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Chromatic Numbers of Exact Distance Graphs^{*}

Jan van den Heuvel,[†] H.A. Kierstead[‡] and Daniel A. Quiroz^{†§}

Abstract

For any graph $G = (V, E)$ and positive integer p , the *exact distance- p graph* $G^{[2p]}$ is the graph with vertex set V , which has an edge between vertices x and y if and only if x and y have distance p in G . For odd p , Nešetřil and Ossona de Mendez proved that for any fixed graph class with *bounded expansion*, the chromatic number of $G^{[2p]}$ is bounded by an absolute constant.

Using the notion of *generalised colouring numbers*, we give a much simpler proof for the result of Nešetřil and Ossona de Mendez, which at the same time gives significantly better bounds. In particular, we show that for any graph G and odd positive integer p , the chromatic number of $G^{[2p]}$ is bounded by the weak $(2p - 1)$ -colouring number of G . For even p , we prove that $\chi(G^{[2p]})$ is at most the weak $(2p)$ -colouring number times the maximum degree.

For odd p , the existing lower bound on the number of colours needed to colour $G^{[2p]}$ when G is planar is improved. Similar lower bounds are given for K_t -minor free graphs.

Key Words: *bounded expansion, chromatic number, exact distance graphs, generalised colouring numbers, planar graphs*

1 Introduction and Main Results

1.1 Powers, exact powers, and exact distance graphs

All graphs in this paper are assumed to be finite, undirected, simple and without loops. For a graph $G = (V(G), E(G))$ (or just (V, E) if the graph under consideration is clear) and vertices $x, y \in V$, let $d_G(x, y)$ denote the distance between x and y in G , that is, the number of edges contained in a shortest path between x and y .

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[†] Department of Mathematics, London School of Economics and Political Science, London WC2A 2AE, UK.

[‡] School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287, USA.

[§] Current affiliation: Centro de Modelamiento Matemático, Universidad de Chile, Santiago, Chile.

Email: j.van-den-heuvel@lse.ac.uk, kierstead@asu.edu, dquiroz@dim.uchile.cl.

For a positive integer p , the p -th power graph $G^p = (V, E^p)$ of G is the graph with V as its vertex set and E^p contains the edge xy if and only if $d_G(x, y) \leq p$. Problems related to the chromatic number $\chi(G^p)$ of power graphs G^p were first considered by Kramer and Kramer [14, 15] in 1969 and have enjoyed significant attention ever since. It is clear that for $p \geq 2$ any power of a star is a clique, and hence there are not many classes of graphs for which $\chi(G^p)$ can be bounded by a constant. An easy argument shows that for a graph G with maximum degree $\Delta(G) \geq 3$ we have

$$\chi(G^p) \leq 1 + \Delta(G^p) \leq 1 + \Delta(G) \cdot \sum_{i=0}^{p-1} (\Delta(G) - 1)^i \in \mathcal{O}(\Delta(G)^p).$$

However, there are many classes of graphs for which it is possible to find much better upper bounds. Recall that a graph G is k -degenerate if every subgraph of G contains a vertex of degree at most k .

Theorem 1.1 (Agnarsson & Halldórsson [1]).

Let k and p be positive integers. There exists a constant $c = c(k, p)$ such that for every k -degenerate graph G we have $\chi(G^p) \leq c \cdot \Delta(G)^{\lfloor p/2 \rfloor}$.

In this result, the exponent on $\Delta(G)$ is best possible (see below). In particular, $\chi(G^2)$ is at most linear in $\Delta(G)$ for planar graphs G . Wegner [27] conjectured that every planar graph G with $\Delta(G) \geq 8$ satisfies $\chi(G^2) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor + 1$, and gave examples that show this bound would be tight. The conjecture has attracted considerable attention since it was stated in 1977. For more on this conjecture we refer the reader to [2, 16].

In [22, Section 11.9], Nešetřil and Ossona de Mendez define the notion of *exact power graph*. Let $G = (V, E)$ be a graph and p a positive integer. The *exact p -power graph* $G^{\natural p}$ has V as its vertex set, and xy is an edge in $G^{\natural p}$ if and only if there is in G a path of length p (i.e. with p edges) between the vertices x and y (the path need not be a shortest path). Similarly, they define the *exact distance- p graph* $G^{\lfloor p \rfloor}$ as the graph with V as its vertex set, and xy as an edge if and only if $d_G(x, y) = p$. Since obviously $E(G^{\lfloor p \rfloor}) \subseteq E(G^{\natural p}) \subseteq E(G^p)$, we have $\chi(G^{\lfloor p \rfloor}) \leq \chi(G^{\natural p}) \leq \chi(G^p)$.

For planar graphs G , Theorem 1.1 gives that the exact p -power graphs $G^{\natural p}$ satisfy $\chi(G^{\natural p}) \in \mathcal{O}(\Delta(G)^{\lfloor p/2 \rfloor})$. This result is best possible, even for outerplanar graphs, as the following examples show. For $k \geq 2$ and $p \geq 4$, let $T_{k, \lfloor p/2 \rfloor}$ be the k -regular tree of radius $\lfloor \frac{1}{2}p \rfloor$ with root v . We say that a vertex z is at level ℓ if $d(v, z) = \ell$. For every edge xy between vertices at levels ℓ and $\ell + 1$ for some $\ell \geq 1$, we do the following: if p is even, then add a path of length $\ell + 1$ between x and y ; if p is odd, then add paths of length $\ell + 1$ and $\ell + 2$ between x and y . Call the resulting graph $G_{k,p}$. It is straightforward to check that $\Delta(G_{k,p}) \leq 2k$ for even p , that $\Delta(G_{k,p}) \leq 3k$ for odd p , and that there is a path of length p between any two vertices at level $\lfloor \frac{1}{2}p \rfloor$. Since there are $k(k-1)^{\lfloor p/2 \rfloor - 1}$ vertices at level $\lfloor \frac{1}{2}p \rfloor$, this immediately means that $\chi(G_{k,p}^{\natural p}) \geq k(k-1)^{\lfloor p/2 \rfloor - 1} \in \Omega(\Delta(G_{k,p})^{\lfloor p/2 \rfloor})$.

Surprisingly, for exact distance graphs, the situation is quite different. In that case we can prove that for planar graphs G and odd p we have $\chi(G^{\lfloor p \rfloor}) \in \mathcal{O}(1)$, while for even p we have $\chi(G^{\lfloor p \rfloor}) \in \mathcal{O}(\Delta(G))$. These bounds are actually special cases of the following more

general results. We will recall the concept of a *graph class with bounded expansion* in the next subsection.

Theorem 1.2.

Let \mathcal{K} be a class of graphs with bounded expansion.

- (a) *Let p be an odd positive integer. Then there exists a constant $C = C(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ we have $\chi(G^{\lfloor p \rfloor}) \leq C$.*
- (b) *Let p be an even positive integer. Then there exists a constant $C' = C'(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ we have $\chi(G^{\lfloor p \rfloor}) \leq C' \cdot \Delta(G)$.*

We will give two proofs of part (a). The two proofs give incomparable bounds. Also, both proofs are considerably shorter and provide better bounds than the original proof of part (a) of Nešetřil and Ossona de Mendez [22, Theorem 11.8]. Theorem 1.2 (b) is new, as far as we are aware.

As we showed above, if we consider exact powers instead of exact distance graphs, then we need to use bounds involving $\Delta(G)$ if we want to bound $\chi(G^{\lfloor p \rfloor})$, even for odd p and if G is planar. However, by adding the condition that G has sufficiently large *odd girth* (length of a shortest odd cycle), $\chi(G^{\lfloor p \rfloor})$ can be bounded without reference to $\Delta(G)$, for odd p . It follows from Theorem 1.2 (a) that this is possible if the odd girth is at least $2p + 1$. This is because odd girth at least $2p + 1$ guarantees that if there is a path of length p between u and v , then any shortest uv -path has odd length. With some more care we can reprove the following.

Theorem 1.3 (Nešetřil & Ossona de Mendez [22, Theorem 11.7]).

Let \mathcal{K} be a class of graphs with bounded expansion and let p be an odd positive integer. Then there exists a constant $M = M(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ with odd girth at least $p + 1$ we have $\chi(G^{\lfloor p \rfloor}) \leq M$.

Theorem 1.2(a) is quite surprising, since already for planar graphs G , the exact distance graphs $G^{\lfloor p \rfloor}$ can be very dense. To see this, for $i \geq 2$ let L_i be obtained from the complete graph K_4 by subdividing each edge $i - 1$ times (i.e. by replacing each edge by a path of length i). For $k \geq 1$, form $L_{i,k}$ by adding four sets of k new vertices to L_i and joining all k vertices in the same set to one of the vertices of degree three in L_i . See Figure 1 for a sketch of $L_{1,k}$.

It is easy to check that $L_{i,k}$ is a planar graph with $4 + 6(i - 1) + 4k$ vertices, while $L_{i,k}^{\lfloor i(i+2) \rfloor}$ has $6k^2$ edges. So for fixed i and large k , the graph $L_{i,k}^{\lfloor i(i+2) \rfloor}$ has approximately $3/4$ times the number of edges of the complete graph on the same number of vertices. Apart from having unbounded density, the graphs $L_{i,k}^{\lfloor i(i+2) \rfloor}$ have unbounded colouring number (and even unbounded *list chromatic number*), since $L_{i,k}^{\lfloor i(i+2) \rfloor}$ contains a complete bipartite graph $K_{k,k}$ as an (induced) subgraph. This makes the fact that these graphs have bounded chromatic number even more surprising.

It is interesting to see what actual upper and lower bounds we can get for the chromatic numbers of $G^{\lfloor p \rfloor}$ for G from some specific classes of graphs and for specific values of (odd) p . Using the proof in [22], it follows that for $p = 3$ and for planar graphs G we can get the upper

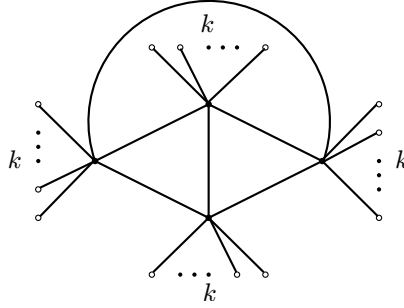


Figure 1: A graph $L_{1,k}$ such that $L_{1,k}^{[k^3]}$ has edge density approximately $3/4$.

bound $\chi(G^{[k^3]}) \leq 5 \cdot 2^{20,971,522}$ (see also Subsection 1.3). On the other hand, [22, Exercise 11.4] gives an example of a planar graph G with $\chi(G^{[k^3]}) = 6$.

Our new proof of Theorem 1.2 (a) already gives a much smaller upper bound for $\chi(G^{[k^3]})$ for planar graph G . By a more careful analysis, we can reduce that upper bound even further, giving the bound in the following result. We also managed to increase the lower bound, although by one only. Details can be found in Section 4.

Theorem 1.4.

- (a) For every planar graph G we have $\chi(G^{[k^3]}) \leq 105$.
- (b) There exists a planar graph G_5 such that $\chi(G_5^{[k^3]}) = 7$.

For outerplanar graphs G we have that $\chi(G^{[k^3]}) \leq 10$, while there exists an outerplanar graph G_4 such that $\chi(G_4^{[k^3]}) = 5$ (see the results in Sections 3 and 4).

1.2 Generalised colouring numbers and main results

When solving an optimisation problem it is often useful to preorder the input so as to minimise some parameter. One such parameter is the *colouring number* $\text{col}(G)$ of a graph G . This is the minimum integer k such that there is a linear ordering L of V such that every vertex y has at most $k - 1$ neighbours x with $x <_L y$. (So the colouring number is one more than the *degeneracy* of a graph.) It is well-known and easy to see that the chromatic number $\chi(G)$ of a graph G satisfies $\chi(G) \leq \text{col}(G)$. Although this bound is far from being tight in many cases, it is often used to show that a specific class of graphs has bounded chromatic number.

Different generalisations of the colouring number can be found in the literature. Chen and Schelp [5] proved that the class of planar graphs has linear Ramsey number by also controlling, for all vertices v , the number of smaller vertices that can be reached by a path of length two, whose middle vertex is larger than v . Various versions of their idea were applied by Kierstead and Trotter [11], Kierstead [9], and Zhu [28] to problems concerning the game chromatic number of graphs and gave rise to the 2-colouring number defined below. In their study of oriented game chromatic number of graphs, Kierstead and Trotter [12] considered paths of length four with different configurations of “large” internal vertices, which later motivated the notions of 4-colouring number and weak 4-colouring number. Kierstead and Yang [13]

bounded the game colouring number in terms of the 2-colouring number, and Kierstead and Kostochka [10] applied game colouring number to a (non-game) packing problem.

All of these notions are encompassed in the concepts of the *k-colouring number* and the *weak k-colouring number* of a graph, both of which were first introduced by Kierstead and Yang [13].

Let $G = (V, E)$ be a graph, L a linear ordering of V , and k a positive integer. We say that a vertex $x \in V$ is *k-accessible* from $y \in V$ if $x <_L y$ and there exists an xy -path P of length at most k such that $y <_L z$ for all internal vertices z of P . Similarly, if all internal vertices z of P satisfy the less restrictive condition that $x <_L z$, then we say that x is *weakly k-accessible* from y . Let $R_{L,k}(y)$ be the set of vertices that are k -accessible from y , and $Q_{L,k}(y)$ the set of vertices that are weakly k -accessible from y . The *k-colouring number* $\text{col}_k(G)$ and *weak k-colouring number* $\text{wcol}_k(G)$ of a graph G are defined as follows:

$$\begin{aligned}\text{col}_k(G) &= 1 + \min_L \max_{y \in V} |R_{L,k}(y)|, \\ \text{wcol}_k(G) &= 1 + \min_L \max_{y \in V} |Q_{L,k}(y)|.\end{aligned}$$

If we allow paths of any length (but still have restrictions on the position of the internal vertices), we get $R_{L,\infty}(y)$, $Q_{L,\infty}(y)$, the ∞ -colouring number $\text{col}_\infty(G)$ and the *weak ∞ -colouring number* $\text{wcol}_\infty(G)$.

We now state the main results of this paper.

Theorem 1.5.

- (a) For every odd positive integer p and every graph G we have $\chi(G^{\lfloor \frac{1}{2}p \rfloor}) \leq \text{wcol}_{2p-1}(G)$.
- (b) For every even positive integer p and every graph G we have $\chi(G^{\lfloor \frac{1}{2}p \rfloor}) \leq \text{wcol}_{2p}(G) \cdot \Delta(G)$.

Theorem 1.6.

Let p be an odd positive integer and G a graph. Set $q = \text{wcol}_p(G)$.

- (a) We have $\chi(G^{\lfloor \frac{1}{2}p \rfloor}) \leq (\lfloor \frac{1}{2}p \rfloor + 2)^q$.
- (b) If G has odd girth at least $p + 1$, then $\chi(G^{\lfloor \frac{1}{2}p \rfloor}) \leq (\lfloor \frac{1}{2}p \rfloor + 2)^q$.

An interesting aspect of generalised colouring numbers is that these invariants can also be seen as gradations between the colouring number $\text{col}(G)$ and two important minor monotone invariants, namely the *tree-width* $\text{tw}(G)$ and the *tree-depth* $\text{td}(G)$ (which is the minimum height of a depth-first search tree for a supergraph of G , see [20]). More explicitly, for every graph G we have the following relations.

Proposition 1.7.

- (a) $\text{col}(G) = \text{col}_1(G) \leq \text{col}_2(G) \leq \dots \leq \text{col}_\infty(G) = \text{tw}(G) + 1$;
- (b) $\text{col}(G) = \text{wcol}_1(G) \leq \text{wcol}_2(G) \leq \dots \leq \text{wcol}_\infty(G) = \text{td}(G)$.

The equality $\text{col}_\infty(G) = \text{tw}(G) + 1$ was first proved in [6]. The equality $\text{wcol}_\infty(G) = \text{td}(G)$ is [22, Lemma 6.5].

Relations between the two sets of numbers exist as well. Clearly, $\text{col}_1(G) = \text{wcol}_1(G)$ and $\text{col}_k(G) \leq \text{wcol}_k(G)$. For the converse, Kierstead and Yang [13] proved that $\text{wcol}_k(G) \leq$

$(\text{col}_k(G))^k$. Note that this means that if one of the generalised colouring numbers is bounded for a class of graphs (for some k), then so is the other one.

Shortly after Nešetřil and Ossona de Mendez [21] introduced the notion of *classes with bounded expansion*, Zhu provided, in [29], a way of characterising these classes in terms of the weak k -colouring numbers. We will use this characterisation as a definition.

Definition 1.8.

A class of graphs \mathcal{K} has *bounded expansion* if and only if there exist constants c_k , $k = 1, 2, \dots$ such that $\text{wcol}_k(G) \leq c_k$ for all k and all $G \in \mathcal{K}$.

By this definition, Theorem 1.2 (a) follows directly from both Theorems 1.5 (a) and 1.6 (a).

We give the proofs of Theorems 1.5 and 1.6 in the next section. The proof of Theorem 1.6 actually proves a stronger result. For two graphs $G = (V, E)$ and $G' = (V, E')$ on the same vertex set, define $G \cup G' = (V, E \cup E')$. Then the upper bound in both parts of Theorem 1.6 holds for $\chi(G^{\lfloor 1 \rfloor} \cup G^{\lfloor 3 \rfloor} \cup \dots \cup G^{\lfloor p \rfloor})$ and $\chi(G^{\lfloor 1 \rfloor} \cup G^{\lfloor 3 \rfloor} \cup \dots \cup G^{\lfloor p \rfloor})$, respectively.

A natural question is if for even p we can generalise the bound in Theorem 1.5 (b) by a similar bound $\chi(G^{\lfloor 2 \rfloor} \cup G^{\lfloor 4 \rfloor} \cup \dots \cup G^{\lfloor p \rfloor}) \leq C \cdot \Delta(G)$, where C depends on the generalised colouring numbers. But this is not possible. Let $T_{\Delta,2}$ be the Δ -regular tree of radius 2. Then we have $\text{wcol}_1(T_{\Delta,2}) = 1$ and $\text{wcol}_k(T_{\Delta,2}) = 2$ for all $k \geq 2$. It is easy to check that $\chi(T_{\Delta,2}^{\lfloor 2 \rfloor}) = \chi(T_{\Delta,2}^{\lfloor 4 \rfloor}) = \Delta$, but $\chi(T_{\Delta,2}^{\lfloor 2 \rfloor} \cup T_{\Delta,2}^{\lfloor 4 \rfloor}) = \Delta(\Delta - 1) + 1$. These examples generalise to larger distances.

The results in Theorem 1.6 are best possible in the sense that they give upper bounds of $\chi(G^{\lfloor p \rfloor})$ and $\chi(G^{\lfloor p \rfloor})$ that depend on $\text{wcol}_p(G)$ only, whereas no such results are possible that depend on $\text{wcol}_k(G)$ with $k < p$. To see this, for $n, p \geq 2$ let $S_{n,p}$ be the $(p - 1)$ -subdivision of the complete graph K_n (that is, the graph formed by replacing the edges of K_n by paths of length p). Then we obviously have $\chi(S_{n,p}^{\lfloor p \rfloor}) = n$. On the other hand we have $\text{wcol}_{p-1}(S_{n,p}) \leq p + 1$. To verify this, order the vertices of $S_{n,p}$ as follows. First order the branch vertices (the vertices in the original clique), and then order the subdivision vertices in any way. Clearly, each branch vertex will not weakly $(p - 1)$ -access any other vertex. An internal vertex of a subdivided edge can only weakly $(p - 1)$ -access the other p vertices on the path that replaced the edge (including the two end-vertices of the path). So for fixed odd $p \geq 3$ we cannot bound $\chi(S_{n,p}^{\lfloor p \rfloor})$ by an expression that involves $\text{wcol}_{p-1}(S_{n,p})$ only.

The bound on the odd girth in Theorem 1.6 (b) is also best possible. To show this, for $k, p \geq 1$ let $A_{k,p}$ be formed by taking the path P_{p-1} of length $p - 2$, and adding k new vertices that are adjacent to both end-vertices of P_{p-1} only. It is clear that if p is odd, then $A_{k,p}$ has odd girth p . Since between any of the k extra vertices there is a path of length p , we have $\chi(A_{k,p}^{\lfloor p \rfloor}) \geq k$. The ordering obtained by taking the two end-vertices of P_{p-1} first, and then ordering the other vertices in any way, shows that $\text{wcol}_p(A_{k,p}) \leq p - 1$. So for fixed odd $p \geq 3$ we cannot bound $\chi(A_{k,p}^{\lfloor p \rfloor})$ by an expression that involves $\text{wcol}_p(A_{k,p})$ only.

Nešetřil and Ossona de Mendez [22, Section 11.9.3] give examples that even if we replace “there exists a path of length p between x and y ” by “there exists an *induced* path of length p between x and y ” in the definition of $G^{\lfloor p \rfloor}$, it is not possible to reduce the bound on the odd girth in Theorem 1.6 (b).

Finally, we point out a connection between the bound on $\chi(G^{\natural 1} \cup G^{\natural 3} \cup \dots \cup G^{\natural p})$ in the proof of Theorem 1.6 (b) and results of Naserasr et al. [19]. For a positive integer p and graph $G = (V, E)$, let the p -th walk power $G^{(p)}$ of G be the graph with vertex set V and where xy is an edge if and only if there exists a walk of length p between x and y . It is easy to see (see also Lemma 2.3) that for odd p , if G has odd girth at least $p + 1$, then for any two vertices $x, y \in V(G)$ there exists a walk of length p between x and y if and only if there exists a path of odd length at most p between x and y . Hence for odd p , if G has odd girth at least $p + 1$, then $G^{(p)}$ is isomorphic to $\chi(G^{\natural 1} \cup G^{\natural 3} \cup \dots \cup G^{\natural p})$. So it follows from [19, Theorem 13] that for odd p there exist planar graphs G with odd girth at least $p + 1$ such that $\chi(G^{\natural 1} \cup G^{\natural 3} \cup \dots \cup G^{\natural p}) = \chi(G^{(p)}) \geq 2^{p+1}$.

1.3 Explicit upper bounds

The upper bounds obtained by Nešetřil and Ossona de Mendez in their proof of Theorem 1.2 (a) are very large, even for $p = 3$. Their proof relies on the concept of p -centred colourings of graphs. A (proper) colouring of a graph G is a p -centred colouring if for each connected induced subgraph H of G , either one colour appears exactly once on H or H gets at least p colours. This is what is proved in [22].

Theorem 1.9 (Nešetřil & Ossona de Mendez [22]).

Let p be an odd positive integer. If a graph G has a p -centred colouring that uses at most $N = N(p)$ colours, then $\chi(G^{\natural p}) \leq N2^{N2^N}$.

Given a graph G , the *star chromatic number* $\chi_s(G)$ is the smallest number of colours needed to properly colour G such that every two colours induce a star forest (a forest where every component is isomorphic to a *star* $K_{1,m}$). It is easy to see that a colouring of a graph is 3-centred if and only if every two colours induce a star forest. Albertson et al. [3] showed that the star chromatic number of planar graphs is at most 20, and there exist planar graphs with star chromatic number equal to 10. This means that the best upper bound known for $\chi(G^{\natural 3})$ for planar graphs G given by Theorem 1.9 is $5 \cdot 2^{20,971,522}$, while the best possible upper bound for planar graphs that can be found using that theorem directly is $5 \cdot 2^{10,241}$.

An alternative bound can be obtained from Theorem 1.9 using the following result.

Theorem 1.10 (Zhu [29]).

Every graph G has a p -centred colouring that uses at most $\text{wcol}_{2p-2}(G)$ colours.

Corollary 1.11.

Let p be an odd positive integer and G a graph. Setting $W = \text{wcol}_{2p-2}(G)$ we have $\chi(G^{\natural p}) \leq W2^{W2^W}$.

More recently, Stavropoulos [26] improved Corollary 1.11.

Theorem 1.12 (Stavropoulos [26]).

For every odd integer $p \geq 3$ and every graph G we have $\chi(G^{\natural p}) \leq \text{wcol}_{2p-3}(G)2^{\text{wcol}_{2p-3}(G)}$.

The best upper bound known for the weak colouring numbers of planar graphs is given by the following result.

Theorem 1.13 (Van den Heuvel et al. [8]).

For every positive integer k and planar graph G we have $wcol_k(G) \leq \binom{k+2}{2} \cdot (2k+1)$.

So $wcol_2(G) \leq 30$ and $wcol_3(G) \leq 70$ for planar G , which, when combined with Corollary 1.11, unfortunately gives a worse bound for $\chi(G^{\lfloor k/3 \rfloor})$ than the one using the star chromatic number obtained earlier. Theorems 1.12 and 1.13 together give $\chi(G^{\lfloor k/3 \rfloor}) \leq 70 \cdot 2^{70}$ for every planar graph G , while combining Theorems 1.5 (a) and 1.13 already gives the significantly better upper bound $\chi(G^{\lfloor k/3 \rfloor}) \leq 231$. In Section 3 we will show that this bound can be lowered further to 105.

The remainder of this paper is organised as follows. In the next section we prove our main results, Theorems 1.5 and 1.6. We use the results from that section in Section 3 to find explicit upper bounds for the chromatic number of exact distance graphs for some specific classes of graphs, including graphs with bounded genus, graphs with bounded tree-width, and graphs without a specified complete minor. In Section 4 we describe the graph promised in Theorem 1.4 (b). We close with a number of open problems and directions for further study.

2 Proofs of the main results

We need a few more definitions. For a positive integer k , we denote $[k] = \{1, 2, \dots, k\}$. For a vertex $v \in V$, we will denote by $N^k(y)$ the k -th neighbourhood of y , that is, the set of vertices different from v with distance at most k from v ; and we set $N^k[v] = N^k(v) \cup \{v\}$. As is standard, we write $N(v)$ for $N^1(v)$.

2.1 Proof of Theorem 1.5

For later use, we actually prove a slightly stronger result, which involves a more technical variant of the generalised colouring numbers. Let $G = (V, E)$ be a graph, L a linear ordering of V , and k a positive integer. For a vertex $y \in V$, let $D_{L,k}(y)$ be the set of vertices x such that there is an xy -path $P_x = z_0, \dots, z_s$, with $x = z_0, y = z_s$, of length $s \leq k$, such that x is the minimum vertex in P_x with respect to L , and such that $y \leq_L z_i$ for $\lfloor \frac{1}{2}k \rfloor + 1 \leq i \leq s$. We define the *distance- k -colouring number* $dcol_k(G)$ of a graph G as follows:

$$dcol_k(G) = 1 + \min_L \max_{y \in V} |D_{L,k}(y)|.$$

Since $R_{L,k}(y) \subseteq D_{L,k}(y) \subseteq Q_{L,k}(y)$ for every ordering L , distance k and vertex y , we obtain $col_k(G) \leq dcol_k(G) \leq wcol_k(G)$. On the other hand, we also have $Q_{L, \lfloor k/2 \rfloor + 1}(y) \subseteq D_{L,k}(y)$, which implies that $wcol_{\lfloor k/2 \rfloor + 1}(G) \leq dcol_k(G)$.

We will prove the following sharpening of Theorem 1.5.

Theorem 2.1.

- (a) For every odd positive integer p and every graph G we have $\chi(G^{\lfloor p/2 \rfloor}) \leq dcol_{2p-1}(G)$.
- (b) For every even positive integer p and every graph G we have $\chi(G^{\lfloor p/2 \rfloor}) \leq dcol_{2p}(G) \cdot \Delta(G)$.

Proof. (a) For an odd positive integer p and graph $G = (V, E)$, set $q = \text{dcol}_{2p-1}(G)$ and let L be an ordering of V that witnesses $\max_{y \in V} |D_{L, 2p-1}(y)| = q - 1$. Moving along the ordering L we assign to each vertex $y \in V$ a colour $a(y) \in [q]$ that is different from $a(x)$ for all $x \in D_{L, 2p-1}(y)$. Next, define $\mu(y)$ as the minimum vertex with respect to L of the vertices in $N^{\lfloor p/2 \rfloor}[y]$, and define $h : V \rightarrow [q]$ by $h(y) = a(\mu(y))$. We claim that h is a q -colouring of $G^{\lfloor p \rfloor}$.

Consider an edge $uv \in E(G^{\lfloor p \rfloor})$. So there exists a path $P = z_0, z_1, \dots, z_p$ with $z_0 = u$ and $z_p = v$. Clearly, $N^{\lfloor p/2 \rfloor}[u] \cap N^{\lfloor p/2 \rfloor}[v] = \emptyset$, and hence $\mu(u) \neq \mu(v)$. Without loss of generality, assume $\mu(u) <_L \mu(v)$. Since $\mu(u), z_{\lfloor p/2 \rfloor} \in N^{\lfloor p/2 \rfloor}[u]$, there exists a path S_1 between $\mu(u)$ and $z_{\lfloor p/2 \rfloor}$ of length at most $2 \lfloor \frac{1}{2} p \rfloor = p - 1$ such that $V(S_1) \subseteq N^{\lfloor p/2 \rfloor}[u]$. Similarly, there exists a path S_2 between $z_{\lfloor p/2 \rfloor + 1}$ and $\mu(v)$ of length at most $p - 1$ such that $V(S_2) \subseteq N^{\lfloor p/2 \rfloor}[v]$. Since $N^{\lfloor p/2 \rfloor}[u] \cap N^{\lfloor p/2 \rfloor}[v] = \emptyset$ and $z_{\lfloor p/2 \rfloor}, z_{\lfloor p/2 \rfloor + 1} \in E$, we can combine these paths to a path S between $\mu(u)$ and $\mu(v)$ of length at most $2p - 1$.

Note that if we write $S = w_0, w_1, \dots, w_t$ with $w_0 = \mu(u)$ and $w_t = \mu(v)$, then the vertices w_i for $\lfloor \frac{1}{2} k \rfloor + 1 \leq i \leq t$ all lie on S_2 , hence are in $N^{\lfloor p/2 \rfloor}[v]$. Since $\mu(v)$ is the minimum vertex in $N^{\lfloor p/2 \rfloor}[v]$, we have $\mu(v) \leq_L w_i$ for those w_i . Thus S witnesses that $\mu(u) \in D_{L, 2p-1}(\mu(v))$. We conclude that $h(u) = a(\mu(u)) \neq a(\mu(v)) = h(v)$, as required.

(b) For an even positive integer p and graph $G = (V, E)$, set $q = \text{dcol}_{2p}(G)$ and let L be an ordering of V that witnesses $\max_{y \in V} |D_{L, 2p}(y)| = q - 1$. Moving along the ordering L we assign to each vertex $y \in V$ a colour $a(y) \in [q]$ that is different from $a(x)$ for all $x \in D_{L, 2p}(y)$. Additionally, for each vertex y , choose an injective function $c_y : N(y) \rightarrow [\Delta(G)]$.

Next, define $\mu(y)$ as the minimum vertex with respect to L of the vertices in $N^{p/2}[y]$. We also choose an arbitrary vertex in $N(\mu(y)) \cap N^{p/2-1}(y)$; call it $\beta(y)$. To each vertex y we assign as its colour the pair $(a(\mu(y)), c_{\mu(y)}(\beta(y)))$. It is clear that this colouring uses at most $q \cdot \Delta(G)$ colours, and we claim that it is a proper colouring of $G^{\lfloor p \rfloor}$.

Consider an edge $uv \in E(G^{\lfloor p \rfloor})$. First suppose that $\mu(u) \neq \mu(v)$. Then we can follow the proof of part (a) to conclude that $a(\mu(u)) \neq a(\mu(v))$, and hence the colours of u and v differ in the first coordinate.

So we are left with the case $\mu(u) = \mu(v)$. Since $d_G(u, v) = p$, we have that $\mu(v) \in N^{p/2}(u) \cap N^{p/2}(v)$, while $N^{p/2-1}(u) \cap N^{p/2-1}(v) = \emptyset$. This means that $\beta(u) \neq \beta(v)$. Together with the fact that $\beta(u), \beta(v) \in N(\mu(v))$, we obtain that $c_{\mu(v)}(\beta(u)) \neq c_{\mu(v)}(\beta(v))$. This gives that the colours of u and v differ in the second coordinate, which completes the proof. \square

2.2 Proof of Theorem 1.6

In the proof of Theorem 1.6 we use the following lemmas.

Lemma 2.2.

Let $G = (V, E)$ be a graph and L a linear ordering of V . Let x, y, z be distinct vertices in G . If x is weakly k -accessible from y , and z is weakly ℓ -accessible from y , then x is weakly $(k + \ell)$ -accessible from z or z is weakly $(k + \ell)$ -accessible from x .

Proof. Since x is weakly k -accessible from y , there is a path $x, v_1, v_2, \dots, v_{r-1}, y$ of length $r \leq k$ for which all internal vertices v_i satisfy $x <_L v_i$. Also, since z is weakly ℓ -accessible

from y , there is a path $y, u_1, u_2, \dots, u_{s-1}, z$ of length $s \leq \ell$ for which all internal vertices u_j satisfy $z <_L u_j$. Then, if $x <_L z$, there is an xz -path of length at most $k + \ell$ with all internal vertices greater than x in L ; hence, x is weakly $(k + \ell)$ -accessible from z . Similarly, if $z <_L x$, then z is weakly $(k + \ell)$ -accessible from x . \square

Lemma 2.3.

Let p be a positive integer and G a graph with odd girth at least $p + 1$

- (a) Every closed walk of odd length has length at least $p + 1$.
- (b) Let x, y be different vertices and W a walk between x and y of length $r \leq p$. Then there exists a path between x and y of length $s \leq r$ such that s and r have the same parity.

Proof. The proof of (a) is straightforward, since a closed walk of odd length contains a cycle of odd length. For (b), let $W = w_0, \dots, w_r$, with $x = w_0$ and $y = w_r$. If W itself is not a path, then some vertex z appears more than once in W . The part of W between the first and last appearances of z is a closed walk W' of length $t \leq r$. Using (a) we obtain that t must be even. Hence, if we remove W' from W , we get a shorter walk between x and y of length $r - t \equiv r \pmod{2}$. Additionally, the resulting walk has fewer vertices that appear more than once than W does. Hence, if we do not immediately obtain a path, we can repeat this procedure inductively until we obtain an xy -path with the desired property. \square

Proof of Theorem 1.6.

For both parts of the theorem we use the same colouring. Let L be an ordering of V such that $\max_{y \in V} |Q_{L,p}(y)| = q - 1$. We first create an auxiliary colouring $a(y) \in [q]$ by moving along the ordering L , and assigning to each vertex $y \in V$ a colour $a(y) \in [q]$ that is different from $a(x)$ for all $x \in Q_{L,p}(y)$. Next, for a vertex $x \in Q_{L, \lfloor p/2 \rfloor}(y)$, let $d'_y(x)$ be the minimum integer k such that x is weakly k -accessible from y , and set $d'_y(y) = 0$.

Define the function $b_y : [q] \rightarrow [\lfloor \frac{1}{2}p \rfloor] \cup \{-1, 0\}$ as follows. For a colour $c \in [q]$, let

$$b_y(c) = \begin{cases} d'_y(x), & \text{if there exists an } x \in Q_{L, \lfloor p/2 \rfloor}(y) \cup \{y\} \text{ with } a(x) = c; \\ -1, & \text{otherwise.} \end{cases}$$

By Lemma 2.2 and the definition of $a(x)$, we see that if $x \in Q_{L, \lfloor p/2 \rfloor}(y) \cup \{y\}$ satisfies $a(x) = c$, then x is the only vertex in $Q_{L, \lfloor p/2 \rfloor}(y) \cup \{y\}$ with colour c . That implies that b_y is well defined.

The number of possible functions $b_y : [q] \rightarrow [\lfloor \frac{1}{2}p \rfloor] \cup \{-1, 0\}$ is $(\lfloor \frac{1}{2}p \rfloor + 2)^q$. We will prove that labelling each vertex $y \in V$ with b_y gives a proper colouring for the graphs and situations described in parts (a) and (b) of the theorem. It is more convenient to do part (b) first.

(b) Consider two vertices u, v for which there exists a path of length p between u and v . Without loss of generality we assume $u <_L v$. If u is weakly p -accessible from v in L , then we know that $a(u) \neq a(v)$, and hence $b_u(a(u)) = 0 \neq b_v(a(u))$.

So we are left with the case in which u is not weakly p -accessible from v in L . Let k be the length of the shortest odd-length path between u and v . We obviously have $k \leq p$. Because u is not weakly p -accessible from v in L , we also have $k \neq 1$, hence $k \geq 3$. Let $P = z_0, z_1, z_2, \dots, z_{k-1}, z_k$ be a path of length k between $u = z_0$ and $v = z_k$. Let z_ℓ be the

vertex of P that is minimum with respect to the ordering L . Since $u <_L v$, we get that $z_\ell \neq v$, and, since u is not weakly p -accessible from v , we see that $z_\ell \neq u$. Therefore, z_ℓ is weakly ℓ -accessible from u and weakly $(k - \ell)$ -accessible from v .

First consider the case that $\ell < k - \ell$. Then $\ell < \frac{1}{2}k$. We want to prove that $d'_u(z_\ell) = \ell$. For this, assume that $d'_u(z_\ell) = m < \ell$. Hence there is a path A between u and z_ℓ of length m . If ℓ and m have different parity, then the union of A and the path z_0, z_1, \dots, z_ℓ gives a closed walk of odd length $m + \ell < 2\ell < k \leq p$, which contradicts Lemma 2.3 (a). So m and ℓ have the same parity. Now if we replace in the path P the part z_0, z_1, \dots, z_ℓ with A , we get a walk between u and v of length $k - \ell + m < k$, hence with odd length. By Lemma 2.3 (b), this walk contains a path between u and v of odd length at most $k - \ell + m < k$, which contradicts the choice of P .

So we know that $d'_u(z_\ell) = \ell$. Notice that since there is a path of length $k - \ell$ between z_ℓ and v , we have that $d'_v(z_\ell) \leq k - \ell \leq p - \ell$. Since $\ell < \frac{1}{2}k \leq \frac{1}{2}p$, we have that $z_\ell \in Q_{L, \lfloor p/2 \rfloor}(u)$, and hence $b_u(a(z_\ell)) = \ell$.

Now consider a vertex $x \in Q_{L, \lfloor p/2 \rfloor}(v)$ with $d'_v(x) = \ell$. We first prove that $x \neq z_\ell$. For suppose this is not the case, then there is a path from v to z_ℓ of length ℓ . Together with the part of $z_\ell, z_{\ell+1}, \dots, z_k = v$ from the path P , this gives a closed walk of length $k \leq p$. Since k is odd, this contradicts Lemma 2.3 (a).

Since $d'_v(x) = \ell$, $d'_v(z_\ell) \leq p - \ell$ and $x \neq z_\ell$, by Lemma 2.2 we get that x is weakly p -accessible from z_ℓ or z_ℓ is weakly p -accessible from x . This gives $a(x) \neq a(z_\ell)$, which implies, by choice of x , that $b_v(a(z_\ell)) \neq \ell$.

If $k - \ell < \ell$, we can prove in a similar way that $b_u \neq b_v$, which completes the proof of part (b) of the theorem.

(a) This time we consider two vertices u, v that have distance k in G , for some odd integer $k \leq p$. (To prove the statement, it would be enough to prove the case $k = p$, but we prefer to give the proof of a more general statement.) We can more or less follow the proof of part (b) above, working with a shortest path $P = z_0, z_1, z_2, \dots, z_{k-1}, z_k$ between $u = z_0$ and $v = z_k$.

Since P is a shortest path, we immediately get that $d'_u(z_\ell) = d_G(u, z_\ell) = \ell$ and $d'_v(z_\ell) = d_G(v, z_\ell) = p - \ell$. This also means that $x \neq z_\ell$, since $d_G(v, x) \leq d'_v(x) = \ell < p - \ell$. For the remainder, the proofs are exactly the same. \square

The proofs of Theorem 1.6 (a) and (b) above give results that are stronger than the statements in the theorem. We already discussed in Subsection 1.2 that in fact we prove upper bounds on $\chi(G^{\lfloor p/2 \rfloor} \cup G^{\lfloor p/3 \rfloor} \cup \dots \cup G^{\lfloor p/p \rfloor})$ and $\chi(G^{\lfloor p/2 \rfloor} \cup G^{\lfloor p/3 \rfloor} \cup \dots \cup G^{\lfloor p/p \rfloor})$. Additionally, in part (a) we could replace the condition that we add an edge uv to $G^{\lfloor p/2 \rfloor}$ if $d_G(u, v) = p$, i.e. “there is a shortest path of length p between u and v ”, by the weaker condition “there is a path P of length p between u and v such that any shorter path between those vertices is internally disjoint from P ”.

3 Explicit upper bounds on the chromatic number of exact distance graphs

In this section we use Theorem 2.1 (a) to find explicit upper bounds for the chromatic number of exact distance graphs for certain types of graphs, including planar graphs, graphs with bounded tree-width, and graphs without a complete minor. Obtaining these bounds involves finding upper bounds for the distance- k -colouring numbers $\text{dcol}_k(G)$. More explicitly, we will prove the following results.

Theorem 3.1.

Let k be a positive integer.

(a) For every planar graph G we have $\text{dcol}_k(G) \leq \binom{\lfloor k/2 \rfloor + 3}{2} \cdot (2k + 1) - k$.

(b) For every graph G with genus g we have $\text{dcol}_k(G) \leq \left(2g + \binom{\lfloor k/2 \rfloor + 3}{2}\right) \cdot (2k + 1) - k$.

Theorem 3.2.

Let k and t be positive integers. For every graph G with tree-width at most t we have $\text{dcol}_k(G) \leq \binom{t + \lfloor k/2 \rfloor + 1}{t}$.

Theorem 3.3.

Let k and t be positive integers with $t \geq 4$. For every K_t -minor free graph G we have $\text{dcol}_k(G) \leq \binom{t + \lfloor k/2 \rfloor - 1}{t - 2} \cdot (t - 3)(2k + 1)$.

Since outerplanar graphs G have tree-width at most 2, combining Theorems 2.1 (a) and 3.2 gives $\chi(G^{\lfloor k/3 \rfloor}) \leq 10$. Similarly, from Theorem 3.1 we see that for planar graphs G we have $\chi(G^{\lfloor k/3 \rfloor}) \leq 105$, while for graphs G embeddable on the torus we have $\chi(G^{\lfloor k/3 \rfloor}) \leq 127$.

We will prove those theorems in the remainder of this section. They are based on the methods developed in Van den Heuvel et al. [8] to obtain bounds for the generalised colouring numbers.

3.1 Graphs with bounded tree-width

Recall that Proposition 1.7 tells us that $\text{col}_\infty(G) = \text{tw}(G) + 1$. In [6], Grohe et al. provided a sharp upper bound for the weak colouring numbers $\text{wcol}_k(G)$ of a graph G in terms of its tree-width. The following result is implicit in the proof of [6, Theorem 4.2].

Lemma 3.4 (Grohe et al. [6]).

Let G be a graph and L a linear ordering of $V(G)$ with $\max_{y \in V(G)} |R_{L,\infty}(y)| \leq t$. For every positive integer k and vertex $y \in V(G)$ we have $|Q_{L,k}(y)| \leq \binom{t+k}{t} - 1$.

Although we can define tree-width of a graph in terms of its ∞ -colouring number, in order to prove Theorem 3.2 we shall make use of a better known definition which is in terms of

k-trees. A *k*-tree is a graph which is either a clique of size $k + 1$ or is obtained from a smaller *k*-tree by adding a vertex adjacent to k vertices which are pairwise adjacent. The *tree-width* of a graph G is the smallest k such that G is a subgraph of a *k*-tree.

Let G be a *k*-tree. For a fixed way of constructing G from a $(k + 1)$ -clique K_0 we obtain a linear ordering L of $V(G)$ as follows. Let the vertices of K_0 be the smallest in the ordering, and order them in an arbitrary way. Then for $y \notin K_0$ we let $x <_L y$ if x was added to the *k*-tree before y . We call this a *simplicial ordering*. For $y \notin K_0$ we note that, by definition of L , $R_{L,1}(y)$ induces a *k*-clique.

Proof of Theorem 3.2. Since $\text{dcol}_k(G)$ cannot decrease if we add edges, we may assume that G is a *k*-tree. Let L be a simplicial ordering derived as above, where we started with some $(k + 1)$ -clique K_0 in G . While in general we have $Q_{L, \lfloor k/2 \rfloor + 1}(y) \subseteq D_{L,k}(y)$, we shall prove that our choice of G and L implies $Q_{L, \lfloor k/2 \rfloor + 1}(y) = D_{L,k}(y)$, for every $k \geq 1$ and $y \in V(G)$.

Our first step in this direction will be proving that every vertex $y \in V(G)$ satisfies $R_{L,1}(y) = R_{L,\infty}(y)$. Notice that if $y \in V(G)$ belongs to K_0 , then $R_{L,\infty}(y)$ only contains vertices in K_0 and, since K_0 induces a clique in G , all of these vertices belong to $R_{L,1}(y)$. So consider some $y \notin K_0$. From the construction of a *k*-tree, it follows that removing $R_{L,1}(y)$ disconnects the graph, and that the component C_y containing y satisfies $y <_L z$ for all $z \in C_y$, $z \neq y$. This tells us that any xy -path with $x <_L y$ and $y <_L z$ for all internal vertices z must have its interior in C_y . In turn, this implies that for such a path to exist we must have $x \in R_{L,1}(y)$. This shows $R_{L,1}(y) = R_{L,\infty}(y)$.

Suppose $x, y \in V(G)$ satisfy $x \in D_{L,k}(y)$ for some integer $k \geq 1$. By the definition of $D_{L,k}(y)$, we have that there is an xy -path $P = z_0, \dots, z_s$, with $x = z_0$, $y = z_s$, of length $s \leq k$, such that x is the minimum vertex in P with respect to L , and such that $y \leq_L z_i$ for $\lfloor \frac{1}{2}k \rfloor + 1 \leq i \leq s$. Let $0 \leq d \leq s$ be the largest index such that $z_d <_L y$. The subpath z_d, \dots, z_s of P guarantees that that $z_d \in R_{L,\infty}(y)$. Since $R_{L,1}(y) = R_{L,\infty}(y)$, we know that $z_d \in N(y)$. By the definition of P and choice of d , we also know that $d \leq \lfloor \frac{1}{2}k \rfloor$. Therefore, the path z_0, \dots, z_d, z_s is an xy -path of length at most $\lfloor \frac{1}{2}k \rfloor + 1$ with no other restriction than the one that x is its minimum vertex with respect to L . This means that $x \in Q_{L, \lfloor k/2 \rfloor + 1}(y)$. Since the choice of x, y and k was arbitrary, we have that $Q_{L, \lfloor k/2 \rfloor + 1}(y) = D_{L,k}(y)$ for every integer $k \geq 1$ and every $y \in V(G)$.

Since our ordering satisfies $t \geq R_{L,1}(y) = R_{L,\infty}(y)$, the bound on $D_{L,k}(y)$ follows from Lemma 3.4. \square

It is possible to modify the examples in Grohe et al. [6] to show that the upper bounds on $\text{dcol}_k(G)$ in Theorem 3.2 for graphs with tree-width at most t are best possible.

3.2 Graphs with excluded complete minors

In order to provide upper bounds for the generalised colouring numbers for graphs that exclude a fixed minor, Van den Heuvel et al. [8] constructed ordered vertex partitions where each part has neighbours in only a bounded number of earlier parts and the intersection of each part with the *k*-neighbourhood of an earlier part is also bounded. We will make use of these decompositions for our proofs as well.

A *decomposition* of a graph G is a sequence $\mathcal{H} = (H_1, \dots, H_\ell)$ of non-empty subgraphs of G such that the vertex sets $V(H_1), \dots, V(H_\ell)$ partition $V(G)$. The decomposition \mathcal{H} is *connected* if each H_i is connected.

Let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a decomposition of a graph G , i a positive integer, and C a component of $G - \bigcup_{1 \leq j \leq i} V(H_j)$. We define the *i -th separating number of C* as $s_i(C) = |\{j \in [i] \mid E(C, H_j) \neq \emptyset\}|$, where $E(C, H_j)$ is the set of edges with one end-vertex in C and the other end-vertex in H_j . Let $w_i(\mathcal{H}) = \max s_i(C)$, where the maximum is taken over all components C of $G - \bigcup_{1 \leq j < i} V(H_j)$. We define the *width* of \mathcal{H} as $W(\mathcal{H}) = \max_{1 \leq i \leq \ell} w_i(\mathcal{H})$.

Let G be a graph, let $H \subseteq G$ be a *connected* subgraph of G , and let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function. We say that H *f -spreads on G* if, for every $k \in \mathbb{N}$ and $v \in V(G)$, we have

$$|N^k[v] \cap V(H)| \leq f(k).$$

We say a decomposition \mathcal{H} is *f -flat* if each H_i f -spreads on $G - \bigcup_{1 \leq j < i} V(H_j)$. We say \mathcal{H} is a *flat decomposition* if \mathcal{H} is an f -flat decomposition for some function $f : \mathbb{N} \rightarrow \mathbb{N}$.

Van den Heuvel et al. [8] related the width of a connected decomposition to the tree-width of the minor obtained by contracting each part.

Lemma 3.5 (Van den Heuvel et al. [8]).

Let G be a graph, and let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a connected decomposition of G of width at most t . By contracting each (connected) subgraph H_i to a single vertex, we obtain a graph H with ℓ vertices and tree-width at most t .

The proof of the lemma in [8] shows the power of generalised colouring numbers. It actually gives a short argument that the contracted graph H satisfies $\text{col}_\infty(H) \leq t + 1$. The bound on the tree-width then follows by Proposition 1.7. Moreover, the proof shows that the ordering L of $V(H)$ obtained by setting $H_i <_L H_j$ if $i < j$ satisfies $\max_{1 \leq i \leq \ell} |R_{L, \infty}(H_i)| \leq t$. Using this property we can prove that if the decomposition from which H was obtained is f -flat, then we can find an upper bound on $\text{dcol}_k(G)$ in terms of $f(k)$.

Lemma 3.6.

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ and let t, k be positive integers. For every graph G that admits a connected f -flat decomposition of width at most t we have $\text{dcol}_k(G) \leq \binom{t + \lfloor k/2 \rfloor + 1}{t} \cdot f(k)$.

Proof. The proof of this lemma is similar to that of [8, Lemma 3.5]. Let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a connected f -flat decomposition of G of width t . Since \mathcal{H} is connected, we know, by Lemma 3.5, that contracting the subgraphs in \mathcal{H} leads to a graph H with tree-width at most t . We identify the vertices of H with the subgraphs H_i , and define a linear ordering L on $V(H)$ by setting $H_i <_L H_j$ if $i < j$. By the proof of [8, Lemma 3.1] we get that L satisfies $\max_{1 \leq i \leq \ell} |R_{L, \infty}(H_i)| \leq t$. Using Lemma 3.4 this implies that $|Q_{L, \lfloor k/2 \rfloor + 1}(H_i)| \leq \binom{t + \lfloor k/2 \rfloor + 1}{t} - 1$ for any vertex $H_i \in V(H)$. Arguing as in the proof of Theorem 3.2, we see that for every $H_i \in V(H)$ we have $|D_{L, k}(H_i)| \leq \binom{t + \lfloor k/2 \rfloor + 1}{t} - 1$.

From L we define an ordering L' on $V(G)$ in the following way. For $u \in H_i$ and $v \in H_j$ with $i \neq j$, we let $u <_{L'} v$ if $i < j$. Then, for every $1 \leq i \leq \ell$, we order the vertices of H_i in

any order. It is easy to see that any vertex $v \in H_i$ satisfies

$$D_{L',k}(v) \subseteq N^k[v] \cap (H_i \cup \{H_j \mid H_j \in D_{L,k}(H_i)\}).$$

Hence, we have that there are at most $\binom{t+\lfloor k/2 \rfloor + 1}{t}$ subgraphs among H_1, \dots, H_ℓ in G that contain vertices from $D_{L',k}(v)$. Since \mathcal{H} is f -flat, we know that the intersection of each of these subgraphs with $N^k[v]$ is at most $f(k)$. Finally, since $D_{L',k}(v)$ is a proper subset of $N^k[v]$ (as $v \notin D_{L',k}(v)$), the result follows. \square

Also in [8], it was proved that graphs that do not contain a complete graph as a minor have flat decompositions of small width.

Lemma 3.7 (Van den Heuvel et al. [8]).

Let $t \geq 4$ and let $f : \mathbb{N} \rightarrow \mathbb{N}$ be the function $f(k) = (t - 3)(2k + 1)$. For every K_t -minor free graph G we have that there is a connected f -flat decomposition of G with width at most $t - 2$.

Combining Lemmas 3.6 and 3.7 immediately gives Theorem 3.3.

We say a path is *optimal* if it is a shortest path between its end-points. The following easy result states that a decomposition $\mathcal{H} = (H_1, \dots, H_\ell)$ in which each subgraph H_i is an optimal path in $G - \bigcup_{1 \leq j < i} V(H_j)$ is f -flat for $f(k) = 2k + 1$. We call such a decomposition an *optimal-path decomposition*.

Lemma 3.8 (Van den Heuvel et al. [8]).

Let G be a graph, y be a vertex of G , and P be an optimal path in G . Then P contains at most $2k + 1$ vertices of the closed k -neighbourhood $N^k[y]$ of y .

Optimal-path decompositions of small width were found in [8] for planar graphs.

Lemma 3.9 (Van den Heuvel et al. [8]).

Every maximal planar graph has an optimal-path decomposition of width at most 2.

This lemma allows us, through Lemma 3.6, to prove Theorem 3.1.

Proof of Theorem 3.1. We begin by proving part (a). Since $\text{dcol}_k(G)$ cannot decrease when edges are added, we may assume that G is maximal planar. By Lemma 3.9, there exists an optimal-path decomposition $\mathcal{H} = (H_1, \dots, H_\ell)$ of G of width at most 2. The proof of Lemma 3.6 tells us that since G admits a connected decomposition of width at most 2, there is an ordering L' of $V(G)$ such that at most $\binom{\lfloor k/2 \rfloor + 3}{2}$ subgraphs among H_1, \dots, H_ℓ contain vertices from $D_{L',k}(v)$, for every integer $k \geq 1$ and $v \in V(G)$. This ordering is obtained from an ordering L of the subgraphs H_1, \dots, H_ℓ , where vertices in the same subgraph are ordered in an arbitrary way. This time we have that each subgraph H_i is an optimal path. We order each H_i in its path order. Take $y \in V(G)$. Then $y \in V(H_i)$ for some $1 \leq i \leq \ell$. Lemma 3.8 tells us that an optimal-path decomposition is $(2k + 1)$ -flat. Therefore, $D_{L',k}(y)$ contains at most $2k + 1$ vertices of each of the at most $\binom{\lfloor k/2 \rfloor + 3}{2} - 1$ subgraphs, other than H_i , which intersect $D_{L',k}(y)$. Meanwhile, $D_{L',k}(y)$ contains at most k vertices of H_i , since we have ordered the optimal path H_i in its path order. We find that every vertex y in G satisfies

$$|D_{L',k}(y)| \leq \left(\binom{\lfloor k/2 \rfloor + 3}{2} - 1 \right) \cdot (2k + 1) + k = \binom{\lfloor k/2 \rfloor + 3}{2} \cdot (2k + 1) - k - 1,$$

which concludes the proof of part (a).

The proof of part (b) is similar to the proof of [8, Theorem 1.5 (a)]. We assume $g > 0$, as otherwise the result holds by Theorem 3.1 (a). It is well known (see e.g. [18, page 111] and the proof of [24, Theorem 1]) that a graph of genus $g > 0$ contains a non-separating cycle C that consists of two optimal paths and such that $G - C$ has genus $g - 1$. We construct a linear order L of $V(G)$ in the following way. The first vertices in L will be the vertices in such a cycle C . If after removing that cycle the genus of the resulting graph is greater than 0, then we choose another such cycle, make its vertices the next ones in the ordering, and remove the cycle. We repeat this process inductively until the resulting graph is a planar graph G' . The vertices of G' are placed at the end of L , ordered in the way that gives the bound on $\text{dcol}_k(G')$ from Theorem 3.1 (a).

Lemma 3.8 tells us that for any vertex y and optimal path P we have $|N^k[y] \cap V(P)| \leq 2k + 1$ for every k . Hence $|D_{L,k}(y) \cap V(P)| \leq 2k + 1$ for every vertex y and optimal path P . It follows that for any vertex y in G , the set $D_{L,k}(y)$ can have at most $2g(2k + 1)$ vertices on the removed cycles. (Each of the two optimal paths that form a cycle is optimal after the earlier cycles are removed, and vertices cannot belong to $D_{L,k}(y)$ through vertices in older cycles.) Only a vertex x in the planar graph G' can have other vertices of G' in $D_{L,k}(x)$ and Theorem 3.1 (a) gives us a bound on the number of such vertices. Hence, we obtain that every vertex y in G satisfies

$$|D_{L,k}(y)| \leq 2g \cdot (2k + 1) + \binom{\lfloor k/2 \rfloor + 3}{2} \cdot (2k + 1) - k - 1.$$

The result follows. □

4 A lower bound on the chromatic number of exact distance-3 graphs of planar graphs

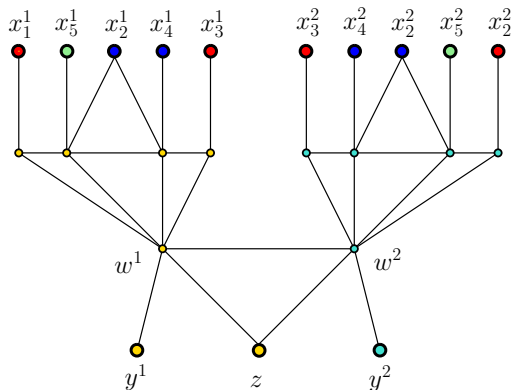


Figure 2: An outerplanar graph G_4 with $\chi(G_4^{[4^3]}) = 5$.

In [22, Exercise 11.4] a planar graph G such that $\chi(G^{[4^3]}) = 6$ is given (see also [23]). As we will prove below, the outerplanar graph G_4 in Figure 2 satisfies $\chi(G_4^{[4^3]}) = 5$. We will use

that graph to construct a planar graph G_5 such that $\chi(G_5^{[3]}) = 7$.

Theorem (Theorem 1.4 (b)).

There is a planar graph G_5 such that $\chi(G_5^{[3]}) = 7$.

Proof. We will prove first that $\chi(G_4^{[3]}) = 5$, using the vertex labelling provided in Figure 2. Consider a proper colouring of $G_4^{[3]}$. Note that $C^1 = x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_1^1$ and $C^2 = x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1^2$ form disjoint 5-cycles $G_4^{[3]}$. Hence, the vertices in $V(C^1) \cup V(C^2)$ need at least 3 colours. Given that $V(C^1) \cup V(C^2) \subseteq N(z)$ in $G_4^{[3]}$, if we use more than 3 colours on $V(C^1) \cup V(C^2)$, then we already use at least 5 colours. So assume that the vertices in $V(C^1) \cup V(C^2)$ are coloured with 3 colours only. Since $V(C^i) \subseteq N(y^i)$ in $G_4^{[3]}$ for $i = 1, 2$, and $y^1 y^2 \in E(G_4^{[3]})$, we need at least 2 extra colours. So we always use at least 5 colours in a proper colouring of $G_4^{[3]}$. Figure 2 gives a colouring of G_4 with 5 colours which is a proper colouring of $G_4^{[3]}$. This shows that $\chi(G_4^{[3]}) = 5$.

Now let F_1 and F_2 be two disjoint copies of G_4 . Let H be a path on 5 vertices, disjoint from F_1 and F_2 , with vertices $y'_1, w'_1, z', w'_2, y'_2$ in that order, together with the edge $w'_1 w'_2$. (This is exactly the graph formed by the vertices $\{y^1, w^1, z, w^2, y^2\}$ in Figure 2.) The graph G_5^- has vertex set and edge set:

$$\begin{aligned} V(G_5^-) &= V(F_1) \cup V(F_2) \cup V(H); \\ E(G_5^-) &= E(F_1) \cup E(F_2) \cup E(H) \cup \{b_1 w'_1 \mid b_1 \in V(F_1)\} \cup \{b_2 w'_2 \mid b_2 \in V(F_2)\}. \end{aligned}$$

Finally, the graph G_5 is obtained from G_5^- by subdividing once all the edges of the form $b_1 w'_1$ and $b_2 w'_2$ (replacing each edge by a path of length 2). Since G_4 is outerplanar, it is easy to check that G_5 is planar.

If $u, v \in V(F_1)$ and P is a uv -path in G_5 but $V(P) \not\subseteq V(F_1)$, then $w'_1 \in V(P)$. Thus the length of P is at least 4. We conclude that if two vertices u, v have distance 3 in G_5 , then any shortest uv -path has all its vertices in $V(F_1)$. Therefore, the number of colours needed to colour the vertices of F_1 in $G_5^{[3]}$ is 5, and the same applies to F_2 . We now can argue as in the proof of $\chi(G_4^{[3]}) = 5$ above to reach the conclusion $\chi(G_5^{[3]}) = 7$. \square

Since the graph G_4 in Figure 2 is outerplanar, it does not have K_4 as a minor. Also, the graph G_5 we constructed above is planar, so does not have K_5 as a minor. We can iterate the construction to obtain graphs G_t that are K_t -minor free, for $t \geq 4$, and for which $\chi(G_t^{[3]}) \geq 2(t-2) + 1$. To obtain G_{t+1} from G_t , we take two copies of G_t , one copy of the graph H from above, and add paths of length 2 between all vertices in the first copy of G_t and w'_1 , and between all vertices in the second copy of G_t and w'_2 . It is straightforward to check that if G_t is K_t -minor free, then G_{t+1} is K_{t+1} -minor free, and that $G_{t+1}^{[3]}$ needs at least 2 more colours than $G_t^{[3]}$ does.

The property that for $t \geq 5$ there exists a graph G that is K_t -minor free and satisfies $\chi(G^{[3]}) \geq 2(t-2) + 1$ does not extend to $t = 3$. To see this, note that the only graphs that are K_3 -minor free are acyclic graphs (i.e. forests), which implies they are bipartite. And for bipartite graphs G we have that $G^{[3]}$ is bipartite as well (in fact, even the exact p -power graph $G^{\uparrow p}$ is bipartite for every odd p), hence $\chi(G^{[3]}) \leq 2$.

Notice that one can construct the graph G_4 of Figure 2 (and the graphs G_t for $t \geq 4$) by using operations similar to those of used in the Hajós construction [7]. Consider the graph S induced on G_4 by $(N(w^1) \setminus w^2) \cup \{w_1, x_1^1, x_2^2, \dots, x_5^1\}$. The main connected component of the graph $S^{\lfloor 3 \rfloor}$ consists of a cycle and two apex vertices, z and y_1 , that are adjacent to all the vertices in the cycle. One can obtain G_4 by taking two copies of S , identifying the two vertices that correspond to z , and adding an edge between the two vertices that correspond to w_1 . In the exact distance-3 graph, we see that one of the apex vertices has been identified, while those that correspond to y_1 have been joined by an edge. However, the operation of deletion, used in the Hajós construction, is not used in our construction. This is mainly because we want to obtain a graph with chromatic number strictly larger than that of the parts it is formed of.

5 Discussion and open problems

In this paper we give bounds on the chromatic number of exact distance graphs for some classes of graphs. In general, the difference between the best lower and upper bounds is still quite large, so we can't really claim we have an insight of what the correct best possible bounds are.

When considering odd distances, one, trivial, example for which there are tight bounds is the class of bipartite graphs. We noted at the end of Section 4 that every bipartite graph G satisfies $\chi(G^{\lfloor 2p \rfloor}) \leq \chi(G^{\lfloor p \rfloor}) \leq 2$ for every odd p .

Since our upper bounds are expressed in terms of generalised colouring numbers they increase with the distance. In contrast, we do not provide lower bounds which increase with the distance. Because of the difficulty in providing lower bounds which depend on the distance, the following question, attributed to Van den Heuvel and Naserasr, was asked in [22, Section 11.9.3] (see also [23]): “Is there a constant C such that for every odd integer p and every planar graph G we have $\chi(G^{\lfloor 2p \rfloor}) \leq C$?” Very recently, Bousquet et al. [4] gave a negative answer to this question by constructing a sequence of outerplanar graphs U_3, U_5, \dots such that for every odd $p \geq 3$ we have $\chi(U_p^{\lfloor 2p \rfloor}) \in \Omega(\frac{p}{\log p})$. In Section 3 we proved that if G has tree-width at most t then $\chi(G^{\lfloor 2p \rfloor}) \in \mathcal{O}(p^{t-1})$. This means that graphs G of tree-width at most 2 satisfy $\chi(G^{\lfloor 2p \rfloor}) \in \mathcal{O}(p)$. Therefore, for graphs of tree-width at most 2 (which includes outerplanar graphs), our upper bounds are close to having the right order in terms of the distance.

As we mentioned in Section 1, the proof of Theorem 1.6 actually gives that for a class of graphs \mathcal{K} with bounded expansion we can find a constant $N = N(\mathcal{K}, p)$ such that $\chi(G^{\lfloor 1 \rfloor} \cup G^{\lfloor 3 \rfloor} \cup \dots \cup G^{\lfloor 2p \rfloor}) \leq N$. There are constructions that show that this constant must grow with p , even if \mathcal{K} is the class of outerplanar graphs. One such construction appears in [23]. A very simple one, which we sketch in Figure 3, can be found in [25].

For a graph G , a natural generalisation of $G^{\lfloor 1 \rfloor} \cup G^{\lfloor 3 \rfloor} \cup \dots \cup G^{\lfloor 2p \rfloor}$ is the graph G^{odd} , which has the same vertex set as G , and xy is an edge in G^{odd} if and only if x and y have odd distance. Both constructions in the previous paragraph tell us that for outerplanar graphs G the chromatic number of G^{odd} can be arbitrarily large because the clique number $\omega(G^{odd})$ can



Figure 3: Outerplanar graphs G for which $\omega(G^{odd})$, and hence $\chi(G^{odd})$, can be arbitrarily large.

be arbitrarily large. This motivates the following open problem of Thomassé, which appears in [22, Section 11.9.3] (see also [23]).

Problem 5.1 ([22, Problem 11.2]).

Is there a function f such that for every planar graph G we have $\chi(G^{odd}) \leq f(\omega(G^{odd}))$?

Another area that is ripe for further research is the chromatic number of exact distance graphs with even distance, for specific classes of graphs. Theorem 1.5 (b) gives a first result for even distances. There is very little we know about the dependencies between $\chi(G^{\lfloor p \rfloor})$ and $\text{wcol}_p(G)$ for even p .

It is well-known, and easy to prove (see, e.g., [17]), that for every graph G we have $\chi(G^2) \leq (2\text{col}(G) - 3) \cdot \Delta(G)$, hence certainly $\chi(G^{\lfloor 2 \rfloor}) \leq (2\text{col}(G) - 3) \cdot \Delta(G)$. This suggests that there might exist a function φ such that $\chi(G^{\lfloor p \rfloor}) \leq \varphi(\text{wcol}_{p-1}(G)) \cdot \Delta(G)$, or even $\chi(G^{\lfloor p \rfloor}) \leq \varphi(\text{wcol}_{p/2}(G)) \cdot \Delta(G)$. We have not been able to prove such a result. Neither do we know what the best value of $r(p)$ should be such that a result of the form $\chi(G^{\lfloor p \rfloor}) \leq \varphi(\text{wcol}_{r(p)}(G)) \cdot \Delta(G)$ is possible for even p .

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