2$ (right). The shaded entries are potential 1-entries, the dots represent the three 1-entries $e_1$, $e_2$ and $e_3$.

Suppose $P$ is covered by rows $r_1$ and $r_2$ and column $c_3$. Choose $c_1$ and $c_2$ to be as large as possible, and $r_3$ to be as small as possible, i.e., the choice maximizes $c_1 + c_2 - r_3$. Together with the absence of patterns from $Q$, this means that apart from the 1-entries $e_1$, $e_2$ and $e_3$, all the remaining 1-entries must be inside the intervals $\alpha$, $\beta$ and $\gamma$ or at the positions $a$, $b$ or $c$ depicted in Figure 13 (center). Moreover, if $a = 1$ then $\beta$ is empty. Therefore, $P$ is an upside-down copy of a matrix of Type 2, with the role of column $c$ played by $c_1$ if $a = 0$ or by $c_2$ if $a = 1$.

Let us now suppose that $P$ is covered by rows $r_1$ and $r_3$ and column $c_2$. See Figure 13 (right). Suppose $c_1$ is largest possible and $c_3$ smallest possible, i.e., $c_1 - c_3$ is maximized. We make no assumptions about $r_2$, to keep the configuration symmetric. All the 1-entries are in the intervals $\alpha$, $\beta$, $\gamma$ and $\delta$ or at the positions $a$ and $b$ depicted in the figure. Since $P$ contains $\overline{D}_2$, at least one of $a$ and $b$ is a 1-entry. Suppose without loss of generality that $a = 1$. Then $\alpha$ is empty. If $\delta$ is nonempty, then $b = 0$, and $P$ is a Type 2 matrix rotated 90 degrees clockwise. Otherwise $\delta$ is empty and $P$ is a rotated Type 1 matrix.

The cases when $P$ can be covered by three columns, or by two columns and a row, are symmetric to the cases handled so far by a 90-degree rotation.

We now have all the ingredients to complete the proof of our main result.

Theorem 3.18. Every pattern $P \in \Av_\prec (Q)$ is row-bounding.

Proof. Choose a $P \in \Av_\prec (Q)$. By Proposition 3.6, either $P$ can be covered by three lines, or it avoids $D_2$, or it avoids $\overline{D}_2$. If $P$ avoids one of the two patterns of size 2, then it is row-bounding by Lemma 3.16. If it can be covered by two lines, it is row-bounding by Lemmas 3.13, 3.14 and 3.15. Finally, if $P$ contains both $D_2$ and $\overline{D}_2$ and cannot be covered by two lines, Lemma 3.17 shows that, up to symmetry, $P$ corresponds to a matrix of Type 1 or Type 2. We therefore need to argue that the matrices of these two types, as well as their transposes, are row-bounding. See Figure 14.
If $P$ is of Type 1, its 1-entries in column $c$ or in column $c'$ are row-bounding by Corollary 3.9, and those in row $r'$ are row-bounding by Lemma 3.10, Type 1.

If $P$ is the transpose of a Type 1 matrix, then its 1-entries in columns $r$ and $r''$ are row-bounding by Corollary 3.9, and those in column $r'$ by Lemmas 3.11 and 3.12.

If $P$ is of Type 2, the 1-entries in row $r'$ and in column $c'$ are row-bounding by Lemma 3.8 and Corollary 3.9, and those in row $r$ are row-bounding by Lemma 3.10.

Finally, if $P$ is the transpose of a Type 2 matrix, the 1-entries in column $r'$ and in row $c'$ are row-bounding by Lemma 3.8 and Corollary 3.9, and the remaining 1-entries are covered by Lemmas 3.11 and 3.12. \hfill $\Box$

Theorems 3.4 and 3.18 together imply Theorem 3.1.

4. Further directions and open problems

**Boundedness of non-principal classes.** So far, we have only considered principal classes of matrices, i.e., the classes determined by a single forbidden pattern. It is natural to ask to what extent our results generalize to arbitrary minor-closed classes of matrices, or at least to classes determined by a finite number of forbidden patterns.

All our row-boundedness results for principal classes are based on the study of row-bounding 1-entries in a pattern $P$. This approach extends straightforwardly to the setting of multiple forbidden patterns. In particular, for a set $\mathcal{F}$ of patterns, a pattern $P \in \mathcal{F}$ and a 1-entry $e$ of $P$, we say that $e$ is row-bounding in $Av_\prec (\mathcal{F})$ if each row of a matrix $M \in Av_\prec (\mathcal{F})$ has only a bounded number of 0-runs critical for $e$ with respect to $P$. Note that if $\mathcal{F}$ is finite, then $Av_\prec (\mathcal{F})$ is row-bounded if and only if each 1-entry of each pattern $P \in \mathcal{F}$ is row-bounding in $Av_\prec (\mathcal{F})$. 

Figure 14: Illustration of the proof of Theorem 3.18. The symbols indicate the criteria used to prove row-boundedness of the 1-entries in the two types of patterns of Lemma 3.17.
Figure 15: Left: illustration that $\text{Av}_{\leq} (\mathcal{F})$ has unbounded column-complexity relative to the entry $e = (2, 1)$ of $P$. Right: illustration of the proof that $\text{Av}_{\leq} (\mathcal{F})$ is row-bounded.

Note also that, by definition, if $e$ is a 1-entry of $P$ that is row-bounding in $\text{Av}_{\leq} (P)$, then for every set of patterns $\mathcal{F}$ that contains $P$, the entry $e$ is also row-bounding in $\text{Av}_{\leq} (\mathcal{F})$, since $\text{Av}_{\leq} (\mathcal{F})$ is a subclass of $\text{Av}_{\leq} (P)$. Therefore, all the criteria for row-bounding entries that we derived in Subsection 3.2 are applicable to non-principal classes as well.

We have seen in Corollary 3.2 that a principal class is row-bounded if and only if it is column-bounded. Our next example shows that this property does not generalize to non-principal classes.

**Proposition 4.1.** For the set of patterns $\mathcal{F} = \{D_4, P\}$ with

$$P = \begin{pmatrix} 
1 & 1 \\
1 & 1 
\end{pmatrix} \text{ and } D_4 = \begin{pmatrix} 
1 & 1 \\
1 & 1 
\end{pmatrix},$$

the class $\text{Av}_{\leq} (\mathcal{F})$ is row-bounded but not column-bounded.

**Proof.** To prove that $\text{Av}_{\leq} (\mathcal{F})$ is not column-bounded, we apply the transpose of the construction of Theorem 3.4, and observe that the constructed matrix avoids $D_4$ (see Figure 15 (left)).

To prove that the class $\text{Av}_{\leq} (\mathcal{F})$ is row-bounded, observe first that all the 1-entries in $D_4$ are row-bounding by Lemma 3.10, the leftmost and the rightmost 1-entry of $P$ are row-bounding by Corollary 3.9, and the 1-entry $(3, 2)$ of $P$ is row-bounding by Lemma 3.11. It thus remains to show that the entry $e = (1, 2)$ of $P$ is row-bounding in $\text{Av}_{\leq} (\mathcal{F})$.

We will show that each matrix $M \in \text{Av}_{\leq} (\mathcal{F})$ has at most two 0-runs critical for $e$ in any given row. Refer to Figure 15 (right). For contradiction, suppose that there are three 0-runs $z_1 < z_2 < z_3$ in a row $r$ of $M$. Let $g_1$ be a 1-entry of $M$ that lies in row $r$ between $z_1$ and $z_2$, let $g_2$ be a 1-entry of $M$ in row $r$ between $z_2$ and $z_3$, and let $f$ be a 0-entry in $z_3$ critical for the entry $e$. Let $\phi$ be an embedding of $P$ into $M \Delta f$ with $\phi(e) = f$.

Consider the three 1-entries $h_1 = \phi(2, 1)$, $h_2 = \phi(3, 2)$ and $h_3 = \phi(4, 3)$. If $h_1$ is in a column strictly to the right of $g_1$, then $g_1$ forms an image of $D_4$ with the three $h_i$s, a contradiction. If, on the other hand, $h_1$ is not to the right of $g_1$, then $h_1$ is strictly to the left of $g_2$, and $g_2$ forms an image of $P$ with the three $h_i$s (recall that $h_3$ is to the right of $f = \phi(1, 2)$, and therefore also to the right of $g_2$). This shows that $\text{Av}_{\leq} (\mathcal{F})$ is row-bounded. \qed
Recall from Corollary 3.3, that any principal subclass of a bounded principal class is again bounded. The example of Proposition 4.1 shows that this result does not generalize to non-principal classes: indeed, the class \( Av_{\prec} (D_4) \) is bounded by Theorem 3.1 (or by Corollary 2.4), while its subclass \( Av_{\prec} (\mathcal{F}) \) is not bounded.

On the positive side, it is not hard to show that row-boundedness (and therefore also boundedness) is closed under union and intersection of classes.

**Proposition 4.2.** If \( C_1 \) and \( C_2 \) are row-bounded classes of matrices, then the classes \( C_1 \cup C_2 \) and \( C_1 \cap C_2 \) are row-bounded as well.

**Proof.** Let \( K_i \) be the row-complexity of the class \( C_i \), for \( i \in \{1, 2\} \). Since every matrix that is critical for \( C_1 \cup C_2 \) is also critical for \( C_1 \) or for \( C_2 \), we observe that \( C_1 \cup C_2 \) has row-complexity at most \( \max \{K_1, K_2\} \). In particular, \( C_1 \cup C_2 \) is row-bounded.

Let us argue that \( C_1 \cap C_2 \) is row-bounded as well. We claim that \( C_1 \cap C_2 \) has row-complexity at most \( K := K_1 + K_2 \). For contradiction, suppose that there is a matrix \( M \) critical for \( C_1 \cap C_2 \) with row-complexity at least \( K + 1 \).

Let \( r \) be a row of \( M \) with maximum complexity, let \( z_1, z_2, \ldots, z_{K+1} \) be a sequence of 0-runs in this row, and let \( f_i \) be a 0-entry in \( z_i \). By criticality of \( M \), we know that for each \( i \in [K+1] \), the matrix \( M \Delta f_i \) does not belong to \( C_1 \) or does not belong to \( C_2 \).

In particular, there are either at least \( K_1 + 1 \) values of \( i \) for which \( M \Delta f_i \) is not in \( C_1 \), or at least \( K_2 + 1 \) values of \( i \) for which \( M \Delta f_i \) is not in \( C_2 \). Suppose without loss of generality that the former situation occurs. Let \( M^+ \) be a critical matrix for the class \( C_1 \) that dominates the matrix \( M \). If \( f_i \) is a 0-entry of \( M \) such \( M \Delta f_i \) is not in \( C_1 \), then \( f_i \) is also a 0-entry of \( M^+ \). It follows that \( M^+ \) has at least \( K_1 + 1 \) 0-runs in row \( r \), which is impossible, since \( K_1 \) is the row-complexity of \( C_1 \).

In contrast with Proposition 4.2, an intersection of two unbounded classes is not necessarily unbounded, as we will now show. Consider the two patterns \( Q_1 = (\bullet \bullet \bullet) \) and \( Q_2 = (\bullet \bullet \bullet) \), and recall from Theorem 3.1 that both \( Av_{\prec} (Q_1) \) and \( Av_{\prec} (Q_2) \) are unbounded classes.

**Proposition 4.3.** The class \( Av_{\prec} (\{Q_1, Q_2\}) = Av_{\prec} (Q_1) \cap Av_{\prec} (Q_2) \) is bounded.

**Proof.** Let us first show that every 1-entry of the two patterns \( Q_1 \) and \( Q_2 \) is row-bounding for \( C := Av_{\prec} (\{Q_1, Q_2\}) \). For a 1-entry that belongs to the first or the last column of either pattern, this follows from Corollary 3.9.

Consider the 1-entry \( e = (1, 2) \) of the pattern \( Q_1 \). We claim that each row in a matrix \( M \in C \) has at most two 0-runs critical for \( e \). Suppose that a matrix \( M \in C \) has a row \( r \) with three 0-runs \( z_1 < z_2 < z_3 \) critical for \( e \). Let \( f_i \) be a 0-entry in \( z_i \) critical for \( e \), and let \( g_i \) be a 1-entry in row \( r \) between \( z_i \) and \( z_{i+1} \), for \( i \in \{1, 2\} \).

For \( i \in \{1, 2, 3\} \), let \( \phi_i \) be an embedding of \( Q_1 \) into \( M \Delta f_i \) that maps \( e \) to \( f_i \). Consider the three 1-entries \( h_1 = \phi_1 (2, 1), h_2 = \phi_3 (3, 3), \) and \( h_3 = \phi_3 (3, 3) \). Let \( p_i \) and \( q_i \) be the row and the column containing \( h_i \).

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Note that $g_1$ cannot be to the left of column $q_2$, since then $h_1$, $g_1$, and $h_2$ would form an image of $Q_1$. It follows that $g_1$ is in the column $q_2$ or to the right of it, and consequently, we have $q_2 < q_3$. Moreover, if $p_3 > p_1$, then $h_1$, $g_2$, and $h_3$ form an image of $Q_1$, so $p_2$ is no larger than $p_1$ and hence $p_3 < p_2$. But then $h_2$, $g_2$, and $h_3$ form an image of $Q_2$, a contradiction.

By symmetry, the 1-entry $(1, 2)$ of $Q_2$ is row-bounding as well, and therefore $\mathcal{C}$ is row-bounded.

Let us now argue that $\mathcal{C}$ is column-bounded. It is enough to show that the 1-entry $e' = (2, 1)$ of $Q_1$ is column-bounding for $\mathcal{C}$, the rest follows from symmetry and from Corollary 3.9. Suppose that a matrix $M \in \mathcal{C}$ has a column $c$ with three 0-runs critical for $e'$. In particular, column $c$ contains a 0-entry $f'$ critical for $e'$ such that below $f'$, there are at least two 1-entries $g_1'$ and $g_2'$ in column $c$ of $M$. Suppose that $g_1'$ is above $g_2'$.

Let $\phi$ be an embedding of $Q_1$ into $M \Delta f'$ with $\phi(e') = f'$. Define $h'_2 = \phi(3, 3)$ and $h'_3 = \phi(1, 2)$. Let $r'$ be the row containing $h'_2$. If $g_1'$ is above row $r'$, then $g_1'$, $h'_1$ and $h'_2$ form a copy of $Q_1$, and if $g_1'$ is not above row $r'$, then $g_2'$ is below row $r'$ and $g_2'$, $h'_1$ and $h'_2$ form a copy of $Q_2$, a contradiction.

**Open problems.** A natural question arising from our results is to extend the dichotomy of Theorem 3.1 to non-principal classes of matrices.

**Problem 4.4.** For which sets $\mathcal{F}$ of patterns is the class $Av_{\leq} (\mathcal{F})$ row-bounded? Can we characterize such sets $\mathcal{F}$, at least when $\mathcal{F}$ is finite?

The notion of complexity we used in this paper is quite crude, in the sense that it only takes into account single lines of the corresponding matrix. It is reasonable to expect that matrices from a class of unbounded complexity possess nontrivial properties that could be revealed by a more refined approach.

**Problem 4.5.** Is there a refinement of our complexity notion that would provide nontrivial insight into the structure of critical matrices in unbounded classes?

Throughout the paper, we focused on distinguishing bounded classes from unbounded ones. We made no attempts to obtain tight estimates for the actual value of the complexity of a bounded class. This might be a line of research worth pursuing.

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**Figure 16:** Illustration of the row-boundedness (left) and column-boundedness (right) of $Av_{\leq} ((Q_1, Q_2))$. 

[Diagram showing row-boundedness and column-boundedness]
Problem 4.6. What is the highest possible value of \( r (Av_{\leq} (P)) \), over all row-bounding patterns \( P \) of a given size \( k \times \ell \)? For which pattern is this maximum attained?

By Observation 2.5, if \( P^+ \) is a pattern obtained by adding an empty row or column to the boundary of a pattern \( P \), then \( Av_{\leq} (P) \) has the same complexity as \( Av_{\leq} (P^+) \), and the avoiders of \( P^+ \) can be easily described in terms of the avoiders of \( P \).

It is, however, more challenging to deal with a pattern \( P^+ \) obtained by inserting an empty line into the interior of a pattern \( P \). Theorem 3.1 implies that \( P \) is bounding if and only if \( P^+ \) is bounding, but we are not aware of any direct proof of this.

Problem 4.7. Let \( P^+ \) be a pattern obtained from a pattern \( P \) by inserting a new empty row or column to an arbitrary position inside \( P \). Can we bound \( r (Av_{\leq} (P^+)) \) in terms of \( r (Av_{\leq} (P)) \)? Can we describe the avoiders of \( P^+ \) in terms of the avoiders of \( P \)? If \( \mathcal{F} \) is a set of patterns and \( \mathcal{F}^+ \) a set of patterns obtained by inserting empty rows and columns to the patterns in \( \mathcal{F} \), is it true that \( Av_{\leq} (\mathcal{F}^+) \) is bounded if and only if \( Av_{\leq} (\mathcal{F}) \) is?

Acknowledgements

This research was supported by project 16-01602Y of the Czech Science Foundation (GAČR), and by the Neuron Foundation for the Support of Science.

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