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WEAK TAIL CONDITIONS FOR LOCAL MARTINGALES[*]

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The following conditions are necessary and jointly sufficient for an arbitrary càdlàg local martingale to be a uniformly integrable martingale: (A) The weak tail of the supremum of its modulus is zero; (B) its jumps at the first-exit times from compact intervals converge to zero in L^1 on the events that those times are finite; and (C) its almost sure limit is an integrable random variable.

1. Introduction. Let $(\Omega, \mathscr{F}, \mathfrak{F}, \mathsf{P})$ be a filtered probability space with a right-continuous filtration $\mathfrak{F} = (\mathscr{F}_t)_{t\geq 0}$, and let $\mathscr{S}$ and $\mathscr{S}_{\mathsf{f}}$ denote the families of stopping times and finite-valued stopping times on $(\Omega, \mathscr{F}, \mathfrak{F}, \mathsf{P})$. Unless indicated otherwise, stochastic processes are defined on $(\Omega, \mathscr{F}, \mathfrak{F}, \mathsf{P})$ and are adapted to $\mathfrak{F}$, and all stochastic processes are assumed to be real-valued and càdlàg. The family of local martingales is denoted by $\mathscr{M}_{\mathsf{loc}}$, while $\mathscr{M}$ denotes the family of uniformly integrable martingales.[1] Similarly, $\mathscr{M}_{\mathsf{c,loc}}$ and $\mathscr{M}_{\mathsf{c}}$ denote the families of continuous local martingales and continuous uniformly integrable martingales, respectively. The strict inclusion $\mathscr{M} \subsetneq \mathscr{M}_{\mathsf{loc}}$ gives rise to the following problem.

PROBLEM 1. Given $M \in \mathscr{M}_{\mathsf{loc}}$, formulate necessary and sufficient conditions for determining whether $M \in \mathscr{M}$.

Since $M \in \mathscr{M}_{\mathsf{loc}}$ is a martingale if and only if $M^t := M_{t\wedge\cdot} \in \mathscr{M}$, for all $t \geq 0$, any solution to Problem 1 implicitly solves the following problem as well.

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[1]Our notion of local martingales corresponds with that of Jacod and Shiryaev (2003, Definitions I.1.33 and I.1.45), which implies that $\mathsf{E}(|M_0|) < \infty$, for all $M \in \mathscr{M}_{\mathsf{loc}}$. Several authors, including Protter (2005, Section I.6) and Revuz and Yor (1999, Definition IV.1.5), allow for the possibility that the initial component of a local martingale may be non-integrable. This additional generality would add a technical overhead to the results that follow, without offering any compensating advantages.

Problem 2. Given $M \in \mathscr{M}_{\mathsf{loc}}$, formulate necessary and sufficient conditions for determining whether M is a martingale.

Problems 1 and 2 have been the focus of a sustained research effort for over fifty years. Girsanov (1960) set the ball rolling, by enquiring about conditions for determining whether an exponential local martingale $\mathscr{E}(L) \in \mathscr{M}_{\mathsf{loc}}$ is a (uniformly integrable) martingale, for any given $L \in \mathscr{M}_{\mathsf{loc}}$. This restricted version of the problems above derives its importance from the widespread use of equivalent changes of probability measure in Mathematical Finance and Stochastic Control Theory, where exponential martingales play the role of density processes. Novikov (1972) famously demonstrated that $\mathscr{E}(L) \in \mathscr{M}_{\mathsf{c}}$ if

$$\mathsf{E}\Big(\mathrm{e}^{\frac{1}{2}\langle L\rangle_\infty}\Big) < \infty,$$

for all $L \in \mathscr{M}_{\mathsf{c,loc}}$, where $\langle L\rangle_\infty := \langle L\rangle_{\infty-}$, while Kazamaki (1977) demonstrated that $\mathscr{E}(L) \in \mathscr{M}_{\mathsf{c}}$ if $(\mathrm{e}^{L_t/2})_{t\geq 0}$ is a uniformly integrable submartingale. Alternative sufficient (and sometimes also necessary) characterisations of (uniformly integrable) exponential martingales were obtained by Lépingle and Mémin (1978a,b), Okada (1982), Kazamaki and Sekiguchi (1983), Engelbert and Schmidt (1984), Stummer (1993), Kallsen and Shiryaev (2002), Cheridito et al. (2005), Protter and Shimbo (2008), Blei and Engelbert (2009), Mayerhofer et al. (2011), Mijatović and Urusov (2012), Ruf (2013b), Larsson and Ruf (2014) and Blanchet and Ruf (2016).

Delbaen and Schachermayer (1995) and Sin (1998) reinforced the importance of Problem 2 for Mathematical Finance, by giving examples of models where discounted security prices are strict local martingales under a risk-neutral probability measure. The observation that fundamental no-arbitrage relationships, such as put-call parity, are violated in such models attracted a lot of interest, with models of this type subsequently interpreted as descriptions of asset price bubbles. Prominent contributions to this literature include Cox and Hobson (2005), Heston et al. (2007), Jarrow et al. (2007, 2010), Hulley (2010), Protter (2013), Ruf (2013a) and Carr et al. (2014). In this setting, a solution to Problem 2 allows one to distinguish between bubbles and non-bubbles.

Rao (1969) initiated an interesting approach to Problem 1 that focuses on the weak tails of the suprema of the moduli of local martingales, as well as the weak tails of their quadratic variations. He considered a continuous martingale $M = (M_t)_{t\geq 0}$ satisfying $\sup_{t\geq 0} \mathsf{E}(|M_t|) < \infty$, in which case Doob's martingale convergence theorem ensures that the almost sure limit $M_\infty := M_{\infty-}$ exists and satisfies $\mathsf{E}(|M_\infty|) < \infty$. Let

$$\tau_\lambda := \inf\{t \geq 0 \,|\, |M_t| > \lambda\} \tag{1.1}$$

denote the first exit-time from the compact interval $[-\lambda, \lambda]$, for all $\lambda \geq 0$. Since $M^{\tau_\lambda} := M_{\tau_\lambda \wedge \cdot}$ is a bounded martingale, and hence also a uniformly integrable martingale, for all $\lambda \geq 0$, it follows that

$$\mathsf{E}(M_0^{\tau_\lambda}) = \mathsf{E}(M_\infty^{\tau_\lambda}) = \lambda \mathsf{P}\Big(\sup_{t\geq 0} |M_t| > \lambda\Big) + \mathsf{E}\Big(\mathbf{1}_{\{\sup_{t\geq 0} |M_t| \leq \lambda\}} M_\infty\Big).$$

Finally, an application of the dominated convergence theorem yields

$$\lim_{\lambda\uparrow\infty} \lambda \mathsf{P}\Big(\sup_{t\geq 0} |M_t| > \lambda\Big) = \mathsf{E}(M_0) - \mathsf{E}(M_\infty),$$

whence $M \in \mathscr{M}$ if and only if $\lim_{\lambda\uparrow\infty} \lambda \mathsf{P}\big(\sup_{t\geq 0} |M_t| > \lambda\big) = 0$. Azema et al. (1980) derived this result by means of a similar argument. In addition, they showed that $M \in \mathscr{M}$ if and only if $\lim_{\lambda\uparrow\infty} \lambda \mathsf{P}\big(\langle M\rangle_\infty^{1/2} \geq \lambda\big) = 0$, where $\langle M\rangle_\infty := \langle M\rangle_{\infty-}$. Novikov (1981) independently obtained the same characterisations of uniformly integrable martingales, in the context of first-passage problems. Elworthy et al. (1997, 1999) and Takaoka (1999) extended the results above, to obtain weak tail characterisations of uniformly integrable martingales within the class of continuous local martingales, provided the processes in question satisfy certain integrability requirements. Further generalisations were obtained by Novikov (1997) and Liptser and Novikov (2006), while Kaji (2007, 2008, 2009) derived weak tail characterisations of uniformly integrable martingales within the class of locally square-integrable martingales. Once again, the processes must satisfy a variety of additional integrability conditions in order for the results to be applicable.

We contribute to the literature surveyed above by presenting three conditions that are shown to be necessary and jointly sufficient for determining whether an arbitrary local martingale is a uniformly integrable martingale. As opposed to previous characterisations of uniformly integrable martingales, which apply only to specific classes of local martingales, our conditions are universally applicable. As such, they represent the culmination of a research effort instigated by Girsanov (1960). In detail, we provide the following solution for Problem 1.

THEOREM 1.1. *Let $M \in \mathscr{M}_{\mathsf{loc}}$. Then $M \in \mathscr{M}$ if and only if the following* *three conditions hold simultaneously:*

$$\varliminf_{\lambda\uparrow\infty} \lambda \mathsf{P}\left(\sup_{t\geq 0} |M_t| > \lambda\right) = 0; \tag{A}$$

$$\lim_{\lambda\uparrow\infty} \mathsf{E}\left(\mathbf{1}_{\{\tau_\lambda<\infty\}}|\Delta M_{\tau_\lambda}|\right) = 0; \qquad \textit{and} \tag{B}$$

$$\mathsf{E}\left(\varliminf_{t\uparrow\infty} |M_t|\right) < \infty, \tag{C}$$

where $\Delta M \coloneqq M - M_-$ is the jump process associated with M.

Condition (A) generalises Rao's (1969) weak tail condition. Several studies recognise that the jumps of a local martingale $M \in \mathscr{M}_{\mathsf{loc}}$ must be constrained in some way, in order for it to be a uniformly integrable martingale (see e.g. Liptser and Novikov 2006 and Kaji 2008). Condition (B) does this by controlling jumps that increase $|M|$. Together, Conditions (A) and (B) ensure that $M_\infty \coloneqq M_{\infty-}$ exists and satisfies $M_\infty \in \mathbb{R}$ (see Lemma 2.2). When they are combined with Condition (C), it follows that $\mathsf{E}(|M_\infty|) < \infty$. Ruf (2015) showed that $M \in \mathscr{M}$ if and only if $\mathsf{E}(M_\tau) = \mathsf{E}(M_0)$, for all $\tau \in \mathscr{S}$, and Condition (C) holds. In the presence of Condition (C), the former criterion (which is too abstract to verify in practice) is thus equivalent to Conditions (A) and (B) together.

As mentioned previously, a solution for Problem 1 also provides a solution for Problem 2, since a local martingale is a martingale if and only if it is a uniformly integrable martingale when stopped at arbitrary deterministic times. Based on this observation, we obtain the following solution for Problem 2.

COROLLARY 1.2. *Let $M \in \mathscr{M}_{\mathsf{loc}}$. Then M is a martingale if and only if the following three conditions hold simultaneously:*

$$\varliminf_{\lambda\uparrow\infty} \lambda \mathsf{P}\left(\sup_{s\in[0,t]} |M_s| > \lambda\right) = 0; \tag{A$'$}$$

$$\lim_{\lambda\uparrow\infty} \mathsf{E}\left(\mathbf{1}_{\{\tau_\lambda\leq t\}}|\Delta M_{\tau_\lambda}|\right) = 0; \qquad \textit{and} \tag{B$'$}$$

$$\mathsf{E}(|M_t|) < \infty, \tag{C$'$}$$

for all $t \geq 0$.

The remainder of the article is structured as follows. We prove Theorem 1.1 in Section 2, after which Section 3 demonstrates the minimality of Conditions (A)–(C), by presenting three examples of local martingales that are not uniformly integrable martingales due to the selective failure of precisely one of those conditions.

2. The Proof of Theorem 1.1. In the lead-up to the proof of Theorem 1.1, we first explore some of the consequences of Conditions (A)–(C). To begin with, recall that a continuous local martingale that is stopped when first it leaves a compact interval is a bounded local martingale, and hence also a uniformly integrable martingale. The following lemma generalises this observation.

LEMMA 2.1. *Let $M \in \mathscr{M}_{\mathsf{loc}}$ satisfy Condition (B). Then $M^{\tau_\lambda} \in \mathscr{M}$, for all $\lambda \geq 0$.*

PROOF. Condition (B) guarantees the existence of a $\lambda_* \geq 0$, such that $\mathsf{E}\big(\mathbf{1}_{\{\tau_\lambda<\infty\}}|\Delta M_{\tau_\lambda}|\big) < \infty$, for all $\lambda \geq \lambda_*$, from which it follows that

$$\mathsf{E}\bigg(\sup_{t\geq 0}|M_t^{\tau_\lambda}|\bigg) \leq \mathsf{E}(|M_0|) + \lambda + \mathsf{E}\big(\mathbf{1}_{\{\tau_\lambda<\infty\}}|\Delta M_{\tau_\lambda}|\big) < \infty,$$

for all $\lambda \geq \lambda_*$. Given $\lambda \geq \lambda_*$, an application of the dominated convergence theorem then yields

$$\begin{aligned}\lim_{K\uparrow\infty}\sup_{\sigma\in\mathscr{S}_{\mathsf{f}}}\mathsf{E}\bigg(\mathbf{1}_{\{|M_\sigma^{\tau_\lambda}|\geq K\}}|M_\sigma^{\tau_\lambda}|\bigg) &\leq \lim_{K\uparrow\infty}\mathsf{E}\bigg(\mathbf{1}_{\{\sup_{t\geq 0}|M_t^{\tau_\lambda}|\geq K\}}\sup_{t\geq 0}|M_t^{\tau_\lambda}|\bigg)\\ &= 0,\end{aligned}$$

since $|M_\sigma^{\tau_\lambda}| \leq \sup_{t\geq 0}|M_t^{\tau_\lambda}|$, for all $\sigma \in \mathscr{S}_{\mathsf{f}}$. In other words, M^{τ_λ} is a local martingale belonging to class (D) (see e.g. Jacod and Shiryaev, 2003, Definition I.1.46), and is thus a uniformly integrable martingale (see e.g. Jacod and Shiryaev, 2003, Proposition I.1.47). On the other hand, if $\lambda \in [0, \lambda_*]$, then $\tau_\lambda \leq \tau_{\lambda_*}$, whence $M^{\tau_\lambda} = M^{\tau_{\lambda_*}\wedge\tau_\lambda} \in \mathscr{M}$, since $M^{\tau_{\lambda_*}} \in \mathscr{M}$ and the family of uniformly integrable martingales is stable under stopping. □

Next, we establish two useful facts about local martingales for which Conditions (A) and (B) hold, one of which is that such processes possess real-valued almost-sure limits.

LEMMA 2.2. *Let $M \in \mathscr{M}_{\mathsf{loc}}$ satisfy Conditions (A) and (B). Then*

$$\varliminf_{\lambda\uparrow\infty}\mathsf{E}\big(\mathbf{1}_{\{\tau_\lambda<\infty\}}|M_\infty^{\tau_\lambda}|\big) = 0.$$

Moreover, the almost sure limit $M_\infty := M_{\infty-}$ exists and satisfies $M_\infty \in \mathbb{R}$.

PROOF. Note that the almost sure limit $M_\infty^{\tau_\lambda} \coloneqq M_{\infty-}^{\tau_\lambda}$ exists and satisfies $M_\infty^{\tau_\lambda} \in \mathbb{R}$, for all $\lambda \geq 0$, as a result of Lemma 2.1. Now observe that

$$\begin{aligned}
&\lim_{\lambda\uparrow\infty} \mathsf{E}\Big(\mathbf{1}_{\{\tau_\lambda<\infty\}}|M_\infty^{\tau_\lambda}|\Big) \leq \lim_{\lambda\uparrow\infty} \mathsf{E}\Big(\mathbf{1}_{\{\tau_\lambda<\infty\}}(|M_0| + \lambda + |\Delta M_{\tau_\lambda}|)\Big) \\
&\leq \lim_{\lambda\uparrow\infty} \mathsf{E}\Big(\mathbf{1}_{\{\tau_\lambda<\infty\}}|M_0|\Big) + \lim_{\lambda\uparrow\infty} \lambda \mathsf{P}\left(\sup_{t\geq 0}|M_t| > \lambda\right) + \lim_{\lambda\uparrow\infty} \mathsf{E}\Big(\mathbf{1}_{\{\tau_\lambda<\infty\}}|\Delta M_{\tau_\lambda}|\Big) \\
&= 0,
\end{aligned}$$

by virtue of the dominated convergence theorem and a direct application of Conditions (A) and (B). Given $\lambda \geq 0$, it also follows that

$$\mathbf{1}_{\{\tau_\lambda=\infty\}} M_{\infty-} = \mathbf{1}_{\{\tau_\lambda=\infty\}} M_{\infty-}^{\tau_\lambda} = \mathbf{1}_{\{\tau_\lambda=\infty\}} M_\infty^{\tau_\lambda} \in \mathbb{R},$$

whence $\{M_{\infty-} \in \mathbb{R}\} \supseteq \{\tau_\lambda = \infty\}$. Consequently,

$$\mathsf{P}(M_{\infty-} \in \mathbb{R}) \geq \lim_{\lambda\uparrow\infty} \mathsf{P}(\tau_\lambda = \infty) = 1,$$

since Condition (A) implies that $\lim_{\lambda\uparrow\infty} \mathsf{P}(\tau_\lambda < \infty) = 0$. That is to say, the almost sure limit $M_\infty \coloneqq M_{\infty-}$ exists and satisfies $M_\infty \in \mathbb{R}$. □

Finally, we establish a convergence result that will be used in the proof of Theorem 1.1 below to show that Conditions (A)–(C) are sufficient for a local martingale to be a uniformly integrable martingale.

LEMMA 2.3. *Let $M \in \mathscr{M}_{\mathsf{loc}}$ satisfy Conditions* (A)–(C). *Then the almost sure limit $M_\infty \coloneqq M_{\infty-}$ exists and*

$$\lim_{\lambda\uparrow\infty} \mathsf{E}(|M_\infty^{\tau_\lambda} - M_\infty|) = 0.$$

PROOF. An application of the dominated convergence theorem gives

$$\mathsf{E}\Big(\lim_{\lambda\uparrow\infty} \mathbf{1}_{\{\tau_\lambda<\infty\}}\Big) = \lim_{\lambda\uparrow\infty} \mathsf{P}(\tau_\lambda < \infty) = 0,$$

by virtue of Condition (A), from which it follows that $\lim_{\lambda\uparrow\infty} \mathbf{1}_{\{\tau_\lambda<\infty\}} = 0$. Another application of the dominated convergence theorem then yields

$$\lim_{\lambda\uparrow\infty} \mathsf{E}\Big(\mathbf{1}_{\{\tau_\lambda<\infty\}}|M_\infty|\Big) = 0,$$

since Lemma 2.2 and Condition (C) ensure that $M_\infty \coloneqq M_{\infty-}$ exists and satisfies $\mathsf{E}(|M_\infty|) < \infty$. Finally, we observe that

$$\begin{aligned}
\lim_{\lambda\uparrow\infty} \mathsf{E}(|M_\infty^{\tau_\lambda} - M_\infty|) &= \lim_{\lambda\uparrow\infty} \mathsf{E}\Big(\mathbf{1}_{\{\tau_\lambda<\infty\}}|M_\infty^{\tau_\lambda} - M_\infty|\Big) \\
&\leq \lim_{\lambda\uparrow\infty} \mathsf{E}\Big(\mathbf{1}_{\{\tau_\lambda<\infty\}}|M_\infty^{\tau_\lambda}|\Big) + \lim_{\lambda\uparrow\infty} \mathsf{E}\Big(\mathbf{1}_{\{\tau_\lambda<\infty\}}|M_\infty|\Big) = 0,
\end{aligned}$$

by virtue of Lemma 2.2 and the previous argument. □

We now prove Theorem 1.1. The first part of the proof shows that every uniformly integrable martingale satisfies Conditions (A)–(C), while the second part uses Lemma 2.3 to demonstrate that any local martingale satisfying those three conditions is a uniformly integrable martingale.

PROOF OF THEOREM 1.1. ($\Rightarrow$) Suppose $M \in \mathscr{M}$, in which case Condition (C) holds immediately, since the almost sure limit $M_\infty := M_{\infty-}$ exists and satisfies $\mathsf{E}(|M_\infty|) < \infty$. Moreover, $|M|$ is a uniformly integrable submartingale, which implies that

$$\begin{aligned} \mathsf{E}(|M_\infty|) \geq \mathsf{E}(|M_{\tau_\lambda}|) &= \mathsf{E}\big(\mathbf{1}_{\{\tau_\lambda < \infty\}}|M_{\tau_\lambda}|\big) + \mathsf{E}\big(\mathbf{1}_{\{\tau_\lambda = \infty\}}|M_\infty|\big) \\ &\geq \lambda \mathsf{P}(\tau_\lambda < \infty) + \mathsf{E}\big(\mathbf{1}_{\{\tau_\lambda = \infty\}}|M_\infty|\big), \end{aligned} \tag{2.1}$$

for all $\lambda \geq 0$. Next, by applying the monotone convergence theorem, followed by Doob's maximal inequalities, we obtain

$$\begin{aligned} \mathsf{E}\Big(\lim_{\lambda\uparrow\infty} \mathbf{1}_{\{\tau_\lambda = \infty\}}\Big) = \lim_{\lambda\uparrow\infty} \mathsf{P}(\tau_\lambda = \infty) &= 1 - \lim_{\lambda\uparrow\infty} \mathsf{P}\bigg(\sup_{t\geq 0}|M_t| > \lambda\bigg) \\ &\geq 1 - \lim_{\lambda\uparrow\infty} \frac{\mathsf{E}(|M_\infty|)}{\lambda} = 1, \end{aligned} \tag{2.2}$$

from which $\lim_{\lambda\uparrow\infty} \mathbf{1}_{\{\tau_\lambda = \infty\}} = 1$ follows. Combining this with (2.1) gives

$$\begin{aligned} \lim_{\lambda\uparrow\infty} \lambda \mathsf{P}\bigg(\sup_{t\geq 0}|M_t| > \lambda\bigg) &= \lim_{\lambda\uparrow\infty} \lambda \mathsf{P}(\tau_\lambda < \infty) \\ &\leq \mathsf{E}(|M_\infty|) - \lim_{\lambda\uparrow\infty} \mathsf{E}\big(\mathbf{1}_{\{\tau_\lambda = \infty\}}|M_\infty|\big) = 0, \end{aligned}$$

by an application of the monotone convergence theorem. In other words, Condition (A) holds. Finally, the inequality $|\Delta M_{\tau_\lambda}| \leq 2|M_{\tau_\lambda}|$, for all $\lambda \geq 0$, together with the fact that $|M|$ is a uniformly integrable submartingale, yield

$$\begin{aligned} \lim_{\lambda\uparrow\infty} \mathsf{E}\big(\mathbf{1}_{\{\tau_\lambda < \infty\}}|\Delta M_{\tau_\lambda}|\big) \leq 2 \lim_{\lambda\uparrow\infty} \mathsf{E}\big(\mathbf{1}_{\{\tau_\lambda < \infty\}}|M_{\tau_\lambda}|\big) &\leq 2 \lim_{\lambda\uparrow\infty} \mathsf{E}\big(\mathbf{1}_{\{\tau_\lambda < \infty\}}|M_\infty|\big) \\ &= 2\mathsf{E}\Big(\lim_{\lambda\uparrow\infty} \mathbf{1}_{\{\tau_\lambda < \infty\}}|M_\infty|\Big) \\ &= 0, \end{aligned}$$

by virtue of the dominated convergence theorem, since (2.2) implies that $\lim_{\lambda\uparrow\infty} \mathbf{1}_{\{\tau_\lambda < \infty\}} = 0$, and $\mathsf{E}(|M_\infty|) < \infty$. That is to say, Condition (B) holds.

($\Leftarrow$) Suppose $M \in \mathscr{M}_{\mathsf{loc}}$ satisfies Conditions (A)–(C), in which case it follows from Lemma 2.3 that the almost sure limit $M_\infty := M_{\infty-}$ exists and satisfies

$$\lim_{n\uparrow\infty} \mathsf{E}\Big(\Big|M_\infty^{\tau_{\lambda_n}} - M_\infty\Big|\Big) = 0,$$

for some increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers satisfying $\lambda_n \uparrow \infty$. Now fix $t \geq 0$ and $A \in \mathscr{F}_t$, and define

$$A_m := A \cap \{M_t \geq 0\} \cap \{\tau_{\lambda_m} > t\},$$

for each $m \in \mathbb{N}$. It follows that

$$\lim_{n\uparrow\infty} \mathsf{E}\Big(\mathbf{1}_{A_m}\Big|M_\infty^{\tau_{\lambda_n}} - M_\infty\Big|\Big) = 0,$$

for each $m \in \mathbb{N}$, whence

$$\begin{aligned}\mathsf{E}(\mathbf{1}_{A_m} M_\infty) = \lim_{n\uparrow\infty} \mathsf{E}\Big(\mathbf{1}_{A_m} M_\infty^{\tau_{\lambda_n}}\Big) = \lim_{n\uparrow\infty} \mathsf{E}\Big(\mathbf{1}_{A_m} M_t^{\tau_{\lambda_n}}\Big) &= \lim_{n\uparrow\infty} \mathsf{E}(\mathbf{1}_{A_m} M_t) \\ &= \mathsf{E}(\mathbf{1}_{A_m} M_t),\end{aligned}$$

since $M^{\tau_{\lambda_n}} \in \mathscr{M}$, for each $n \in \mathbb{N}$, as a consequence of Lemma 2.1, and $\mathbf{1}_{A_m} M_t^{\tau_{\lambda_n}} = \mathbf{1}_{A_m} M_t$, for each $n \geq m$, by the construction of A_m. Combining this with the fact that $\lim_{m\uparrow\infty} \tau_{\lambda_m} = \infty$ gives

$$\begin{aligned}\mathsf{E}\Big(\mathbf{1}_{A\cap\{M_t\geq 0\}} M_\infty\Big) = \lim_{m\uparrow\infty} \mathsf{E}(\mathbf{1}_{A_m} M_\infty) &= \lim_{m\uparrow\infty} \mathsf{E}(\mathbf{1}_{A_m} M_t) \\ &= \mathsf{E}\Big(\mathbf{1}_{A\cap\{M_t\geq 0\}} M_t\Big),\end{aligned}$$

where the first equality follows from the dominated convergence theorem, since Condition (C) implies that $\mathsf{E}(|M_\infty|) < \infty$, while the second equality follows from the monotone convergence theorem. A similar argument reveals that

$$\mathsf{E}\Big(\mathbf{1}_{A\cap\{M_t<0\}} M_\infty\Big) = \mathsf{E}\Big(M_t \mathbf{1}_{A\cap\{M_t<0\}}\Big).$$

Consequently, $\mathsf{E}(\mathbf{1}_A M_\infty) = \mathsf{E}(\mathbf{1}_A M_t)$, from which we may conclude that $M \in \mathscr{M}$, since $t \geq 0$ and $A \in \mathscr{F}_t$ were chosen arbitrarily. $\square$

3. Three Examples. In this section we construct three examples of local martingales for which precisely one of Conditions (A)–(C) fails (a different one in each case), while the other two hold. In each case, Theorem 1.1 legislates that the process in question cannot be a uniformly integrable martingale. This establishes the minimality of Conditions (A)–(C).

The first example considers a well-known family of continuous local martingales, namely the family of non-negative time-homogeneous regular diffusions in natural scale. Although such processes satisfy Conditions (B) and (C), they cannot be uniformly integrable martingales, since they do not satisfy Condition (A).

EXAMPLE 3.1 (Condition (A) fails). Let $X = (X_t)_{t\geq 0}$ be a non-negative time-homogeneous regular scalar diffusion in natural scale, with state-space $[0,\infty)$ or $(0,\infty)$, depending on its behaviour at the origin. Since such a process is continuous, it trivially satisfies Condition (B). Being in natural scale means that the scale function for X is given by $\mathfrak{s}(x) := x$, for all $x > 0$. This ensures that X is a non-negative P_x–local martingale, for all $x > 0$, and consequently also a non-negative P_x–supermartingale. As a result, it satisfies Condition (C). The fact that X is a non-negative supermartingale imposes constraints on its behaviour at the origin. In particular, the origin is either an absorbing boundary or a natural boundary. In the former case the state space of X is $[0,\infty)$, while it is $(0,\infty)$ in the latter case. Either way, we observe that

$$\mathsf{P}_x\left(\sup_{t\geq 0} X_t > \lambda\right) = \mathsf{P}_x(\tau_\lambda < \infty) = \lim_{l\downarrow 0} \mathsf{P}_x(\tau_\lambda < \tau_l) = \lim_{l\downarrow 0} \frac{\mathfrak{s}(x) - \mathfrak{s}(l)}{\mathfrak{s}(\lambda) - \mathfrak{s}(l)} = \frac{x}{\lambda},$$

for all $x > 0$ and all $\lambda \geq x$, where P_x is the probability measure under which $X_0 = x$.[2] Consequently, we obtain

$$\lim_{\lambda\uparrow\infty} \lambda \mathsf{P}_x\left(\sup_{t\geq 0} X_t > \lambda\right) = x > 0,$$

for all $x > 0$. That is to say, X is not a uniformly integrable martingale, due to the failure of Condition (A). □

Although the example above shows that non-negative time-homogeneous diffusions in natural scale cannot satisfy Condition (A), they can satisfy Condition (A′). In other words, non-negative time-homogeneous diffusions in natural scale can be (non-uniformly integrable) martingales, by virtue of Corollary 1.2. Kotani (2006) and Hulley and Platen (2011) derived purely analytical necessary and sufficient conditions under which such processes are martingales. Those conditions are naturally equivalent to Condition (A′), as demonstrated formally by Hulley and Platen (2011).

[2]There is a slight abuse of notation here, in the sense that τ_λ should be interpreted as the first-exit time (1.1) with M replaced by X, for any $\lambda \geq 0$.

The next example constructs a non-negative pure-jump martingale that is not a uniformly integrable martingale, since it satisfies Conditions (A) and (C), but not Condition (B). Starting with an initial value of one, the process jumps only at integer-valued times, while remaining constant over the intervening intervals. Negative jumps take it to zero, where it is absorbed, while the sizes of successive positive jumps grow combinatorially. To ensure that the resulting process is a martingale, the probabilities of positive jumps decrease very quickly.

Example 3.2 (Condition (B) fails). Suppose $(\Omega, \mathscr{F}, \mathsf{P})$ supports a sequence $(Y_n)_{n\in\mathbb{Z}_+}$ of positive random variables, with $Y_0 = 1$ and

$$\mathsf{P}(Y_n \in \mathrm{d}y) := \frac{(n+1)!}{n}\mathbf{1}_{(n!,(n+1)!]}(y)\frac{1}{y^2}\,\mathrm{d}y, \tag{3.1}$$

for all $y \in \mathbb{R}_+$ and each $n \in \mathbb{N}$, as well as a sequence $(\xi_n)_{n\in\mathbb{Z}_+}$ of Bernoulli random variables, with $\xi_0 = 1$ and

$$\mathsf{P}(\xi_n = 1 \,|\, \xi_0, \cdots, \xi_{n-1}, Y_0, \cdots, Y_{n-1}) := \frac{Y_{n-1}}{\mathsf{E}(Y_n)}\prod_{i=0}^{n-1}\xi_i, \tag{3.2}$$

for each $n \in \mathbb{N}$. Furthermore, we assume that Y_n is independent of $\xi_0, \cdots, \xi_n$ and $Y_0, \cdots, Y_{n-1}$, for each $n \in \mathbb{N}$. The filtration $\mathfrak{F} = (\mathscr{F}_t)_{t\geq 0}$ is determined by $\mathscr{F}_t := \sigma(\xi_n, Y_n \,|\, 0 \leq n \leq \lfloor t\rfloor)$, for all $t \geq 0$, while the process $M = (M_t)_{t\geq 0}$ is specified by

$$M_t := Y_{\lfloor t\rfloor}\prod_{i=0}^{\lfloor t\rfloor}\xi_i,$$

for all $t \geq 0$. It follows that M is adapted to $\mathfrak{F}$, while the boundedness of Y_n, for each $n \in \mathbb{Z}_+$, ensures that $\mathsf{E}(|M_t|) < \infty$, for all $t \geq 0$. Also note that (3.2) implies that $\prod_{i=0}^{n}\xi_i = \xi_n$, for each $n \in \mathbb{Z}_+$, so that we may write $M_t = \xi_{\lfloor t\rfloor}Y_{\lfloor t\rfloor}$, for all $t \geq 0$. This yields the useful identities

$$\mathbf{1}_{\{M_n>0\}} = \mathbf{1}_{\{\xi_n=1\}} = \xi_n, \tag{3.3}$$

for each $n \in \mathbb{Z}_+$. It also allows us to rewrite (3.2) as follows:

$$\mathsf{P}(\xi_n = 1 \,|\, \mathscr{F}_{n-1}) = \frac{M_{n-1}}{\mathsf{E}(Y_n)}, \tag{3.4}$$

for each $n \in \mathbb{N}$. It is now easy to see that M is a martingale, since

$$\begin{aligned}
\mathsf{E}(M_n \,|\, \mathscr{F}_{n-1}) = \mathsf{E}(\xi_n Y_n \,|\, \mathscr{F}_{n-1}) &= \mathsf{E}\Big(\xi_n\mathsf{E}\Big(Y_n \,\Big|\, \sigma(\xi_n) \vee \mathscr{F}_{n-1}\Big) \,\Big|\, \mathscr{F}_{n-1}\Big)\\
&= \mathsf{E}(\xi_n \,|\, \mathscr{F}_{n-1})\mathsf{E}(Y_n)\\
&= \mathsf{P}(\xi_n = 1 \,|\, \mathscr{F}_{n-1})\mathsf{E}(Y_n) = M_{n-1},
\end{aligned}$$

for each $n \in \mathbb{N}$, by virtue of (3.3), (3.4), and the fact that Y_n is independent of $\sigma(\xi_n) \vee \mathscr{F}_{n-1}$. Moreover, since M is non-negative, Condition (C) holds a fortiori. Next, we compute the probability that M is strictly positive at any integer-valued time as follows:

$$\mathsf{P}(M_n > 0) = \mathsf{P}(\xi_n = 1) = \mathsf{E}\big(\mathsf{P}(\xi_n = 1 \,|\, \mathscr{F}_{n-1})\big) = \mathsf{E}\left(\frac{M_{n-1}}{\mathsf{E}(Y_n)}\right) = \frac{1}{\mathsf{E}(Y_n)},$$

for each $n \in \mathbb{N}$, with the help of (3.3), (3.4), and the fact that M is a martingale with $M_0 = 1$. Consequently, given $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\mathsf{P}(M_n > \lambda) = \mathsf{P}(\xi_n Y_n > \lambda) = \mathsf{P}(\xi_n = 1, Y_n > \lambda) &= \mathsf{P}(\xi_n = 1)\mathsf{P}(Y_n > \lambda)\\ &= \frac{\mathsf{P}(Y_n > \lambda)}{\mathsf{E}(Y_n)},\end{aligned}$$

for all $\lambda \geq 0$, since Y_n is independent of ξ_n. Now, given $\lambda > 1$, let $n \in \mathbb{N}$ be the unique positive integer such that $n! < \lambda \leq (n+1)!$. In that case, the previous two identities, together with (3.1), give

$$\begin{aligned}
&\lambda \mathsf{P}\left(\sup_{t\geq 0} |M_t| > \lambda\right) \leq \lambda\big(\mathsf{P}(M_n > \lambda) + \mathsf{P}(M_{n+1} > 0)\big)\\
&= \lambda\left(\frac{\mathsf{P}(Y_n > \lambda)}{\mathsf{E}(Y_n)} + \frac{1}{\mathsf{E}(Y_{n+1})}\right)\\
&\leq \frac{\lambda \mathsf{P}(Y_n > \lambda)}{\mathsf{E}(Y_n)} + \frac{(n+1)!}{\mathsf{E}(Y_{n+1})}\\
&= \lambda\left(\frac{(n+1)!}{n}\int_{\lambda}^{(n+1)!} \frac{1}{y^2}\,\mathrm{d}y\right)\left(\frac{(n+1)!}{n}\int_{n!}^{(n+1)!}\frac{1}{y}\,\mathrm{d}y\right)^{-1}\\
&\qquad\qquad + (n+1)!\left(\frac{(n+2)!}{n+1}\int_{(n+1)!}^{(n+2)!}\frac{1}{y}\,\mathrm{d}y\right)^{-1}\\
&\leq \lambda\left(\frac{(n+1)!}{n}\frac{1}{\lambda}\right)\left(\frac{(n+1)!}{n}\ln(n+1)\right)^{-1} + (n+1)!\left(\frac{(n+2)!}{n+1}\ln(n+2)\right)^{-1}\\
&= \frac{1}{\ln(n+1)} + \frac{n+1}{(n+2)\ln(n+2)} < \frac{2}{\ln(n+1)},
\end{aligned}$$

by virtue of the inclusion $\{\sup_{t\geq 0} M_t > \lambda\} \subseteq \{M_n > \lambda\} \cup \{M_{n+1} > 0\}$. Consequently,

$$\lim_{\lambda\uparrow\infty} \lambda \mathsf{P}\left(\sup_{t\geq 0} |M_t| > \lambda\right) \leq \lim_{n\uparrow\infty} \frac{2}{\ln(n+1)} = 0,$$

which establishes that M satisfies Condition (A). Finally, given $n \in \mathbb{N}$, we use the identities $\xi_{n+1}^2 = \xi_{n+1}$ and $\xi_{n+1}\xi_n = \xi_{n+1}\prod_{i=0}^{n}\xi_i = \prod_{i=0}^{n+1}\xi_i = \xi_{n+1}$ to get

$$\begin{aligned}
\mathsf{E}\big(\mathbf{1}_{\{\tau_{n!}<\infty\}}|\Delta M_{\tau_{n!}}|\big) &= \mathsf{E}\big(\mathbf{1}_{\{M_n>0\}}\Delta M_n\big) = \mathsf{E}(\xi_n \Delta M_n) \\
&= \mathsf{E}\big(\xi_n(\xi_n Y_n - \xi_{n-1}Y_{n-1})\big) = \mathsf{E}\big(\xi_n(Y_n - Y_{n-1})\big) \\
&= \mathsf{E}(M_n) - \mathsf{E}\big(\mathsf{P}(\xi_n = 1 \,|\, \mathscr{F}_{n-1})Y_{n-1}\big) = 1 - \mathsf{E}\left(\frac{M_{n-1}}{\mathsf{E}(Y_n)}Y_{n-1}\right) \\
&\geq 1 - \mathsf{E}\left(\frac{M_{n-1}}{\mathsf{E}(Y_n)}(n-1)!\right) = 1 - \frac{(n-1)!}{\mathsf{E}(Y_n)} \\
&= 1 - (n-1)! \times \left(\frac{(n+1)!}{n}\int_{n!}^{(n+1)!}\frac{1}{y}\,\mathrm{d}y\right)^{-1} \\
&= 1 - (n-1)! \times \left(\frac{(n+1)!}{n}\ln(n+1)\right)^{-1} = 1 - \frac{1}{(n+1)\ln(n+1)},
\end{aligned}$$

with the help of (3.1), (3.3) and (3.4), and the fact that M is a martingale. Hence,

$$\varlimsup_{\lambda\uparrow\infty}\mathsf{E}\big(\mathbf{1}_{\{\tau_\lambda<\infty\}}|\Delta M_{\tau_\lambda}|\big) \geq 1 - \lim_{n\uparrow\infty}\frac{1}{(n+1)\ln(n+1)} = 1,$$

from which we deduce that M does not satisfy Condition (B). So M is a non-negative martingale that satisfies Conditions (A) and (C), but not Condition (B), and is thus not a uniformly integrable martingale. □

Finally, we present an example of a continuous local martingale that satisfies Conditions (A) and (B), but not Condition (C). This elaborates on an example due to Azema et al. (1980).

EXAMPLE 3.3 (Condition (C) fails). Let B be a scalar Brownian motion on $(\Omega, \mathscr{F}, \mathfrak{F}, \mathsf{P})$, and suppose the sigma-algebra $\mathscr{F}_0$ accommodates a discrete random variable Y, whose distribution is determined by

$$\mathsf{P}(Y = n) \coloneqq \frac{c}{n^2\ln(n+2)},$$

for each $n \in \mathbb{N}$, where

$$c \coloneqq \left(\sum_{i=1}^{\infty}\frac{1}{i^2\ln(i+2)}\right)^{-1}.$$

Now let

$$\rho \coloneqq \inf\{t \geq 0 \,|\, |B_t| = Y\}$$

denote the first hitting time of Y by $|B|$, and note that $\rho < \infty$. The definition of Y ensures that

$$n\mathsf{P}(Y \geq n) = n \sum_{j=n}^{\infty} \frac{c}{j^2 \ln(j+2)} \leq \frac{cn}{\ln(n+2)} \sum_{j=n}^{\infty} \frac{1}{j^2} \leq \frac{cn}{\ln(n+2)} \int_{n-1}^{\infty} \frac{1}{x^2}\,\mathrm{d}x = \frac{cn}{(n-1)\ln(n+2)} \leq \frac{2c}{\ln(n+2)},$$

for each $n \in \mathbb{N}$. The martingale $M := B^\rho$ then satisfies Condition (A), since

$$\lim_{\lambda\uparrow\infty} \lambda\mathsf{P}\left(\sup_{t\geq 0} |M_t| > \lambda\right) = \lim_{\lambda\uparrow\infty} \lambda\mathsf{P}\left(\sup_{t\geq 0} |B_t^\rho| > \lambda\right) = \lim_{\lambda\uparrow\infty} \lambda\mathsf{P}(|B_\rho| > \lambda) = \lim_{n\uparrow\infty} n\mathsf{P}(Y \geq n) = 0.$$

Moreover, M satisfies Condition (B), by virtue of its continuity. Based on these observations, Lemma 2.2 ensures that $M_\infty := M_{\infty-}$ exists and satisfies $M_\infty = B_\rho = \pm Y$. However,

$$\mathsf{E}(|M_\infty|) = \mathsf{E}(Y) = \sum_{n=1}^{\infty} \frac{c}{n \ln(n+2)} = \infty$$

implies that M does not satisfy Condition (C), which implies that it cannot be a uniformly integrable martingale. □

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