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An Investment Model with Switching Costs and the Option to Abandon

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Abstract

We develop a complete analysis of a general entry-exit-scrapping model. In particular, we consider an investment project that operates within a random environment and yields a payoff rate that is a function of a stochastic economic indicator such as the price of or the demand for the project's output commodity. We assume that the investment project can operate in two modes, an "open" one and a "closed" one. The transitions from one operating mode to the other one are costly and immediate, and form a sequence of decisions made by the project's management. We also assume that the project can be permanently abandoned at a discretionary time and at a constant sunk cost. The objective of the project's management is to maximise the expected discounted payoff resulting from the project's management over all switching and abandonment strategies. We derive the explicit solution to this stochastic control problem that involves impulse control as well as discretionary stopping. It turns out that this has a rather rich structure and the optimal strategy can take eight qualitatively different forms, depending on the problems data.

Keywords. Decision analysis, project management, real options, entry-exit-scrapping decisions, optimal switching with discretionary stopping.

1 Introduction

Optimal sequential switching is an area of stochastic control that emerged from financial economics in the context of real options (see Dixit and Pindyck [5] and Trigeorgis [28]). Its numerous applications include the optimal scheduling of production in a real asset such

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as a power plant that can operate in distinct modes, say "open" and "closed", as well as the optimal timing of sequentially investing and disinvesting, e.g., in a given stock. The references Bayraktar and Egami [1], Brekke and Øksendal [2], Carmona and Ludkovski [4], Djehiche, Hamadène and Popier [7], Duckworth and Zervos [8], El Asri [9], El Asri and Hamadène [10], Elie and Kharroubi [11], Gassiat, Kharroubi and Pham [12], Guo and Tomecek [13], Hamadène and Jeanblanc [14], Hamadène and Zhang [15], Johnson and Zervos [17], Korn, Melnyk and Seifried [19], Lumley and Zervos [20], Ly Vath and Pham [21], Martyr [22], Pham [23], Pham, Ly Vath and Zhou [24], René, Campi, Langrené and Pham [25], Song, Yin and Zhang [26], Tang and Yong [27], Tsekrekos and Yannacopoulos [29], Zhang and Zhang [31], and Zhang [32] provide an alphabetically ordered list of important contributions in the area.

In this paper, we derive the complete solution to a problem of optimal sequential switching that incorporates an additional permanent abandonment option. The model that we study goes back to Brennan and Schwartz [3] who considered a firm's decisions to operate, mothball or abandon a mine producing a natural resource. A special case of the model is extensively analysed in Dixit and Pindyck [5, Section 7.2] using heuristic arguments and numerical examples in the context of several real options applications.

To fix ideas, we consider an investment project that operates within a random environment and yields a payoff rate that is a function of a stochastic economic indicator such as the price of or the demand for the project's output commodity. We model this economic indicator by the geometric Brownian motion given by

$$dX_t = bX_t\, dt + \sqrt{2}\sigma X_t\, dW_t, \quad X_0 = x > 0, \tag{1}$$

where b and $\sigma \neq 0$ are given constants and W is a standard Brownian motion. We assume that the investment project can operate in two modes, an "open" one and a "closed" one. The transitions from one operating mode to the other one are immediate and form a sequence of decisions made by the project's management. We use a process Z with values in $\{0, 1\}$ to model such a sequence of decisions. In particular, we assume that $Z_t = 1$ (resp., $Z_t = 0$) if the project is "open" (resp., "closed") at time t. We also denote by $z \in \{0, 1\}$ the project's mode at time 0, so that $Z_0 = z$. The stopping times at which the jumps of Z occur are the intervention times at which the project's operating mode is changed. We assume that the project can be permanently abandoned at a stopping time τ, which is an additional decision variable. With each admissible strategy (Z, τ), we associate the performance criterion

$$\begin{aligned} J_{z,x}(Z, \tau) = \mathbb{E}\Bigg[\int_0^\tau e^{-rs} h(X_s) Z_s\, ds \\ - \sum_{j=1}^\infty e^{-rT_j^1} K_1 \mathbf{1}_{\{T_j^1 \leq \tau\}} - \sum_{j=1}^\infty e^{-rT_j^0} K_0 \mathbf{1}_{\{T_j^0 \leq \tau\}} - e^{-r\tau} K \Bigg], \end{aligned} \tag{2}$$

where (T_j^1) (resp., (T_j^0)) is the sequence of times at which Z jumps from 0 to 1 (resp., from 1 to 0). Here, $h :]0, \infty[\to \mathbb{R}$ models the running payoff resulting from the investment project while this is in its "open" operating mode.[1] The constants $K_1 > 0$ and $K_0 > 0$ are the costs resulting from "switching" the project from its "closed" mode to its "open" one and vice versa, whereas $K \in \mathbb{R}$ is the cost resulting from the decision to permanently abandon it. Note that we allow for K to be negative, which corresponds to a situation where capital can be recovered at abandonment.[2] Also, on the event $\{T_j^\ell = \tau\}$, $\ell = 1, 0$, a cost of $K_\ell + K$ is incurred at time T_j^ℓ, which corresponds to the possibility that the project's operating mode can be switched just before the project is permanently abandoned.[3] The objective is to maximise the performance criterion $J_{z,x}$ over the set Π_z of all admissible strategies (Z, τ). Accordingly, we define the value function v by

$$v(z, x) = \sup_{(Z,\tau)\in\Pi_z} J_{z,x}(Z, \tau), \quad \text{for } (z, x) \in \{0, 1\} \times]0, \infty[. \tag{3}$$

The related special case that arises if $X = W$, $h(x) = x$ and $K > 0$ was solved by Zervos [30]. Although the analysis of this related problem has shed some light on the qualitative nature of the optimal strategy, its impact on the real options theory has been limited by the rather unrealistic assumptions that the underlying economic indicator is a standard Brownian motion rather than a geometric Brownian motion and that the running payoff function h is linear. The existence of an optimal strategy in a more general context with finite time horizon was established by Djehiche and Hamadène [6] using systems of Snell envelopes and viscosity solutions. Despite its fundamental mathematical importance, this result is of rather limited practical use because it does not provide a qualitative characterisation of the optimal strategy or a genuinely practical way of implementing it.

We derive the complete solution to the problem that we study in an explicit form by solving its Hamilton-Jacobi-Bellman (HJB) equation that takes the form of a pair of coupled quasi-variational inequalities. In particular, we identify the five regions that partition the state space $\{0, 1\} \times]0, \infty[$ and characterise the optimal strategy, namely, the "production" region, the "waiting" region, the "switch in" region, the "switch out" region and the "abandonment" region. It turns out that the qualitative nature of the problem's solution is surprisingly rich and can take eight different forms, depending on the problem data. We illustrate the results derived using the choice

$$h(x) = c + x^\vartheta, \quad x > 0, \tag{4}$$

for some constants $c \in \mathbb{R}$, $\vartheta \in]0, n[$[4], and some related numerical calculations (see Examples 1–9).

[1] Using a trivial re-parametrisation, we can allow for the project to yield a constant payoff rate while it is in its "closed" mode (see Remark 1).

[2] For the same reason, it would make sense in some economic applications to allow for at least K_0 to be negative, as long as $K_1 + K_0 > 0$. However, such a relaxation would add most significant complexity and would result in a substantially longer paper.

[3] Although this setting is convenient for the problem's formulation, switching followed by immediate abandonment is never optimal due to the strict positivity of K_ℓ, $\ell = 1, 0$.

[4] The inequality $\vartheta < n$, where n is defined by (15), is essential for the value function to be finite.

The value that may be added by waiting before implementing a certain investment decision is a central feature of the real options theory. In some of the cases that arise in our analysis, *value may be added by waiting before choosing one of two investment actions of a qualitatively different nature, one partially reversible and one totally irreversible*. To the best of our knowledge, such a possibility has not been appreciated in the real options literature. For instance, in Case II.3 in Section 4.2 (see also Figure 6), the part of the "production" region identified by the set $\{1\} \times]\delta, \gamma[$ separates the "abandonment" region from the "switch out" region. In this case, if the initial condition of the state process is in this part of the state space, then it is optimal to take no action before committing to *either* enter a perpetual cycle of operating the investment project by optimally switching it between its two modes *or* permanently abandoning the project, depending on whether the economic indicator X first rises to the level γ or first drops to the level δ. Furthermore, the investment project has infinite lifetime if the initial condition of the state process is in $\{1\} \times [\gamma, \infty[\cup \{0\} \times]0, \infty[$ and finite lifetime with strictly positive probability otherwise. The situation becomes more dramatic in Case III.2 in Section 4.3 (see also Figure 8). In this case, the part of the "production" region identified by the set $\{1\} \times]\delta, \gamma[$ separates the "abandonment" region from the "switch out" region, while the whole "waiting" region $\{0\} \times]\zeta, \alpha[$ separates the "abandonment" region from the "switch in" region. If the initial condition of the state process is in this part of the "production" region (resp., in the "waiting" region), then it is optimal to take no action before committing to *either* switch the investment project to its "closed" mode *or* permanently abandon it (resp., *either* switch the investment project to its "open" mode *or* permanently abandon it). Contrary to the previous case, the investment project's lifetime is always finite with strictly positive probability, and with probability 1 if $\mu - \sigma^2 \leq 0$.

The paper is organised as follows. We formulate the stochastic optimisation problem that we solve in Section 2. In Section 3, we consider the problem's HJB equation, we discuss how it characterises the five regions that determine the optimal strategy and we recall some related implications of the assumptions we make. We present the explicit solution to the stochastic control problem in Section 4. Here, we organise the eight cases that arise in three groups based on the analytical affinity of the different cases. The proofs of Lemmas 1-8 can be found in the complete version of the paper that is available online (`https://arxiv.org/abs/1607.08406`).

2 Problem formulation

We build the model that we study on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions and supporting a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{Z}$ the family of all $(\mathcal{F}_t)$-adapted finite variation càglàd processes Z with values in $\{0, 1\}$, and by $\mathcal{S}$ the set of all $(\mathcal{F}_t)$-stopping times.

As we have discussed in the introduction, we consider an investment project that operates within a random environment and yields a payoff rate that is a function of a stochastic economic indicator that is modelled by the geometric Brownian motion given by (1). We assume that the investment project can operate in two modes, an "open" one and a "closed" one. We use a process $Z \in \mathcal{Z}$ to model such a sequence of decisions: $Z_t = 1$ (resp., $Z_t = 0$) if the project is "open" (resp., "closed") at time t. We also denote by $z \in \{0, 1\}$ the project's mode at time 0, so that $Z_0 = z$. The stopping times at which the jumps of Z occur are the intervention times at which the project's operating mode is changed. If we define recursively

$$
\begin{gathered}
T_1^1 = \inf \{t \geq 0 \mid \ \Delta Z_t = 1\}, \quad T_1^0 = \inf \{t \geq 0 \mid \ \Delta Z_t = -1\}, \\
T_{j+1}^1 = \inf \left\{t > T_j^1 \mid \ \Delta Z_t = 1\right\} \quad \text{and} \quad T_{j+1}^0 = \inf \left\{t > T_j^0 \mid \ \Delta Z_t = -1\right\}, \quad \text{for } j \geq 1,
\end{gathered}
$$

where $\Delta Z_t = Z_{t+} - Z_t$ and we adopt the usual convention that $\inf \emptyset = \infty$, then T_j^1 (resp., T_j^0) are the $(\mathcal{F}_t)$-stopping times at which the project is switched from "closed" to "open" (resp., from "open" to "closed"). We also assume that the project can be permanently abandoned at an $(\mathcal{F}_t)$-stopping time τ. We define the set of all admissible strategies to be

$$
\Pi_z = \left\{(Z, \tau) \mid \ Z \in \mathcal{Z}, \ Z_0 = z, \ \text{and } \tau \in \mathcal{S}\right\}.
$$

With each admissible strategy $(Z, \tau) \in \Pi_z$, we associate the performance criterion given by (2). The objective is to maximise the performance criterion $J_{z,x}$ over Π_z. Accordingly, we define the value function v by (3).

For the resulting optimisation problem to be well-posed in the sense that there are no integrability problems and there are no admissible strategies with payoff equal to ∞, we make the following assumption.

Assumption 1 The running payoff function $h :]0, \infty[\to \mathbb{R}$ is right-continuous and increasing, $\lim_{x\to\infty} h(x) = \infty$, and

$$
\mathbb{E}\left[\int_0^\infty e^{-rt} \left|h(X_t)\right| dt\right] < \infty \tag{5}
$$

for every initial condition $x > 0$. Furthermore, $K_1, K_0 > 0$ and $K \in \mathbb{R}$. ∎

Remark 1 To simplify the exposition, we have assumed that the investment project yields zero payoff while it is in its "closed" mode. In view of the calculation

$$
\begin{aligned}
J_{z,x}(Z, \tau) = \mathbb{E}\bigg[& \int_0^\tau e^{-rs} \left[\bar{h}(X_s) Z_s - C(1 - Z_s)\right] ds \\
& - K_1 \sum_{j=1}^\infty e^{-rT_j^1} \mathbf{1}_{\{T_j^1 \leq \tau\}} - K_0 \sum_{j=1}^\infty e^{-rT_j^0} \mathbf{1}_{\{T_j^0 \leq \tau\}} - e^{-r\tau} \bar{K}\bigg] + \frac{C}{r},
\end{aligned}
$$

where C is a constant, $\bar{h} = h - C$ and $\bar{K} = K + \frac{C}{r}$, we can see that allowing for a constant payoff rate while the project is in its "closed" mode can be accommodated trivially in the model that we study. ∎

3 The Hamilton-Jacobi-Bellman (HJB) equation

In view of standard stochastic control theory that has been developed and used in references we have discussed in the introduction, we expect that the value function of the problem we study is given by

$$v(1,\cdot) = w_1 \quad \text{and} \quad v(0,\cdot) = w_0, \tag{6}$$

where the functions $w_1, w_0 :]0,\infty[\to \mathbb{R}$ satisfy the coupled quasi-variational inequalities

$$\max\Big\{\sigma^2 x^2 w_1''(x) + bxw_1'(x) - rw_1(x) + h(x),\ w_0(x) - w_1(x) - K_0,\ -w_1(x) - K\Big\} = 0, \tag{7}$$

$$\max\Big\{\sigma^2 x^2 w_0''(x) + bxw_0'(x) - rw_0(x),\ w_1(x) - w_0(x) - K_1,\ -w_0(x) - K\Big\} = 0, \tag{8}$$

as well as appropriate growth conditions (see Zervos [30, Theorem 1] for a general verification theorem). In view of the heuristics explaining the structure of this HJB equation, the state space $\{0,1\} \times]0,\infty[$ splits into five pairwise disjoint regions[5]:

(i) *The "production" region* $\{1\} \times \mathcal{P}$, *where* $\mathcal{P}$ *is an open subset of* $]0,\infty[$. Whenever the project is in its "open" mode and the process X takes values in $\mathcal{P}$, it is optimal to keep the project in its "open" mode, which is associated with production. In particular, $\mathcal{P}$ is the set in which the function w_1 satisfies the ODE

$$\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) + h(x) = 0. \tag{9}$$

(ii) *The "waiting" region* $\{0\} \times \mathcal{W}$, *where* $\mathcal{W}$ *is an open subset of* $]0,\infty[$. If the project is in its "closed" mode and the process X takes values in $\mathcal{W}$, then it is optimal to take no action, namely, keep the project on standby. The set $\mathcal{W}$ is characterised by the requirement that w_0 satisfies the ODE

$$\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) = 0. \tag{10}$$

(iii) *The "switch out" region* $\{1\} \times \mathcal{S}_{\text{out}}$, *where* $\mathcal{S}_{\text{out}}$ *is a closed subset of* $]0,\infty[$. If the project is in its "open" mode, then it is optimal to switch it to its "closed" mode as soon as X takes values in $\mathcal{S}_{\text{out}}$. The set $\mathcal{S}_{\text{out}}$ is characterised by the identity

$$w_1(x) = w_0(x) - K_0 \quad \text{for all } x \in \mathcal{S}_{\text{out}}. \tag{11}$$

(iv) *The "switch in" region* $\{0\} \times \mathcal{S}_{\text{in}}$, *where* $\mathcal{S}_{\text{in}}$ *is a closed subset of* $]0,\infty[$. It is optimal to switch the project from its "closed" to its "open" mode as soon as X takes values in $\mathcal{S}_{\text{in}}$. In this case,

$$w_0(x) = w_1(x) - K_1 \quad \text{for all } x \in \mathcal{S}_{\text{in}}. \tag{12}$$

[5] In the description of the five possible regions, we characterise subsets of $]0,\infty[$ as open or closed relative to the topology on $]0,\infty[$ that is the trace of the usual topology on $\mathbb{R}$, for instance, $]0,a] =]0,\infty[\setminus]a,\infty[$ and $[a,\infty[=]0,\infty[\setminus]0,a[$ are closed sets.

(v) *The "abandonment" region* $\{0\} \times \mathcal{A}_0 \cup \{1\} \times \mathcal{A}_1$*, where* $\mathcal{A}_0$*,* $\mathcal{A}_1$ *are closed subsets of* $]0, \infty[$. It is optimal to abandon permanently the project as soon as the state process hits the abandonment region. Accordingly,

$$w_i(x) = -K \quad \text{for all } x \in \mathcal{A}_i \text{ and } i = 0, 1. \tag{13}$$

The tactics associated with these regions exhaust all possible control actions. Therefore,

$$\mathcal{P} \cup \mathcal{S}_{\text{out}} \cup \mathcal{A}_1 = \mathcal{W} \cup \mathcal{S}_{\text{in}} \cup \mathcal{A}_0 = \,]0, \infty[.$$

We will solve the control problem that we study by identifying these regions and deriving appropriate explicit solutions to the HJB equation (7)–(8). To this end, we will use the following facts. It is well-known that the general solution to the Euler's ODE (10) is given by

$$w(x) = Ax^m + Bx^n, \tag{14}$$

for some constants $A, B \in \mathbb{R}$, where the constants $m < 0 < n$ are defined by

$$m, n = \frac{1}{2\sigma^2}\left[\sigma^2 - b \mp \sqrt{(b - \sigma^2)^2 + 4\sigma^2 r}\right]. \tag{15}$$

If $h :]0, \infty[\to \mathbb{R}$ is a function satisfying the integrability condition in (5), then a particular solution to the ODE (9) is the function $R_h :]0, \infty[\to \mathbb{R}$ given by

$$\begin{aligned} R_h(x) &= \frac{1}{\sigma^2(n-m)}\left[x^m \int_0^x s^{-m-1} h(s)\, ds + x^n \int_x^\infty s^{-n-1} h(s)\, ds\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-rs} h(X_s)\, ds\right]. \end{aligned} \tag{16}$$

A straightforward calculation reveals that

$$R_h'(x) = \frac{1}{\sigma^2(n-m)}\left[mx^{m-1} \int_0^x s^{-m-1} h(s)\, ds + nx^{n-1} \int_x^\infty s^{-n-1} h(s)\, ds\right]. \tag{17}$$

Furthermore, for a choice of h as in Assumption 1,

$$R_h \text{ is increasing,} \tag{18}$$

$$h(0) := \lim_{x \downarrow 0} h(x) = r \lim_{x \downarrow 0} R_h(x) \quad \text{and} \quad \lim_{x \to \infty} R_h(x) = \infty, \tag{19}$$

$$\lim_{T \to \infty} e^{-rT}\, \mathbb{E}\Big[\big|R_h(X_T)\big|\Big] = 0 \tag{20}$$

$$\text{and} \quad \mathbb{E}\left[\int_0^T e^{-2rt} X_t^2 \big|R_h'(X_t)\big|^2\, dt\right] < \infty \quad \text{for all } T > 0. \tag{21}$$

All of these claims regarding the function R_h as well as several more general results can be found in Knudsen, Meister and Zervos [18, Section 4], and Johnson and Zervos [16].

Example 1 If h is the function given by (4), then Assumption 1 holds true if and only if $\vartheta \in \,]0, n[$, in which case,

$$R_h(x) = -\frac{x^\vartheta}{\sigma^2\vartheta^2 + (b-\sigma^2)\vartheta - r} + \frac{c}{r}.$$

We will illustrate our results numerically for the choices

$$b = 0, \quad \sigma = 1, \quad r = 2, \quad \vartheta = 1 \quad \text{and} \quad K_1 = K_0 = \frac{1}{2},$$

which are associated with

$$m = -1, \quad n = 2 \quad \text{and} \quad R_h(x) = \frac{1}{2}x + \frac{c}{2}.$$

4 The solution to the control problem

We now derive the solution to the stochastic control problem formulated in Section 2 by identifying the sets $\mathcal{P}$, $\mathcal{W}$, $\mathcal{S}_{\text{out}}$, $\mathcal{S}_{\text{in}}$, $\mathcal{A}_1$, $\mathcal{A}_0$ we have discussed in the previous section and deriving appropriate solutions to the HJB equation (7)–(8) using (9)–(13). To this end, we first note that, if the investment project is in its "open" mode at time 0 and is never switched to its "closed" mode or abandoned, then it will yield a total expected discounted payoff equal to $R_h(x)$ (see (16)). On the other hand, if the project is "closed" at time 0 and is never switched to its "open" operating mode or abandoned, then it will yield 0 total expected discounted payoff. Since R_h is increasing and $\lim_{x\to\infty} R_h(x) = \infty$ (see (18) and (19)), it should be optimal to operate the project in its "open" mode whenever the process X takes sufficiently high values. It follows that there exists $M > 0$ such that

$$]M, \infty[\,\subseteq \mathcal{P} \quad \text{and} \quad]M, \infty[\,\subseteq \mathcal{S}_{\text{in}}.$$

If $\mathcal{A}_1 \neq \emptyset$ (resp, $\mathcal{A}_0 \neq \emptyset$), then $\mathcal{A}_1 = \,]0, \delta]$ (resp., $\mathcal{A}_0 = \,]0, \zeta]$) for some $\delta > 0$ (resp., $\zeta > 0$) because R_h is increasing. Furthermore, in view of the smoothness of a solution to the HJB equation (7)–(8) that is required to identify it with the control problem's value function and the analysis in the previous section, we expect that the "abandonment" region does not have any common boundary points with either the "switch in" region or the "switch out" region.

In light of these observations, we will show that the production and the waiting regions $\mathcal{P}$ and $\mathcal{W}$ have the general forms

$$\mathcal{P} = \,]\delta, \gamma[\, \cup \,]\beta, \infty[\quad \text{and} \quad \mathcal{W} = \,]\zeta, \alpha[, \tag{22}$$

for some $0 \leq \delta \leq \gamma \leq \beta < \infty$ and $0 \leq \zeta \leq \alpha < \infty$ (see Figures 1-8), where we adopt the usual convention that, e.g., $]0, 0[\,= \emptyset$. In view of the solutions to the ODEs (9), (10) given in the previous section, the solution to the HJB equation (7)–(8) should be such that

$$w_1(x) = \begin{cases} R_h(x), & \text{for all } x \in \,]0, \infty[, \text{ if } \delta = \gamma = \beta = 0 \\ Ax^m + R_h(x), & \text{for all } x \in \,]\beta, \infty[, \text{ if } \gamma < \beta \text{ or } 0 < \delta = \gamma = \beta \\ \Gamma_1 x^m + \Gamma_2 x^n + R_h(x), & \text{for all } x \in \,]\delta, \gamma[, \text{ if } 0 < \delta < \gamma < \beta \end{cases} \Bigg\} \tag{23}$$

and

$$w_0(x) = \begin{cases} Bx^n, & \text{for all } x \in \,]0, \alpha[, \text{ if } \zeta = 0 < \alpha \\ \Delta_1 x^m + \Delta_2 x^n, & \text{for all } x \in \,]\zeta, \alpha[, \text{ if } 0 < \zeta < \alpha \end{cases} \Bigg\}, \tag{24}$$

for some constants A, Γ_1, Γ_2, B, Δ_1 and Δ_2 because these are the only choices that are consistent with the requirements of the verification theorem that we will use to identify the solution to (7)–(8) with the control problem's value function.

To determine free-boundary points such as δ, γ, β, ζ, α appearing in (22) and constants such as A, Γ_1, Γ_2, B, Δ_1, Δ_2 appearing in (23)–(24), we will use the C^1 continuity that we expect the functions w_1, w_0 to have. In particular, we will require that w_1, w_0 should be C^1 at every boundary point separating any two of the five regions. Using the expressions (16), (17) and the identity $\sigma^2 mn = -r$, we will then derive appropriate systems of equations for the unknown parameters. We will only provide the results of these calculations because they are straightforward to replicate.

We have organised the presentation of the possible cases arising by splitting them in three groups. Group I includes the cases in which it is not optimal to switch or abandon the project if this is in its "open" mode. Group II contains all cases where it may be optimal to switch or abandon the project if this is in its "open" mode but abandonment is not optimal if the project is in its "closed" mode. Finally, Group III includes all remaining cases. The proofs of Lemmas 1-8 can be found in the complete version of the paper that is available online (`https://arxiv.org/abs/1607.08406`).

4.1 Group I: taking action is not optimal whenever the project is in its "open" operating mode ($\mathcal{P} = \,]0, \infty[$)

All cases in this group are such that $\mathcal{P} = \,]0, \infty[$ and are associated with a solution to the HJB equation (7)–(8) such that

$$w_1(x) = R_h(x) \quad \text{for all } x > 0. \tag{25}$$

Case I.1 (Figure 1) In this case, it is optimal to immediately switch the investment project to its "open" mode if it is originally "closed". Accordingly,

$$\mathcal{P} = \mathcal{S}_{\text{in}} = \,]0, \infty[\quad \text{and} \quad \mathcal{W} = \mathcal{S}_{\text{out}} = \mathcal{A}_0 = \mathcal{A}_1 = \emptyset,$$

and the functions w_1 and w_0 given by (25) and

$$w_0(x) = R_h(x) - K_1, \quad \text{for } x > 0, \tag{26}$$

should satisfy the HJB equation (7)–(8).

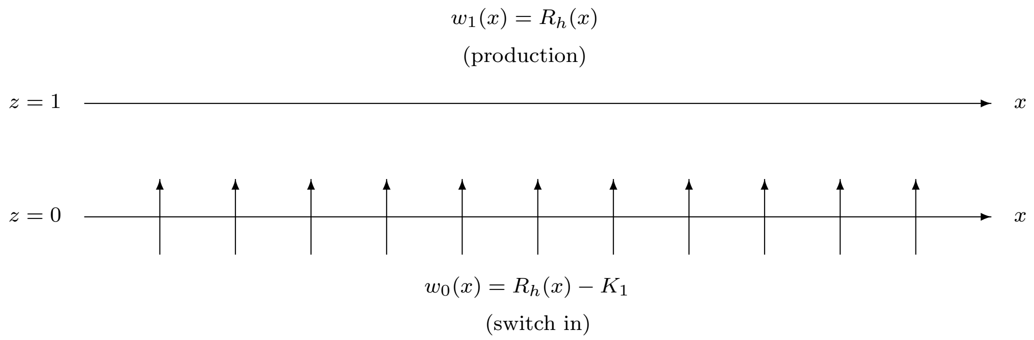

Figure 1. Illustration of the regions determining the optimal strategy in the context of Case I.1

Lemma 1 *The increasing functions w_1, w_0 defined by (25), (26) satisfy the HJB equation (7)–(8) if and only if*

$$\max\{rK_1,\, rK_1 - rK\} \leq h(0).$$

Example 2 If h is the function given by (4) and the problem data is as in Example 1, then this case characterises the optimal strategy if and only if $K \in \mathbb{R}$ and $\max\{1,\, 1 - rK\} \leq c$.

Case I.2 (Figure 2) In this case, it is optimal to switch the investment project to its "open" mode if it is originally "closed" as long as the process X takes sufficiently high values. In particular, there exists a boundary point $\alpha > 0$ such that, if the project starts in its "closed" mode, then it is optimal to wait for all long as X takes values strictly less than α and switch the project to its "open" mode as soon as X takes a value exceeding α. Accordingly,

$$\mathcal{P} =]0, \infty[, \quad \mathcal{W} =]0, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{S}_{\text{out}} = \mathcal{A}_0 = \mathcal{A}_1 = \emptyset.$$

In view of (12) and (23)–(24), the functions w_1 and w_0 given by (25) and

$$w_0(x) = \begin{Bmatrix} Bx^n, & \text{if } x < \alpha \\ R_h(x) - K_1, & \text{if } x \geq \alpha \end{Bmatrix} \tag{27}$$

should satisfy the HJB equation (7)–(8).

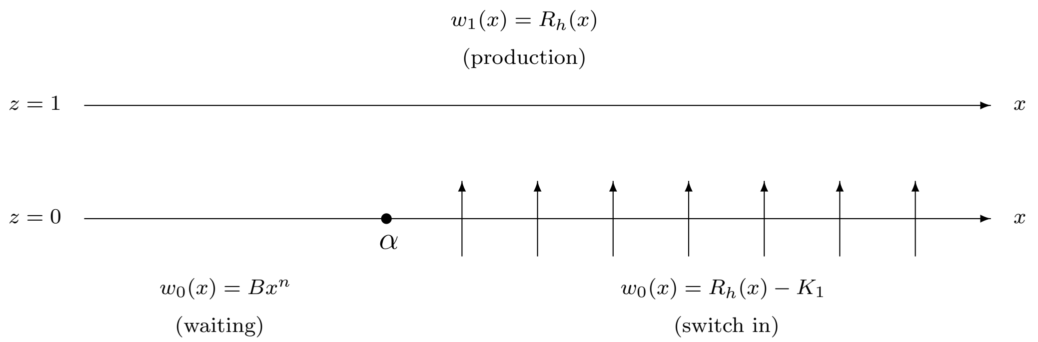

Figure 2. Illustration of the regions determining the optimal strategy in the context of Case I.2

The requirement that w_0 should be C^1 at α yields the expressions

$$B = \frac{1}{\sigma^2(n-m)} \int_\alpha^\infty s^{-n-1} \big[h(s) - rK_1\big] \, ds \tag{28}$$

$$\text{and} \quad \int_0^\alpha s^{-m-1} \big[h(s) - rK_1\big] \, ds = 0. \tag{29}$$

Lemma 2 *Equation (29) has a unique solution $\alpha > 0$ and the functions w_1, w_0 defined by (25), (27), for $B > 0$ given by (28), are increasing and satisfy the HJB equation (7)–(8) if and only if*

$$0 \leq K \quad \textit{and} \quad \max\{-rK_0, \, -rK\} \leq h(0) < rK_1.$$

Example 3 If h is the function given by (4), then (28) and (29) are equivalent to

$$\alpha = \left(-\frac{(\vartheta - m)(rK_1 - c)}{m} \right)^{-1/m} \quad \text{and} \quad B = \frac{\alpha^{-n}}{\sigma^2(n-m)} \left(\frac{\alpha^\vartheta}{n - \vartheta} - \frac{rK_1 - c}{n} \right).$$

If the problem data is as in Example 1, then this case characterises the optimal strategy if and only if $0 \leq K$ and $\max\{-1, \, -rK\} \leq c < 1$. In particular, if $c = \frac{1}{2}$, then

$$\alpha = 1 \quad \text{and} \quad B = \frac{1}{4}.$$

Case I.3 (Figure 3) This case differs from the previous one by the fact that abandoning the investment project if it is in its "closed" mode and the process X takes values below a given threshold level ζ becomes optimal. Accordingly,

$$\mathcal{P} = \,]0, \infty[, \quad \mathcal{A}_0 = \,]0, \zeta], \quad \mathcal{W} = \,]\zeta, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{S}_{\text{out}} = \mathcal{A}_1 = \emptyset,$$

and, in view of (12)–(13) and (23)–(24), the required solution to the HJB equation (7)–(8) should be given by the function w_1 defined by (25) and the function w_0 defined by

$$w_0(x) = \begin{cases} -K, & \text{if } x \leq \zeta \\ \Delta_1 x^m + \Delta_2 x^n, & \text{if } \zeta < x < \alpha \\ R_h(x) - K_1, & \text{if } x \geq \alpha \end{cases} \Bigg\}. \tag{30}$$

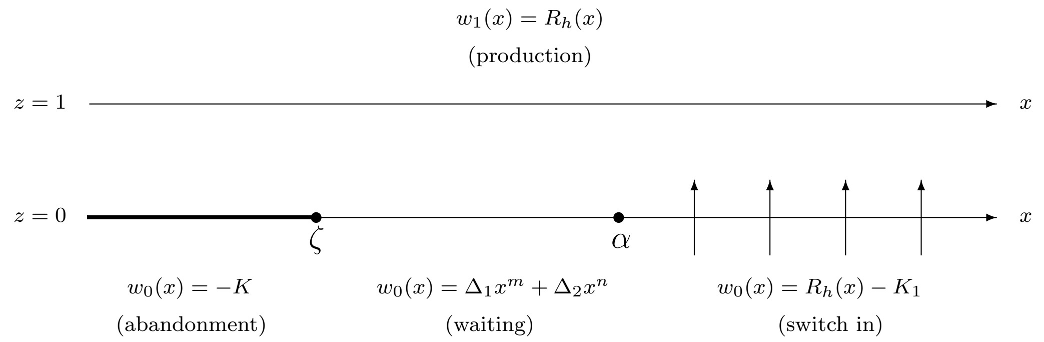

Figure 3. Illustration of the regions determining the optimal strategy in the context of Case I.3

To determine the free-boundary points ζ, α and the parameters Δ_1, Δ_2, we require that w_0 should be C^1, which yields the expressions

$$f_1(\zeta, \alpha) := m \int_0^\alpha s^{-m-1} \big[h(s) - rK_1\big] \, ds - rK\zeta^{-m} = 0, \tag{31}$$

$$f_2(\zeta, \alpha) := n \int_\alpha^\infty s^{-n-1} \big[h(s) - rK_1\big] \, ds + rK\zeta^{-n} = 0, \tag{32}$$

$$\Delta_1 = \frac{rK\zeta^{-m}}{\sigma^2 m(n-m)} \quad \text{and} \quad \Delta_2 = -\frac{rK\zeta^{-n}}{\sigma^2 n(n-m)}. \tag{33}$$

Lemma 3 *The system of equations (31)–(32) has a unique solution (ζ, α) such that $0 < \zeta < \alpha$ and the functions w_1, w_0 defined by (25), (30), for $\Delta_1 > 0$, $\Delta_2 > 0$ given by (33), are increasing and satisfy the HJB equation (7)–(8) if and only if*

$$K < 0 \quad \textit{and} \quad -rK \leq h(0) < rK_1 - rK.$$

Example 4 If h is the function given by (4), then the system of equations (31)–(32) takes the form

$$(rK_1 - c)\alpha^{-m} + \frac{m}{\vartheta - m}\alpha^{\vartheta - m} - rK\zeta^{-m} = 0,$$
$$(rK_1 - c)\alpha^{-n} - \frac{n}{n - \vartheta}\alpha^{-(n-\vartheta)} - rK\zeta^{-n} = 0.$$

If the problem data is as in Example 1, then this case characterises the optimal strategy if and only if $K < 0$ and $-rK \leq c < 1 - rK$. In particular, if $K = -\frac{1}{2}$ and $c = 1$, then

$$\zeta = 2^{-\frac{1}{3}}, \quad \alpha = 2^{\frac{1}{3}} \quad \text{and} \quad \Delta_1 = \Delta_2 = 2^{-\frac{1}{3}} \times 3^{-1}.$$

4.2 Group II: taking action may be optimal if the project is in its "open" mode but abandonment is not optimal whenever the project is in its "closed" operating mode ($\mathcal{P} \neq \,]0, \infty[$ and $\mathcal{A}_0 = \emptyset$)

We now consider cases that complement the ones in the previous group and are characterised by the non-optimality of abandonment whenever the project is in its "closed" mode. In all of these cases, $\mathcal{W} = \,]0, \alpha[$ and $\mathcal{S}_{\text{in}} = [\alpha, \infty[$. Otherwise, the cases are differentiated by the arrangement of the optimal tactics whenever the project is in its "open" mode.

Case II.1 (Figure 4) In this case, sequential switching of the investment project from "open" to "closed" and vice versa is optimal, and abandonment is not part of the optimal strategy. Whenever the project is in its "open" (resp., "closed") mode, it is optimal to stay there for as long as the process X takes values above (resp., below) a given threshold β (resp., α) and switch to its "closed" (resp., "open") mode as soon as X takes values below (resp., above) the threshold β (resp., α). Of course, for such a strategy to be well-defined, we must have $\beta < \alpha$. Accordingly,

$$\mathcal{S}_{\text{out}} = \,]0, \beta], \quad \mathcal{P} = \,]\beta, \infty[, \quad \mathcal{W} = \,]0, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{A}_0 = \mathcal{A}_1 = \emptyset.$$

In view of (11)–(12) and (23)–(24), we can see that the required solution to the HJB equation (7)–(8) should be given by the functions defined by

$$w_1(x) = \begin{Bmatrix} Bx^n - K_0, & \text{if } x \leq \beta \\ Ax^m + R_h(x), & \text{if } x > \beta \end{Bmatrix} \tag{34}$$

$$\text{and} \quad w_0(x) = \begin{Bmatrix} Bx^n, & \text{if } x < \alpha \\ Ax^m + R_h(x) - K_1, & \text{if } x \geq \alpha \end{Bmatrix}. \tag{35}$$

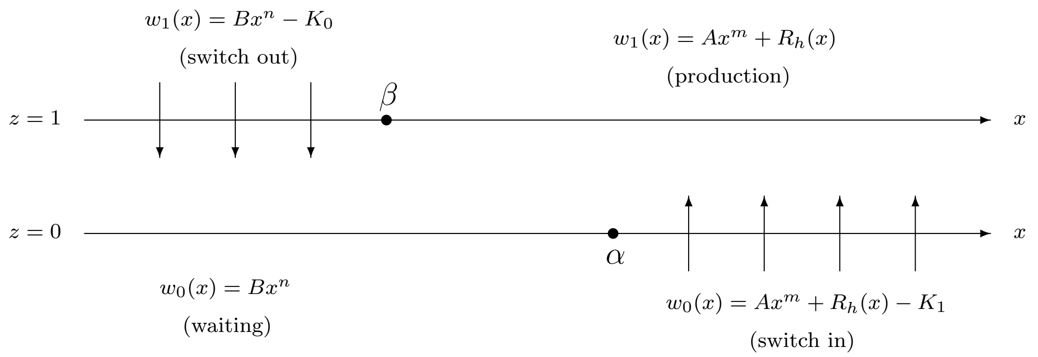

Figure 4. Illustration of the regions determining the optimal strategy in the context of Case II.1

To determine the free-boundary points β, α and the parameters A, B, we once again require that the functions w_1, w_0 should be C^1, which yields the expressions

$$A = -\frac{1}{\sigma^2(n-m)} \int_0^\beta s^{-m-1} \big[h(s) + rK_0\big]\, ds, \tag{36}$$

$$B = \frac{1}{\sigma^2(n-m)} \int_\alpha^\infty s^{-n-1} \big[h(s) - rK_1\big]\, ds, \tag{37}$$

and the system of equations

$$m \int_\beta^\alpha s^{-m-1} h(s)\, ds + rK_0 \beta^{-m} + rK_1 \alpha^{-m} = 0, \tag{38}$$

$$n \int_\beta^\alpha s^{-n-1} h(s)\, ds + rK_0 \beta^{-n} + rK_1 \alpha^{-n} = 0. \tag{39}$$

Lemma 4 *The system of equations (38)–(39) has a unique solution (β, α) such that $0 < \beta < \alpha$ and the functions w_1, w_0 defined by (34), (35), for $A > 0$, $B > 0$ given by (36), (37), are increasing and satisfy the HJB equation (7)–(8) if and only if*

$$K_0 \le K \quad \textit{and} \quad h(0) < -rK_0.$$

Example 5 If h is the function given by (4), then the system of equations (38)–(39) takes the form

$$(rK_1 - c)\alpha^{-m} + (rK_0 + c)\beta^{-m} + \frac{m}{\vartheta - m}\left(\alpha^{\vartheta - m} - \beta^{\vartheta - m}\right) = 0,$$
$$(rK_1 - c)\alpha^{-n} + (rK_0 + c)\beta^{-n} - \frac{n}{n - \vartheta}\left(\alpha^{-(n-\vartheta)} - \beta^{-(n-\vartheta)}\right) = 0,$$

while

$$A = \frac{\beta^{-m}}{\sigma^2(n-m)}\left(\frac{rK_0 + c}{m} - \frac{\beta^\vartheta}{\vartheta - m}\right) \quad \text{and} \quad B = \frac{\alpha^{-n}}{\sigma^2(n-m)}\left(\frac{\alpha^\vartheta}{n - \vartheta} - \frac{rK_1 - c}{n}\right).$$

If the problem data is as in Example 1, then this case characterises the optimal strategy if and only if $\frac{1}{2} \leq K$ and $c < -1$. In particular, if $c = -2$, then

$$\beta \simeq 0.537, \quad \alpha \simeq 5.866, \quad A \simeq 0.131 \quad \text{and} \quad B \simeq 0.042.$$

Case II.2 (Figure 5) Abandoning the project if this is in its “open” mode and the state process X takes values below a given threshold $\delta_\dagger$ instead of switching it to its “closed” mode is the difference between this case and the previous one.[6] Accordingly,

$$\mathcal{A}_1 =]0, \delta_\dagger], \quad \mathcal{P} =]\delta_\dagger, \infty[, \quad \mathcal{W} =]0, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{S}_{\text{out}} = \mathcal{A}_0 = \emptyset,$$

and the functions defined by

$$w_1(x) = \begin{Bmatrix} -K, & \text{if } x \leq \delta_\dagger \\ Ax^m + R_h(x), & \text{if } x > \delta_\dagger \end{Bmatrix} \tag{40}$$

$$\text{and} \quad w_0(x) = \begin{Bmatrix} Bx^n, & \text{if } x < \alpha \\ Ax^m + R_h(x) - K_1, & \text{if } x \geq \alpha \end{Bmatrix} \tag{41}$$

should provide a solution to the HJB equation (7)–(8).

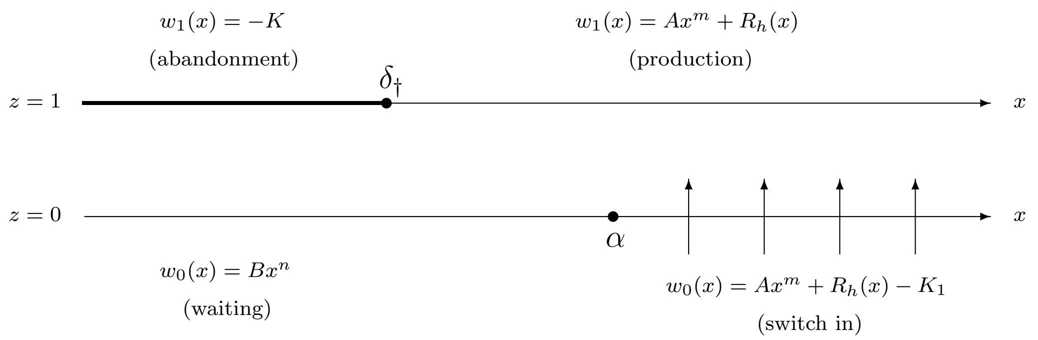

Figure 5. Illustration of the regions determining the optimal strategy in the context of Case II.2

Requiring that w_1, w_0 should be C^1, we obtain the expressions

$$A = -\frac{1}{\sigma^2(n-m)} \int_0^{\delta_\dagger} s^{-m-1} \big[h(s) + rK\big]\, ds, \tag{42}$$

$$B = \frac{1}{\sigma^2(n-m)} \int_\alpha^\infty s^{-n-1} \big[h(s) - rK_1\big]\, ds, \tag{43}$$

[6] We use the notation $\delta_\dagger$ rather than the simpler δ because this point will appear in assumptions that we will make in later cases.

and the system of equations

$$\int_{\delta_\dagger}^{\infty} s^{-n-1}\big[h(s) + rK\big]\, ds = 0, \tag{44}$$

$$f(\delta,\alpha) := m\int_{\delta_\dagger}^{\alpha} s^{-m-1}\big[h(s) - rK_1\big]\, ds + r(K_1 + K)\delta_\dagger^{-m} = 0. \tag{45}$$

The following result involves the point

$$K_0^\star = -K_1 - \frac{m\hat{x}^m}{r}\int_{\hat{x}}^{\alpha} s^{-m-1}\big[h(s) - rK_1\big]\, ds, \tag{46}$$

where $\hat{x}$ solves the equation

$$m\hat{x}^m\int_{\hat{x}}^{\alpha} s^{-m-1}\big[h(s) - rK_1\big]\, ds - n\hat{x}^n\int_{\hat{x}}^{\alpha} s^{-n-1}\big[h(s) - rK_1\big]\, ds = 0. \tag{47}$$

Lemma 5 *The system of equations (44)–(45) has a unique solution $(\delta_\dagger, \alpha)$ such that $0 < \delta_\dagger < \alpha$, while equation (47) has a unique solution $\hat{x} \in \,]\delta_\dagger, \alpha[$. Given these solutions, the functions w_1, w_0 defined by (40), (41), for $A > 0$, $B > 0$ given by (42), (43), are increasing and satisfy the HJB equation (7)–(8) if and only if*

$$0 \le K$$

and

$$\Big(K < K_0 \text{ and } -rK_0 \le h(0) < -rK\Big) \quad \text{or} \quad \Big(K_0^\star \le K_0 \text{ and } h(0) < -rK_0\Big),$$

where $K_0^\star \in \,]K, -r^{-1}h(0)[$, which depends on all problem data except K_0, is defined by (46).

Example 6 If h is the function given by (4), then the system of equations (44)–(45) takes the form

$$\delta_\dagger^\vartheta = -\frac{(n-\vartheta)(c + rK)}{n},$$

$$(rK_1 - c)\alpha^{-m} + \frac{m}{\vartheta - m}\alpha^{\vartheta - m} + (rK + c)\delta_\dagger^{-m} - \frac{m}{\vartheta - m}\delta_\dagger^{\vartheta - m} = 0,$$

while

$$A = -\frac{\vartheta(rK + c)\delta_\dagger^{-m}}{r(\vartheta - m)} \quad \text{and} \quad B = \frac{\alpha^{-n}}{\sigma^2(n-m)}\left(\frac{\alpha^\vartheta}{n-\vartheta} - \frac{rK_1 - c}{n}\right).$$

The critical point $K_0^\star$ defined by (46) admits the expression

$$K_0^\star = -K_1 + \frac{rK_1 - c}{r}\left[1 - \left(\frac{\hat{x}}{\alpha}\right)^m\right] - \frac{m}{r(\vartheta - m)}\hat{x}^m\big(\alpha^{\vartheta - m} - \hat{x}^{\vartheta - m}\big) \in \,]K, -r^{-1}c[,$$

where $\hat{x}$ is the unique solution to the equation

$$(rK_1 - c)\left[\left(\frac{\hat{x}}{\alpha}\right)^m - \left(\frac{\hat{x}}{\alpha}\right)^n\right] + \frac{m}{\vartheta - m}\alpha^{\vartheta - m}\hat{x}^m + \frac{n}{n - \vartheta}\alpha^{-(n-\vartheta)}\hat{x}^n - \frac{\vartheta(n-m)}{(n-\vartheta)(\vartheta - m)}\hat{x}^\vartheta = 0.$$

If the problem data is as in Example 1, then this case characterises the optimal strategy if and only if either ($0 \leq K < \frac{1}{2}$ and $-1 \leq c < -rK$) or ($0 \leq K$, $K_0^\star \leq \frac{1}{2}$ and $c < -1$). If $K = 0$ and $c = -1$, then

$$\delta_\dagger = \frac{1}{2}, \quad \alpha = 2 + \frac{\sqrt{13}}{2}, \quad A = \frac{1}{8} \quad \text{and} \quad B \simeq 0.065.$$

while, if $K = \frac{1}{4}$ and $c = -2$, then

$$\hat{x} \simeq 0.808, \quad K_0^\star = 0.276, \quad \delta_\dagger = \frac{3}{4}, \quad \alpha = 3 + \frac{\sqrt{117}}{4}, \quad A = \frac{9}{32} \quad \text{and} \quad B \simeq 0.043.$$

Case II.3 (Figure 6) The last case in this group is a hybrid of the previous two. If the investment project is initially in its "open" mode and the initial value x of the process X is greater than a threshold γ or it is initially in its "closed" mode, then it is optimal to follow the same strategy as in Case II.1, which is determined by two thresholds $\beta < \alpha$ such that $\gamma < \beta$. In this case, the project is sequentially switched from "open" to "closed" and vice versa, and it is never abandoned. On the other hand, if the project is initially in its "open" mode and the initial value x of X is strictly less than γ, then it is optimal to abandon the project as soon as X falls below another threshold $\delta < \gamma$ before hitting γ. Otherwise, it is optimal to switch the project to its "closed" mode if X rises to γ before hitting δ, and then maintain the sequential switching strategy defined by β and α. Accordingly,

$$\mathcal{A}_1 =]0, \delta], \quad \mathcal{P} =]\delta, \gamma[\cup]\beta, \infty[, \quad \mathcal{S}_{\text{out}} = [\gamma, \beta],$$
$$\mathcal{W} =]0, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{A}_0 = \emptyset.$$

In view of (11)–(13) and (23)–(24), we can see that the required solution to the HJB equation (7)–(8) should be given by the functions defined by

$$w_1(x) = \begin{Bmatrix} -K, & \text{if } x \leq \delta \\ \Gamma_1 x^m + \Gamma_2 x^n + R_h(x), & \text{if } \delta < x < \gamma \\ Bx^n - K_0, & \text{if } \gamma \leq x \leq \beta \\ Ax^m + R_h(x), & \text{if } x > \beta \end{Bmatrix} \tag{48}$$

$$\text{and} \quad w_0(x) = \begin{Bmatrix} Bx^n, & \text{if } x < \alpha \\ Ax^m + R_h(x) - K_1, & \text{if } x \geq \alpha \end{Bmatrix}. \tag{49}$$

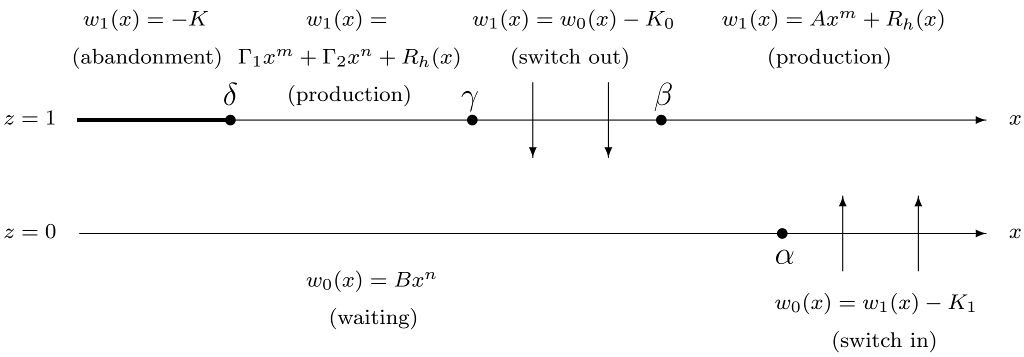

Figure 6. Illustration of the regions determining the optimal strategy in the context of Case II.3

To determine Γ_1, Γ_2, A, B, δ, γ, β and α we require that w_1, w_0 should be C^1 at the free-boundary points δ, γ, β and α. In view of this requirement, we can verify that δ, γ, β and α should satisfy the equations (38), (39),

$$F_1(\delta, \gamma) := m \int_\delta^\gamma s^{-m-1} \bigl[h(s) + rK_0\bigr] \, ds + r(K - K_0)\delta^{-m} = 0 \tag{50}$$

$$\text{and} \quad F_2(\delta, \gamma) := n \int_\delta^\gamma s^{-n-1} \bigl[h(s) + rK_0\bigr] \, ds + r(K - K_0)\delta^{-n} + n \int_\beta^\infty s^{-n-1} \bigl[h(s) + rK_0\bigr] \, ds = 0, \tag{51}$$

while A, B, Γ_1 and Γ_2 should be given by (36), (37),

$$\Gamma_1 = -\frac{1}{\sigma^2(n-m)} \int_0^\gamma s^{-m-1} \bigl[h(s) + rK_0\bigr] \, ds \tag{52}$$

$$\text{and} \quad \Gamma_2 = -\frac{1}{\sigma^2(n-m)} \int_\gamma^\beta s^{-n-1} \bigl[h(s) + rK_0\bigr] \, ds. \tag{53}$$

Lemma 6 *The system of equations (38), (39), (50) and (51) has a unique solution* $(\delta, \gamma, \beta, \alpha)$ *such that* $0 < \delta < \gamma < \beta < \alpha$ *and the functions* w_1, w_0 *defined by (48), (49), for* $A > 0$, $B > 0$, $\Gamma_1 > 0$, $\Gamma_2 > 0$ *given by (36), (37), (52), (53), are increasing and satisfy the HJB equation (7)–(8) if and only if*

$$0 \le K, \quad h(0) < -rK_0 \quad \textit{and} \quad K < K_0 < K_0^\star,$$

where $K_0^\star \in \left]K, -r^{-1}h(0)\right[$*, which depends on all problem data except* K_0*, is as in Lemma 5.*

We note that the conditions of this result can all hold true only if $h(0) < 0$.

Example 7 If h is the function given by (4), then the system of equations (50)–(51) takes the form

$$(rK_0 + c)(\gamma^{-m} - \delta^{-m}) - \frac{m}{\vartheta - m}(\gamma^{\vartheta-m} - \delta^{\vartheta-m}) + r(K_0 - K)\delta^{-m} = 0,$$

$$(rK_0 + c)(\delta^{-n} - \gamma^{-n} + \beta^{-n}) + \frac{n}{n-\vartheta}(\delta^{-(n-\vartheta)} - \gamma^{-(n-\vartheta)} + \beta^{-(n-\vartheta)}) - r(K_0 - K)\delta^{-n} = 0,$$

while

$$\Gamma_1 = \frac{\gamma^{-m}}{\sigma^2(n-m)}\left(\frac{rK_0 + c}{m} - \frac{\gamma^{\vartheta}}{\vartheta - m}\right)$$

$$\text{and} \quad \Gamma_2 = -\frac{1}{\sigma^2(n-m)}\left[\frac{rK_0 + c}{n}\left(\gamma^{-n} - \beta^{-n}\right) + \frac{1}{n-\vartheta}\left(\gamma^{-(n-\vartheta)} - \beta^{-(n-\vartheta)}\right)\right].$$

If the problem data is as in Example 1, then this case characterises the optimal strategy if and only if $0 \le K < \frac{1}{2}$, $c < -1$ and $\frac{1}{2} < K_0^\star$, where $K_0^\star$ is as in Example 6. In particular, if $K = \frac{5}{11}$ and $c = -4$, then

$$\hat{x} \simeq 1.706, \quad K_0^\star \simeq 0.524, \quad \delta \simeq 0.279, \quad \gamma \simeq 1.348, \quad \beta \simeq 1.740, \quad \alpha \simeq 9.194,$$
$$A \simeq 1.235, \quad B \simeq 0.026, \quad \Gamma_1 \simeq 1.045 \quad \text{and} \quad \Gamma_2 \simeq 0.054.$$

4.3 Group III: the remaining cases

We now consider the remaining cases. These are characterised by the fact that it may be optimal to abandon the investment project when this is in its "closed" mode.

Case III.1 (Figure 7) This case is the modification of Case II.2 (see Figure 5) that arises if abandonment when the project is in its "closed" mode becomes part of the optimal tactics. In this case,

$$\mathcal{A}_1 =]0, \delta_\dagger], \quad \mathcal{P} =]\delta_\dagger, \infty[, \quad \mathcal{A}_0 =]0, \zeta], \quad \mathcal{W} =]\zeta, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{S}_{\text{out}} = \emptyset,$$

and the functions defined by

$$w_1(x) = \begin{Bmatrix} -K, & \text{if } x \le \delta_\dagger \\ Ax^m + R_h(x), & \text{if } x \ge \delta_\dagger \end{Bmatrix} \tag{54}$$

$$\text{and} \quad w_0(x) = \begin{Bmatrix} -K, & \text{if } x \le \zeta \\ \Delta_1 x^m + \Delta_2 x^n, & \text{if } \zeta \le x \le \alpha \\ Ax^m + R_h(x) - K_1, & \text{if } x \ge \alpha \end{Bmatrix} \tag{55}$$

should provide a solution to the HJB equation (7)–(8).

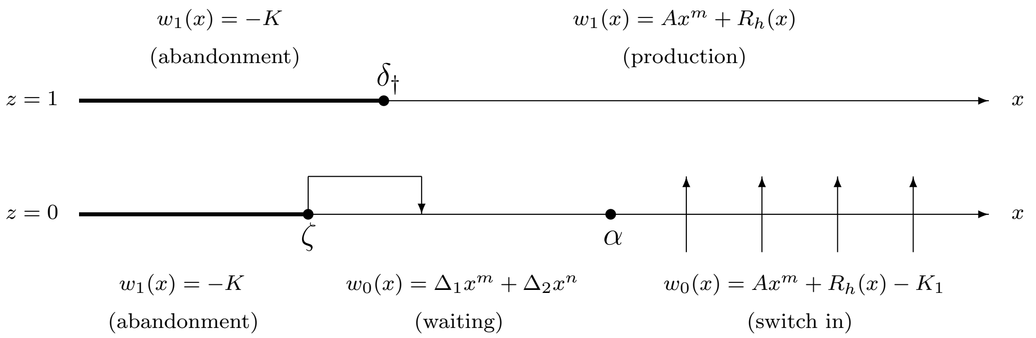

Figure 7. Illustration of the regions determining the optimal strategy in the context of Case III.1 (ζ can be smaller as well as larger than $\delta_\dagger$)

To determine A, Δ_1, Δ_2, $\delta_\dagger$, ζ and α we require that w_1, w_0 should be C^1 at the free-boundary points $\delta_\dagger$, ζ and α. In view of this requirement, we can verify that $\delta_\dagger$, ζ and α should satisfy the system of equations

$$G_1(\delta_\dagger, \zeta, \alpha) := m \int_{\delta_\dagger}^{\alpha} s^{-m-1} \big[h(s) - rK_1\big]\, ds + r(K_1 + K)\delta_\dagger^{-m} - rK\zeta^{-m} = 0 \tag{56}$$

$$\text{and} \quad G_2(\delta_\dagger, \zeta, \alpha) := -n \int_{\delta_\dagger}^{\alpha} s^{-n-1} \big[h(s) - rK_1\big]\, ds - r(K_1 + K)\delta_\dagger^{-n} + rK\zeta^{-n} = 0, \tag{57}$$

where $\delta_\dagger$ is given by (44), while, A, Δ_1 and Δ_2 should be given by (42),

$$\Delta_1 = A + \frac{1}{\sigma^2(n-m)} \int_0^{\alpha} s^{-m-1} \big[h(s) - rK_1\big]\, ds = \frac{rK\zeta^{-m}}{\sigma^2 m(n-m)} \tag{58}$$

$$\text{and} \quad \Delta_2 = \frac{1}{\sigma^2(n-m)} \int_{\alpha}^{\infty} s^{-n-1} \big[h(s) - rK_1\big]\, ds = -\frac{rK\zeta^{-n}}{\sigma^2 n(n-m)}. \tag{59}$$

The following result involves the equation

$$G_2\big(\delta_\dagger, \delta_\dagger, \alpha(K_1); K_1\big) = 0, \tag{60}$$

for K_1, in which we make explicit the dependence of α and G_2 on K_1 (note that $\delta_\dagger$ does not depend on K_1). Also, it involves the point

$$K_0^\dagger = -K_1 - \frac{n\hat{x}^n}{r} \int_{\hat{x}}^{\alpha} s^{-n-1} \big[h(s) - rK_1\big]\, ds, \tag{61}$$

where $\hat{x}$ solves the equation

$$m\hat{x}^m \int_{\hat{x}}^{\alpha} s^{-m-1} \big[h(s) - rK_1\big]\, ds - n\hat{x}^n \int_{\hat{x}}^{\alpha} s^{-n-1} \big[h(s) - rK_1\big]\, ds = 0. \tag{62}$$

Lemma 7 *The system of equations (44), (56) and (57) has a unique solution* $(\delta_\dagger, \zeta, \alpha)$ *such that* $0 < \delta_\dagger \wedge \zeta \leq \delta_\dagger \vee \zeta < \alpha$. *If* $h(\delta_\dagger) < 0$, *then there exists a unique solution* $K_1^\dagger > 0$ *to (60) that depends on all of the problem data except* K_1, K_0. *If* $h(\delta_\dagger) < 0$ *and* $K < K_1^\dagger$, *then equation (62) has a unique solution* $\hat{x} \in \,]\delta_\dagger, \alpha[$ *and the point* $K_0^\dagger > 0$ *depends on all of the problem data except* K_0. *Furthermore,* $\lim_{K_1 \uparrow K_1^\dagger} K_0^\dagger(K_1) = 0$, *and the free-boundary points* ζ *and* $\delta_\dagger$, *which do not depend on* K_0, *are such that*

$$\begin{aligned} & 0 < \zeta < \delta_\dagger \quad \text{if } h(\delta_\dagger) < 0 \text{ and } K_1 < K_1^\dagger, && (63) \\ & 0 < \zeta = \delta_\dagger \quad \text{if } h(\delta_\dagger) < 0 \text{ and } K_1 = K_1^\dagger && (64) \\ \text{and} \quad & 0 < \delta_\dagger < \zeta \quad \text{if } h(\delta_\dagger) \geq 0 \text{ or } \Big(h(\delta_\dagger) < 0 \text{ and } K_1 > K_1^\dagger\Big). && (65) \end{aligned}$$

The functions w_1, w_0 *defined by (54), (55), for* $A > 0$, $\Delta_1 > 0$, $\Delta_2 > 0$ *given by (42), (58), (59), are increasing and satisfy the HJB equation (7)–(8) if and only if*

$$K < 0, \quad h(0) < -rK$$

and

$$\begin{gathered} \Big(-rK_0 \leq h(0)\Big) \quad \text{or} \quad \Big(h(0) < -rK_0 \text{ and } h(\delta_\dagger) \geq 0\Big) \\ \text{or} \quad \Big(h(0) < -rK_0,\ h(\delta_\dagger) < 0 \text{ and } K_1 \geq K_1^\dagger\Big) \\ \text{or} \quad \Big(h(0) < -rK_0,\ h(\delta_\dagger) < 0,\ K_1 < K_1^\dagger \text{ and } K_0 \geq K_0^\dagger\Big). \end{gathered}$$

Example 8 If h is the function given by (4), then the system of equations (56)–(57) takes the form

$$\begin{aligned} (rK_1 - c)(\alpha^{-m} - \delta_\dagger^{-m}) + \frac{m}{\vartheta - m}(\alpha^{\vartheta - m} - \delta_\dagger^{\vartheta - m}) + r(K_1 + K)\delta_\dagger^{-m} - rK\zeta^{-m} = 0, \\ (rK_1 - c)(\alpha^{-n} - \delta_\dagger^{-n}) - \frac{n}{n - \vartheta}(\alpha^{-(n-\vartheta)} - \delta_\dagger^{-(n-\vartheta)}) + r(K_1 + K)\delta_\dagger^{-n} - rK\zeta^{-n} = 0, \end{aligned}$$

where $\delta_\dagger$ admits the expression given in Example 6. The equation (60) that the critical point $K_1^\dagger$ satisfies if $h(\delta_\dagger) < 0$ takes the form

$$c\big[\alpha^{-n}(K_1) - \delta_\dagger^{-n}\big] + \frac{n}{n - \vartheta}\big[\alpha^{-(n-\vartheta)}(K_1) - \delta_\dagger^{-(n-\vartheta)}\big] - K_1\alpha^{-n}(K_1) = 0,$$

while that critical point $K_0^\star$ defined by (61) if $h(\delta_\dagger) < 0$ admits the expression

$$K_0^\star = -K_1 + \frac{rK_1 - c}{r}\left[1 - \left(\frac{\hat{x}}{\alpha}\right)^n\right] + \frac{n}{r(n - \vartheta)}\hat{x}^n(\alpha^{-(n-\vartheta)} - \hat{x}^{-(n-\vartheta)}),$$

where $\hat{x}$ is the unique solution to the equation

$$(rK_1 - c)\left[\left(\frac{\hat{x}}{\alpha}\right)^m - \left(\frac{\hat{x}}{\alpha}\right)^n\right] + \frac{m}{\vartheta - m}\alpha^{\vartheta - m}\hat{x}^m + \frac{n}{n - \vartheta}\alpha^{-(n-\vartheta)}\hat{x}^n - \frac{\vartheta(n-m)}{(n-\vartheta)(\vartheta - m)}\hat{x}^\vartheta = 0.$$

If the problem data is as in Example 1, then this case characterises the optimal strategy if and only if ($K < 0$ and $-1 \leq c < -rK$) or ($K < 0$ and $rK \leq c < -1$) or ($K < 0$, $c < \min\{-1, rK\}$ and $K_1^\dagger \leq \frac{1}{2}$) or ($K < 0$, $c < \min\{-1, rK\}$, $\frac{1}{2} < K_1^\dagger$ and $K_0^\dagger \leq \frac{1}{2}$). If $K = -\frac{1}{2}$ and $c = 0$, then

$$\delta_\dagger = \frac{1}{2}, \quad \zeta \simeq 1.283, \quad \alpha \simeq 2.678, \quad A = \frac{1}{8}, \quad \Delta_1 \simeq 0.428 \quad \text{and} \quad \Delta_2 \simeq 0.101,$$

if $K = -1$ and $c = -\frac{3}{2}$, then

$$\delta_\dagger = \frac{7}{4}, \quad \zeta \simeq 2.625, \quad \alpha \simeq 5.250, \quad A = \frac{49}{32}, \quad \Delta_1 \simeq 1.750 \quad \text{and} \quad \Delta_2 \simeq 0.048,$$

if $K = -\frac{1}{2}$ and $c = -\frac{3}{2}$, then

$$K_1^\dagger \simeq 0.007, \quad \delta_\dagger = \frac{5}{4}, \quad \zeta \simeq 1.798, \quad \alpha \simeq 4.771,$$
$$A = \frac{25}{32}, \quad \Delta_1 \simeq 0.599 \quad \text{and} \quad \Delta_2 \simeq 0.052,$$

while, if $K = -\frac{1}{2}$ and $c = -4$, then

$$K_1^\dagger \simeq 0.595, \quad \hat{x} \simeq 2.542, \quad K_0^\dagger \simeq 5 \times 10^{-4}, \quad \delta_\dagger = \frac{5}{2}, \quad \zeta \simeq 2.440, \quad \alpha \simeq 8.336,$$
$$A = \frac{25}{8}, \quad \Delta_1 \simeq 0.813 \quad \text{and} \quad \Delta_2 \simeq 0.028.$$

Case III.2 (Figure 8) This case is the modification of Case II.3 that arises when it is optimal to abandon the project when this is in its "closed" mode and the process X takes sufficiently low values. In this case,

$$\mathcal{A}_1 =]0, \delta], \quad \mathcal{P} =]\delta, \gamma[\cup]\beta, \infty[, \quad \mathcal{S}_{\text{out}} = [\gamma, \beta],$$
$$\mathcal{A}_0 =]0, \zeta], \quad \mathcal{W} =]\zeta, \alpha[\quad \text{and} \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[,$$

and the required solution to the HJB equation (7)–(8) should be given by the functions

$$w_1(x) = \begin{Bmatrix} -K, & \text{if } x \leq \delta \\ \Gamma_1 x^m + \Gamma_2 x^n + R_h(x), & \text{if } \delta < x < \gamma \\ \Delta_1 x^m + \Delta_2 x^n - K_0, & \text{if } \gamma \leq x \leq \beta \\ A x^m + R_h(x), & \text{if } x > \beta \end{Bmatrix} \tag{66}$$

$$\text{and} \quad w_0(x) = \begin{Bmatrix} -K, & \text{if } x \leq \zeta \\ \Delta_1 x^m + \Delta_2 x^n, & \text{if } \zeta < x < \alpha \\ A x^m + R_h(x) - K_1, & \text{if } x \geq \alpha \end{Bmatrix}. \tag{67}$$

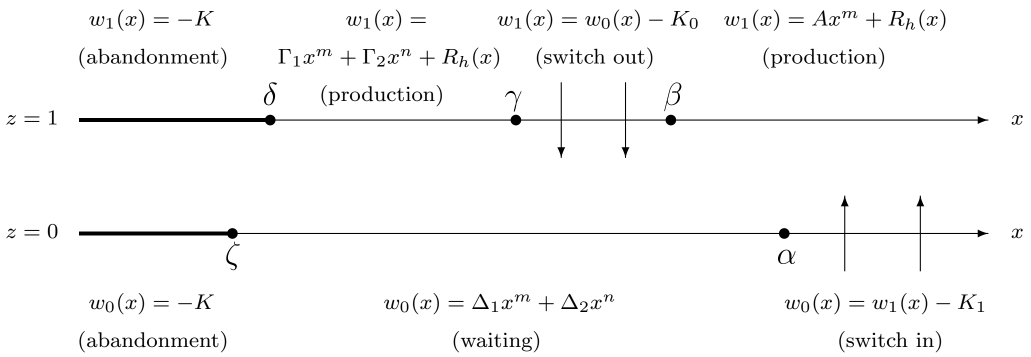

Figure 8. Illustration of the regions determining the optimal strategy in the context of Case III.2

Once again, we specify Γ_1, Γ_2, A, Δ_1, Δ_2, ζ, δ, γ, β and α by requiring that the functions w_1, w_0 should be C^1. This requirement implies that the free-boundary points ζ, δ, γ, β and α should satisfy the system of equations given by (38), (39),

$$\begin{aligned} G_3(\delta,\gamma,\beta) &:= n\int_\delta^\infty s^{-n-1}\big[h(s)+rK\big]\,ds - n\int_\gamma^\beta s^{-n-1}\big[h(s)+rK_0\big]\,ds \\ &= 0, \qquad (68)\\ G_4(\zeta,\beta) &:= n\int_\beta^\infty s^{-n-1}\big[h(s)+rK_0\big]\,ds + rK\zeta^{-n} \\ &= 0 \qquad (69)\\ \text{and}\quad G_5(\zeta,\delta,\gamma) &:= m\int_0^\gamma s^{-m-1}\big[h(s)+rK_0\big]\,ds - m\int_0^\delta s^{-m-1}\big[h(s)+rK\big]\,ds - rK\zeta^{-m} \\ &= 0, \qquad (70) \end{aligned}$$

while the constants Γ_1, Γ_2, A, Δ_1, Δ_2 should be given by

$$\Gamma_1 = -\frac{1}{\sigma^2(n-m)}\int_0^\delta s^{-m-1}\big[h(s)+rK\big]\,ds, \qquad (71)$$

$$\Gamma_2 = -\frac{1}{\sigma^2(n-m)}\int_\delta^\infty s^{-n-1}\big[h(s)+rK\big]\,ds, \qquad (72)$$

$$\Delta_1 = \frac{rK\zeta^{-m}}{\sigma^2 m(n-m)}, \qquad \Delta_2 = -\frac{rK\zeta^{-n}}{\sigma^2 n(n-m)} \qquad (73)$$

$$\text{and}\quad A = \Delta_1 - \frac{1}{\sigma^2(n-m)}\int_0^\alpha s^{-m-1}\big[h(s)-rK_1\big]\,ds. \qquad (74)$$

Lemma 8 *The system of equations (38), (39), (68), (69), (70) has a unique solution $(\delta,\gamma,\beta,\alpha)$ such that $0<\delta<\gamma<\beta<\alpha$ and the functions w_1, w_0 defined by (66), (67), for*

$\Gamma_1 > 0$, $\Gamma_2 > 0$, $A > 0$, $\Delta_1 > 0$, $\Delta_2 > 0$ given by (71)–(74), are increasing and satisfy the HJB equation (7)–(8) if and only if

$$K < 0, \quad h(0) < -rK_0, \quad h(\delta_\dagger) < 0, \quad K_1 < K_1^\dagger \quad \text{and} \quad K_0 < K_0^\dagger,$$

where $\delta_\dagger > 0$ is the unique solution to (44), and $K_1^\dagger > 0$ (resp., $K_0^\dagger > 0$), which depends on all problem data except K_1, K_0 (resp., K_0) is as in Lemma 7.

Example 9 If h is the function given by (4), then the system of equations (56)–(57) takes the form

$$(rK + c)\delta^{-n} + \frac{n}{n-\vartheta}(\delta^{-(n-\vartheta)} - \gamma^{-(n-\vartheta)} + \beta^{-(n-\vartheta)}) + (rK_0 + c)(\beta^{-n} - \gamma^{-n}) = 0,$$

$$(rK_0 + c)\beta^{-n} + \frac{n}{n-\vartheta}\beta^{-(n-\vartheta)} + rK\zeta^{-n} = 0,$$

$$(rK + c)\delta^{-m} - (rK_0 + c)\gamma^{-m} - \frac{m}{\vartheta - m}(\delta^{\vartheta-m} - \gamma^{\vartheta-m}) - rK\zeta^{-m} = 0,$$

while

$$\Gamma_1 = -\frac{\delta^{-m}}{\sigma^2(n-m)}\left(-\frac{rK+c}{m} + \frac{\delta^\vartheta}{\vartheta - m}\right), \quad \Gamma_2 = -\frac{\delta^{-n}}{\sigma^2(n-m)}\left(\frac{rK+c}{n} + \frac{\delta^\vartheta}{n-\vartheta}\right)$$

$$\text{and} \quad A = \Delta_1 - \frac{\alpha^{-m}}{\sigma^2(n-m)}\left(\frac{rK_1 - c}{m} + \frac{\alpha^\vartheta}{\vartheta - m}\right).$$

If the problem data is as in Example 1, then this case characterises the optimal strategy if and only if $K < 0$, $c < \min\{-1, rK\}$, $\frac{1}{2} < K_1^\dagger$ and $\frac{1}{2} < K_0^\dagger$. If $K = -\frac{1}{50}$ and $c = -13$, then

$$K_1^\dagger \simeq 392.048, \quad \hat{x} \simeq 9.756, \quad K_0^\dagger \simeq 0.501,$$
$$\zeta \simeq 0.806, \quad \delta \simeq 6.514, \quad \gamma \simeq 7.924, \quad \beta \simeq 7.942, \quad \alpha \simeq 22.275,$$
$$\Gamma_1 \simeq 21.242, \quad \Gamma_2 \simeq 5 \times 10^{-5}, \quad \Delta_1 \simeq 0.011, \quad \Delta_2 \simeq 0.010 \quad \text{and} \quad A \simeq 21.266.$$

4.4 The main result

The following table summarises the conditions on the problem data that determine the optimality of each of the cases that we have studied in Sections 4.1-4.3. An inspection of the table reveals that these mutually exclusive conditions exhaust the whole range of possible problem data. Therefore, Lemmas 1-8 provide a complete solution to the HJB equation (7)–(8).

Conditions on $K_1 > 0$, $K_0 > 0$, $K \in \mathbb{R}$ and $h(\cdot)$		Case	w_1, w_0
$0 \le K$	$rK_1 \le h(0)$	I.1, Lemma 1	(25), (26)
	$\max\{-rK_0,\, -rK\} \le h(0) < rK_1$	I.2, Lemma 2	(25), (27)
	$K_0 \le K$ and $h(0) < -rK_0$	II.1, Lemma 4	(35), (34)
	$K < K_0$ and $-rK_0 \le h(0) < -rK$	II.2, Lemma 5	(40), (41)
	$K < K_0^\star \le K_0$ and $h(0) < -rK_0$	II.2, Lemma 5	(40), (41)
	$K < K_0 < K_0^\star$ and $h(0) < -rK_0$	II.3, Lemma 6	(48), (49)
$K < 0$	$rK_1 - rK \le h(0)$	I.1, Lemma 1	(25), (26)
	$-rK \le h(0) < rK_1 - rK$	I.3, Lemma 3	(25), (30)
	$-rK_0 \le h(0) < -rK$	III.1, Lemma 7	(54), (55)
	$h(0) < -rK_0$ and $\big[h(\delta_\dagger) \ge 0$ or $\big(h(\delta_\dagger) < 0$ and $K_1 \ge K_1^\dagger\big)$ or $\big(h(\delta_\dagger) < 0,\ K_1 < K_1^\dagger$ and $K_0 \ge K_0^\dagger\big)\big]$	III.1, Lemma 7	(54), (55)
	$h(0) < -rK_0$, $h(\delta_\dagger) < 0$, $K_1 < K_1^\dagger$ and $K_0 < K_0^\dagger$	III.2, Lemma 8	(66), (67)

Theorem 9 *Consider the stochastic optimal control problem formulated in Section 2 and suppose that Assumption 1 holds true. The value function v is given by (6), where w_1, w_0 are as in Lemmas 1-8. In each of the possible cases arising, the optimal strategy (Z°, τ°) is as discussed in the proof below.*

Proof. Given any initial condition $(z, x) \in \{0,1\} \times \,]0,\infty[$ and any strategy $(Z, \tau) \in \Pi_z$, the monotone convergence theorem and (5) in Assumption 1 imply that $\lim_{m\to\infty} J_{z,x}(Z, \tau \wedge T_m) = J_{z,x}(Z, \tau)$ for every sequence of times (T_m) such that $T_m \to \infty$. By construction, there exists a constant $C > 0$ such that

$$\big|w(z,x)\big| \le C\big(1 + \big|R_h(x)\big|\big) \quad \text{and} \quad \big|w_x(z,x)\big| \le C\big(1 + \big|R_h'(x)\big|\big) \quad \text{for all } x > 0,$$

where $w(z,x) = zw_1(x) + (1-z)w_0(x)$. These estimates, (20) and (21) imply that

$$\lim_{T\to\infty} \mathbb{E}\Big[e^{-rT}\big|w(Z_T, X_T)\big|\Big] = 0,$$

and that the process M defined by

$$M_T = \int_0^T e^{-rt} X_t w_x(Z_t, X_t)\, dW_t$$

is a square integrable martingale for every switching strategy $Z \in \mathcal{Z}$. Furthermore, w_1, w_0 are C^1 as well as C^2 outside a finite set, and they satisfy the HJB equation (7)–(8) in the classical sense. In view of these observations, we can see that Theorem 1 in Zervos [30] implies that $w = v$ as long as there exists an optimal strategy (Z°, τ°), namely, a switching strategy $Z^\circ \in \mathcal{Z}$ such that

$$\begin{aligned} \sigma^2 X_t^2 w_{xx}(Z_t^\circ, X_t) + bX_t w_x(Z_t^\circ, X_t) - rw(Z_t^\circ, X_t) + Z_t^\circ h(X_t) &= 0, \\ \big[w(1, X_t) - w(0, X_t) - K_1\big](\Delta Z_t^\circ)^+ &= 0 \\ \text{and} \quad \big[w(0, X_t) - w(1, X_t) - K_0\big](\Delta Z_t^\circ)^- &= 0, \end{aligned}$$

for all $t \leq \tau^\circ$, where

$$\tau^\circ = \inf\left\{t \geq 0 \mid \ w(Z_t^\circ, X_t) = -K\right\}.$$

Such a switching strategy is constructed in Duckworth and Zervos [8, Theorem 5] and Zervos [30, Theorem 1] for Cases I.1, I.2, II.1, II.2 and II.3. For the remaining cases, it can be constructed using similar arguments. □

5 Conclusion

In this paper, we considered a general entry-exit-scrapping model with positive switching costs. We fully characterised the optimal switching and abandonment strategy by deriving an explicit solution to the control problem's HJB equation. It turned out that the optimal strategy can take eight qualitatively different forms, depending on the problem data. The analysis of these cases gives rise to the observation that value may be added by waiting before choosing between two investment actions of a qualitatively different nature (one partially reversible and one totally irreversible). Furthermore, it suggests that having "waiting" regions to separate regions of the state space associated with different types of actions should be a generic rather than an exceptional property of the optimal strategy in real option models.

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