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Local martingales in discrete time

Vilmos Prokaj [*]   Johannes Ruf [†]

Abstract

For any discrete-time P–local martingale S there exists a probability measure $Q \sim P$ such that S is a Q–martingale. A new proof for this result is provided. The core idea relies on an appropriate modification of an argument by Chris Rogers, used to prove a version of the fundamental theorem of asset pricing in discrete time. This proof also yields that, for any $\varepsilon > 0$, the measure Q can be chosen so that $dQ/dP \leq 1 + \varepsilon$.

1 Introduction and related literature

Let $(\Omega, \mathscr{F}, P)$ denote a probability space equipped with a discrete-time filtration $(\mathscr{F}_t)_{t \in \mathbb{N}_0}$, where $\mathscr{F}_t \subset \mathscr{F}$. Moreover, let $S = (S_t)_{t \in \mathbb{N}_0}$ denote a d-dimensional P–local martingale, where $d \in \mathbb{N}$. Then there exists a probability measure Q, equivalent to P, such that S is a Q–martingale. This follows from more general results that relate appropriate no-arbitrage conditions to the existence of an equivalent martingale measure; see Dalang et al. (1990) and Schachermayer (1992) for the finite-horizon case and Schachermayer (1994) for the infinite-horizon case. These results are sometimes baptized fundamental theorems of asset pricing.

More recently, Kabanov (2008) and Prokaj and Rásonyi (2010) have provided a direct proof for the existence of such a measure Q; see also Section 2 in Kabanov and Safarian (2009). The proof in Kabanov (2008) relies on deep functional analytic results, e.g., the Krein-Šmulian theorem. The proof in Prokaj and Rásonyi (2010) avoids functional analysis but requires non-trivial measurable selection techniques.

As this note demonstrates, in one dimension, an important but special case, the Radon-Nikodym derivative $Z_\infty = dQ/dP$ can be explicitly constructed. Moreover, in higher dimensions, the measurable selection results can be simplified. This is done here by appropriately modifying an ingenious idea of Rogers (1994).

More precisely, the following theorem will be proved in Section 3.

Theorem 1.1. *For all $\varepsilon > 0$, there exists a uniformly integrable P–martingale $Z = (Z_t)_{t \in \mathbb{N}_0}$, bounded from above by $1 + \varepsilon$, with $Z_\infty = \lim_{t \uparrow \infty} Z_t > 0$, such that ZS is a P–martingale and such that $E_P[Z_t |S_t|^p] < \infty$ for all $t \in \mathbb{N}_0$ and $p \in \mathbb{N}$.*

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The fact that the bound on Z can be chosen arbitrarily close to 1 seems to be a novel observation. Considering a standard random walk S directly yields that there is no hope for a stronger version of Theorem 1.1 which would assert that ZS is not only a P–martingale but also a P–uniformly integrable martingale.

A similar version of the following corollary is formulated in Prokaj and Rásonyi (2010); it would also be a direct consequence of Kabanov and Stricker (2001). To state it, let us introduce the total variation norm $\|\cdot\|$ for two equivalent probability measures Q_1, Q_2 as

$$\|Q_1 - Q_2\| = E_{Q_1}\left[|{}^{dQ_2}\!/{}_{dQ_1} - 1|\right].$$

Corollary 1.2. *For all $\varepsilon > 0$, there exists a probability measure Q, equivalent to P, such that S is a Q–martingale, $\|P - Q\| < \varepsilon$, and $E_Q[|S_t|^p] < \infty$ for all $t \in \mathbb{N}_0$ and $p \in \mathbb{N}$.*

To reformulate Corollary 1.2 in more abstract terms, let us introduce the spaces

$$\mathcal{Q}_l = \{Q \sim P : S \text{ is a } Q\text{–local martingale}\} ;$$
$$\mathcal{Q}^p = \{Q \sim P : S \text{ is a } Q\text{–martingale with } E_Q[|S_t|^p] < \infty \text{ for all } t \in \mathbb{N}_0\}, \qquad p > 0.$$

Then Corollary 1.2 states that the space $\bigcap_{p \in \mathbb{N}} \mathcal{Q}^p$ is dense in $\mathcal{Q}_l$ with respect to the total variation norm $\|\cdot\|$.

Proof of Corollary 1.2. Consider the P–uniformly integrable martingale Z of Theorem 1.1, with ε replaced by ${}^{\varepsilon}\!/_2$. Then the probability measure Q, given by ${}^{dQ}\!/_{dP} = Z_\infty$, satisfies the conditions of the assertion. Indeed, we only need to observe that

$$E_P[|Z_\infty - 1|] = 2E_P[(Z_\infty - 1)\mathbf{1}_{\{Z_\infty > 1\}}] \leq \varepsilon,$$

where we used that $E_P[Z_\infty - 1] = 0$ and the assertion follows. $\qquad\square$

2 Generalized conditional expectation and local martingales

For sake of completeness, we review the relevant facts related to local martingales in discrete time. To start, note that for a sigma algebra $\mathscr{G} \subset \mathscr{F}$ and a nonnegative random variable Y, not necessarily integrable, we can define the so called generalized conditional expectation

$$E_P[Y \mid \mathscr{G}] = \lim_{k \uparrow \infty} E_P[Y \wedge k \mid \mathscr{G}].$$

Next, for a general random variable W with $E_P[|W| \mid \mathscr{G}] < \infty$, but not necessarily integrable, we can define the generalized conditional expectation

$$E_P[W \mid \mathscr{G}] = E_P[W^+ \mid \mathscr{G}] - E_P[W^- \mid \mathscr{G}].$$

For a stopping time τ and a stochastic process X we write X^τ to denote the process obtained from stopping X at time τ.

Definition 2.1. *A stochastic process $S = (S_t)_{t \in \mathbb{N}_0}$ is*

- *a P–martingale if $E_P[|S_t|] < \infty$ and $E_P[S_{t+1} \mid \mathscr{F}_t] = S_t$ for all $t \in \mathbb{N}_0$;*

- *a P–local martingale if there exists a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times such that $\lim_{n \uparrow \infty} \tau_n = \infty$ and $S^{\tau_n} \mathbf{1}_{\{\tau_n > 0\}}$ is a P–martingale;*

- *a P–generalized martingale if $E_P[|S_{t+1}| \mid \mathscr{F}_t] < \infty$ and $E_P[S_{t+1} \mid \mathscr{F}_t] = S_t$ for all $t \in \mathbb{N}_0$.*

Proposition 2.2. *Any P–local martingale is a P–generalized martingale.*

This proposition dates back to Theorem II.42 in Meyer (1972); see also Theorem VII.1 in Shiryaev (1996). Its reverse direction would also be true but will not be used below. A direct corollary of the proposition is that a P–local martingale S with $\mathsf{E}_\mathsf{P}[|S_t|] < \infty$ for all $t \in \mathbb{N}_0$ is indeed a P–martingale.

For sake of completeness, we will provide a proof of the proposition here.

Proof of Proposition 2.2. Let S denote a P–local martingale. Fix $t \in \mathbb{N}_0$ and a localization sequence $(\tau_n)_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$, we have, on the event $\{\tau_n > t\}$,

$$\mathsf{E}_\mathsf{P}[|S_{t+1}| \mid \mathscr{F}_t] = \lim_{k\uparrow\infty} \mathsf{E}_\mathsf{P}[|S_{t+1}| \wedge k \mid \mathscr{F}_t] = \lim_{k\uparrow\infty} \mathsf{E}_\mathsf{P}[|S_{t+1}^{\tau_n}| \wedge k \mid \mathscr{F}_t] = \mathsf{E}_\mathsf{P}[|S_{t+1}^{\tau_n}| \mid \mathscr{F}_t] < \infty.$$

Since $\lim_{n\uparrow\infty} \tau_n = \infty$, we get $\mathsf{E}_\mathsf{P}[|S_{t+1}| \mid \mathscr{F}_t] < \infty$.

The next step we only argue for the case $d = 1$, for sake of notation, but the general case follows in the same manner. As above, again for fixed $n \in \mathbb{N}$, on the event $\{\tau_n > t\}$, we get

$$\mathsf{E}_\mathsf{P}[S_{t+1} \mid \mathscr{F}_t] = \lim_{k\uparrow\infty} \left(\mathsf{E}_\mathsf{P}[S_{t+1}^+ \wedge k \mid \mathscr{F}_t] - \mathsf{E}_\mathsf{P}[S_{t+1}^- \wedge k \mid \mathscr{F}_t] \right)$$
$$= \lim_{k\uparrow\infty} \mathsf{E}_\mathsf{P}[(S_{t+1}^{\tau_n} \wedge k) \vee (-k) \mid \mathscr{F}_t] = S_t.$$

Thanks again to $\lim_{n\uparrow\infty} \tau_n = \infty$, the assertion follows. $\square$

Example 2.3. Assume that $(\Omega, \mathscr{F}, \mathsf{P})$ supports two independent random variables U and θ such that U is uniformly distributed on $[0, 1]$, and $\mathsf{P}[\theta = -1] = \frac{1}{2} = \mathsf{P}[\theta = 1]$. Moreover, let us assume that $\mathscr{F}_0 = \{\emptyset, \Omega\}$, $\mathscr{F}_1 = \sigma(U)$, and $\mathscr{F}_t = \sigma(U, \theta)$ for all $t \in \mathbb{N} \setminus \{1\}$. Then the stochastic process $S = (S_t)_{t \in \mathbb{N}_0}$, given by $S_t = \frac{\theta}{U} \mathbf{1}_{t \geq 2}$ is easily seen to be a P–generalized martingale and a P–local martingale with localization sequence $(\tau_n)_{n \in \mathbb{N}}$ given by

$$\tau_n = 1 \times \mathbf{1}_{\{1/U > n\}} + \infty \times \mathbf{1}_{\{1/U \leq n\}}.$$

However, we have $\mathsf{E}_\mathsf{P}[|S_2|] = \mathsf{E}_\mathsf{P}[1/U] = \infty$; hence S is not a P–martingale.

Now, consider the process $Z = (Z_t)_{t \in \mathbb{N}_0}$, given by $Z_t = \mathbf{1}_{t=0} + 2U\mathbf{1}_{t \geq 1}$. A simple computation shows that Z is a strictly positive P–uniformly integrable martingale. Moreover, since $Z_t S_t = 2\theta \mathbf{1}_{t \geq 2}$, we have $\mathsf{E}_\mathsf{P}[Z_t|S_t|] \leq 2$ for all $t \in \mathbb{N}_0$ and ZS is a P–martingale. If we require the Radon-Nikodym to be bounded by a constant $1 + \varepsilon \in (1, 2]$, we could consider $\widehat{Z} = (\widehat{Z}_t)_{t \in \mathbb{N}_0}$ with $\widehat{Z}_t = \mathbf{1}_{t=0} + \frac{(U \wedge \varepsilon)}{(\varepsilon - \varepsilon^2/2)} \mathbf{1}_{t \geq 1}$. This illustrates the validity of Theorem 1.1 in the context of this example.

To see a difficulty in proving Theorem 1.1, let us consider a local martingale $S' = (S'_t)_{t \in \mathbb{N}_0}$ with two jumps instead of one; for example, let us define

$$S'_t = \left(\mathbf{1}_{\{U > 1/2\}} - \mathbf{1}_{\{U < 1/2\}} \right) \mathbf{1}_{t \geq 1} + \frac{\theta}{U} \mathbf{1}_{t \geq 2}.$$

Again, it is simple to see that this specification makes S' indeed a P–local and P–generalized martingale. However, now we have $\mathsf{E}_\mathsf{P}[Z_1 S'_1] = \frac{1}{2} \neq 0$; hence ZS' is not a P–martingale. Similarly, neither is $\widehat{Z}S'$. Nevertheless, as Theorem 1.1 states, there exists a uniformly integrable P–martingale Z' such that $Z'S'$ is a P–martingale.

More details on the previous example are provided in Ruf (2018).

3 Proof of Theorem 1.1

In this section, we shall provide the proof of this note's main result. Its overall structure resembles Theorem 1.3 in Prokaj and Rásonyi (2010). The main novelty lies in Lemma 3.1, where the ideas of Rogers (1994) are adapted to obtain an equivalent martingale measure together with the required integrability condition (see Lemmata 3.2 and 3.3). In contrast, the construction of the equivalent martingale measure in Prokaj and Rásonyi (2010) is based on Dalang et al. (1990).

Lemma 3.1. *Let* Q *denote some probability measure on* $(\Omega, \mathscr{F})$*, let* $\mathscr{G}, \mathscr{H}$ *be sigma algebras with* $\mathscr{G} \subset \mathscr{H} \subset \mathscr{F}$*, let* W *denote a* $\mathscr{H}$*–measurable* d*-dimensional random vector with*

$$\mathsf{E}_{\mathsf{Q}}[|W| \,|\, \mathscr{G}] < \infty \qquad and \qquad \mathsf{E}_{\mathsf{Q}}[W \,|\, \mathscr{G}] = 0. \tag{3.1}$$

Suppose that $(\alpha_k)_{k \in \mathbb{N}}$ *is a bounded family of* $\mathscr{H}$*–measurable random variables with* $\lim_{k\uparrow\infty} \alpha_k = 1$*. Then for any* $\varepsilon > 0$ *there exists a family* $(V_k)_{k \in \mathbb{N}}$ *of random variables such that*

(i) V_k *is* $\mathscr{H}$*–measurable and takes values in* $(1 - \varepsilon, 1)$ *for each* $k \in \mathbb{N}$*;*

(ii) $\lim_{k\uparrow\infty} \mathbf{1}_{\{\mathsf{E}_{\mathsf{Q}}[V_k \alpha_k W \,|\, \mathscr{G}] = 0\}} = 1$*.*

We shall provide two proofs of this lemma, the first one applies only to the case $d = 1$, but avoids the technicalities necessary for the general case.

Proof of Lemma 3.1 in the one-dimensional case. With the convention $^0/_0 := 1$, define, for each $k \in \mathbb{N}$, the random variable

$$C_k = \frac{\mathsf{E}_{\mathsf{Q}}[\alpha_k W^+ \,|\, \mathscr{G}]}{\mathsf{E}_{\mathsf{Q}}[\alpha_k W^- \,|\, \mathscr{G}]}$$

and note that

$$\lim_{k\uparrow\infty} |C_k - 1| = \left| \frac{\mathsf{E}_{\mathsf{Q}}[W^+ \,|\, \mathscr{G}]}{\mathsf{E}_{\mathsf{Q}}[W^- \,|\, \mathscr{G}]} - 1 \right| = \frac{1}{\mathsf{E}_{\mathsf{Q}}[W^- \,|\, \mathscr{G}]} \left| \mathsf{E}_{\mathsf{Q}}[W^+ \,|\, \mathscr{G}] - \mathsf{E}_{\mathsf{Q}}[W^- \,|\, \mathscr{G}] \right| = 0.$$

Next, set

$$V_k = (1 - \varepsilon) \vee \left(\mathbf{1}_{\{W \geq 0\}} (1 \wedge C_k^{-1}) + \mathbf{1}_{\{W < 0\}} (1 \wedge C_k) \right),$$

and note that on the event $\{1 - \varepsilon \leq C_k \leq {}^1/_{(1-\varepsilon)}\} \in \mathscr{G}$ we indeed have $\mathsf{E}_{\mathsf{Q}}[V_k \alpha_k W \,|\, \mathscr{G}] = 0$, which concludes the proof. $\qquad \square$

Proof of Lemma 3.1 in the general case. The proof is similar to the proof of the Dalang–Morton–Willinger theorem based on utility maximisation, see Rogers (1994) and Delbaen and Schachermayer (2006, Section 6.6) for detailed exposition. But instead of using the exponential utility, we choose a strictly convex function (the negative of the utility) which is smooth and whose derivative takes values in $(1 - \varepsilon, 1)$. Indeed, in what follows we fix the convex function

$$f(a) = a \left(1 + \frac{\varepsilon}{\pi} \left(\arctan(a) - \frac{\pi}{2} \right) \right), \qquad a \in \mathbb{R}.$$

Then f is smooth and a direct computation shows that f is convex with derivative f' taking values in the interval $(1 - \varepsilon, 1)$.

We formulated the statement with generalized conditional expectations. However, changing the probability appropriately with a $\mathscr{G}$–measurable density we can assume, without loss of generality, that $W \in \mathsf{L}^1(\mathsf{Q})$. Indeed, the probability measure Q', given by

$$\frac{d\mathsf{Q}'}{d\mathsf{Q}} = \frac{e^{-\mathsf{E}_{\mathsf{Q}}[|W| \,|\, \mathscr{G}]}}{\mathsf{E}_{\mathsf{Q}}[e^{-\mathsf{E}_{\mathsf{Q}}[|W| \,|\, \mathscr{G}]}]},$$

satisfies that $W \in \mathsf{L}^1(\mathsf{Q}')$. Moreover, the (generalized) conditional expectations with respect to $\mathscr{G}$ are the same under Q and Q'. Hence, in what follows, we assume that $|W|$ is an integrable random variable.

For W there is a maximal $\mathscr{G}$–measurable orthogonal projection R of $\mathbb{R}^d$ such that $RW = 0$ almost surely. The maximality of R means that for any $\mathscr{G}$–measurable vector variable U which is orthogonal to W almost surely we have $RU = U$. We shall use this property at the end of this proof, such that on the event $\{RU \neq U\}$ the scalar product $W \cdot U$ is non-zero with zero conditional mean so its conditional law is non-degenerate. The idea behind the construction of R is to consider the space of $\mathscr{G}$–measurable vector variables orthogonal to W almost surely, and "take an orthonormal basis over each $\omega \in \Omega$" in a $\mathscr{G}$–measurable way. For details of the proof, see Proposition 2.4 in Rogers (1994) or Section 6.2 in Delbaen and Schachermayer (2006). The orthocomplement of the range of R is called the predictable range of W.

Let B now denote the d–dimensional Euclidean unit ball and set $\alpha_\infty = 1$. For each $k \in \mathbb{N} \cup \{\infty\}$, consider the random function (or field) h_k over B, defined by the formula

$$h_k(u, \cdot) = h_k(u) = \mathsf{E}_\mathsf{Q}[f(\alpha_k W \cdot u) \,|\, \mathscr{G}] + \frac{1}{2}|Ru|^2 \qquad \text{for all } u \in B.$$

Since f is continuous, for each $k \in \mathbb{N} \cup \{\infty\}$, h_k has a version that is continuous in u for each $\omega \in \Omega$; see Lemma A.1 below. Then for each compact subset C of B and each $k \in \mathbb{N} \cup \{\infty\}$ there is a $\mathscr{G}$–measurable random vector U_k^C taking values in C such that $h_k(U_k^C) = \min_{u \in C} h_k(u)$. This is a kind of measurable selection; for sake of completeness we give an elementary proof below in Lemma A.3.

Next, for each $k \in \mathbb{N}$, let U_k be a $\mathscr{G}$–measurable minimiser of h_k in the unit ball B and define

$$V_k = f'(\alpha_k W \cdot U_k).$$

With this definition, (i) follows directly. For (ii) we prove below that

$$\mathsf{E}_\mathsf{Q}[V_k \alpha_k W \,|\, \mathscr{G}] + RU_k = 0, \qquad \text{on } \{|U_k| < 1\}, \quad k \in \mathbb{N}; \tag{3.2}$$

$$\lim_{k \uparrow \infty} U_k = 0, \qquad \text{almost surely.} \tag{3.3}$$

Then, on the event $\{|U_k| < 1\}$, (3.2) and the $\mathscr{G}$–measurability of R yield

$$|\mathsf{E}_\mathsf{Q}[V_k \alpha_k W \,|\, \mathscr{G}]|^2 = -\mathsf{E}_\mathsf{Q}[V_k \alpha_k W \,|\, \mathscr{G}] \cdot RU_k = -\mathsf{E}_\mathsf{Q}[V_k \alpha_k RW \,|\, \mathscr{G}] \cdot U_k = 0,$$

giving us (ii).

Thus, in order to complete the proof it suffices to argue (3.2)–(3.3). For (3.2), note that h_k is continuously differentiable almost surely for each $k \in \mathbb{N}$, see Lemma A.2 below; moreover, its derivative at the minimum point U_k, which equals the left-hand side of (3.2), must be zero when U_k is inside the ball B.

For (3.3) observe that h_∞ has a unique minimiser over B which is the zero vector. To see this, observe that

$$h_\infty(u) = \mathsf{E}_\mathsf{Q}[f(W \cdot (I - R)u) \,|\, \mathscr{G}] + \frac{1}{2}|Ru|^2,$$

where I denotes the d-dimensional identity matrix. So to see that the zero vector is the unique minimiser it is enough to show that $\inf_{|u| \geq \delta} h_\infty(u) > 0 = h_\infty(0)$ almost surely for any $\delta \in (0, 1]$. Let U be a $\mathscr{G}$–measurable minimiser of h_∞ over $\{u : |u| \in [\delta, 1]\}$. Then

$$\mathsf{E}_\mathsf{Q}[f(W \cdot (I - R)U) \,|\, \mathscr{G}] > 0, \qquad \text{on } \{(I - R)U \neq 0\};$$
$$|RU|^2 \geq \delta^2 > 0, \qquad \text{on } \{(I - R)U = 0\}.$$

The first part follows from the strict convexity of f in conjunction with Jensen's inequality, taking into account that $\mathsf{E}_{\mathsf{Q}}[W \mid \mathscr{G}] = 0$ and that $W \cdot (I - R)U$ has non-trivial conditional law on $\{(I - R)U \neq 0\}$ by the maximality of R. Whence $\inf_{|u| \geq \delta} h_\infty(u) > 0 = h_\infty(0)$, as required.

Finally, as $\lim_{k \uparrow \infty} \alpha_k = 1$ and f is Lipschitz continuous we have

$$\lim_{k \uparrow \infty} \sup_{u \in B} |h_k(u) - h_\infty(u)| = \lim_{k \uparrow \infty} \sup_{u \in B \cap \mathbb{Q}^d} |h_k(u) - h_\infty(u)| = 0 \qquad \text{almost surely.}$$

Hence, any $\mathscr{G}$–measurable sequence $(U_k)_{k \in \mathbb{N}}$ of minimisers of h_k converges to zero, the unique minimiser of h_∞, almost surely. This shows (3.3) and completes the proof. $\square$

Lemma 3.2. *Let Q denote some probability measure on $(\Omega, \mathscr{F})$, let $\mathscr{G}, \mathscr{H}$ be sigma algebras with $\mathscr{G} \subset \mathscr{H} \subset \mathscr{F}$, let Y denote a one-dimensional random variable with $Y \geq 0$ and $\mathsf{E}_{\mathsf{Q}}[Y \mid \mathscr{H}] < \infty$, and let W denote a $\mathscr{H}$–measurable d-dimensional random vector such that (3.1) holds. Then, for any $\varepsilon > 0$, there exists a random variable z such that*

(i) *z is $\mathscr{H}$–measurable and takes values in $(0, 1 + \varepsilon)$;*

(ii) *$\mathsf{Q}[z < 1 - \varepsilon] < \varepsilon$;*

(iii) *$\mathsf{E}_{\mathsf{Q}}[z \mid \mathscr{G}] = 1$;*

(iv) *$\mathsf{E}_{\mathsf{Q}}[zW \mid \mathscr{G}] = 0$;*

(v) *$\mathsf{E}_{\mathsf{Q}}[zY \mid \mathscr{G}] < \infty$.*

Proof. For each $k \in \mathbb{N}$, define the $(0, 1]$–valued, $\mathscr{H}$–measurable random variable

$$\alpha_k = \mathbf{1}_{\{\mathsf{E}_{\mathsf{Q}}[Y|\mathscr{H}] \leq k\}} + \frac{1}{\mathsf{E}_{\mathsf{Q}}[Y \mid \mathscr{H}]} \mathbf{1}_{\{\mathsf{E}_{\mathsf{Q}}[Y|\mathscr{H}] > k\}}$$

and note that $\lim_{k \uparrow \infty} \alpha_k = 1$. Lemma 3.1 now yields the existence of a family $(V_k)_{k \in \mathbb{N}}$ of $\mathscr{H}$–measurable random variables such that $V_k \in \left(1/(1 + \varepsilon/2), 1\right)$ and $\lim_{k \uparrow \infty} \mathbf{1}_{\{\mathsf{E}_{\mathsf{Q}}[V_k \alpha_k W | \mathscr{G}] = 0\}} = 1$. Note that this yields a $\mathscr{G}$–measurable random variable K, taking values in $\mathbb{N}$, such that $\mathsf{E}_{\mathsf{Q}}[V_K \alpha_K W \mid \mathscr{G}] = 0$, $\mathsf{E}_{\mathsf{Q}}[V_K \alpha_K \mid \mathscr{G}] > 1/(1 + \varepsilon)$, and $\mathsf{Q}[\mathsf{E}_{\mathsf{Q}}[Y \mid \mathscr{H}] > K] < \varepsilon$. Setting now

$$z = \frac{V_K \alpha_K}{\mathsf{E}_{\mathsf{Q}}[V_K \alpha_K \mid \mathscr{G}]}$$

yields a random variable with the claimed properties. $\square$

Lemma 3.3. *Fix $n \in \mathbb{N}_0$, let Q denote some probability measure on $(\Omega, \mathscr{F})$ such that S is a Q–local martingale, and let Y denote a one-dimensional random variable with $Y \geq 0$ and $\mathsf{E}_{\mathsf{Q}}[Y \mid \mathscr{F}_n] < \infty$. Then, for each $\varepsilon > 0$, there exists a probability measure Q', equivalent to Q, with density $Z^{(n)} = \mathrm{d}\mathsf{Q}'/\mathrm{d}\mathsf{Q}$ such that*

(i) *$Z^{(n)} \in (0, 1 + \varepsilon)$;*

(ii) *$\mathsf{Q}[Z^{(n)} < 1 - \varepsilon] < \varepsilon$;*

(iii) *S is a Q'–local martingale;*

(iv) *$\mathsf{E}_{\mathsf{Q}'}[Y] < \infty$.*

Proof. In this proof, we use the convention $\mathscr{F}_{-1} = \{\emptyset, \Omega\}$ and $\Delta S_0 = 0$. Set $\widetilde{\varepsilon} > 0$ be sufficiently small such that

$$(n+1)\widetilde{\varepsilon} \leq \varepsilon, \qquad (1+\widetilde{\varepsilon})^{n+1} \leq 1+\varepsilon, \qquad (1-\widetilde{\varepsilon})^{n+1} \geq 1-\varepsilon.$$

We shall construct a sequence $(z_0, \cdots, z_n)$ iteratively starting with z_n and proceeding backward until z_0 such that for each $t = 0, 1, \ldots, n$,

$$z_t \leq 1+\tilde{\varepsilon}, \quad \mathsf{Q}[z_t < 1-\tilde{\varepsilon}] < \tilde{\varepsilon}, \quad \mathsf{E}_\mathsf{Q}[z_t \mid \mathscr{F}_{t-1}] = 1, \quad \mathsf{E}_\mathsf{Q}[z_t \Delta S_t \mid \mathscr{F}_{t-1}] = 0, \qquad (3.4)$$

and

$$\mathsf{E}_\mathsf{Q}[Y \prod_{i=t}^{n} z_i \mid \mathscr{F}_{t-1}] < \infty. \qquad (3.5)$$

For $t = n$ we apply Lemma 3.2 with ε replaced by $\widetilde{\varepsilon}$ and with $\mathscr{G} = \mathscr{F}_{n-1}$, $\mathscr{H} = \mathscr{F}_n$, and $W = \Delta S_n$. We have $\mathsf{E}[Y \mid \mathscr{H}] < \infty$ by assumption and $\mathsf{E}_\mathsf{Q}[|W| \mid \mathscr{G}] < \infty$ and $\mathsf{E}_\mathsf{Q}[W \mid \mathscr{G}] = 0$ by Proposition 2.2. Hence, Lemma 3.2 provides us an appropriate z_n satisfying (3.4) and (3.5) for $t = n$.

For $0 \leq t < n$ assume that we have random variables $z_{t+1}, \cdots, z_n$ satisfying (3.4) and (3.5), in particular, $\mathsf{E}_\mathsf{Q}[Y \prod_{i=t+1}^{n} z_i \mid \mathscr{F}_t] < \infty$. We now obtain a random variable z_t by again applying Lemma 3.2, with ε replaced by $\widetilde{\varepsilon}$ and with $\mathscr{G} = \mathscr{F}_{t-1}$, $\mathscr{H} = \mathscr{F}_t$, $W = \Delta S_t$, and Y replaced by $Y \prod_{i=t+1}^{n} z_i$.

With the family $(z_0, \cdots, z_n)$ now given, let us define $Z^{(n)} = \prod_{i=0}^{n} z_i$ and Q' by $\mathrm{d}\mathsf{Q}'/\mathrm{d}\mathsf{Q} = Z^{(n)}$. With this definition of $Z^{(n)}$ (i),(ii), and (iv) are clear by the choice of $\tilde{\varepsilon}$. To argue that S is a Q'–local martingale, let τ be an $(\mathscr{F}_t)_{t \geq 0}$ stopping time such that the stopped process S^τ is a martingale. Then $\Delta S_t \mathbf{1}_{\tau \geq t}$ is Q' integrable random vector as $Z^{(n)}$ is bounded from above. Moreover, Bayes' rule yields

$$\mathsf{E}_{\mathsf{Q}'}[\Delta S_t \mathbf{1}_{\{\tau \geq t\}} \mid \mathscr{F}_{t-1}] = \frac{\mathsf{E}_\mathsf{Q}[Z^{(n)} \Delta S_t \mathbf{1}_{\{\tau \geq t\}} \mid \mathscr{F}_{t-1}]}{\mathsf{E}_\mathsf{Q}[Z^{(n)} \mid \mathscr{F}_{t-1}]} = \mathbf{1}_{\{\tau \geq t\}} \mathsf{E}_\mathsf{Q}[z_t \Delta S_t \mid \mathscr{F}_{t-1}] = 0.$$

So any sequence of stopping times that localizes S under Q also localizes it under Q'. This shows (iii); hence the lemma is proven. $\square$

Proof of Theorem 1.1. We inductively construct a sequence $(\mathsf{Q}^{(n)})_{n \in \mathbb{N}_0}$ of probability measures, equivalent to P, and a sequence $(\varepsilon^{(n)})_{n \in \mathbb{N}_0}$ of positive reals using Lemma 3.3. To start, set $\mathsf{Q}^{(-1)} = \mathsf{P}$. Now, fix $n \in \mathbb{N}_0$ for the moment and suppose that we have $\mathsf{Q}^{(n-1)}$ and $(\varepsilon^{(m)})_{0 \leq m < n}$ such that $\prod_{m=0}^{n-1}(1+\varepsilon^{(m)}) < 1+\varepsilon$. Choose $\varepsilon^{(n)}$ to be sufficiently small such that $\prod_{m=0}^{n}(1+\varepsilon^{(m)}) < 1+\varepsilon$, and for any $A \in \mathscr{F}$ with $\mathsf{Q}^{(n-1)}[A] \leq \varepsilon^{(n)}$ we have $\mathsf{P}[A] < 2^{-n}$. Then apply Lemma 3.3 with ε replaced by $\varepsilon^{(n)}$, and with $\mathsf{Q} = \mathsf{Q}^{(n-1)}$ and $Y = \mathrm{e}^{|S_n|}$ to obtain a probability measure $\mathsf{Q}^{(n)}$ with density $Z^{(n)}$, that is $\mathrm{d}\mathsf{Q}^{(n)} = Z^{(n)}\mathrm{d}\mathsf{Q}^{(n-1)} = (\prod_{m=0}^{n} Z^{(m)})\mathrm{d}\mathsf{P}$.

Due to the fact

$$\mathsf{P}\left[|1 - Z^{(n)}| > \varepsilon^{(n)}\right] \leq 2^{-n} \quad \text{as} \quad \mathsf{Q}^{(n-1)}\left[|1 - Z^{(n)}| > \varepsilon^{(n)}\right] \leq \varepsilon^{(n)},$$

the Borel-Cantelli lemma yields $\sum_{n \in \mathbb{N}_0} |1 - Z^{(n)}| < \infty$; hence the infinite product $Z_\infty = \prod_{n=0}^{\infty} Z^{(n)}$ converges and is positive P–almost surely. It is clear that $Z_\infty \leq 1+\varepsilon$.

We define the probability measure Q by $\mathrm{d}\mathsf{Q}/\mathrm{d}\mathsf{P} = Z_\infty$ and denote the corresponding density process by $Z_t = \mathsf{E}_\mathsf{P}[Z_\infty \mid \mathscr{F}_t]$, for each $t \in \mathbb{N}_0$. As $\prod_{m>t} Z^{(m)} < 1+\varepsilon$ we have $\mathsf{Q} \leq (1+\varepsilon)\mathsf{Q}^{(t)}$ and as a result

$$\mathsf{E}_\mathsf{P}\left[Z_t e^{|S_t|}\right] = \mathsf{E}_\mathsf{Q}\left[e^{|S_t|}\right] \leq (1+\varepsilon)\mathsf{E}_{\mathsf{Q}^{(t)}}\left[e^{|S_t|}\right] < \infty$$

by the choice of $Q^{(t)}$; hence $E_P[Z_t|S_t|^p] < \infty$ for all $t, p \in \mathbb{N}_0$.

It remains to argue that ZS is a P–martingale or, equivalently, that S is a Q–martingale. Since we already have established $E_Q[|S_t|] < \infty$ for all $t \in \mathbb{N}_0$, it suffices to fix $t \in \mathbb{N}$ and to prove that $E_Q[S_t \mid \mathscr{F}_{t-1}] = S_{t-1}$. To this end, recall that S is a $Q^{(n)}$–local martingale for each $n \in \mathbb{N}_0$ by Lemma 3.3(iii) and note that dominated convergence, Bayes formula, and Proposition 2.2 yield

$$E_Q[S_t \mid \mathscr{F}_{t-1}]Z_{t-1} = E_P[S_t Z_\infty \mid \mathscr{F}_{t-1}] = \lim_{n\uparrow\infty} E_P\left[S_t \prod_{m=0}^{n} Z^{(m)} \,\middle|\, \mathscr{F}_{t-1}\right]$$

$$= \lim_{n\uparrow\infty} E_{Q^{(n)}}[S_t \mid \mathscr{F}_{t-1}] \left.\frac{dQ^{(n)}}{dP}\right|_{\mathscr{F}_{t-1}} = S_{t-1} \lim_{n\uparrow\infty} E_P\left[\prod_{m=0}^{n} Z^{(m)} \,\middle|\, \mathscr{F}_{t-1}\right]$$

$$= S_{t-1}Z_{t-1}.$$

This completes the proof. □

A Appendix

In this appendix, we provide some measurability results necessary for the proof of Lemma 3.1. We write $C(K)$ for the space of continuous functions over some metric space (K, m) and equip $C(K)$ with the supremum norm.

When a random variable takes values in an abstract measurable space we call it a random element from that space. In all cases below, the measurable space is a metric space equipped with its Borel σ-algebra, the σ-algebra generated by the open sets. In particular, ξ is a random element from $C(K)$ if and only if $\xi(u)$ is a random variable for each u and $u \mapsto \xi(u, \omega)$ is continuous for each $\omega \in \Omega$.

Lemma A.1. *Let $\mathscr{G}$ be a sigma algebra with $\mathscr{G} \subset \mathscr{F}$ and let ξ be a random element in $C(K)$, where (K, m) is a compact metric space. Suppose that $E_P[\sup_{u \in K}|\xi(u)|] < \infty$ and let $\eta(u) = E_P[\xi(u) \mid \mathscr{G}]$ for all $u \in K$. Then $(\eta(u))_{u \in K}$ has a continuous modification.*

Proof. Let D be a countable dense subset of K. We show that there is $\Omega' \in \mathscr{G}$ with full probability such that $(\eta(u))_{u \in D}$ is uniformly continuous over D on Ω'. Then we can define

$$\tilde{\eta}(u) = \begin{cases} \lim_{\substack{u_n \to u \\ u_n \in D}} \eta(u_n) & \text{on } \Omega', \\ 0 & \text{otherwise.} \end{cases}$$

It is a routine exercise to check that $\tilde{\eta}$ is well defined and a continuous modification of η.

One way to get Ω' is the following. Let μ be the modulus of continuity of ξ, that is,

$$\mu(\delta) = \sup_{u, u' \in K, \, m(u, u') \le \delta} |\xi(u) - \xi(u')|, \qquad \delta > 0.$$

Obviously $\mu(\delta) \to 0$ everywhere as $\delta \downarrow 0$. Dominated convergence, in conjunction with the bound $\mu \le 2 \sup_{u \in K}|\xi(u)|$, yields $\tilde{\mu}(\delta) = E[\mu(\delta) \mid \mathscr{G}] \to 0$ as $\delta \downarrow 0$ almost surely. Now define

$$\Omega' = \left\{\lim_{n\uparrow\infty} \tilde{\mu}\left(\frac{1}{n}\right) = 0\right\} \cap \left(\bigcap_{n\in\mathbb{N}} \bigcap_{u, u' \in D, \, m(u, u') \le 1/n} \left\{|\eta(u) - \eta(u')| \le \tilde{\mu}\left(\frac{1}{n}\right)\right\}\right).$$

Clearly Ω' has full probability and the claim is proved. □

In the setting of Lemma A.1 when $K \subset \mathbb{R}^d$ and ξ is a random element in $C^1(K)$ then under mild conditions $\eta(u) = \mathsf{E}[\xi(u) \mid \mathscr{G}]$ has a version taking values in $C^1(K)$. This is the content of the next lemma. Recall that a function f defined on K belongs to $C^1(K)$ if f is continuous and there is a continuous $\mathbb{R}^d$–valued function on K which agrees with the gradient f' of f in the interior of K.

Lemma A.2. *Let $\mathscr{G}$ be a sigma algebra with $\mathscr{G} \subset \mathscr{F}$ and let ξ be a random element in $C^1(K)$, where $K \subset \mathbb{R}^d$ is a compact subset set. Suppose that*

$$\mathsf{E}_\mathsf{P}\left[\sup_{u \in K}|\xi(u)|\right] + \mathsf{E}_\mathsf{P}\left[\sup_{u \in K}|\xi'(u)|\right] < \infty$$

and let $\eta(u) = \mathsf{E}_\mathsf{P}[\xi(u) \mid \mathscr{G}]$ for all $u \in K$. Then $(\eta(u))_{u \in K}$ has a version taking values in $C^1(K)$ and the continuous version of $(\mathsf{E}[\xi'(u) \mid \mathscr{G}])_{u \in K}$ gives the gradient of η almost surely.

Proof. By Lemma A.1 both $\eta(u) = \mathsf{E}[\xi(u) \mid \mathscr{G}]$ and $\eta'(u) = \mathsf{E}[\xi'(u) \mid \mathscr{G}]$ have continuous versions. We prove that, apart from a null set, η' is indeed the gradient of η. To this end, let D be a countable dense subset of the interior of K and denote by $I(a, b)$ a directed segment going from a to b, for each $a, b \in K$. Then, by assumption, for $a, b \in D$, with $I(a, b) \subset \operatorname{int} K$ we get

$$\eta(b) - \eta(a) = \mathsf{E}[\xi(a) - \xi(b) \mid \mathscr{G}] = \mathsf{E}\left[\int_{I(a,b)} \xi'(u)\mathrm{d}u \,\Bigg|\, \mathscr{G}\right] = \int_{I(a,b)} \eta'(u)\mathrm{d}u, \quad \text{almost surely.}$$

Hence, there exists an event $\Omega' \in \mathscr{G}$ with $\mathsf{P}[\Omega'] = 1$ such that

$$\eta(b, \omega) - \eta(a, \omega) = \int_{I(a,b)} \eta'(u, \omega)du, \quad \text{for all } a, b \in D, \text{ with } I(a, b) \subset \operatorname{int} K \text{ and } \omega \in \Omega'.$$

By continuity this identity extends to all $a, b \in \operatorname{int} K$ with $I(a, b) \subset \operatorname{int} K$ on Ω'. Using again the continuity of $\eta'(., \omega)$ yields that η' is indeed the gradient of η on Ω'. $\qquad\square$

Lemma A.3. *Let (K, m) be a compact metric space and η a random element in $C(K)$. Then there is a measurable minimiser of η, that is, a random element U in K such that $\eta(U) = \min_{u \in K} \eta(u)$.*

Proof. To shorten the notation, for each $x \in K$ and $\delta \geq 0$, let

$$B(x, \delta) = \{u \in K \,:\, m(u, x) \leq \delta\}, \quad \eta(x, n) = \min\{\eta(u) \,:\, u \in B(x, 2^{-n})\}.$$

For each $n \in \mathbb{N}$ let D_n be a finite 2^{-n}-net in K; that is, $K \subset \bigcup_{x \in D_n} B(x, 2^{-n})$. For each $n \in \mathbb{N}$ fix an order of the finite set D_n. We shall use the fact that for any closed set $F \subset K$ the minimum over F, that is, $\min_{u \in F} \eta(u)$, is a random variable. This follows easily since a continuous function on a metric space is Borel measurable, and $C(K) \ni f \mapsto \inf_{u \in F} f(u)$ depends continuously on f, it is even Lipschitz continuous.

We construct a sequence $(U_n)_{n \in \mathbb{N}}$ of random elements in K by recursion, such that

- $\eta(U_n, n) = \min_{u \in K} \eta(u)$, and

- $m(U_n, U_{n+1}) \leq 2^{-n} + 2^{-(n+1)}$.

Then $(U_n)_{n \in \mathbb{N}}$ has a limit U which is a measurable minimiser of η over K. To see that U is indeed a minimiser, observe that for each $r > 0$ there is an n such that $B(U_n, 2^{-n}) \subset B(U, r)$, hence

$$\min_{u \in K} \eta(u) \leq \min_{u \in B(U,r)} \eta(u) \leq \min_{u \in B(U_n, 2^{-n})} \eta(u) = \eta(U_n, n) = \min_{u \in K} \eta(u).$$

That is, the minimum of η over the closed ball around U with an arbitrary small positive radius r agrees with the global minimum of η. Letting $r \to 0$ the continuity of η yields that $\eta(U) = \min_{u \in K} \eta(u)$.

We now construct the sequence $(U_n)_{n \in \mathbb{N}}$. For $n = 1$ let U_1 be the first element in

$$\left\{ v \in D_1 \ : \ \eta(v, 1) = \min_{u \in K} \eta(u) \right\}.$$

Since this set is not empty, U_1 is well defined. Moreover, U_1 takes values in the finite set $D_1 = \{v_1, \ldots, v_k\}$, and the levelset $\{U_1 = v_\ell\} = A_\ell \setminus \cup_{i < \ell} A_i$, where $A_i = \{\eta(v_i, 1) = \min_{u \in K} \eta(u)\}$, is obviously an event, as $\eta(v_i, 1)$ and $\min_{u \in K} \eta(u)$ are random variables. So U_1 is measurable, that is, a random element from K.

If $U_1, \ldots, U_n$ are defined for some $n \in \mathbb{N}$ set U_{n+1} to be the first element in

$$\left\{ v \in D_{n+1} \ : \ \eta(v, n+1) = \min_{u \in K} \eta(u), \ m(v, U_n) \leq 2^{-n} + 2^{-(n+1)} \right\}$$

This set is not empty as

$$B(U_n, 2^{-n}) \subset \bigcup_{\substack{v \in D_{n+1} \\ m(v, U_n) \leq 2^{-n} + 2^{-(n+1)}}} B(v, 2^{-(n+1)}),$$

so U_{n+1} is well defined and its measurability is obtained similarly to that of U_1. We conclude that the sequence with the above properties exists and its limit is a measurable minimiser. $\qquad\square$

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