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Bounding the size of an almost-equidistant set in Euclidean space

Andrey Kupavskii∗ Nabil H. Mustafa†‡ Konrad J. Swanepoel§

Abstract

A set of points in $d$-dimensional Euclidean space is almost equidistant if among any three points of the set, some two are at distance 1. We show that an almost-equidistant set in $\mathbb{R}^d$ has cardinality $O(d^{4/3})$.

1 Introduction

A set of lines through the origin of Euclidean $d$-space $\mathbb{R}^d$ is almost orthogonal if among any three of the lines, some two are orthogonal. Erdős asked (see [12]) what is the largest cardinality of an almost-orthogonal set of lines in $\mathbb{R}^d$? By taking the union of two sets of $d$ pairwise orthogonal lines, we see that $2d$ is possible. Rosenfeld [12] showed that $2d$ is the maximum by considering the eigenvalues of the Gramian of the unit vectors spanning the lines. His result was subsequently given simpler proofs by Pudlák [11] and Deaett [6].

In this note we consider the analogous notion obtained by replacing orthogonal pairs of lines by pairs of points at unit distance. A subset $V$ of Euclidean $d$-space $\mathbb{R}^d$ is almost equidistant if among any three points in $V$, some two are at Euclidean distance 1. We investigate the largest size, which we denote by $f(d)$, of an almost-equidistant set in $\mathbb{R}^d$.

Although asking for the size of this function is a very natural question, it seems to be harder than the question of Erdős, which can be reformulated as asking for the largest size of an almost-equidistant set on a sphere of radius $1/\sqrt{2}$ in $\mathbb{R}^d$. Before stating our main result, we give an overview of what is known about $f(d)$.

Bezdek, Naszódi and Visy [5] showed that $f(2) \leq 7$, and István Talata (personal communication, 2007) showed that the only almost-equidistant set in $\mathbb{R}^2$ with 7 points is the Moser spindle. Györey [7] showed that $f(3) \leq 10$ and that there is a unique almost-equidistant set of 10 points in $\mathbb{R}^3$, a configuration originally considered by Nechushtan [9].

The Moser spindle can be generalized to higher dimensions, giving an almost-equidistant set of $2d + 3$ points in $\mathbb{R}^d$ [4]. (We mention that Bezdek and Langi [4] considered the variant of Erdős’s problem where the radius of the sphere is arbitrary instead of $1/\sqrt{2}$.)

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construction of Larman and Rogers [8] shows that \( f(5) \geq 16 \). Since there does not exist a set of \( d+2 \) points in \( \mathbb{R}^d \) that are pairwise at distance 1, it follows that \( f(d) \leq R(d+2,3) - 1 \), where the Ramsey number \( R(a,b) \) is the smallest \( n \) such that whenever each edge of the complete graph on \( n \) vertices is coloured blue or red, there is either a blue clique of size \( a \) or a red clique of size \( b \). Ajtai, Komlós, and Szemerédi [1] showed \( R(k,3) = O(k^2 / \log k) \), which implies the asymptotic upper bound \( f(d) \leq O(d^2 / \log d) \). Balko, Pór, Scheuer, Swanepoel, and Valtr [3] generalized the Nechushtan configuration to higher dimensions, giving \( f(d) \geq 2d + 4 \) for all \( d \geq 3 \). They also obtained the asymptotic upper bound \( f(d) = O(d^3 / 2) \) by an argument based on Deaett’s paper [6]. Using computer search and ad hoc geometric arguments, they obtained the following bounds for small \( d \):

\[
\begin{align*}
\ f(4) & \leq 13, \\
\ f(5) & \leq 20, \\
\ f(6) & \leq 18, \\
\ f(7) & \leq 26, \\
\ f(8) & \geq 24.
\end{align*}
\]

Polyanskii [10] subsequently improved the asymptotic upper bound to \( f(d) = O(d^{13/9}) \).

In this note we obtain a further improvement to the upper bound.

**Theorem 1.** An almost-equidistant set in \( \mathbb{R}^d \) has cardinality \( O(d^{4/3}) \).

Its proof is based on the approach of [3] to show the upper bound \( O(d^{3/2}) \), which is in turn based on Deaett’s proof [6] of Rosenfeld’s result. Before proving Theorem 1 in Section 3, we establish notation and collect some lemmas in the next section.

## 2 Preliminaries

We denote the Euclidean norm of \( x \in \mathbb{R}^d \) by \( \|x\| \) and the inner product of \( x, y \in \mathbb{R}^d \) by \( \langle x, y \rangle \). The cardinality of a finite set \( A \) is denoted by \( |A| \). We call a finite non-empty subset \( C \) of \( \mathbb{R}^d \) (the vertex set of) a unit simplex if the distance between any two points in \( C \) equals 1. It has already been mentioned in the Introduction that if \( C \) is a unit simplex then \( |C| \leq d + 1 \). Given any finite \( V \subset \mathbb{R}^d \), we define the unit-distance graph \( G = (V, E) \) on \( V \) to be the graph with \( vw \in E \) iff \( \|v - w\| = 1 \). Thus, \( C \subset V \) is a unit simplex iff it is a clique in \( G \). We denote the set of neighbours of \( v \in V \) in \( G \) by \( N(v) \).

The following well-known lemma gives a lower bound for the rank of a square matrix in terms of its entries [2, 6, 11].

**Lemma 1.** For any non-zero \( n \times n \) symmetric matrix \( A = [a_{i,j}] \),

\[
\text{rank}(A) \geq \frac{(\sum_{i,j} a_{i,j})^2}{\sum_{i,j} a_{i,j}^2}.
\]

For the sake of completeness, we include the proofs of the following three lemmas on the vertices and centroids of unit simplices.

**Lemma 2.** Let \( C \) be a unit simplex with centroid \( c = \frac{1}{|C|} \sum_{v \in C} v \). Then

\[
\|v - c\|^2 = \frac{1}{2} \left( 1 - \frac{1}{|C|} \right) \quad \text{for all } v \in C,
\]

and

\[
\langle v - c, v' - c \rangle = -\frac{1}{2|C|} \quad \text{for all distinct } v, v' \in C.
\]
Proof. We may translate $C$ so that $c$ is the origin $o$. Write $C = \{p_1, \ldots, p_k\}$. By symmetry, $\alpha := \|p_i\|^2$ is independent of $i$, and $\beta := \langle v_i, v_j \rangle$ ($i \neq j$) is independent of $i$ and $j$. Then
\[
0 = \left\| \sum_{i=1}^{k} p_i \right\|^2 = k\alpha + k(k-1)\beta
\]
and
\[
1 = \|p_i - p_j\|^2 = 2\alpha - 2\beta.
\]
Solving these two linear equations in $\alpha$ and $\beta$, we obtain $\alpha = \frac{1}{2} - \frac{1}{2k}$ and $\beta = \frac{1}{2k}$. $\square$

Lemma 3. Let $C$ be a unit simplex with centroid $c = \frac{1}{|C|} \sum_{v \in C} v$, and let $F \subset C$ be a unit simplex with centroid $f = \frac{1}{|F|} \sum_{v \in F} v$. Then
\[
\|c - f\|^2 = \frac{1}{2} \left( \frac{1}{|F|} - \frac{1}{|C|} \right).
\]

Proof. Let $k := |C|$, $\ell := |F|$. Then
\[
\|f - c\|^2 = \left\| \frac{1}{\ell} \sum_{v \in F} (v - c) \right\|^2 = \frac{1}{\ell^2} \left( \ell \cdot \frac{1}{2} \left( 1 - \frac{1}{k} \right) - \frac{\ell(\ell-1)}{2k} \right) \text{ by Lemma 2}
\]
\[
= \frac{1}{2} \left( \frac{1}{\ell} - \frac{1}{k} \right),
\]
$\square$.

Lemma 4. Let $A$ and $B$ be disjoint unit simplices with centroids $a = \frac{1}{|A|} \sum_{v \in A} v$ and $b = \frac{1}{|B|} \sum_{v \in B} v$, respectively, such that $A \cup B$ is also a unit simplex. Then
\[
\|a - b\|^2 = \frac{1}{2} \left( \frac{1}{|A|} + \frac{1}{|B|} \right).
\]

Proof. Let $C := A \cup B$ have centroid $c$. Then by Lemma 3, $\|a - c\|^2 = \frac{1}{2} \left( \frac{1}{|A|} - \frac{1}{|C|} \right)$ and $\|b - c\|^2 = \frac{1}{2} \left( \frac{1}{|B|} - \frac{1}{|C|} \right)$. It follows that
\[
\|a - b\|^2 = (\|a - c\| + \|b - c\|)^2
\]
\[
= \frac{1}{2} \left( \frac{1}{|A|} + \frac{1}{|B|} - \frac{2}{|C|} \right) + \sqrt{ \left( \frac{1}{|A|} - \frac{1}{|C|} \right) \left( \frac{1}{|B|} - \frac{1}{|C|} \right) }
\]
\[
= \frac{1}{2} \left( \frac{1}{|A|} + \frac{1}{|B|} \right).
\]
$\square$.

3 Proof of Theorem 1

Let $G$ be the unit-distance graph of a given almost-equidistant set $V$. Then the complement of $G$ is $K_3$-free, and the non-neighbours of any vertex form a unit simplex. Let $C$ be a clique of maximum cardinality in $G$. Write $k = |C|$. Each $v \in V \setminus C$ is a non-neighbour of some point in $C$, and it follows that $|V| \leq |C| + |C| k = k^2 + k$. Thus, without loss of generality, $k > d^{2/3}$. 

3
We split $V$ up into two parts, each to be bounded separately. Let

$$N = \left\{ v \in V : |N(v) \cap C| \geq k - k^{4/3}d^{-2/3} \right\}.$$ 

Note that $k^{4/3}d^{-2/3} = O(d^{-1/3})k$. We first bound the complement of $N$. Consider the set

$$X = \{(u, v) \in C \times V \setminus N : uv \notin E(G)\}.$$

For each $v \in V \setminus N$, there are more than $k^{4/3}d^{-2/3}$ points $u \in C$ such that $u \notin N(v)$, hence $|X| > k^{4/3}d^{-2/3}|V \setminus N|$. On the other hand, for each $u \in C$, the set of non-neighbours of $u$ forms a clique, so has cardinality at most $k$, and $|X| \leq |C| k = k^2$. It follows that

$$|V \setminus N| < k^{2/3}d^{2/3}. \quad (1)$$

Next, we estimate $|N|$. Without loss of generality, $\frac{1}{k} \sum_{v \in C} v = 0$ and $N = \{v_1, \ldots, v_n\}$. We want to apply Lemma 1 to the $n \times n$ matrix $A = ([v_i, v_j])$, which has rank at most $d$.

**Claim 1.** For each $i = 1, \ldots, n$, $\|v_i\|^2 = \frac{1}{2} + O(k^{-1/3}d^{-1/3})$, and for each $v_i v_j \in E(G)$, $\langle v_i, v_j \rangle = O(k^{-1/3}d^{-1/3})$.

**Proof of Claim 1.** Let $C_i := N(v_i) \cap C$, $k_i := |C_i|$, and $c_i := \frac{1}{k_i} \sum_{v \in C_i} v$. Then $k_i \geq k - k^{4/3}d^{-2/3}$. By Lemma 4 applied to $A = \{v_i\}$ and $B = C_i$, $\|v_i - c_i\|^2 = \frac{1}{2} \left(1 + \frac{1}{k_i}\right)$, hence $\|v_i - c_i\| = \frac{1}{\sqrt{2}} + O(k^{-1})$. By Lemma 3 applied to $C$ and $F = C_i$,

$$\|c_i\| = \sqrt{\frac{1}{2} \left(1 - \frac{1}{k_i}\right)} \leq \sqrt{\frac{1}{2} \left(\frac{1}{k - k^{4/3}d^{-2/3}} - \frac{1}{k}\right)} = O(k^{-1/3}d^{-1/3}).$$

By the triangle inequality,

$$\|v_i\| = \|v_i - c_i\| + O(\|c_i\|) = \frac{1}{\sqrt{2}} + O(k^{-1/3}d^{-1/3}),$$

and $\|v_i\|^2 = \frac{1}{2} + O(k^{-1/3}d^{-1/3})$. Also, $2 \langle v_i, v_j \rangle = \|v_i\|^2 + \|v_j\|^2 - 1 = O(k^{-1/3}d^{-1/3}).$ \hfill \Box

**Claim 2.** For each $i = 1, \ldots, n$,

$$\sum_{v_i v_j \notin E(G)} \langle v_i, v_j \rangle^2 = O(k^{2/3}d^{-1/3}).$$

**Proof of Claim 2.** The non-neighbours $N \setminus N(v_i)$ of $v_i$ form a unit simplex with cardinality $t := |N \setminus N(v_i)| \leq k$ and with centroid $c$, say. If $t = d + 1$, remove one point $v_j$ from the unit simplex, which decreases the sum by $\langle v_i, v_j \rangle^2 = O(1)$. Thus, without loss of generality, $t \leq d$, and there exists a point $p \in \mathbb{R}^d$ such that $p - c$ is orthogonal to the affine hull of $N \setminus N(v_i)$, $\|p - c\| = 1/\sqrt{2t}$, the set $\{v_j - p : v_j \in N \setminus N(v_i)\}$ is orthogonal, and $\|v_j - p\| = 1/\sqrt{2}$ for each non-neighbour $v_j$ of $v_i$. Then, by the finite Bessel inequality,

$$\sum_{v_j \in N \setminus N(v_i)} \langle v_i, v_j - p \rangle^2 \leq \frac{1}{2} \|v_i\|^2.$$
hence by applying Cauchy–Schwarz a few times,

\[ \sum_{v_j \in N \setminus N(v_i)} \langle v_i, v_j \rangle^2 = \sum_{v_j \in N \setminus N(v_i)} \left( \langle v_i, v_j - p \rangle + \langle v_i, p - c \rangle + \langle v_i, c \rangle \right)^2 \]

\[ \leq 3 \sum_{v_j \in N \setminus N(v_i)} \left( \langle v_i, v_j - p \rangle^2 + \langle v_i, p - c \rangle^2 + \langle v_i, c \rangle^2 \right) \]

\[ \leq 3 \left( \frac{1}{2} \|v_i\|^2 + t \|v_i\|^2 \|p - c\|^2 + t \|v_i\|^2 \|c\|^2 \right) \]

\[ \leq 3 \left( \|v_i\|^2 + t \|v_i\|^2 \|c\|^2 \right). \]  

(2)

By Claim 1, \( \|v_i\|^2 = \frac{1}{2} + O(k^{-1/3}d^{-1/3}) \) and

\[ \|c\|^2 = \left\| \frac{1}{t} \sum_{v_j \in N \setminus N(v_i)} v_j \right\|^2 = \frac{1}{t^2} \left( \sum_{v_j \in N \setminus N(v_i)} \|v_j\|^2 + \sum_{v_j, v_j' \in N \setminus N(v_i) \atop v_j \neq v_j'} \langle v_j, v_j' \rangle \right) \]

\[ \leq \frac{1}{t^2} \left( t \left( \frac{1}{2} + O(k^{-1/3}d^{-1/3}) \right) + t(t - 1)O(k^{-1/3}d^{-1/3}) \right) \]

\[ = \frac{1}{2t} + O(k^{-1/3}d^{-1/3}). \]

Therefore,

\[ t \|c\|^2 = \frac{1}{2} + O(tk^{-1/3}d^{-1/3}) = O(k^{2/3}d^{-1/3}). \]

Substitute this back into (2) to finish the proof of Claim 2.

We now finish the proof of the theorem. By Claim 2,

\[ \sum_{j=1}^{n} \langle v_i, v_j \rangle^2 = \|v_i\|^4 + \sum_{v_j \in N(v_i)} \langle v_i, v_j \rangle^2 + O(k^{2/3}d^{-1/3}) \]

\[ = nO(k^{-2/3}d^{-2/3}) + O(k^{2/3}d^{-1/3}) \quad \text{by Claim 1}. \]

Also by Claim 1, \( \sum_{i=1}^{n} \|v_i\|^2 = \Omega(n) \). Therefore, by Lemma 1,

\[ d \geq \text{rank}(A) \geq \frac{\left( \sum_{i=1}^{n} \|v_i\|^2 \right)^2}{\sum_{i,j=1}^{n} \langle v_i, v_j \rangle^2} = \frac{\Omega(n^2)}{n \left( nO(k^{-2/3}d^{-2/3}) + O(k^{2/3}d^{-1/3}) \right)}, \]

hence \( n = O(nk^{-2/3}d^{1/3}) + O(k^{2/3}d^{2/3}) \). Since \( O(k^{-2/3}d^{1/3}) = o(1) \), it follows that \( |N| = n = O(k^{2/3}d^{2/3}) \). Recalling (1), we obtain that

\[ |V| = |N| + |V \setminus N| = O(k^{2/3}d^{2/3}) = O(d^{4/3}). \]

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References


