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Decentralized Bargaining in Matching Markets: Efficient Stationary Equilibria and the Core*

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Abstract

This paper studies market clearing in matching markets. The model is non-cooperative, fully decentralized, and in Markov strategies. Workers and firms bargain with each other to determine who will be matched to whom and at what terms of trade. Once a worker-firm pair reach agreement they exit the market. Alternative possible matches affect agents' bargaining positions. We ask when do such markets clear efficiently and find that inefficiencies – mismatch and delay – often feature. Mismatch occurs whenever an agent's bargaining position is at risk of deteriorating. Delay occurs whenever agents expect their bargaining position to improve. Delay can be extensive and structured with vertically differentiated markets endogenously clearing from the top down.

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1 Introduction

We study thin matching markets, particularly labor markets, featuring decentralized negotiations that involve heterogeneous agents from both sides of the market. We capture one dimension of heterogeneity through constraints restricting who can match to whom. Firms might be able to employ only workers they have interviewed; some positions might only be filled through referrals; and some people may simply be unqualified for some positions. On top of this, we allow for variability in how well-suited different workers are to fill different vacancies. We take these matching constraints and heterogeneities as given, and ask when decentralized negotiations can clear markets efficiently.

We assume players exit the market once they reach an agreement. Exit shapes the set of alternative matches available to players remaining in the market. We contend that in many decentralized labor markets agreements are reached sequentially, and the market context in which the remaining workers and firms bargain evolves. For instance, when negotiating with a firm, a worker might use the possibility of taking a position with another firm to achieve a higher wage. But this bargaining stance may be undermined if that position is filled by a different worker. When the changing market context affects the terms that are agreed in equilibrium, we find that markets fail to clear efficiently. People delay when they expect the market to evolve in their favor, and match inefficiently when they expect the market to evolve against them.

In our model there are multiple buyers, multiple sellers, matching is one-to-one, and each buyer-seller pair generates some pair-specific surplus if matched. Time is infinite, and in each period a single agent is selected at random to make a proposal. The proposer chooses an unmatched player and offers a split of the surplus that the pair would generate if matched. If the offer is accepted, the pair exits the market. If it is rejected, the pair remains in the market. While non-stationarities created by matched pairs exiting complicate matters, the endogenous evolution of the market is central to our results. We find it is this that creates scope for bargaining frictions, and we investigate the role of market evolution in driving inefficiencies, both mismatch (inefficient matching) and delay.

We study the Markov perfect equilibria (MPE) in which strategies only depend on the set of players who remain in the market. Restricting attention to such stationary equilibria is common in the bargaining literature,[1] can be motivated on grounds of complexity,[2] and has

[1]See, for example, Rubinstein and Wolinsky (1985, 1990), Gale (1987), Polanski and Winter (2010), Abreu and Manea (2012b).

[2]Maskin and Tirole (2001) argue for considering MPE on the basis of complexity. Further, the conclusions of Bhaskar, Mailath and Morris (2013), which are reached in a more general setting and can be applied to our model, imply that only MPE are purifiable. Finally, Sabourian (2004) provides additional motivation specific to bargaining in markets.

received some experimental support.[3]

Our primary focus is on the existence of an efficient MPE (that is, an equilibrium in which buyers and sellers are matched to maximize total surplus) when players are patient (or equivalently, interactions are frequent). We consider equilibria that are efficient for sufficiently patient players, as well as limiting equilibria that only become efficient as the discount factor converges to 1. Interpreting the probability that a player is selected to make an offer as the bargaining power of that player, suppose the surplus generated by an efficiently matched pair is split in proportion to these bargaining powers. We refer to these payoffs as agents' *Rubinstein payoffs*, as they would obtain in the limit if all efficient pairs bargained bilaterally.[4] Our first main result establishes that an efficient MPE exists for sufficiently patient players if and only if Rubinstein payoffs are in the core of the market (that is, if no pair of players can deviate and benefit by matching to each other when all players receive their Rubinstein payoffs). Moreover, if this condition holds, all players receive their Rubinstein payoffs in the limit as they become arbitrarily patient. If it is a mutual best response for players to ignore the market context and bargain bilaterally with each other when discount factors are sufficiently high, there is an MPE in which players do so. As a result, players match efficiently. If not, there is no efficient MPE when players are sufficiently patient. Thus, whenever the market context matters and players' bargaining positions evolve as others exit the market, there does not exist an efficient MPE.

To gain some intuition for why there is no efficient MPE when Rubinstein payoffs are outside the core, note that if all players were to just bargain bilaterally with their core matches they would receive their Rubinstein payoffs, and there would then be at least one player i who would prefer to deviate and reach agreement with some other player j. If so, in order to preserve efficiency, this alternative match would have to serve as a binding outside option, bounding i's payoff while not being exercised. However, if i and j never agreed, then i's efficient partner would benefit by waiting for j to exit the market and by agreeing with i only when his bargaining position decays. Thus, in such instances, there is no equilibrium in which all players agree with their efficient partners with certainty.

This intuition also identifies a limitation of the aforementioned efficiency result. If j never exits before i in equilibrium, then i's efficient partner cannot weaken i's bargaining position by delaying. So when there is no danger of binding alternative matching opportunities being lost,

[3]Agranov and Elliott (2017) investigate bargaining in a laboratory experiment in an environment closely related to the one we study. The MPE organize the data well and substantially better than standard alternative theories and in particular efficient perfect equilibria. They find empirical support for several predictions we make in this paper.

[4]We refer to these payoffs as Rubinstein payoffs as they are the unique limiting equilibrium outcome in bilateral bargaining settings with random proposers, the classical generalization to random proposers of the bilateral bargaining setting with alternating offers analysed in Rubinstein (1982).

we might expect there to be an MPE that is inefficient away from the limit, but that exhibits vanishingly small inefficiencies in the limit as players become arbitrarily patient. Indeed, in a two-player setting, Sutton (1986) establishes that, as agents become perfectly patient, outside options can bound payoffs from below while being exercised with a probability converging to zero. Investigating this possibility, we look for MPE that exhibit vanishing inefficiencies in the limit. We begin by studying MPE that exhibit no delay or mismatch in the limit. In such equilibria, agents may provide binding alternative matching opportunities and never exit the market because in equilibrium they are unmatched. In this case, a modified version of our main result continues to hold in the limit, and MPE will exhibit no delay or mismatch in the limit only if suitably modified Rubinstein payoffs belong to the core. Permitting delay in the limit, there is another way in which binding alternative matching opportunities might affect payoffs without resulting in mismatch in the limit. This requires players to endogenously exit in sequence. In particular, all players except one efficient pair may choose to delay with probability one in the limit while waiting for this pair to reach agreement and exit the market. If so, the agreeing pair of players would have alternative matching opportunities that are never lost before their exit. Consequently, their bargaining positions would not evolve; and alternative matches can bound the payoffs of this pair of players while being exercised with a vanishingly small probability in the limit. It might be reasonable to postulate that no such equilibrium would ever exist as it requires delay in the limit from pairs of players who expect to be matched with probability one in the limit. Perhaps surprisingly, endogenous delay of this form, resulting in sequential exit from the market, can occur in equilibrium. With four players and equal bargaining powers we find necessary and sufficient conditions for such an MPE to exist. The market must be highly vertically differentiated and clear from the top. This is consistent with anecdotal evidence from high-skill labor markets. In sports and in the movie industry, markets are sometimes reported to be held up until a star is matched.

Delay is possible in our model despite information being perfect because the order of play is random. As time progresses and matched pairs exit the market, the strength of players' bargaining positions evolves stochastically. Equilibrium delay stems from favorable beliefs about the market evolution (for instance, beliefs about tempting alternative matches for their bargaining partners exiting the market). In limiting equilibria with sequential exit, such favorable beliefs are driven by vanishingly small probabilities of an inefficient match occurring, which increase the expected payoffs of all delaying players.

Related Literature: We study decentralized bargaining in thin markets. The prototypical market we intend to speak to is a labor market for high skill individuals. Such markets are inherently thin, and characterized by heterogeneities and by decentralized negotiations. Our approach is closest to the literature analyzing non-cooperative bargaining in thin markets.

This literature takes the coalitional bargaining approach, but restricts the coalitions which can generate surplus and reach agreement to pairs of players. As there are large literatures considering coalitional approaches to non-cooperative bargaining and bargaining in large markets, we do not attempt a complete review of these. Instead, we just highlight some of the most closely related work.

Because of the additional generality, coalitional bargaining models are typically a better fit for political negotiations and committee decision making. The closest papers to ours in this literature, Moldovanu and Winter (1995) and Okada (2011), also link cooperative and non-cooperative approaches. Moldovanu and Winter consider a bargaining model with no discounting and deterministic proposer orders. They find conditions on the proposer order for a core outcome to be reached in a stationary equilibrium.[5] The indeterminacy of equilibrium bargaining outcomes caused by the lack of discounting is critical for their analysis. Our conclusions rule out this indeterminacy by studying the limit of a model in which delay costs vanish, and can nevertheless relate players' bargaining power (or equivalently, proposal probability) to the existence of an efficient MPE. Like us, Okada finds conditions under which no efficient MPE exists, and relates these conditions to the core. However, in the assignment economies we consider, the conditions he identifies are generically violated when there are two or more players on each side of the market, implying that an efficient equilibrium never exists.[6] In contrast, an efficient MPE exists for a positive measure subset of the parameter space in our decentralized bargaining model. Moreover, even when such conditions fail, we show that there can be equilibria with vanishing inefficiencies as delay costs become small.

A vast literature has considered decentralized bargaining in large markets, meaning either that the number of players is infinite or that agreeing players are replaced by exact replicas.[7] Seminal work includes Rubinstein and Wolinsky (1985), Gale (1987), and Binmore and Herrero (1988). The literature has focused on deriving conditions for equilibrium outcomes to approximate competitive equilibria (or equivalently, core outcomes) when the frictions get small. Lauerman (2013) provides a tight characterization of when these two outcomes can be expected to coincide. Most papers in this literature study steady state outcomes,[8] but Moreno and Wooders (2002) is an exception. As in some equilibria of our model, they find delay can occur in the limit. But unlike our model, equilibrium outcomes in their setting are always competitive as players become infinitely patient.

[5]Their main conclusion establishes that a core outcome is reached when there is a stationary equilibrium that holds for any proposer order. In our model, except in trivial cases, an efficient equilibrium never exists for all proposer probabilities.

[6]See Section 5 of the online appendix for a more detailed comparison.

[7]See Manea (2013) for how the replica assumption relates to steady state outcomes in large markets.

[8]Some examples, with a particular focus on network bargaining, include Atakan (2010), Manea (2011) and Polanski and Lazarova (2014).

The most closely related work to ours models non-cooperative bargaining without replacement in thin markets. This literature includes Rubinstein and Wolinsky (1990), Corominas-Bosch (2004), Gale and Sabourian (2006), Polanski (2007), Polanski and Winter (2010), Abreu and Manea (2012a, 2012b), Kanoria et al (2014), and Polanski and Vega Redondo (2014). These papers embed different degrees of coordination into their bargaining protocols. Corominas-Bosch (2004) investigates the existence of competitive equilibria in markets with homogeneous surpluses (a link in the bipartite network indicates that the two players would generate a unit of surplus if matched) and alternating non-exclusive offers. The setup differs from the one considered here, and requires a high degree of coordination both at the offer stage (as players on one side propose simultaneously to everyone on the other side of the market) and at the acceptance stage (as more than one assignment may be possible). Polanski (2007) also considers a setting with homogeneous surpluses and strong coordination (as a maximum matching is used to select which players bargain bilaterally each period); and links subgame perfect equilibrium outcomes to the Dulmage-Mendelsohn decomposition of the bipartite network.

The closest papers to ours are Gale and Sabourian (2006) and Abreu and Manea (2012a, 2012b). Gale and Sabourian (2006) differs from us insofar as all players are simultaneously matched into pairs before an agent in each pair is selected to be the proposer with equal probability. They include heterogeneous surpluses, but assume that different sellers have identical objects to sell, so that a given buyer generates the same surplus with all sellers. They provide an example in which all MPE payoffs are non-competitive and, therefore, the market outcome is inefficient.

Abreu and Manea (2012a, 2012b) consider environments with homogeneous surpluses in which players cannot necessarily be partitioned into buyers and sellers (implying that a core match might not exist or be unique). One of the extensions of Abreu and Manea (2012a) analyzes a protocol close to the one we consider, but does not restrict attention to stationary equilibria. Their conclusions prove the existence of non-stationary equilibria that converge to efficiency as the time elapsed between offers vanishes. Abreu and Manea (2012b), like us, focuses on limiting stationary equilibria, but in the context of a bargaining protocol in which players are randomly paired to bargain.[9] As in Gale and Sabourian (2006), an important contribution of their paper is to provide examples in which all MPE are inefficient. While both Gale and Sabourian (2006) and Abreu and Manea (2012b) identify interesting and important features of market inefficiencies, neither provides general conditions to ensure that an efficient limiting stationary equilibrium exists or does not exist. But, such conditions are important in order to assess the extent of bargaining frictions in markets. To fulfill this goal, we select

[9] Each period, a link is selected according to some probability distribution, and then a player on that link is selected with equal probability to propose.

a protocol that favours the existence of efficient equilibria, and introduce a slightly stronger notion of efficiency (as we explain in Section 5).

In protocols that select links to determine the proposer, when an inefficient link is selected, players must either disagree or match inefficiently. Allowing players to choose to whom they make an offer prevents delay and mismatch from being necessary features of equilibrium play and simplifies the characterization of efficient MPE. In the conclusions, we make this point explicitly by discussing an example contained in Abreu and Manea (2012b). Given the possibility of inefficiency, we select a bargaining protocol that is predisposed to admit an efficient MPE. To clarify the role of bargaining frictions, we further restrict attention to environments in which the surplus maximizing match exists and is unique. This alleviates coordination problems that might arise,[10] and is the generic case whenever the market can be partitioned into two sides. Our results clarify that disagreement is possible even between players who are matched in the unique efficient match, and that the multiplicity of equilibria is driven by the underlying coordination game and not by the multiplicity of core matches. As is customary in the literature, we allow bargaining frictions, represented by the time elapsed between offers, to get small. Despite making these modeling choices, we find inefficiencies to be a common feature in these market and we find conditions for these to occur.

Roadmap: The next two sections introduce the economy (Section 2) and the directed-search bargaining protocol analyzed (Section 3). Section 4 defines solution concepts and presents the baseline characterization. Section 5 introduces our efficiency criteria and relates them to welfare. Several examples preview the main conclusions in Section 6. All the main contributions on stationary equilibrium welfare are in Section 7. The relationship to the search literature and alternative bargaining protocols are discussed in Section 8. All the proofs of propositions are in the appendix, while the proofs of remarks and several additional robustness checks can be found in the online appendix.

2 The Assignment Economy

An *assignment economy* consists of a set of players $N = \{1, ..., n\}$ and an n by n matrix S characterizing the surplus that can be generated by any two players in the economy. The ij entry of S, $s_{ij} \geq 0$, denotes the surplus generated when players i and j are matched. The surplus matrix S can be interpreted as a network. The network is assumed to be undirected (so that $s_{ij} = s_{ji}$ for any $i, j \in N$) and bipartite (so that, for some partition (P_1, P_2) of the set of players N, $s_{ij} = 0$ whenever $i, j \in P_k$ for $k \in \{1, 2\}$). The two assumptions imply that the surplus generated in a match is independent of the identity of the player who initiates the

[10] Section 2 in the online appendix for a more detailed comparison on this point.

match, and that surplus can be generated only by players of different types. By assumption, workers generate surplus only with firms, men generate surplus only with women, and buyers generate surplus only with sellers.

A *match* is a map $\mu : N \to N$ such that $\mu(\mu(i)) = i$ for any $i \in N$. If $\mu(i) = i$, we say that player i is unmatched. If $\mu(i) = j$, then i and j generate surplus s_{ij}. Let $M(N)$ denote the set of possible matches for a given set of players N. An *efficient match* η for an assignment economy S is a match that maximizes surplus

$$\sum_{i\in N} s_{i\eta(i)} = \max_{\mu\in M(N)} \left\{ \sum_{i\in N} s_{i\mu(i)} \right\}.$$

The *core* of the market consists of the set of match and payoff vector pairs (μ, u) satisfying:

$$[1] \quad u_i + u_{\mu(i)} = s_{i\mu(i)} \text{ for any } i \in N,$$
$$[2] \quad u_i + u_j \geq s_{ij} \text{ for any } i, j \in N.$$

Shapley and Shubik (1971) establish that any core match is an efficient match, and that a unique efficient match exists when no two positive links have the same value.[11] As the condition for uniqueness is generic, our analysis restricts attention to economies with a unique efficient match. Thus, throughout the analysis we refer to the unique efficient match η as the *core match*.

Although condition [2] rules only out the existence of profitable pairwise-deviations, Shapley and Shubik (1971) establish that this suffices to rule out the existence of profitable coalitional-deviations. The lowest and the highest payoff that player i can receive in a core outcome will be denoted by $\underline{u}_i$ and $\overline{u}_i$.

3 Matching and Bargaining

The analysis considers a non-cooperative, infinite-horizon bargaining protocol in which players choose whom to bargain with. All players discount the future by a common factor $\delta \in (0, 1)$. At the beginning of the game, all players are active, but they can become inactive as the game unfolds. In every period, a single player $i \in N$ is selected at random to be the proposer, with probability $p_i > 0$. If proposer i is active, he can make an offer to at most one other active player. We adopt as a convention that a player failing to make an offer chooses to offer to himself. An offer from player i to a player $j \neq i$ consists of a surplus split $x_{ji} \in [0, s_{ij}]$, where x_{ji} denotes the amount of surplus generated by the new match, s_{ij}, that he intends to

[11]Formally, the efficient match is unique if $s_{ij} > 0$ implies $s_{kl} \neq s_{ij}$ for all $kl \neq ij$.

leave to j. The player receiving the offer then has a binary choice, to accept (1) or reject (0) the offer. If j rejects the offer, both players remain active, and the game moves to the next stage. Otherwise, players i and j become inactive, and their final payoffs are determined by the discounted value of the shares that they have agreed upon. In particular, the value at the beginning of the game to players i and j of reaching an agreement x_{ji} at stage t is

$$u_j = \delta^{t-1} x_{ji} \quad \text{and} \quad u_i = \delta^{t-1}(s_{ij} - x_{ji}).$$

In the next stage the proposer is selected according to the same probability distribution.[12] If an inactive player is selected the game moves to the subsequent period. The game ends when the surplus generated by any pair of active players is zero. The structure of the game is common knowledge among players. Information is perfect. Thus, all players observe any offer previously made and the corresponding acceptance decision.

Histories and Strategies: Denote the set of histories at date t observed by any player after the new proposer has been selected by $H^t = N \times [N^2 \times \mathbb{R}_+ \times \{0, 1\}]^{t-1}$. Such histories consist of the identity of the current proposer, the identities of past proposers, whom they offered to, the offer they made and whether the offer was accepted or rejected. Denote the set of histories of length t observed after an offer has been made by $R^t = N \times \mathbb{R}_+ \times H^t$. Let $R = \cup_t R^t$ and $H = \cup_t H^t$. Finally, let H_i denote the subset of histories in H in which player i is the proposer, and let R_i denote the subset of histories in R in which player i is the responder.

We say that player $i \in N$ is *active* at history $h \in H$ if player i has never accepted an offer and has never made an offer that was accepted. For any history $h \in H$, let $A(h) \subseteq N$ denote the set of active players after history h. Throughout, $\Delta(\cdot)$ denotes the simplex of a finite set. The strategy of an active player $i \in A(h)$ when making an offer consists of a pair of functions, ρ_i and χ_i, such that

$$\rho_i(h) \in \Delta(A(h)) \quad \text{and} \quad \chi_i(h) \in \mathbb{R}_+^{|A(h)|} \quad \text{for } h \in H_i.$$

The first map $\rho_i(h)$ describes the probability distribution over players who may receive an offer from i at any given history, while the second map $\chi_i(h)$ identifies the amount of surplus that i would offer to each potential partner. The strategy of an active player $i \in A(h)$ when receiving an offer instead consists a single function, α_i, such that

$$\alpha_i(h) \in [0, 1] \quad \text{for } h \in R_i.$$

[12] Results are unaffected by updating proposal probabilities conditional on being active. However, we opted to keep the expected time to propose of each player stationary across periods.

The map $\alpha_i(h)$ describes the probability that an offer is accepted. Strategy profiles are denoted by omitting the dependence on players, $(\rho, \chi, \alpha) = \{\rho_i, \chi_i, \alpha_i\}_{i \in N}$.

4 MPE Existence and Characterization

The analysis restricts attention to stationary Markov perfect equilibria in which strategies depend only on the set of active players in the game.

Definition 1 *A subgame perfect equilibrium (ρ, χ, α) is a Markov perfect equilibrium (MPE) if strategies coincide whenever active player sets coincide. That is, for any two histories $h, h' \in H$ such that $A(h) = A(h')$:*

[1] $\rho(h) = \rho(h')$ *and* $\chi(h) = \chi(h')$,

[2] $\alpha(i, x|h) = \alpha(i, x|h')$ *for any offer* $(i, x) \in N \times \mathbb{R}_+$.

Strategies are stationary as calendar date is not part of the Markov state. As we only consider stationary MPE, we often omit the word "stationary" and make the dependence on the active player set explicit (thereby omitting the dependence on histories). Notation $\left(\rho^\delta, \chi^\delta, \alpha^\delta\right)$ will occasionally be used to clarify that equilibrium strategies may also depend on the discount factor δ. But, we omit this dependence when redundant.

Some of the results consider MPE behavior in the limit as the discount factor converges to 1. To simplify the discussion we introduce a notion of limiting equilibrium.

Definition 2 *A limiting Markov perfect equilibrium (LMPE) $(\bar{\rho}, \bar{\chi}, \bar{\alpha})$ is the limit of a selection $\left\{\rho^\delta, \chi^\delta, \alpha^\delta\right\}_{\delta=0}^1$ from the MPE correspondence as δ converges to 1.*

Throughout the text the expression *equilibrium* will refer to an MPE, and the expression *limiting equilibrium* will refer to an LMPE. In order to simplify notation, we invoke the following two conventions for all $i, j \in A$

$$A_{-i} = A \backslash \{i\} \text{ and } A_{-ij} = A \backslash \{i, j\}.$$

For any MPE (ρ, χ, α) and any set of players $A \subseteq N$, let $\pi_{ij}(A)$ denote the *agreement probability* between players $i \in A$ and $j \in A_{-i}$ when i is selected to make an offer,

$$\pi_{ij}(A) = \underbrace{\rho_i(j|A)}_{\Pr(i \text{ offers to } j)} \cdot \underbrace{\alpha_j(i, \chi_i(j|A)|A)}_{\Pr(j \text{ accepts})},$$

and let $\pi_{ii}(A)$ denote the probability that i does not reach agreement when selected to make an offer,

$$\pi_{ii}(A) = 1 - \textstyle\sum_{j \in A_{-i}} \pi_{ij}(A).$$

Also, let $V_i(A)$ denote the expected payoff – or equivalently *value* – of an active player i at the beginning of a subgame in which the set of active players is A, and let $v_i(A)$ denote the MPE value of an active player i when he is chosen to be the proposer.

We begin by proving equilibrium existence and by providing a preliminary characterization of equilibrium bargaining values. For convenience, let $p_A = \sum_{j \in A} p_j$. The characterization allows for mixed strategy equilibria. Fix an active player set A and consider any Markovian strategy profile (ρ, χ, α) and its associated values and agreement probabilities $(\pi, V) \in [\Delta(A) \times \mathbb{R}]^{|A|}$, where we omit the dependence on A for clarity. As in numerous bargaining models, subgame perfection dictates that a proposer never offers to another player more than that player's present discounted value of staying in the game. As players can choose whom to bargain with, proposers necessarily offer to those players who leave them with the highest surplus, $\text{argmax}_{j \in A_{-i}} \{s_{ij} - \delta V_j\}$, whenever such surplus exceeds the value of remaining unmatched, δV_i. Formally, we define the value of proposing at active player set A for a player $i \in A$ by

$$v_i = \max\{\delta V_i, \max_{j \in A_{-i}}\{s_{ij} - \delta V_j\}\}.$$

It follows that for any active player set $A \subseteq N$, MPE values $V(A)$ for any player $i \in A$ must be a fixed point of the following system of value equations

$$V_i = \underbrace{p_i v_i}_{i \text{ proposes}} + \sum_{j \in A_{-i}} p_j [\underbrace{(\pi_{ji} + \pi_{jj}) \delta V_i}_{j \text{ agrees with } i \text{ or delays}} + \underbrace{\sum_{k \in A_{-ij}} \pi_{jk} \delta V_i(A_{-jk})}_{j \text{ agrees with } k \neq i,j}] + \underbrace{(1 - p_A) \delta V_i}_{\text{no player proposes}},$$

for some profile of agreement probabilities $\pi(A)$ satisfying

$$\begin{aligned} \pi_{ij} &= 0 \quad \text{if} \quad v_i > s_{ij} - \delta V_j \text{ and } j \neq i, \\ \pi_{ii} &= 0 \quad \text{if} \quad v_i > \delta V_i. \end{aligned} \tag{1}$$

Proposition 1 *An MPE exists. Moreover, $\{\pi(A), V(A)\}_{A \subseteq N}$ is a profile of MPE values and agreement probabilities if and only if it solves system (1) at any active player set $A \subseteq N$.*

Existence is proved by applying Kakutani's fixed point theorem. The result extends Proposition 1 and Lemma 1 in Abreu and Manea (2012b) to environments in which players are allowed to choose whom to offer to and in which the surplus generated in a match depends on the identity of the players. While MPE are not unique, MPE values are uniquely determined by MPE agreement probabilities.

The result implies that no player $i \in A$ can delay with an active player set A in equilibrium if there exists a player $j \in A$ such that $\delta V_i + \delta V_j < s_{ij}$. Thus, in any MPE displaying on-path delay it must be that $\delta V_i + \delta V_j \geq s_{ij}$ at some on-path subgame. Moreover, as V_i and V_j are a discounted weighted average of the possible future agreements i and j can reach on-path, if i

and j delay they must collectively expect higher payoffs from delaying and letting the market evolve than from reaching agreement now.

5 Efficiency, Welfare and Delay

Next, we introduce the two efficiency criteria and the notion of delay that will be analyzed in the following sections. Let E denote the set of unmatched players in the core of the original assignment economy, $E = \{i \in N | \eta(i) = i\}$, and let $C(N)$ denote the set of possible active player sets that may arise as core matches are removed from the game,

$$C(N) = \{ A | \, A = \cup_{i \in M} \{i, \eta(i)\} \cup E \quad \text{for some} \quad M \subseteq N \} \, .$$

We are interested at active player sets in $C(N)$ as only such subgames can arise with positive probability in equilibria in which players eventually match efficiently. The properties of the core imply that the core partner of every player must coincide at all active player sets $A \in C(N)$.

Consider a social planner who is able to impose terms of trade and agreement probabilities, but is otherwise constrained by the environment of the game. For a high enough discount factor, this constrained social planner will implement only efficient matches and will do so at the first available opportunity. An MPE with these features is said to be strongly efficient. It requires that every player who is matched in the core of the assignment economy agrees on a division of surplus with his core partner at the very first opportunity. One way in which surplus can be lost is through delay. However, when delay costs are small (that is, when δ is close to 1) little surplus is dissipated by deferring agreements. We therefore also consider a weaker efficiency criterion which only requires that nobody ever matches with players other than their core partner. We refer to these MPE as weakly efficient MPE. Results then establish that all players must eventually agree on a division of surplus with their core partners in any such MPE.[13]

Definition 3 *Consider an MPE (ρ, χ, α). If for all $A \in C(N)$:*

- $\pi_{i\eta(i)}(A) = 1$ *for all $i \in A$, the MPE is strongly efficient;*
- $\pi_{i\eta(i)}(A) + \pi_{ii}(A) = 1$ *for all $i \in A$, the MPE is weakly efficient.*

[13] In terms of utilitarian welfare, for all δ sufficiently high, strongly efficient MPE maximize the ex-ante sum of expected payoffs, whereas weakly efficient MPE will not unless they are also strongly efficient. Even in strongly efficient MPE, however, the sum of values is necessarily below total surplus, as it takes time for the core match to form. Moreover, in a strongly efficient MPE all active player sets in $C(N)$ obtain with positive probability. But, this is not the case for weakly efficient MPE, as the market may clear sequentially.

Neither efficiency criterion is satisfied when an inefficient match obtains with positive probability. As we assume $\delta < 1$, this may rule out instances in which an inefficient match occurs with vanishingly small probability as $\delta \to 1$. To address these, we apply the two efficiency criteria to the limiting equilibria. For convenience, given a profile MPE $\left(\rho^{\delta}, \chi^{\delta}, \alpha^{\delta}\right)$ for all $\delta < 1$, define limiting agreement probabilities as $\bar{\pi}_{ij}(A) = \lim_{\delta \to 1} \pi_{ij}^{\delta}(A)$ for all $i, j \in A$ and the limiting values as $\bar{V}_i(A) = \lim_{\delta \to 1} V_i^{\delta}(A)$ for all $i \in A$ (when these limits exist).

Definition 4 *Consider an LMPE $(\bar{\rho}, \bar{\chi}, \bar{\alpha})$. If for all $A \in C(N)$:*

- $\bar{\pi}_{i\eta(i)}(A) = 1$ *for all $i \in A$, the LMPE is strongly efficient;*
- $\bar{\pi}_{i\eta(i)}(A) + \bar{\pi}_{ii}(A) = 1$ *for all $i \in A$, the LMPE is weakly efficient.*

While we establish that both strongly and weakly efficient LMPE generate the same surplus in the limit, it is instructive to separate them for the purpose of classifying limiting efficient equilibrium play. The strong and weak efficiency taxonomy parses efficiency loss through inefficient matching versus inefficient delay. When applying our efficiency criteria to LMPE, it is worthwhile noting that active player sets outside $C(N)$ may now occur with positive probability for all $\delta < 1$.

As customary, refer to the sum of ex-ante values, $\sum_{i \in N} V_i(N)$, as *utilitarian welfare*. The next proposition establishes that utilitarian welfare converges to total surplus, $\sum_{i \in N} s_{i\eta(i)}$, as $\delta \to 1$ in any weakly efficient LMPE. This motivates our efficiency criterion by showing that no welfare can be lost from delay in any such equilibrium.

Proposition 2 *Any weakly efficient LMPE maximizes surplus,*

$$\textstyle\sum_{i \in N} \bar{V}_i(N) = \sum_{i \in N} s_{i\eta(i)}.$$

The result is intuitive and relies on delay costs vanishing at a sufficiently fast rate as $\delta \to 1$. Its proof also establishes why in any MPE all players cannot simultaneously delay with positive probability at some active player set. Since weakly efficient LMPE maximize utilitarian welfare, Proposition 2 implies that these equilibria are always "asymptotically efficient" as defined in Abreu and Manea (2012a, 2012b). In principle though, asymptotically efficient LMPE may exists in which players match inefficiently at active player sets that belong to $C(N)$, but that do not materialize on the equilibrium path.[14] Limiting weak efficiency refines asymptotic efficiency by requiring the equilibrium to be efficient in the limit at any active player set in

[14] If the short side of the market has at most two players, $\min_i |P_i| = 2$, then all LMPE must be weakly efficient in any core subgame $A \neq N$. Thus, all asymptotically efficient equilibria must be weakly efficient LMPE. Moreover, any asymptotically efficient equilibrium, in which all players in $A \backslash E$ agree with positive probability in every subgame $A \in C(N)$, must be a weakly efficient LMPE.

$C(N)$, and not just at those active player sets which are reached with positive probability on path.[15] In doing so, it rules out asymptotically efficient equilibria which are sustained by the threat of inefficient matching at some core subgame, instead requiring the consistent selection of efficient equilibria throughout all subgames that can be reached following efficient matching. Relative to earlier studies, the stronger welfare criterion allows a novel approach which entails: disciplining limiting agreement probabilities in all core subgames; and then exploiting the recursive structure of stationary equilibria to derive stronger implications on efficient equilibrium payoffs and on their existence.

Our notion of equilibrium delay requires the existence of a player with a positive value who chooses to forgo the option to make an acceptable offer with positive probability.

Definition 5 *An MPE (ρ, χ, α) displays delay if for some $A \subseteq N$ and some player $i \in A$*

$$V_i(A) > 0 \quad \textit{and} \quad \pi_{ii}(A) > 0.$$

The definition applies only to players with a positive value, as it is immediate that players with zero continuation value might well prefer to disagree. In Section 6, we present two examples in which a player with a positive continuation value chooses to delay on the equilibrium path.

6 Examples

Before proceeding to the main analysis, consider a few examples to illustrate the model, the solution concepts, the efficiency definitions and to preview some of the main conclusions. The first example establishes that equilibrium mismatch can occur. The second shows how mismatch inefficiencies can occur for $\delta < 1$, but disappear in the limit; so that there is a strongly efficient limiting equilibrium. The third demonstrates on-path equilibrium delay, and the fourth shows a weakly efficient limiting equilibrium in which players delay and endogenously exit the market in a fixed sequence. These examples can be skipped.

Example 1: Consider an assignment economy populated by four players who propose with equal probabilities. Surpluses in the market are as depicted in Panel I of Figure 1.
The unique efficient assignment matches player a to b and player c to d whenever $y < 200$, while it matches only player a to d when $y > 200$. Multiple core assignments exist at $y = 200$. Proposition 1 can be used to derive MPE payoffs and strategies in this game for any discount factor. To make the discussion more transparent, suppose that the discount factor is close to

[15] Formally, asymptotically efficient MPE require the conditions defining weakly efficient LMPE to hold only for active player sets $A \in C(N)$ that materialize with positive probability on the equilibrium path, rather than requiring the same conditions but for all $A \in C(N)$.

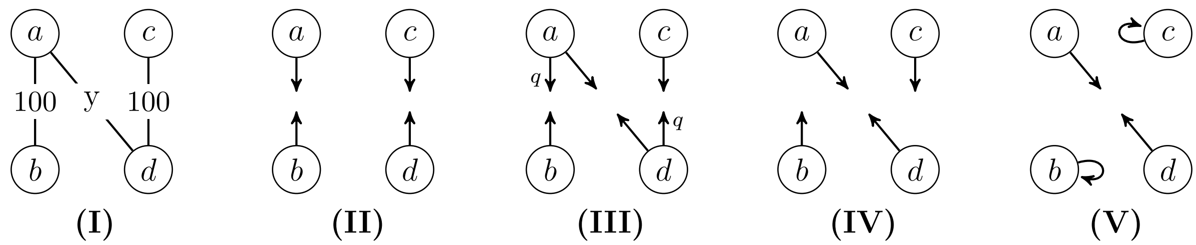

Figure 1: Panel I displays the assignment economy. MPE agreement probabilities $\pi_{ij}(N)$ are shown: in Panel II for $y \in [0, 100]$; in Panel III for $y \in (100, 143]$; in Panel IV for $y \in [144, 200]$; in Panel V for $y \in (200, \infty)$. An arrow between two players represents a positive agreement probability. A self-arrow represents a positive disagreement probability.

unity. When $y \leq 100$, players only make offers to their core partners. It is then as if each player bargains bilaterally with their efficient partner and all players achieve an LMPE payoff of 50. Given this no player is ever tempted to offer to anyone other than their core match. These equilibrium offer strategies for active players $A = \{a, b, c, d\}$ are shown in Panel II of Figure 1.

For values of $y \in (100, 200)$, bilateral bargaining cannot be a solution. Indeed, if everyone only offered to their efficient match, players a and d would both have a profitable deviation to offer to each other. When $y \in (100, 1000/7)$, players a and d randomize in equilibrium between offering to their respective core matches and bargaining with each other (Panel III of Figure 1). By offering to each other with positive probability, a and d reduce the continuation values of their efficient partners. In equilibrium they do this until they are indifferent between offering to each other and to their efficient partners. As y increases, this requires the strong players to offer to each other with higher probability, and at $y = 1000/7$ they reach the corner solution in which indifference requires them to offer to each other with probability 1. As y grows further to $y \in [1000/7, 200)$, players a and d continue to offer only to each other, and still accept offers made by their respective core matches (Panel IV of Figure 1). There is now mismatch with probability 1/2. Despite this inefficiency, the unique equilibrium is in pure strategies.[16]

The final case is the one in which $y > 200$, and in which the efficient assignment matches player a to d. If so, players a and d continue offering to each other with probability 1. However, b and c stop making offers to players a and d, as any acceptable offer would have to exceed the entire surplus in the relevant relationship (Panel V of Figure 1). This change affects limiting payoffs discontinuously. When $y < 200$, player b always makes an acceptable offer to a, leaving

[16] For $y \in (100, 1000/7)$, in the limit as $\delta \to 1$ players a and d make offers to their respective core partners with probability $q = (2\sqrt{2y^2 - 600y + 50000} - y)/(200 - y) \in (0, 1)$, the unique LMPE payoff of players a and d amounts to $V_a = (y + 50 + 50q)/(3 + q)$, while that of players b and c amounts to $V_b = V_a - y + 100$. For $y \in [1000/7, 200)$, the LMPE payoff of a and d further increases to $V_a = (y + 50)/3$, whereas that of b and c decreases to $V_b = (400 - y)/12$.

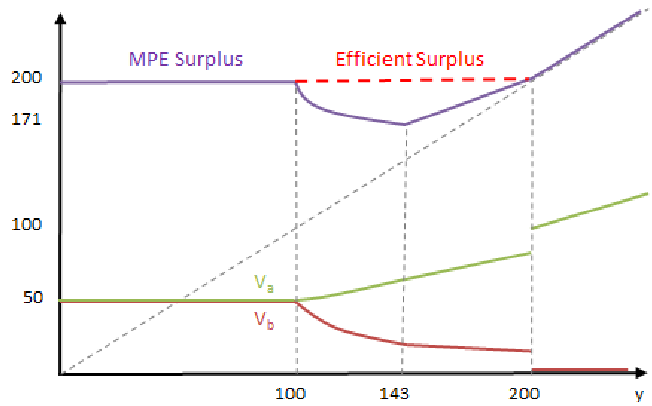

Figure 2: The plot depicts the payoffs, the MPE surplus and the efficient surplus as a function of $s_{ad} = y$ for Example 1. The payoff of players a and d is denoted by V_a, whereas V_b denotes the payoff of b and c.

c to bargain bilaterally with d with probability $1/4$. Thus, c gets a limiting payoff of 50 with probability $1/4$. For $y > 200$, however, b stops making acceptable offers to a, and so c receives a payoff of 0 with certainty. Note that this discontinuity occurs precisely at the value of y for which the core match is not unique. Figure 2 depicts LMPE values and surplus for all y.

Example 2: The next example shows that alternative matches which cannot be lost can act like outside options and bound payoffs while being exercised with probability 0 as players become arbitrarily patient.

Figure 3: Panel I displays the assignment economy, while Panel II displays MPE agreement. The limiting equilibrium shown is strongly efficient as $\lim_{\delta\to 1} q^\delta = 0$.

Consider the three-player market depicted in Panel I of Figure 3. The unique core match of the market matches players e and f, leaving c unmatched. Assume again that players propose with equal probability and that discount factors are sufficiently close to unity. If so, players e and f offer to each other with probability 1 in the unique MPE, whereas player c offers to player f with probability $q^\delta \in (0, 1)$, where $\lim_{\delta\to 1} q^\delta = 0$. Although $q^\delta \to 0$ in the limit, the mere presence of player c significantly affects bargaining outcomes. Players e and f would share the 10 units of surplus evenly were they to bargain in solitude. However, because c never exits the market, he acts like an outside option for f. Indeed, player f extracts the same limiting surplus that he would get were he to bargain in solitude with player e while having access to an outside option with value 8. The limiting payoffs converge to 8 for player f, to 2 for player e, and to 0 for player c. Even though player c does not make an acceptable offer in the limit, the equilibrium does not display delay by our definition, because the payoff of player c equals exactly 0 for all δ sufficiently close to 1. While the equilibrium described is not strongly or weakly efficient for $\delta < 1$, because there is then positive probability of mismatch, in the limit that probability converges to zero, and so the limiting equilibrium is strongly efficient.[17]

Example 3: In example 1, we saw that mismatch could arise when players feared loosing valuable alternative matches. The next example shows that on-path delay can occur when players expect the market to evolve in their favor.

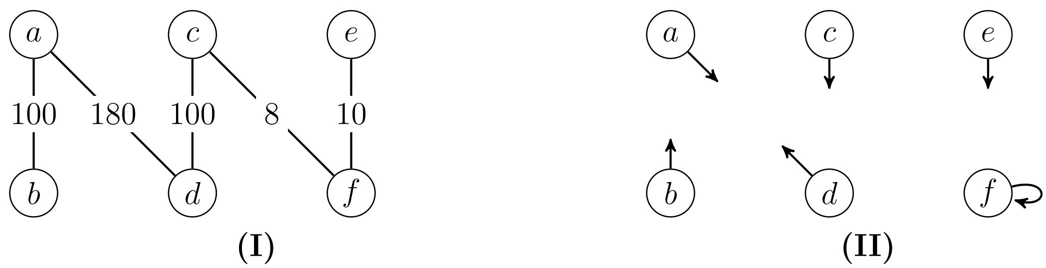

Figure 4: Panel I displays the assignment economy, while Panel II displays MPE agreement. Player f delays with probability 1.

Consider the six-player assignment economy depicted in Panel I of Figure 4, in which agents are selected to propose with equal probability. We show an equilibrium exists in which f delays making offers with probability 1 when selected to propose if all other players are still active in the market. Panel II of Figure 4 shows the equilibrium offer strategies in this MPE. To solve this game, we use backward induction. Under the proposed equilibrium, if the protocol selects agent e as the first proposer, agent e makes an offer to agent f that will be accepted. If so, the remaining subgame coincides precisely with the game discussed in Example 1, so we know the MPE payoffs for all the remaining players in the subgame. If agent c is selected as the first proposer and agrees with d, then in the following subgame agents e and f bargain bilaterally, as do agents a and b. If agents a or d are selected as the first proposers instead, then agent b must remain unmatched, while agents c, e, and f are left in precisely the subgame we considered in Example 2. Finally, if agent b is the first proposer, he agrees with a, and players c, d, e, and f are left in a subgame. While we have not solved this subgame yet, in the unique MPE all players offer to their efficient partner like in Example 1. Limit payoffs for c, d, e and f are then $50, 50, 5$ and 5. With these subgames in mind, it is easy to write down

[17] For δ close to 1, in the unique MPE of this example we have that $V_c(N) = 0$, $V_e(N) = (26\delta - 24)/\delta$, $V_f(N) = 8/\delta$ and $q = (27\delta^2 - 63\delta + 36)/(13\delta^2 - 12)$.

the value equations for the six agents and solve them. For instance, the value equation for agent c simply amounts to

$$V_c(N) = p[\underbrace{2\delta V_c(\mathcal{E}_2)}_{a \text{ or } d \text{ propose}} + \underbrace{\delta V_c(c,d)}_{b \text{ proposes}} + \underbrace{(100 - \delta V_d(N))}_{c \text{ proposes}} + \underbrace{\delta V_c(\mathcal{E}_1)}_{e \text{ proposes}} + \underbrace{\delta V_c(N)}_{f \text{ delays}}],$$

where $V_c(\mathcal{E}_i)$ denotes the value of player c in Example $i \in \{1, 2\}$. Solving the value functions establishes that no player has a profitable deviation from the proposed strategies and that player f must delay for all sufficiently high values of δ. Taking limits as $\delta \to 1$ the payoffs of the six players converge to $V(N) = (55/3, 230/3, 230/3, 55/3, 13/2, 7/2)$. Agents a through d achieve the same limiting values as in Example 1. The additional option available to c (of matching with f) does not improve c's terms of trade as it never binds. Nevertheless, the option of matching to c incentivizes f to delay. There is positive probability that a and d will reach agreement first, and in this case f's bargaining position with e improves. While such threats are factored into the limiting payoff of e, and f ends up indifferent between delaying and making an offer to e when selected to propose first, f must delay with certainty to extract the maximum possible equilibrium value out of his potential future outside option.[18]

Example 4: In example 1, alternative matches which were lost with positive probability did not act like outside options; in example 2 instead, alternative matches which never exited the market did act like outside options with limiting patience. The final example shows that there is another way in which matches can act as outside options without distorting trade. There can be sequential exit, in that all but one pair of players delay with probability 1 in the limit. For those players everyone else waits for, alternative matches never exit the market before them and can act like outside options.

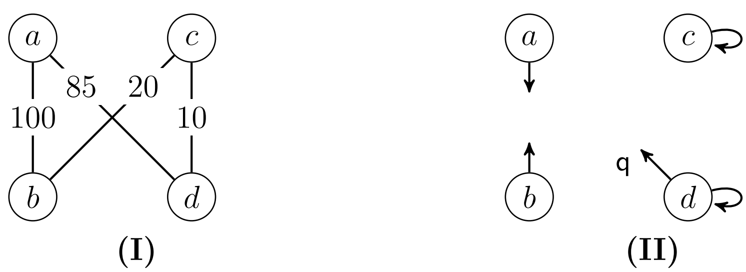

Figure 5: Panel I displays the assignment economy, while Panel II displays MPE agreement. The LMPE features sequential exit. As $\lim_{\delta \to 1} q^\delta = 0$, in the limit c and d wait for a and b to reach an agreement before reaching an agreement themselves.

[18]Delay in this example is driven only by the endogenous evolution of bargaining positions. Players can choose whom to bargain with (which implies that no player has to delay to be matched to his equilibrium partner), and the efficient match is unique (which shuts down possible coordination problems among players). In Section 2 of the online appendix, we show that when multiple efficient matches exist, delay can arise just because players fail to coordinate on one of the efficient matches.

Consider the market depicted in Panel I of Figure 5. This market is vertically differentiated. Both b and d generate a higher surplus with a than c, while both a and c generate more surplus with b than d. Vertical differentiation is so strong that the match between a and b generates ten times more surplus than the match between c and d. The efficient match is assortative, and matches a to b and c to d. There is no strongly or weakly efficient MPE for $\delta < 1$, and no strongly efficient LMPE in this example. There is, however, a weakly efficient LMPE. For high enough $\delta < 1$ there is an MPE in which player c delays with probability 1, player d agrees with a with probability $q^\delta > 0$ and delays with probability $1 - q^\delta$, while a and b always agree with each other. Moreover, $\lim_{\delta \to 1} q^\delta = 0$, so in the limit c and d *both delay* with probability 1 and wait for a and b to reach agreement before bargaining with each other. The market thus clears from the top. The limit payoffs of c and d are 5, a receives 80 and b gets 20.

7 MPE Efficiency and Frictions

We now present the main conclusions on equilibrium welfare. The analysis begins by characterizing payoffs in any efficient MPE and by deriving necessary and sufficient conditions for the existence of such MPE for δ close to 1. These conditions relate the primitives of the bargaining model to the core of the assignment economy. The second part of the section derives similar conclusions for limiting equilibria, and identifies when alternative matches can serve as outside options affecting bargaining outcomes without distorting trade. Broadly, the analysis establishes that inefficiency is a necessary feature of all MPE in which players' bargaining positions evolve as others reach agreement. An efficient MPE exist only when each pair of efficiently matched players can bargain in isolation, ignoring the market context, without having a profitable deviation (outside options provided by alternative matching opportunities cannot bind). The results for limiting efficient MPE provide the same message, but are more subtle. Agents cannot have binding *temporary outside options*, provided by matching opportunities to players who may exit the market before them, but can have binding *permanent outside options*, provided by matching opportunities to players who never exit the market before them.

Efficient Equilibria and Payoffs: To state results, it is useful to introduce three relevant payoff profiles. The first of these identifies the LMPE values that players would achieve while bargaining bilaterally with their core match. For any player $i \in N$, let σ_i denote the *Rubinstein payoff* of player i,

$$\sigma_i = \frac{p_i}{p_i + p_{\eta(i)}} s_{i\eta(i)}.$$

The second profile identifies the highest payoff that players could achieve while offering to players that are unmatched in the core of the assignment economy. For any player $i \in N$, let

ω_i denote the *outside payoff* of player i,

$$\omega_i = \max_{j \in E \cup i} s_{ij}.$$

In the bargaining game, players that are unmatched in the core act as permanent outside options in efficient equilibria, as they never exit the market. The third and final profile identifies the LMPE payoffs that players would achieve while bargaining bilaterally with their core match when facing permanent outside options equal to ω (Shaked and Sutton (1984), Sutton (1986), Binmore and Herrero (1988)). For any player $i \in N$, let $\bar{\sigma}_i$ denote the *shifted Rubinstein payoff*,

$$\bar{\sigma}_i = \begin{cases} \omega_i & \text{if} \quad \omega_i \geq \sigma_i \\ s_{i\eta(i)} - \omega_{\eta(i)} & \text{if} \quad \omega_{\eta(i)} \geq \sigma_{\eta(i)} \\ \sigma_i & \quad\text{otherwise} \end{cases}.$$

Outside options cannot bind for both players in a core match. If they did, an alternative match that generates a weakly higher surplus would be feasible (as outside options are unmatched in the core). But, that would contradict the optimality of the core match or its uniqueness.

While we will identify necessary and sufficient conditions for the existence of an MPE which is efficient, it will be helpful to highlight two potentially separate sources of distortions, namely, inefficient matching and delay in reaching agreements. Both distortions are driven by the endogenous evolution of bargaining power that results from the random order of play. But, whereas mismatch is necessarily a hard friction, as it permanently destroys surplus, delay can be a soft friction, in that its effects on welfare can become negligible when discount factors are sufficiently close to unity. Proposition 3 establishes that delay cannot be the sole source of frictions in the model, as mismatch is necessary for delay. Pinning down weakly efficient equilibria thus amounts to identifying strongly efficient equilibria. The proposition also characterizes equilibrium payoffs in any efficient MPE.

Proposition 3 *Any weakly efficient MPE is strongly efficient. Moreover, in any subgame $A \in C(N)$ of any weakly efficient MPE, payoffs amount to*

$$V_i(A) = \left(\frac{p_i}{(1-\delta) + \delta(p_i + p_{\eta(i)})} \right) s_{i\eta(i)} \quad \textit{for all } i \in A.$$

The proof shows that players never delay in any weakly efficient equilibrium as delay necessarily weakens their bargaining position relative to their core match. Payoffs are then derived by simple manipulation and the observation that behavior in subgames that are off the equilibrium path cannot affect the terms of trade in any equilibrium path subgame, as players could reach such subgames only by exiting the game. As strongly efficient MPE coincide with weakly efficient MPE, henceforth we simply refer to them as *efficient equilibria*. Efficient MPE payoffs are stationary and independent of the set of active players along the equilibrium path, and converge to Rubinstein payoffs. When the cost of delaying is non-negligible, bargaining is efficient only when alternative matches have no effect on outcomes and players achieve the same payoff they would get by bargaining with their efficient match in solitude.

To understand matching incentives in the model consider the case in which delay costs are large. If so, players have strong motive to negotiate only with their preferred bargaining partners as the cost of rejecting offers is extremely high. If so, equilibrium matching could be efficient only if matching with one's core partner would generate at least as much surplus as matching with any other player. Indeed, the existence of an efficient MPE requires players' preferred bargaining partners to coincide with their core partners when delay costs are large. The next remark formalizes these observations. If for some $\varepsilon > 0$ an efficient MPE exists for any $\delta \in (0, 1)$ such that $|x - \delta| \leq \varepsilon$, we say that an efficient MPE exists *for all values of δ close to x*. A *preferred match*[19] μ at an active player set $A \subseteq N$ is a map $\mu : A \to A$ that satisfies

$$s_{i\mu(i)} = \max_{j \in A} s_{ij} \quad \text{for all } i \in A.$$

Remark 1 *For all δ close to 0:*

(a) all MPE maximize utilitarian welfare if the preferred match is unique at all $A \subseteq N$;

(b) an efficient MPE exists if the core match is the unique preferred match at N;

(c) an efficient MPE exists only if the core match is a preferred match at N.

For δ sufficiently high, an MPE maximizes utilitarian welfare if and only if it satisfies our efficiency criterion.[20] However, for low δ this is no longer the case. When delay costs are sufficiently high, maximizing welfare may require matching players contingent on the realization of the sequence of proposers. In particular, for sufficiently low δ, utilitarian welfare is maximized when players agree with their preferred match, as delay costs dominate any allocative efficiency consideration.

To provide a comparison with the high δ case and different bargaining protocols, Remark 1 part (b) considers when the core match will be reached. In our directed bargaining protocol, this occurs when players' preferred match coincides with their core match; in other words, when players prefer to bargain bilaterally with their core match without negotiating with any other partner. In contrast, classical random matching models, such as Gale and Sabourian

[19] A preferred match may not be a match as $\mu(\mu(i)) \neq i$.

[20] When δ is close to 1, an MPE maximizes utilitarian welfare if and only if it is strongly efficient. But by Proposition 3 the set of weakly and strongly efficient MPE coincide.

(2006) and Abreu and Manea (2012b), never implement the core match with probability 1, as in these models players would agree with anyone they meet when δ is sufficiently low.

It is easy to find examples of inefficient equilibria that do not maximize welfare for intermediate costs of delay. However, inefficiencies may be driven by the large costs associated with disagreement. Indeed, one could interpret discounting as the source of matching frictions. However, the next results consider only the case in which delay costs are sufficiently small. Nevertheless, inefficiencies do not vanish.

Proposition 4 *An efficient MPE exists for all δ close to 1:*

(a) if Rubinstein payoffs are in the interior of the core,

$$\sigma_i + \sigma_j > s_{ij} \quad \textit{for all } i, j \in N \textit{ such that } j \neq \eta(i); \tag{2}$$

(b) only if Rubinstein payoffs are in the core,

$$\sigma_i + \sigma_j \geq s_{ij} \quad \textit{for all } i, j \in N. \tag{3}$$

Proposition 4 shows that, whenever Rubinstein payoffs do not belong to the core, players must agree with partners other than their core match with positive probability. When players consider agreeing with their respective core matches, the other active players act as fictitious outside options. But for these outside options to affect bargaining outcomes, these options must sometimes be exercised.[21] Such behavior however necessarily leads to mismatch, surplus dissipation, and possibly delay. Only when Rubinstein payoffs live in the core of the assignment economy, is there an efficient MPE.[22] The sufficient condition for the existence of an efficient MPE is intuitive, but does not guarantee that every Markovian equilibrium is efficient. Indeed, Section 1 of the online appendix presents an example in which condition (2) holds, but in which multiple MPE exist for all δ close to 1. Coordination problems in offer strategies are the source of the multiplicity.[23]

Proposition 4 establishes that bargaining inefficiencies are pervasive when negotiations are

[21]Consider again Example 1, and in particular panel III of Figure 1, so that $y \in (100, 143]$. Suppose an efficient equilibrium is played, and so, by Proposition 3, $q = 1$. A strategy available to b is to reject all offers from a and to delay when selected to be the proposer until c and d exit the market. Doing so will result in a bargaining bilaterally with b in the resulting subgame, and in the limit, b will obtain a payoff of 50. Thus, for a to receive a limiting payoff greater than 50, a must exercise his temporary outside option and inefficiently match to d with positive probability in equilibrium.

[22]Our result does not speak to the non-generic case in which Rubinstein payoffs are on the boundary of the core. In such cases a discount factor equal to 1 may be required to guarantee the existence of an efficient MPE. In Section 2 of the online appendix, we show why no conclusive result is possible in such cases.

[23]It would be compelling to conclude by arguing that if an MPE exists for arbitrarily high and low values of δ that implements the core match, then it also exists for any intermediate value. However, the incentive constraints characterizing such MPE are quadratic in δ and this conclusion does not hold in general.

decentralized and take place in a market context (for instance, if workers' possible alternative vacancies affect the wages they are able to negotiate). In particular, the result implies that bargaining is inefficient whenever the market context matters. In other words, markets are able to clear efficiently only when all players can optimally bargain bilaterally with their efficient partners, ignoring all alternatives. Moreover, these inefficiencies persist even when the discount factor is high and the exogenous frictions imposed by time preferences and sequential play become small. In Section 8, we explore the consequences of Proposition 4 in classical labor market settings, and show that vertical differentiation and increasing differences are not sufficient for the existence of an efficient equilibrium.

To further explore the key conditions in Proposition 4, we apply the definition of Rubinstein payoffs. The existence of an efficient MPE then requires that, for all i and j,

$$\left(\frac{p_i}{p_i + p_{\eta(i)}}\right) s_{i\eta(i)} + \left(\frac{p_j}{p_j + p_{\eta(j)}}\right) s_{j\eta(j)} \geq s_{ij}.$$

An interesting special case is when a social norm determines the relative bargaining power of firms to workers in labor markets. For instance, all agents on one side of the market may propose with the same probability p^1, while all agents on the other side of the market may propose with the same probability p^2. If so, the condition simplifies to

$$\left(\frac{p^1}{p^1 + p^2}\right) s_{i\eta(i)} + \left(\frac{p^2}{p^1 + p^2}\right) s_{j\eta(j)} \geq s_{ij},$$

for all $i \in P_1$ and $j \in P_2$. So, there is an efficient MPE only if, for each worker-firm pair in the economy, a weighted average of the surplus in the worker's efficient match and in the firm's efficient match weakly exceeds the surplus that pair could generate together. Weights capture the bargaining power of workers relative to firms, and both surpluses are weighted equally when $p^1 = p^2$. In many cases, like example 1, there does not exist any values of p^1 and p^2 that satisfy the above condition and so there is no social norm of this form that can eliminate inefficiencies.

The conclusions on efficiency have several implications, which are summarized in the next remark. These imply that: (a) any core payoff can be implemented as an LMPE by appropriately selecting the vector of proposal probabilities; (b) for any pair $\{i, \eta(i)\}$, proportional changes in proposal probabilities cannot affect limiting bargaining outcomes; (c) efficiency is easier to achieve in economies which have a large core; (d) any MPE without on-path delay must lead to agreement on the core match with positive probability. For convenience, say that surpluses S *support more core payoffs than* S' in the strong set order if any core payoff profile in S' is also a core payoff profile in S.[24]

[24]For instance, S supports more core payoffs than S' if for all $i \in N$: $s_{ij} = s'_{ij}$ whenever $j = \eta(i)$; and

Remark 2 *The following are consequences of Proposition 4:*

(a) As $\delta \to 1$, any interior core payoff is an MPE payoff for some probabilities $p \in \Delta(N)$.

(b) If an efficient MPE exists for all δ close to 1 for probabilities p', then it also exists for all δ close to 1 for probabilities p such that $p_i/p_{\eta(i)} = p'_i/p'_{\eta(i)}$ for all $i \in N$.

(c) If an efficient MPE exists for all δ close to 1 for surpluses S', then it also exists for all δ close to 1 for surpluses S that support more core payoffs than S'.

(d) The core match obtains with strictly positive probability in any MPE without on-path delay.

The first part of the result implies that the closure of the set of MPE payoffs that obtain for some proposal probabilities contains the core of the assignment economy. Thus, the core can be spanned by varying proposal probabilities. As the assumptions imposed on the assignment economy imply that the interior of the core is non-empty, for any such surplus matrix it is possible to find proposer probabilities that guarantee the existence of an efficient MPE. By interpreting players' proposal probability as their bargaining power, the second part shows that when delay costs are small a player's bargaining power matters only relative to that of his efficient match in any efficient MPE. The third part implies that economies with larger cores are more likely to result in efficient bargaining outcomes. The final part obtains because in any MPE without on-path delay it is impossible to find a subset of players who prefer to exchange their respective core matches, and thus some players must optimally agree with their efficient match. However, as we saw in Examples 3 and 4, on path delay can occur in equilibrium and the no delay condition is non-trivial.[25]

Limiting Efficiency: Efficient LMPE may differ considerably from efficient MPE. Proposition 4 considers only $\delta < 1$ and so categorizes as inefficient any equilibrium in which mismatch occurs with a vanishingly small probability as δ converges to 1. Moreover, Examples 2 and 4 establish that mismatch can occur in equilibrium with vanishingly small probability. This section studies this possibility asking when inefficiencies can be small in this sense.

The first result of this section extends Proposition 3 showing that strongly efficient LMPE converge to shifted Rubinstein payoffs. Whenever these payoffs differ from Rubinstein payoffs and delay is costless, unmatched players in E can act as permanent outside options without distorting the limiting equilibrium match. In Example 2, for instance, player c had an effect on player f's terms of trade in the limit without ever matching to f. The result also extends the negative efficiency conclusions of Proposition 4 to markets in which delay costs vanish. In

$s_{ij} \leq s'_{ij}$ whenever $j \neq \eta(i)$.

[25] We stress again that Example 2 does not fit our definition of equilibrium delay, as in the unique LMPE the only player who delays has a continuation value equal to zero.

the limit, equilibria cannot be efficient if shifted Rubinstein payoffs are outside the core of the assignment economy.

Proposition 5 *In any strongly efficient LMPE, the payoff of any player $i \in A$ in any equilibrium-path subgame $A \in C(N)$ converges to*

$$\lim_{\delta \to 1} V_i(A) = \bar{\sigma}_i.$$

Moreover, a strongly efficient LMPE exists only if shifted Rubinstein payoffs are in the core,

$$\bar{\sigma}_i + \bar{\sigma}_j \geq s_{ij} \quad \textit{for all } i, j \in N. \tag{4}$$

Core unmatched players can affect the limiting terms of trade without ever agreeing, because they belong to every equilibrium path subgame. Core matched players, instead, cannot play such a role in a strongly efficient LMPE as, in the limit, they exit the game at the first available instance by agreeing with their core match. In addition to demonstrating the robustness of the conclusions previously reached, Proposition 5 uncovers a crucial difference between temporary alternative matches that can be lost as the market evolves and permanent alternative matches that cannot be lost as the market evolves. We term the former temporary outside options and the latter permanent outside options. Furthermore, the result clarifies why bargaining frictions arise endogenously as a strategic response to possible changes in market composition. It is the concern of an alternative match exiting the market, thereby weakening the bargaining position of a player, that induces this player to agree with an inefficient partner even when δ converges to 1. As we have seen in Example 3, similar considerations regarding the evolution of the market can also lead to delay on equilibrium path.

When shifted Rubinstein payoffs are in the interior of the core, they coincide with Rubinstein payoffs by construction. If so, by Proposition 4 an efficient equilibrium exists for any sufficiently high value of δ, and thus a strongly efficient LMPE exists in this case. Strongly efficient LMPE may also exist even when shifted Rubinstein payoffs are on the boundary of the core, as was the case in Example 2. If so, distortions vanish only when the discount factor approaches unity.

Next, we consider weakly efficient LMPE and their properties. The main result establishes that, whereas only core unmatched players can act as permanent outside options in strongly efficient LMPE, all players can potentially act as permanent outside options in some weakly efficient LMPE. However, for this to be the case, the market must clear sequentially, one core match at a time. If so, even players who are ultimately matched can act as permanent outside options by only matching after some other players have matched. To formalize the discussion it is convenient to introduce a notion of sequential agreement.

Definition 6 *A weakly efficient LMPE is a sequential LMPE, if for some $A \in C(N)$ such that $|A \backslash E| \geq 4$ and for some $i \in A \backslash E$*

$$\lim_{\delta \to 1} \pi_{jj}(A) = 1 \text{ for any } j \in A_{-i\eta(i)}. \tag{5}$$

Sequential LMPE display sequential agreement in that all players in the market, except for one pair, delay reaching an agreement until that pair has exited the market. Sequential equilibria require extensive delay to occur despite delay being costly. In particular, pairs who will eventually be matched with probability 1 have to prefer to delay instead of reaching agreement with each other, even though doing so reduces the value of any agreement they can reach. None of the earlier literature, including the examples in Gale and Sabourian (2006) and Abreu and Manea (2012b), feature sequential agreement.

The next result establishes that any weakly efficient LMPE whose limiting payoffs do not converge to shifted Rubinstein payoffs must be sequential. Two LMPE are said to be *payoff equivalent* if the ex-ante limiting values coincide in the two equilibria for all players.

Proposition 6 *Any weakly efficient LMPE that is not payoff equivalent to a strongly efficient LMPE is sequential. Moreover, sequential LMPE exist in some markets.*

An important and immediate implication of Proposition 6 is that when shifted Rubinstein payoffs are outside of the core either there is no efficient LMPE or all efficient LMPE are sequential.[26] Proposition 6 therefore helps pin down when weakly efficient LMPE exist. When exit is sequential, all players remain in the market until a given core match exits, thereby acting effectively as permanent outside options for this match. Proposition 6 further reinforces our central message that inefficiencies are ubiquitous. Indeed, even in a weakly efficient LMPE, outside options cannot affect bargained outcomes without being exercised with strictly positive probability if they are temporary and can be lost on the equilibrium path. Nevertheless, people who are efficiently matched can provide effectively permanent outside options through sequential exit. Although Proposition 6 does not characterize weakly efficient LMPE payoffs, insights in its proof suggest that it should be possible to derive a (not very tractable) payoff set which necessarily contains all weakly efficient LMPE payoffs.[27] If so, by the same logic of Proposition 4, no weakly efficient LMPE would exist whenever such a set does not intersect the core.

[26]By Proposition 5, if a LMPE is payoff equivalent to a strongly efficient LMPE, it must generate shifted Rubinstein payoffs. But, if these payoffs are outside of the core, at least one player has a strict incentive to offer to an inefficient partner.

[27]To do so, consider any subset of players $M \in P_1 \backslash E$ (where $m = |M|$) and consider any order over such players, $M = \{o(1), ..., o(m)\}$ (where $o(i)$ identifies the i^{th} ranked player in M). Define the Rubinstein chain payoffs associated to this order o and this subset M as a payoff profile $u \in \mathbf{R}^{2m}$ (for players in M and their

It is intriguing that sequential exit can occur in equilibrium. The observation conforms with empirical regularities in some matching markets which can clear from the top down. However, delay is a knife-edge phenomenon in most bargaining models without asymmetric information. It might be thought that sequential LMPE will require very specific parameter restrictions on the bargaining problem. To address this issue systematically, we conclude by characterizing the set of sequential LMPE in the context of a 4 player market with equal proposer probabilities. Let $N = \{a, b, c, d\}$ and $p_i = p$ for $i \in N$. To avoid redundancies when stating results we adopt the following labelling *convention*:

- ab and cd are the core matches, $s_{ab} + s_{cd} > s_{ad} + s_{bc}$;
- ab is the most valuable core match, $s_{ab} \geq s_{cd}$;
- ad is the most valuable non-core match, $s_{ad} \geq s_{bc}$.

We also omit the dependence on N when obvious. The final result on efficient LMPE characterizes payoffs in a sequential LMPE, and delivers necessary and sufficient conditions for the existence of such an LMPE.

Remark 3 *Given our convention, if a sequential LMPE exists, then for all δ close to 1*

$$\pi_{ab} = \pi_{ba} = \pi_{cc} = \pi_{da} + \pi_{dd} = 1, \;\; \pi_{da} > 0, \;\; \lim_{\delta \to 1} \pi_{dd} = 1.$$

Moreover, in any such LMPE

$$\begin{array}{ll} \lim_{\delta \to 1} V_a = s_{ad} - \sigma_d & \lim_{\delta \to 1} V_c = \sigma_c \\ \lim_{\delta \to 1} V_b = s_{ab} - s_{ad} + \sigma_d & \lim_{\delta \to 1} V_d = \sigma_d \end{array}$$

Finally, a sequential LMPE exists if and only if

$$s_{ab} > s_{ad} > \frac{s_{ab} + s_{cd}}{2} > s_{bc} > s_{cd} \;\; \textit{and} \;\; \frac{s_{bc} - s_{cd}}{2(s_{ab} - s_{ad})} \geq \frac{s_{bc} + s_{cd}}{s_{ab} + s_{cd}}. \tag{6}$$

respective core partners) such that

$$u_{o(1)} = \bar{\sigma}_{o(1)}, \;\; u_{\eta(o(i))} = s_{o(i)\eta(o(i))} - u_{o(i)} \;\; \text{for} \;\; i \geq 1,$$
$$u_{o(i)} = s_{o(i)\eta(o(i-1))} - u_{\eta(o(i-1))} \;\; \text{for} \;\; i > 1.$$

Now, for any partition of $P_1 \backslash E$ and for any associated order for each element of the partition, define the collection of Rubinstein chain payoffs as a payoff vector in $\mathbf{R}^n$ such that: (1) in every element of the partition payoffs correspond to the chosen Rubinstein chain payoffs; (2) players in E get nothing. Take the union over all partitions of $P_1 \backslash E$ and take the union over all possible orders for each element of the chosen partition; and define such payoff set by Ψ. Our conjecture is that in any weakly efficient LMPE payoffs must belong to Ψ.

The remark pins down agreement probabilities at a high frequency of interaction in any sequential LMPE. In such equilibria, players a and d always reach agreement before c and d. As c and d end up bargaining bilaterally with each other, they have limit payoffs equal to their Rubinstein payoffs. Thus, when players a and d are bargaining, it is as if a had a permanent outside option of value $s_{ad} - \sigma_d$. As $s_{ad} - \sigma_d > \sigma_a$, this outside options binds and a gets a limit payoff of $s_{ad} - \sigma_d$, leaving b with the residual surplus $s_{ab} - (s_{ad} - \sigma_d)$.[28] This equilibrium conforms to previous intuitions. Alternatives within the market can affect the terms of trade only if they remain in the market indefinitely.

Conditions (6) have natural interpretations. Given our labeling convention, the requirement that $s_{ab} > s_{ad} > s_{bc} > s_{cd}$ implies that the market must be vertically differentiated. Moreover, the first match to reach agreement is the most valuable core match. We therefore rationalize top-down sequential exit as a limiting efficient market outcome in a complete information decentralized bargaining game. Delay in bargaining is hard to get, but real world experience suggests that matching markets can occasionally be held up while clearing from the top. Our model delivers such behavior as an equilibrium phenomenon in thin markets without any asymmetric information. The second condition in (6) requires $s_{ad} > (s_{ab} + s_{cd})/2$, or equivalently $s_{ad} > \sigma_a + \sigma_d$. This condition implies that shifted Rubinstein payoffs are outside of the core. If so, by Proposition 5, there is no strongly efficient LMPE, and by Proposition 6 any weakly efficient LMPE must be sequential.

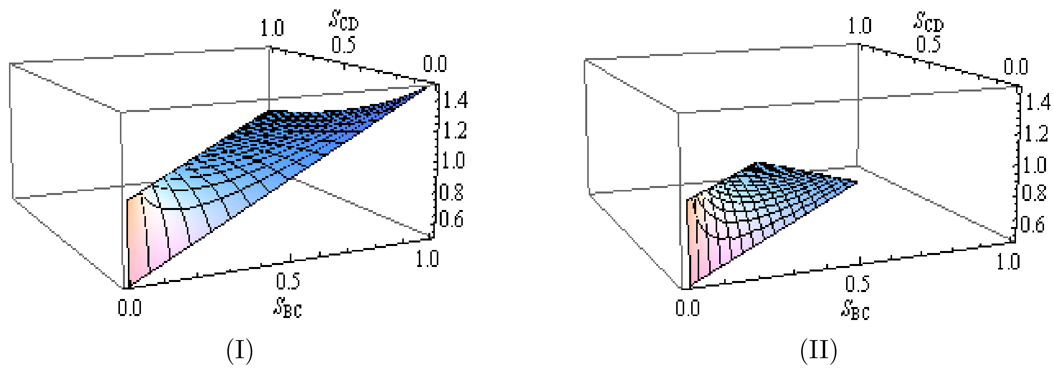

Figure 6: Panel I plots the lower bound for ζ for different combinations of s_{cd} and s_{bc}. As $\zeta < 1$, regions of the parameter space where lower bound is greater than 1 are regions in which no sequential LMPE exists. Panel II shows the lower bound only when $\zeta < 1$.

The final condition in (6) is the hardest to interpret. Although payoffs must be supermodular by the first part of (6), they cannot be log-supermodular by the second part of (6).

[28] In effect b also has a permanent outside option, worth $s_{bc} - \sigma_c$. However, this outside option does not bind. Payoffs are thus pinned down by chains of outside options in any sequential LMPE. These chains are evocative of those discussed in Elliott (2015).

Log-supermodularity, in this context, requires that $s_{ab}s_{cd} \geq s_{ad}s_{bc}$, and the final condition in (6) rules this out.[29] In combination with the other conditions, it requires the market to be highly vertically differentiated (such that proportion of surplus generated by the worst match relative to the best match, s_{cd}/s_{ab}, is small). To see this it is instructive to consider the potential extent of mismatch inefficiencies in the assignment economy (which also captures how supermodular surpluses are). For convenience normalize $s_{ab} = 1$ and define the fraction of potential surplus that is obtained by mismatching as $\zeta = (s_{ad} + s_{bc}) \,/\, (1 + s_{cd}) \in (0, 1)$. The final and key restriction to the parameter space identified in Remark 3 can then be restated in terms of this parameter as requiring

$$\zeta \geq \frac{2(1 + s_{bc})(s_{bc} + s_{cd}) - (1 + s_{cd})(s_{bc} - s_{cd})}{2(s_{bc} + s_{cd})(1 + s_{cd})}.$$

We plot this lower bound on the relative efficiency of the wrong matches in Figure 6. The plot shows that when s_{cd} is relatively large there is no sequential LMPE. More precisely there is a sequential LMPE only if $s_{cd} < 1 - 2s_{bc}$. Since by Remark 3 $s_{bc} > s_{cd}$, a sequential LMPE exists only if $s_{cd}/s_{ab} < 1/3$. So, the less productive core match must be at least three times less productive than the most productive core match. This upper bound on the relative value of s_{cd} becomes much tighter when the potential loss associated with mismatch ζ is at least 5%. Indeed, for $\zeta \leq 0.95$ a similar calculation establishes that $s_{cd}/s_{ab} < 0.133$; so s_{cd} can be at most 13.3% as productive as s_{ab}.[30] We conclude that sequential LMPE only exist in sufficiently vertically differentiated markets, and only in extremely differentiated markets if mismatch generates a considerable amount of inefficiency.

8 Discussion

Assortative Matching: The labor market search literature has extensively studied a particular form of heterogeneity, vertically differentiated markets with assortative matching, as in, for instance, Shimer and Smith (2000), Eeckhout (2006), Smith (2006), Eeckhout and Kircher (2010). To appreciate the content of our efficiency implications we consider this special case of our model.

It will be convenient to introduce some new notation. For this section we refer to the two

[29] The second part of condition (6) can be rewritten as

$$2(s_{ad}s_{bc} - s_{ab}s_{cd}) \geq s_{cd}(s_{ab} + s_{cd} - s_{bc} - s_{ad}) + s_{ab}(s_{bc} - s_{cd}) > 0.$$

[30] Example 4 in Section 6 provides some specific parameter values for which sequential exit occurs. In this example $s_{cd}/s_{ab} = 0.1$ and $\zeta = 0.955$.

sides of the market as workers and firms. Let $W = \{1, ..., w\}$ and $F = \{1, ..., f\}$ denote the sets of workers and firms respectively, and let the surplus generated by worker i and firm j be given by a function $S : W \times F \to \mathbb{R}_+$ satisfying the following conditions:

[C1] $S(i, j) > S(i', j)$ if and only if $i < i'$;

[C2] $S(i, j) > S(i, j')$ if and only if $j < j'$;

[C3] $S(i, j) - S(i, j') > S(i', j) - S(i', j')$ if and only if $i < i'$ and $j < j'$.

Condition C1 requires workers to be vertically differentiated, C2 requires firms to be vertically differentiated, and C3 requires increasing differences in the surpluses that worker-firm pairs can generate. Surplus is generated only in matches between workers and firms. In contrast to our previous notation, there can now be a worker-type i and a firm-type i. Thus typically $S(i, i) \neq 0$ and $S(i, j) \neq S(j, i)$ unless the surplus generated by the i^{th} ranked worker matching to the j^{th} ranked firm is the same as the surplus generated by j^{th} ranked worker matching to the i^{th} ranked firm. Let the set of functions satisfying these conditions be denoted by $\bar{\mathcal{S}}$. It is well known that in such markets the unique core match is the assortative match in which worker k is matched to firm k if $k \leq \min\{w, f\}$, while all the remaining agents are unmatched.

We use our efficiency results to find conditions under which decentralized bargaining would result in an efficient and thus assortative match. For convenience, let the vector p denote the proposal probabilities of firms, where entry p_k is the proposal probability of firm k, and let q denote the proposal probabilities of workers, where entry q_k is the proposal probability of worker k. Thus, a vertically differentiated market is defined by the tuple $\{W, F, S, p, q\}$.

Remark 4 *If $w = f$, $p_k = q_k = p$ for all $k \leq \max\{w, f\}$, and $S(i, j) = S(j, i)$ for all $i, j \leq \min\{w, f\}$, then for all δ close to 1 there is an efficient MPE. However, if at most two of these three conditions hold, there exists a vertically differentiated market for which there is no weakly efficient LMPE.*

Remark 4 shows that, although there are natural conditions under which there is a strongly efficient MPE (the strongest efficiency criterion of the four we consider), these conditions are fairly restrictive and require the market to be highly symmetric. There must be the same number of workers as firms, the k^{th} ranked worker and firm must have the same proposal probabilities, and the surplus generated by the i^{th} ranked worker matching to the j^{th} ranked firm must be the same as the surplus generated by the j^{th} ranked worker matching to the i^{th} ranked firm. When any one of these conditions is not satisfied, there are surpluses $S \in \bar{\mathcal{S}}$ for which there is no weakly efficient LMPE (the weakest efficiency criterion of the four we consider).

Random Matching: The directed search matching protocol considered in our analysis was chosen to minimize frictions. To appreciate the pure delay frictions that arise when players cannot choose whom to bargain with, consider the eight-player line network shown in Figure 7. We provide a brief discussion of such inefficiencies as means of comparison to our model. A comprehensive analysis of the example appears in Section 4 of Abreu and Manea (2012b).

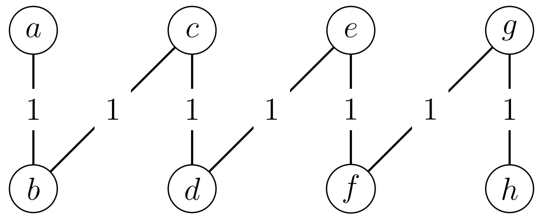

Figure 7: The Eight-Player Line Network

Suppose that matching opportunities are as shown in Figure 7 and that each link is selected with equal probability. An efficient LMPE requires players to disagree with a probability that converges to 1 whenever links bc, de or fg are selected. For this to be the case, the combined continuation values from disagreement of the two players on the link must exceed 1 in the limit, or converge to 1 from above. With the random matching protocol, efficient LMPE exist in the line networks with 4 or 6 players. But, this is not the case in the line network with 8 players. In a four-player line network, the two end players are weak as they get no surplus when the middle players agree. An efficient LMPE exists in which, for δ close to 1, the two middle players exploit such advantage by agreeing with a vanishingly small probability, and in which the continuation values of the end players are diminished to the point where the middle players are indifferent between delaying and agreeing with each other. With eight players, this no longer works. In such networks, players d and e disagree when initially matched. Despite this, their bargaining positions improve relative to the four-player line network as they retain the option to agree with each other in subgames in which their core partners exit. Because of this, players b and c strictly prefer to agree if initially matched for sufficiently high values of δ. Abreu and Manea (2012b) establish in fact that, in the unique LMPE, players b and c must inefficiently agree with probability 1 when matched even when delay costs vanish. If, instead, players were selected to propose with equal probability and were able to choose to whom to agree with, a strongly efficient MPE would always exist as shown in Proposition 7 of the online appendix.

Preventing players from choosing bargaining partners amplifies frictions, as players have to either hold out for their desired partner when presented with an alternative matching opportunity or agree with inefficient partners. We opted for a setting in which players were allowed to choose bargaining partners to diminish the hold-up frictions associated to waiting for the preferred match. Yet, we still found frictions to be common feature of decentralized negotiations because of non-stationarities in the evolution of bargaining power.

Limitations and Evidence: We study the Markov perfect equilibria of a simple bargaining game with many buyers and many sellers, seeking necessary and sufficient conditions for the existence of efficient Markov perfect equilibria. Necessary conditions, however, do not rule out the existence of efficient non-Markovian equilibria,[31] while sufficient conditions do not rule out the existence of inefficient Markov perfect equilibria.[32]

Our protocol is fairly standard and chosen to give the best chance to efficient outcomes while remaining decentralized. To this end, we allow the proposer to choose whom to offer to, study the generic case in which the efficient match is unique, and look at equilibria in which delay costs are small. A key feature of our model is that players' bargaining positions (and, more precisely, their limit payoffs) can change stochastically. We find that these non-stationarities in the evolution of bargaining power are closely linked to inefficiencies, which can include both mismatch and delay. In all weakly efficient LMPE, players' limit payoffs are stationary on the equilibrium path and do not depend on the order in which people are selected to propose. This can occur either because no alternative match provides a binding alternative (Proposition 3), or because binding alternative matches are only provided by those who are unmatched when the market clears efficiently (Proposition 4), or because there is sequential exit and the market endogenously remains stationary while all players wait for a given pair to exit the market before reaching agreements themselves (Proposition 5).

While we view our protocol as natural, many alternative bargaining protocols are equally reasonable. For instance, random matching protocols may describe players bumping into each other at random, while the protocol we study might be a better fit for thin, highly heterogenous markets, in which everyone knows everyone else, and in which search is more likely to be directed. It is not clear whether similar results would hold in the alternative model with random matching. On the one hand, in the limit incentives look very similar to our model, but on the hand, away from the limit random matching forces players to forgo matching inefficiently but immediately, in order to match to their efficient partner later. There are many other alternative protocols. One would be to include the right to make a counter-offer back to the proposer. Another would make a player declining an offer the new proposer. One more would fix a predetermined and commonly known proposer order. We would not expect results close to ours to hold in these environments, as strategically these environments seem fundamentally different.

In practice, interactions in markets are unlikely to be as constrained as any of these bargaining protocols; and players are likely to have much more freedom to endogenously determine,

[31]An efficient subgame perfect equilibrium may always exist, as proven by Abreu and Manea (2012a) for networks with homogeneous surpluses.

[32]We show in Section A of the Online Appendix that inefficient MPE can exist at the same time as efficient MPE.

among other things, who moves when. Indeed, we show that there always exist offer probabilities that generate efficient outcomes. If these are endogenously determined then efficiency might be improved or even restored. Nevertheless, while norms might evolve to affect offer probabilities and increase efficiency, they would need to be tailored to the intricacies of a given market to eliminate inefficiencies (see the discussion following Proposition 4).

Fully endogenizing who makes offers to whom when would come at the cost of tractability. The value of simple theory comes from its ability to provide useful insights in richer settings. Whether our theory, including the equilibrium selection, obtains this goal or not is ultimately an empirical question.

While identifying mismatch empirically is hard because counterfactual productivities are not directly observed, Elliott and Agranov (2017) run a laboratory experiment to circumvent this issue. They begin by studying an experimental protocol that mirrors our bargaining protocol. They find extensive inefficiencies, and show that the Markov perfect equilibrium outcomes correctly predict which markets exhibit mismatch and which exhibit more mismatch than others. However, inefficient matches occur considerably more often than predicted.[33] They then run a second laboratory experiment, but without an experimental protocol. Participants are permitted to make offers to anyone else at any time, accept offers they have received at any time and withdraw offers they have made at any time. They find that inefficiencies remain in the market.[34] Indeed, in this experiment there is not sufficient evidence to reject that inefficiencies are different from the MPE predictions at the 5% level. While this should not be interpreted as evidence that players play the MPE of our bargaining game in an entirely different bargaining environment, it does suggest that in more realistic bargaining situations the inefficiencies we document remain, and that the MPE provide useful intuitions.[35]

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[34] They are positive at the 5% level.

[35] We have argued that inefficiencies in our model are driven by the changing composition of the market. This suggests that allowing people to rematch might reduce or eliminate inefficiencies. Elliott and Arganov (2017) also test this conjecture and find some support for it.

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9 Proof Appendix

Proof of Proposition 1. We first establish the characterization for MPE values, and then proceed to establish existence. Fix a discount factor $\delta \in (0, 1)$. Consider an MPE strategy profile (ρ, χ, α) and its corresponding MPE payoffs $V(A) \in \mathbb{R}^{|A|}$ for any active player set $A \subseteq N$. Fix any subset $A \subseteq N$. By subgame perfection, we know that the acceptance decision by a player $j \in A$ faced with an offer x must be such that he accepts an offer if $x > \delta V_j(A)$, and rejects it if $x < \delta V_j(A)$. Clearly, this implies that it cannot be optimal to offer $x > \delta V_j(A)$ to player j, as the proposer could profitably deviate to an offer in $(\delta V_j(A), x)$. Thus, in any MPE every player would offer at most $\delta V_j(A)$ to player j, and the only offers player j may accept with positive probability are offers of $\delta V_j(A)$ with positive probability. Therefore, a proposer $i \in A$ would make offers with positive probability only to a player j that maximizes his residual payoff $s_{ij} - \delta V_j(A)$. Recall that $\pi_{ij}(A)$ is the joint probability that player i offers $\delta V_j(A)$ to player j and that the offer is accepted, and that $\pi_{ii}(A)$ is the joint probability that i does not agree when proposing. We frequently abuse notation by dropping the dependence of π_{ij} on A where it should not cause confusion. The payoff of any player $k \in A_{-ij}$ at the beginning of the following period is given by $V_k(A_{-ij})$ if an agreement was reached, and by $V_k(A)$ otherwise. Therefore, at a history in which the set of active players is A and in which i is the proposer, the expected payoff of a player $k \in A_{-i}$ must be given by

$$\textstyle\sum_{j \in A_{-ik}} \pi_{ij} \delta V_k(A_{-ij}) + (1 - \sum_{j \in A_{-ik}} \pi_{ij}) \delta V_k(A).$$

When i is chosen to propose, if $\delta[V_i(A) + V_j(A)] < s_{ij}$ for some $j \in A_{-i}$, then i offers with certainty to players j who maximize $s_{ij} - \delta V_j(A)$, and agreement obtains with certainty. The latter observation obtains from the following argument. If $\pi_{ii} > 0$, then the expected payoff conditional on offering $\delta V_j(A)$ to players j who maximize $s_{ij} - \delta V_j(A)$,

$$\sum_{j \in A_{-i}} \pi_{ij}(s_{ij} - \delta V_j(A)) + \left(1 - \sum_{j \in A_{-i}} \pi_{ij}\right) \delta V_i(A),$$

must be strictly smaller than $s_{ij} - \delta V_j(A)$. The payoff conditional on i offering $\delta V_j(A) + \varepsilon$ to j, for $\varepsilon > 0$ is $s_{ij} - \delta V_j(A) - \varepsilon$, as j accepts with probability 1 any offer exceeding $\delta V_j(A)$. Hence, it cannot be optimal to offer more than $\delta V_j(A)$. It also cannot be optimal to offer less than $\delta V_j(A)$ since all such offers are rejected and since $\delta V_i(A) < s_{ij} - \delta V_j(A)$. Thus, if $\pi_{ii} > 0$ and $\delta[V_i(A) + V_j(A)] < s_{ij}$, a profitable deviation always exists. Therefore, $\delta[V_i(A) + V_j(A)] < s_{ij}$ for some $j \in A_{-i}$ implies $\pi_{ii} = 0$. Similarly, $\delta[V_i(A) + V_j(A)] > s_{ij}$ for any $j \in A_{-i}$ implies $\pi_{ii} = 1$. If $\max_{j \in A_{-i}} \{s_{ij} - \delta[V_i(A) + V_j(A)]\} = 0$, then $\pi_{ii} \in [0, 1]$. Thus, any agreement probability $\pi_i \in \Delta(A)$ for player i compatible with equilibrium values $V \in \mathbb{R}^{|A|}$ must belong in

$$\Pi_i(V|A) = \left\{ \pi_i \in \Delta(A) \;\middle|\; \begin{array}{ll} \pi_{ii} = 0 & \text{if} \quad \delta V_i < \max_{j \in A_{-i}} \{s_{ij} - \delta V_j\} \\ \pi_{ik} = 0 & \text{if} \quad s_{ik} - \delta V_k < \max\{\delta V_i, \max_{j \in A_{-i}} \{s_{ij} - \delta V_j\}\} \end{array} \right\}.$$

Next consider the correspondence $f_i(V|A) : \mathbb{R}^{|A|} \rightrightarrows \mathbb{R}^{|A|}$, where for $k \neq i$,

$$f_i(V|A) = \begin{bmatrix} f_{ii}(V|A) \\ f_{ik}(V|A) \\ \ldots \end{bmatrix} = \left\{ \begin{bmatrix} (1 - \pi_{ii}) \max_{j \in A_{-i}} \{s_{ij} - \delta V_j\} + \pi_{ii} \delta V_i \\ \sum_{j \in A_{-ik}} \pi_{ij} \delta V_k(A_{-ij}) + (\pi_{ik} + \pi_{ii}) \delta V_k \\ \ldots \end{bmatrix} \;\middle|\; \pi_i \in \Pi_i(V|A) \right\},$$

where the expressions in the square brackets give the components of a $|A| \times 1$ vector. Let $f_{ik}(V|A)$ denote the k^{th} entry of $f_i(V|A)$. The correspondence $f_{ik}(\cdot|A)$ identifies the set of expected payoffs compatible with our partial equilibrium analysis for a player $k \in A$ and for any history in which A is the set of active players and i is the proposer. Next, define the correspondence

$$F(V|A) = \sum_{i \in A} p_i f_i(V|A) + \left(1 - \sum_{i \in A} p_i\right) \delta V. \tag{7}$$

The k^{th} entry of such a correspondence, $F_k(\cdot|A)$, identifies the set of possible expected payoffs for a player $k \in A$ for any history in which A is the set of active players. Thus, the argument establishes that V is an MPE payoff only if it is a fixed point of the correspondence in (7), $V \in F(V|A)$.

Next, we establish that the converse must hold too. In particular, we argue that if $V(A) \in F(V(A)|A)$ for any subset $A \subseteq N$, then $V(A)$ is an MPE payoff profile for any subgame in which A is the set of active players. At any subgame in which A are the active players, consider a strategy in which any player $i \in A$ chooses $\rho_i(A) = \pi_i$, $\chi_i(j, A) = \delta V_j(A)$, and

$$\alpha_i(j, x, A) = \begin{cases} 1 & \text{if} \quad x \geq \delta V_i(A) \\ 0 & \text{if} \quad x < \delta V_i(A) \end{cases}.$$

For any finite set of players N, the proposed strategy clearly must be an MPE in any subgame in which no more than one player is active, as any such subgame is eventless. By induction, suppose that the proposed strategy is an MPE for any subset of active players of size $k \leq n-1$, in order to show that it is an MPE for any subgame in which the set of active players has size $k+1$. Consider a subgame in which the set of active players A has cardinality $k+1$. Fix an MPE payoff profile $V(A')$ for all subgames in which the cardinality of the set of active players A' does not exceed k. Furthermore, given such values, suppose that we can find a payoff profile $V(A)$ such that $V(A) \in F(V(A)|A)$ (we establish below that such a fixed point exists). If so, no player receiving an offer can profitably deviate from strategy α, as no change in the acceptance rule can strictly increase the payoff. Similarly, given the acceptance rule, the proposer's strategy (ρ, χ) is optimal given that offers are made only to those players who leave the highest residual surplus to the proposer (provided that such surplus exceeds the value of being unmatched). Thus, $V(A)$ is an MPE payoff in any subgame with a set of active players A. Consequently, if $V(A) \in F(V(A)|A)$ for any subset $A \subseteq N$, then $V(A)$ is an MPE payoff profile.

To establish existence, also proceed by induction. Existence follows in subgames in which no more than one player is active, as such subgames are eventless. Assume by induction that an MPE exists for any subset of active players of size $k \leq n-1$, in order to show that it exists for any subgame in which the set of active players A has size $k+1$. If so, consider MPE strategies for all subgames of size k and derive MPE payoffs for all such subgames. Given such values, construct the correspondence $F(\cdot|A)$ as in (7). Observe that the correspondence $\Pi_i(\cdot|A)$ is upper-hemicontinuous with non-empty convex images. Thus, $f_i(\cdot|A)$ is upper-hemicontinuous with non-empty convex images; and so, the correspondence $F(\cdot|A)$ is upper-hemicontinuous with non-empty convex images, as it is a convex combination of the correspondences $f_i(\cdot|A)$ for $i \in A$. By Kakutani's fixed point Theorem $F(\cdot|A)$ has a fixed point. Moreover, such a fixed point is an MPE payoff of this subgame, and can be used to construct consistent MPE strategies and consequently agreement probabilities $\pi \in \Delta(A)^N$ in every subgame, as argued above. ■

Proof of Proposition 2. For convenience, define the limiting agreement probability for a given player $j \in A \backslash E$ as $\beta_j(A) = \lim_{\delta \to 1} p_j \pi_{j\eta(j)}(A)$, and let $\beta_B(A) = \sum_{k \in B} \beta_k(A)$ for any $B \subseteq A$. Recall that $\bar{V}_j(A) = \lim_{\delta \to 1} V_j(A)$. We begin by showing that for any active player set $A \in C(N)$ such that $A \backslash E \neq \emptyset$, there exists a player $i \in A \backslash E$ such that $\beta_i(A) > 0$. This is the case since weak efficiency and $\beta_i(A) = 0$ for all player $i \in A\backslash E$, imply $\bar{\pi}_{ii}(A) = 1$ for all players $i \in A\backslash E$. But, if so, for δ close to 1, any player i would weakly prefer delaying to offering to $\eta(i)$, or equivalently

$$\delta V_i(A) + \delta V_{\eta(i)}(A) \geq s_{i\eta(i)}. \tag{8}$$

This would lead to a contradiction though as the sum of payoffs exceeds total surplus

$$\textstyle\sum_{i\in A} V_i(A) \geq \sum_{i\in A\backslash E} V_i(A) \geq (1/\delta) \sum_{i\in A\cap P_1} s_{i\eta(i)} > \sum_{i\in A\cap P_1} s_{i\eta(i)}$$

where the first and third inequalities are trivial while the second holds by adding the inequalities in (8) and observing that $s_{i\eta(i)} = 0$ if $i \in E$. Thus, in any non-trivial active player set $A \in C(N)$ of any weakly efficient LMPE, there exists a player $i \in A\backslash E$ such that $\beta_i(A) > 0$.

To prove the result, we proceed by induction on the size of the active player set within $C(N)$. We show that in any weakly efficient LMPE $\bar{V}_i(A) + \bar{V}_{\eta(i)}(A) = s_{i\eta(i)}$ for any $i \in A\backslash E$ and any $A = C(N)$. The latter then immediately implies surplus maximization by feasibility. If $E \neq \emptyset$, begin by considering the active player set $E \in C(N)$. If so, any weakly efficient LMPE trivially maximizes surplus as all links are worth zero. Next, consider any active player set $A = E \cup \{i, \eta(i)\}$ for some $i \in N\backslash E$. As the LMPE is weakly efficient, there exists a player $j \in A\backslash E$ such that $\beta_j(A) > 0$. But if so, by taking limits of system (1), we obtain

$$\bar{V}_j(A) = \beta_j(A)(s_{i\eta(i)} - \bar{V}_{\eta(j)}(A)) + (1 - \beta_j(A))\bar{V}_j(A).$$

The latter implies that $\bar{V}_i(A) + \bar{V}_{\eta(i)}(A) = s_{i\eta(i)}$. Finally, by induction assume that any weakly efficient LMPE satisfies $\bar{V}_i(A) + \bar{V}_{\eta(i)}(A) = s_{i\eta(i)}$ for any $i \in A\backslash E$ and any $A \in C(N)$ with cardinality $|A| \leq |E| + 2k$. Consider any set $A \in C(N)$ with cardinality $|A| = |E| + 2(k+1)$. For any player $i \in A\backslash E$, defining $\bar{A}(i) = A_{-i\eta(i)}\backslash E$ and taking limits of system (1) establishes that

$$(\beta_{A\backslash E}(A) - \beta_{\eta(i)}(A))\bar{V}_i(A) = \beta_i(A)\left(s_{i\eta(i)} - \bar{V}_{\eta(i)}(A)\right) + \textstyle\sum_{k\in\bar{A}(i)} \beta_k(A)\bar{V}_i(A_{-k\eta(k)}). \tag{9}$$

By the induction hypothesis, we know that for all $k \in \bar{A}(i)$

$$\bar{V}_i(A_{-k\eta(k)}) + \bar{V}_{\eta(i)}(A_{-k\eta(k)}) = s_{i\eta(i)}.$$

Exploiting this observation while adding equation (9) for player i to that for player $\eta(i)$, implies that

$$\beta_{A\backslash E}(A)(\bar{V}_i(A) + \bar{V}_{\eta(i)}(A)) = \textstyle\sum_{k\in A\backslash E} \beta_k(A) s_{i\eta(i)},$$

or equivalently $\bar{V}_i(A) + \bar{V}_{\eta(i)}(A) = s_{i\eta(i)}$ since by weak efficiency there exists a player $j \in A\backslash E$ such that $\beta_j(A) > 0$. The latter concludes the proof and establishes that any weakly efficient LMPE maximizes surplus. ■

Proof of Proposition 3. We begin by pinning down strongly efficient MPE payoffs. Consider an MPE strategy in which any player $i \in N$ offers to his core match $\eta(i)$ with probability 1 at any active player set $A \in C(N)$. If players follow the prescribed strategy, only core matches are ever consummated, and only subgames $A \in C(N)$ occur on the equilibrium path. As the core match maximizes the total surplus in an assignment economy, the core match of a player does not change when other core pairs exit the market (that is, it coincides at any subgame $A \in C(N)$). By Proposition 1, we know that any proposer $i \in A$ necessarily offers an amount equal to $\delta V_{\eta(i)}(A)$ and that any player $i \in A$ accepts any offer exceeding $\delta V_i(A)$. As players negotiate with only core partners on the equilibrium path, at any $A \in C(N)$ we guess that

$$V_i(A) = V_i(A_{-j\eta(j)}) \text{ whenever } i \in A_{-j\eta(j)}. \tag{10}$$

Thus, at any $A \in C(N)$, equilibrium payoffs for every player $i \in A$ satisfy

$$V_i(A) = p_i(s_{i\eta(i)} - \delta V_{\eta(i)}(A)) + (1 - p_i)\delta V_i(A).$$

Solving the latter equation for player i with the one for player $\eta(i)$ implies that

$$V_i(A) = \frac{p_i}{1 - \delta + \delta p_i} \left(s_{i\eta(i)} - \delta \frac{p_{\eta(i)}}{1 - \delta + \delta p_{\eta(i)}} (s_{i\eta(i)} - \delta V_i(A)) \right),$$

which after some manipulation yields

$$V_i(A) = \frac{p_i}{1 - \delta + \delta p_i + \delta p_{\eta(i)}} s_{i\eta(i)}, \tag{11}$$

which verifies (10) as value functions are unique for given proposal probabilities by the proof of Proposition 1.

To establish the first part then, by contradiction postulate the existence of a weakly efficient MPE that is not strongly efficient. If so, along any equilibrium path, players either agree with their core partner or delay, which implies that any equilibrium-path subgame is associated to an active players A set which belongs to $C(N)$. Formally, such a requirement amounts to finding a fixed point of the MPE characterization in Proposition 1 which satisfies $\pi_{ii}(A) + \pi_{i\eta(i)}(A) = 1$ for any $i \in A$ and any $A \in C(N)$. If such an equilibrium were to exist, an argument equivalent to the first part of the proof would imply that for any $i \in A$ and any

$A \in C(N)$

$$V_i(A) = \frac{p_i \pi_{i\eta(i)}(A)}{1 - \delta + \delta p_i \pi_{i\eta(i)}(A) + \delta p_{\eta(i)} \pi_{\eta(i)i}(A)} s_{i\eta(i)}.$$

But this would give rise to the desired a contradiction as any player i would strictly prefer immediate agreement with his core match rather than disagreement, since $V_i(A)$ strictly increases in $\pi_{i\eta(i)}(A)$. ■

Proof of Proposition 4. First, we establish part (a). Payoffs in any subgame $A \in C(N)$ of an efficient MPE are pinned down by Proposition 3 for any $\delta \in (0, 1)$. We show that complying with efficient strategies yields an equilibrium for any sufficiently high value of δ. Recall that any player $j \in A$ accepts any offer that is worth at least $\delta V_j(A)$. Suppose, by contradiction, that some player $i \in A$ at some subgame $A \in C(N)$ has a profitable deviation which entails offering to $j \neq \eta(i)$ when all players comply with strongly efficient strategies. For such an offer to be profitable for i, at any sufficiently high δ it must be that

$$s_{ij} - \delta V_j(A) > s_{i\eta(i)} - \delta V_{\eta(i)}(A). \tag{12}$$

However, by taking limits, as δ converges to 1, on both sides of this inequality, we obtain

$$s_{ij} - \sigma_j \geq s_{i\eta(i)} - \sigma_{\eta(i)} = \sigma_i.$$

This obviously contradicts the assumption that Rubinstein payoffs are in the interior of the core: $\sigma_i + \sigma_j > s_{ij}$ for all $i, j \in A$ such that $j \neq \eta(i)$. Thus, any player $i \in A$ at any subgame $A \in C(N)$ cannot have a profitable deviation when making offers if the discount factor is sufficiently high, which implies the existence of a strongly efficient MPE for any δ close to 1. Next, we establish part (b). By contradiction, assume that a strongly efficient MPE exists for any δ close to 1, but that $\sigma_i + \sigma_j < s_{ij}$ for some pair $i, j \in N$. Recall that player i has a strictly profitable deviation from a strongly efficient equilibrium if condition (12) holds. Since $\delta V_i \to \sigma_i$ and $\delta V_j \to \sigma_j$, condition (12) must hold for sufficiently high values of δ and player i must have a profitable deviation for any sufficiently high value of δ. ■

Proof of Proposition 5. To pin down LMPE values, for any player $i \in N$, define the outside option partner for player i as follows

$$\lambda(i) = \begin{cases} \arg\max_{j \in E} s_{ij} & \text{if} \quad \omega_i > 0 \\ i & \text{if} \quad \omega_i = 0 \end{cases}.$$

Therefore, $\omega_i = s_{i\lambda(i)}$. An LMPE is strongly efficient if at any active player set $A \in C(N)$, all players $i \notin E$ agree with their core matches $\eta(i)$ with a probability that converges to 1 (that is, $\bar{\pi}_{i\eta(i)}(A) = 1$), and all players $i \in E$ delay with a probability that converges to 1 (that is, $\bar{\pi}_{ii}(A) = 1$). Recall that only subgames $A \in C(N)$ occur on the equilibrium path with positive probability in the limit in a strongly efficient LMPE. Moreover, outside options $\lambda(i)$ must coincide at every subgame $A \in C(N)$ since the core match coincides at any such active player set and since all core unmatched players are active at any such active player set.

To establish that any strongly efficient strategy compatible with equilibrium necessarily yields shifted Rubinstein payoffs as limiting payoffs, we proceed by induction on the size of the active player set within $C(N)$, and show that for any $A \in C(N)$ any strongly efficient LMPE satisfies

$$\bar{V}_j(A) = \bar{\sigma}_j \quad \text{for any} \quad j \in A. \tag{13}$$

First, consider the smallest active player set in $C(N)$, namely, $A = E$, when such a set is not empty. If so, $s_{ij} = 0$ for any $i, j \in E$. Obviously, $V_j(E) = \bar{\sigma}_j = 0$ for any $j \in E$. Next, consider any active player set $A = E \cup \{i, \eta(i)\}$ for some $i \in N \backslash E$. Clearly, not both players in $\{i, \eta(i)\}$ can have binding outside options. If they did, then

$$s_{i\lambda(i)} + s_{\eta(i)\lambda(\eta(i))} \geq s_{i\eta(i)},$$

and an alternative match that generates a weakly higher surplus would be feasible (since both $\lambda(i)$ and $\lambda(\eta(i))$ would be unmatched in the core), thereby contradicting the optimality of the core match or its uniqueness. Without loss of generality, if a player has a binding outside option, let that player be i, so that $\bar{\sigma}_i = \max\{\omega_i, \sigma_i\}$ and $\bar{\sigma}_{\eta(i)} = s_{i\eta(i)} - \bar{\sigma}_i$. Observe that if a player $j \in E$ plays a strategy converging to efficiency, then for sufficiently high δ he must weakly prefer delaying to offering to a player in $\{i, \eta(i)\}$, as $\bar{\pi}_{jj}(A) = 1$. If so, then $v_j(A) = \delta V_j(A)$ and the valuation of such a player necessarily satisfies

$$\bar{V}_j(A) = (1 - p_i - p_{\eta(i)})\bar{V}_j(A) \quad \Rightarrow \quad \bar{V}_j(A) = 0,$$

by the characterization in Proposition 1, the definition of strongly efficient LMPE, and the linearity of the limit operator. Therefore, condition (13) holds for any player $j \in E$. Next, consider player $j \in \{i, \eta(i)\}$. If complying with a strongly efficient strategy is a limiting equilibrium, then for sufficiently high δ it must be that $v_j(A) = s_{j\eta(j)} - \delta V_{\eta(j)}(A)$, as $\bar{\pi}_{j\eta(j)}(A) = 1$. If so, then for any player $k \in E$,

$$s_{j\eta(j)} - \delta V_{\eta(j)}(A) \geq s_{jk} - \delta V_k(A) = s_{jk},$$

which in turn implies that

$$\bar{V}_j(A) = p_j\left(s_{j\eta(j)} - \bar{V}_{\eta(j)}(A)\right) + (1-p_j)\bar{V}_j(A) \quad \Rightarrow \quad \bar{V}_j(A) \geq s_{jk},$$

which establishes that $\bar{V}_j(A) \geq \omega_j$. If indeed $\bar{V}_j(A) > \omega_j$ for any player $j \in \{i, \eta(i)\}$, then for any player $k \in E$ and any δ close to 1, we would have that that $\delta V_k(A) + \delta V_j(A) > s_{jk}$. If so, no player k would ever agree with j. If so, the strategy would be strictly efficient for sufficiently high δ and the result would follow by Proposition 4 as

$$\bar{V}_j(A) = \lim_{\delta \to 1} \frac{p_j}{1-\delta+\delta p_j+\delta p_{\eta(j)}} s_{j\eta(j)} = \sigma_j > \omega_j.$$

Otherwise, suppose that $\bar{V}_j(A) = \omega_j$. If so, taking limits on the characterization in Proposition 1 implies that

$$\bar{V}_{\eta(j)}(A) = p_{\eta(j)}\left(s_{j\eta(j)} - \bar{V}_j(A)\right) + (1-p_{\eta(j)})\bar{V}_{\eta(j)}(A) = s_{j\eta(j)} - \omega_j.$$

The previous observations together imply that $\bar{V}_k(A) = \bar{\sigma}_k$ for any $k \in A$, as $\bar{V}_i(A) = \bar{\sigma}_i = \max\{\omega_i, \sigma_i\}$ and $\bar{V}_{\eta(i)}(A) = s_{i\eta(i)} - \bar{\sigma}_i$.

Next, by induction assume that $\bar{V}_j(A) = \bar{\sigma}_j$ for any $j \in A$ and any active player set $A \in C(N)$ with cardinality $|A| = |E| + 2k$. If so, we show $\bar{V}_j(A) = \bar{\sigma}_j$ for any $j \in A$ and any set $A \in C(N)$ with cardinality $|A| = |E| + 2(k+1)$. Consider such a set A. If a player $j \in E$ complies with a strongly efficient strategy, then $v_j(A) = \delta V_j(A)$ for δ close to 1, and the valuation necessarily satisfies

$$\begin{aligned}\bar{V}_j(A) &= (1-p_{A\setminus E})\bar{V}_j(A) + \textstyle\sum_{k\in A\setminus E} p_k \bar{V}_j(A_{-k\eta(k)}) \\ &= (1-p_{A\setminus E})\bar{V}_j(A) \quad \Rightarrow \quad \bar{V}_j(A) = 0,\end{aligned}$$

where the first equality follows from the characterization in Proposition 1 and the definition of strongly efficient strategy, while the second equality follows from the induction hypothesis. If a player $j \in A\setminus E$ complies with a strongly efficient strategy, then $v_j(A) = s_{j\eta(j)} - \delta V_{\eta(j)}(A)$ for δ close to 1. Thus, defining $\bar{A}(j) = A_{-j\eta(j)}\setminus E$, the valuation necessarily satisfies

$$\begin{aligned}\bar{V}_j(A) &= (1-p_{\bar{A}(j)}-p_j)\bar{V}_j(A) + p_j(s_{j\eta(j)} - \bar{V}_{\eta(j)}(A)) + \textstyle\sum_{k\in \bar{A}(j)} p_k \bar{V}_j(A_{-k\eta(k)}) \\ &= (1-p_{\bar{A}(j)}-p_j)\bar{V}_j(A) + p_j(s_{j\eta(j)} - \bar{V}_{\eta(j)}(A)) + p_{\bar{A}(j)}\bar{\sigma}_j,\end{aligned}$$

where equalities hold for the same reasons stated above. In this case, the limiting value equations for players j and $\eta(j)$ admit a unique solution at

$$\bar{V}_j(A) = \bar{\sigma}_j \text{ and } \bar{V}_{\eta(j)}(A) = \bar{\sigma}_{\eta(j)}.$$

To prove the second part of the result observe that strongly efficiency LMPE mandate play according to the strategies characterized above and payoffs converging to shifted Rubinstein payoffs,

$$\bar{V}_i(A) = \bar{\sigma}_i \text{ for any } i \in A \text{ and any } A \in C(N).$$

Towards a contradiction, suppose that agents complied with these strategies, but that $\bar{\sigma}_i + \bar{\sigma}_j < s_{ij}$ for some pair $i, j \in N$. If so, the definition of shifted Rubinstein payoffs would then immediately imply that $j \notin \{\eta(i), \lambda(i)\}$. If so however, i would have a profitable deviation when selected to make the first offer in the game. Subgame perfection ensures that j would accept any offer greater than $\delta V_j(A)$. Now if the player complied with the prescribed strategy by offering to his core partner, his limiting payoff would amount to

$$\lim_{\delta \to 1} v_i(A) = \bar{\sigma}_i.$$

However, by deviating and offering to j exactly $\delta V_j(A)$, his payoff would increase to

$$\lim_{\delta \to 1} [s_{ij} - \delta V_j(A)] = s_{ij} - \bar{V}_j(A) = s_{ij} - \bar{\sigma}_j > \bar{\sigma}_i.$$

Thus, for any value of δ sufficiently close to 1, player i would have a strict incentive to deviate and make an acceptable offer to j. ■

Proof of Proposition 6. In a weakly efficient LMPE, $\bar{\pi}_{i\eta(i)}(A) + \bar{\pi}_{ii}(A) = 1$ for any player $i \in A$ for every $A \in C(N)$. Thus, all players $i \in E$ delay with a probability converging to 1 (that is, $\bar{\pi}_{ii}(A) = 1$). In the limit, if all players comply with such strategies, only subgames $A \in C(N)$ occur on the equilibrium path with positive probability. Recall that the proof of Proposition 2 established that at any active player set $A \in C(N)$ such that $A \backslash E \neq \emptyset$ of a weakly efficient LMPE, there exists a player $i \in A \backslash E$ such that $\beta_i(A) > 0$ (where $\beta_i(A) = p_i \bar{\pi}_{i\eta(i)}(A)$). To establish that any weakly efficient LMPE that is not strongly efficient must be sequential, we again proceed by induction on the size of the active player set within $C(N)$, and show that there exists $A \in C(N)$ such that only one core match agrees. That is for some $i \in A \backslash E$ such that (5) holds. First, consider the smallest active player set in $C(N)$, namely, $A = E$, when such set is not empty. If so, any weakly efficient LMPE is strongly efficient as the two definitions coincide. Next, consider any active player set $A = E \cup \{i, \eta(i)\}$ for some $i \in N \backslash E$. Clearly, there must be agreement on the core match, that is $\pi_{i\eta(i)}(A) = \pi_{\eta(i)i}(A) = 1$, as $\delta V_i(A) + \delta V_{\eta(i)}(A) < s_{i\eta(i)}$ by feasibility. Thus, any weakly efficient LMPE is strongly efficient.

Next, assume by induction that any weakly efficient LMPE is strongly efficient for any active player set $A \in C(N)$ with cardinality $|A| = |E| + 2k$. Consider any set $A \in C(N)$ with cardinality $|A| = |E| + 2(k+1)$. If a player $j \in E$ complies with a weakly efficient strategy, then $v_j(A) = \delta V_j(A)$ for δ close to 1. If so, the valuation of j necessarily satisfies

$$\begin{aligned}\bar{V}_j(A) &= (1 - \beta_{A\setminus E}(A))\bar{V}_j(A) + \textstyle\sum_{k\in A\setminus E}\beta_k(A)\bar{V}_j(A_{-k\eta(k)}\}) \\ &= (1 - \beta_{A\setminus E}(A))\bar{V}_j(A) \quad \Rightarrow \quad \bar{V}_j(A) = 0,\end{aligned}$$

where the first equality follows by taking limits of value equations and the definition of weakly efficient strategy, where the second equality follows by the induction hypothesis, and where the implication trivially obtains as $\beta_{A\setminus E}(A) > 0$ given that at least 1 core match agrees with positive probability in the limit. If a player $j \in A\setminus E$ complies with a weakly efficient strategy, then for δ close to 1 it must be that $v_j(A) = \max\{\delta V_j(A), s_{j\eta(j)} - \delta V_{\eta(j)}(A)\}$ by weak efficiency. Taking limits of value equations for any $j \in A\setminus E$ while defining $\bar{A}(j) = A_{-j\eta(j)}\setminus E$ establishes that

$$\begin{aligned}\bar{V}_j(A) &= (1 - \beta_{\bar{A}(j)} - \beta_j)\bar{V}_j(A) + \beta_j(A)\left[s_{j\eta(j)} - \bar{V}_{\eta(j)}(A)\right] + \sum_{k\in\bar{A}(j)}\beta_k(A)\bar{V}_j(A_{-k\eta(k)}) \\ &= (1 - \beta_{\bar{A}(j)} - \beta_j)\bar{V}_j(A) + \beta_j(A)\left[s_{j\eta(j)} - \bar{V}_{\eta(j)}(A)\right] + \beta_{\bar{A}(j)}(A)\bar{\sigma}_j, \end{aligned} \tag{14}$$

where the second equality follows by weak efficiency and induction.

If $\beta_i(A) = 0$ for all players $i \in \bar{A}(j)$, then the equilibrium must be sequential by definition.[36] Otherwise, there exists a weakly efficient LMPE in which least two core matches in A reach agreement with positive probability. If so, $\beta_i(A) > 0$ and $\beta_j(A) > 0$ for $i \neq \eta(j)$, and thus $\beta_{\bar{A}(j)} > 0$ for any $j \in A\setminus E$. But, if so, the limiting value equations (14) for players j and $\eta(j)$ admit a unique solution at

$$\bar{V}_j(A) = \bar{\sigma}_j \quad \text{and} \quad \bar{V}_{\eta(j)}(A) = \bar{\sigma}_{\eta(j)}.$$

The weakly efficient LMPE must be payoff equivalent to a strongly efficient LMPE at A thereby fulfilling the induction hypothesis. This establishes that any weakly efficient LMPE that is not strongly efficient must be sequential.

The existence of a sequential LMPE follows by Example 4. ■

[36]If so, $\bar{V}_i(A) = \bar{\sigma}_i$ for all $i \in \bar{A}(j) \cup E$ since $\bar{A}(j) \neq \emptyset$ and $\bar{V}_i(A\setminus\{k, \eta(k)\}) = \bar{\sigma}_j$ for all $k \in A\setminus E$ by induction hypothesis.

Decentralized Bargaining in Matching Markets: Online Appendix

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Abstract

The online appendix discusses: MPE multiplicity; the non-generic cases of core-match multiplicity and of boundary core payoffs; the relationship to Okada (2011); and omitted proofs.

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1 MPE Multiplicity

This short section presents an economy in which condition (2) in Proposition 3 holds, but in which multiple MPE exist for all δ close to 1. Consider the 4-player economy in Figure 7 with $p_a = p_b = 4/10$ and $p_c = p_d = 1/10$.

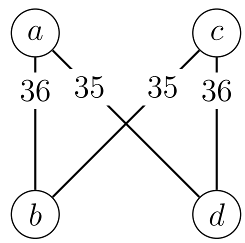

Figure 7: A Four Player Complete Network with Surplus Heterogeneity

The economy clearly satisfies condition (2) as

$$\sigma_a + \sigma_d = \sigma_b + \sigma_c = 36 > 35.$$

Thus, an efficient MPE always exits for all δ close to 1; and consequently a strongly efficient LMPE exists. However, for all δ close to 1, an inefficient MPE also exists with the following proposal probabilities,

$$\pi_{ad} = \pi_{bc} = \pi_{cd} = \pi_{dc} = 1.$$

By setting $V_a = V_b$ and $V_d = V_c$, value equations (1) for the inefficient equilibrium reduce to

$$V_a = \frac{4}{10}(35 - \delta V_c) + \frac{2}{10}\delta V_a(ab) + \frac{4}{10}\delta V_a(ad),$$
$$V_d = \frac{1}{10}(36 - \delta V_d) + \frac{1}{2}\delta V_d + \frac{4}{10}\delta V_d(ad).$$

Solving for subgame values, establishes that

$$V_a = \frac{2(350 - 69\delta - 25\delta^2)}{5(5-\delta)(2-\delta)} \quad \text{and} \quad V_d = \frac{36 - 4\delta}{(5-\delta)(2-\delta)}.$$

Taking limits then implies that $\lim_{\delta \to 1} V_a = 128/5 = 25.6$ and $\lim_{\delta \to 1} V_d = 8$. Limit values then satisfy all the equilibrium incentive constraints, as

$$2V_a > 36, \;\; 2V_d < 36,$$
$$V_a + V_d < 35, \;\; 36 - V_d > 35 - V_a.$$

As incentive constraints are strict and value functions continuous, players strictly prefer to comply with the strategy for all δ close to 1. Thus, the proposed strategy is an MPE all δ close to 1 and so an LMPE. Hence, multiple equilibria may exist even when condition (2) holds and the core match is unique. Intuitively, multiplicity may arise because directed search and partner selection bring about coordination problems as players' bargaining powers are jointly determined by the entire profile of agreement probabilities.

2 Multivalued Core

The complications that arise when the core is multivalued (that is, when multiple matches are efficient) are closely related to those that occur when Rubinstein payoffs are on the boundary of the core, as any core payoff must be on the boundary of the core in such instances. When multiple matches are efficient, each efficient match is associated with a possibly different vector of Rubinstein payoffs. For any efficient match η, let $\sigma^\eta \in \mathbb{R}^{|N|}$ denote the vector of Rubinstein payoffs associated with the efficient match η. Consider an alternative efficient match $\gamma \neq \eta$. Shapley and Shubik (1972) establish that if the pair (η, σ^η) is a core outcome so is the pair (γ, σ^η), in that for all players i, $\sigma_i^\eta + \sigma_{\eta(i)}^\eta = s_{i\eta(i)}$ and $\sigma_i^\eta + \sigma_{\gamma(i)}^\eta = s_{i\gamma(i)}$. As $\eta(i) \neq \gamma(i)$ for some player i, the core outcome (η, σ^η) must be on the boundary of the core as players i and $\gamma(i)$ have a weakly profitable pairwise deviation.

Scenarios in which Rubinstein payoffs are on the boundary of the core payoffs, leads to complications. Our equilibrium construction can lead to payoffs that are outside the core for all $\delta < 1$, such that some player has a profitable deviation to offer inefficiently, even when limit payoffs belong to the core. We illustrate this in the following example.

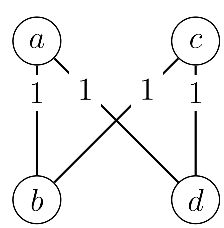

Figure 8: Four Player Complete Network with Unit Surplus

Consider the four player economy in which all matches are possible and generate a surplus of 1 depicted in Figure 8. Suppose first that all players move with equal probability. If so, the match (ab, cd) is efficient and for this match each player's Rubinstein payoff is 1/2. These payoffs belong to the boundary of core. If we attempt the efficient MPE construction we use when players' payoffs are in the interior of the core, with certainty player a would offer to b, b would offer to a, c would offer to d, and d would offer to c. In this example, as is the case with core-interior Rubinstein payoffs, these offer strategies constitute an efficient MPE.

For instance, given these strategies, continuation values for players b and d coincide whenever player a is selected as the proposer. Thus, player a is indifferent between offering to b or d, and offering to b is a best response for a as equilibrium play dictates.

Suppose then that players propose respectively with probabilities

$$p_a = p_b = 3/8 \text{ and } p_c = p_d = 1/8.$$

The match (ab, cd) remains efficient, and for this match each player's Rubinstein payoff is 1/2. Thus, as before, Rubinstein payoffs are on the boundary of the core. However, if we now attempt the efficient MPE construction we use for interior Rubinstein payoffs, we no longer find an equilibrium. By complying with these strategies all player still receive limit payoffs of 1/2, but for all $\delta < 1$ player a has a profitable deviation by offering to d. As d waits longer than b to be matched in expectation, d's continuation value is lower than b's for $\delta < 1$. Hence, a prefers to deviate and offer to d. In the other efficient match (ad, cb), Rubinstein payoffs are 3/4 for a and b and 1/4 for c and d. These Rubinstein payoffs do not belong to the core as c and d have a profitable pairwise deviation. If a were to offer to d, d were to offer to a, b were to offer to c, and c were to offer to b with certainty, player c would have a profitable deviation by offering to d.

This example is intended to illustrate the subtleties that may arise when Rubinstein payoffs are on the boundary of the core. Although there are no strictly profitable deviations in the limit as $\delta \to 1$, there may be strictly profitable deviations for all $\delta < 1$. Whether this happens or not depends on whether the sum of payoffs for each pair of efficiently matched players converges from above or below to the surplus they generate, which in turn depends on the fine details of the game. Nevertheless, there is one canonical case in which there always exists an efficient MPE when Rubinstein payoffs are on the boundary of the core. An assignment economy is said to be *simple* if $s_{ij} \in \{0, 1\}$ for all $i, j \in N$.

Proposition 7 *Consider a simple assignment economy in which all players are selected to propose with equal probability. Then, there exists a strongly efficient MPE if the Rubinstein payoffs associated with an efficient match belong to the core.*

Symmetry in these settings suffices for the existence of strongly efficient MPE. In fact, since all players on one side of the market receive the same payoff and since delay destroys surplus, the sum of payoffs for each pair of efficiently matched players must converge from below to the surplus they generate. In general though, core match multiplicity may lead to discontinuities in equilibrium payoffs which would further complicate efficiency conclusions (as is the case in Example 2 when y equals 200).

To conclude the discussion, we show an example in which core match multiplicity leads to MPE multiplicity and to additional delay frictions. Consider the six-player assignment economy depicted in Panel I of Figure 9, in which agents a and f propose with probability 1/4, whereas all other players propose with probability 1/8. In such an example, the efficient matches are pinned down by the value of parameter y. We consider values of $y \in [2, 3]$.

Figure 9: In Panel I the assignment economy; in Panel II agreement probabilities.

For all values of $y \in [2, 3]$, Rubinstein payoffs do not belong to the core in any core match. We consider whether there can be an equilibrium in which a and f delay making an offer. Suppose that agents c and d agree with each other when proposing. If so, by delaying agent a may end up bargaining bilaterally with agent b provided that players c or d are selected before either b or e. As in this scenario a ends up in a strong position vis-a-vis b, player a could in principle prefer delay. To explore this possibility, we assume that agents a and f delay with probability $1 - q$, and we look for conditions on q and y under which there is an equilibrium with the agreement probabilities shown in Panel II of Figure 9. Finding agents' MPE values in the relevant subgames and taking the limit yields

$$\begin{aligned} \lim\nolimits_{\delta \to 1} V_a(N) = \lim\nolimits_{\delta \to 1} V_f(N) &= \frac{16 + q(17 - 3y)}{24 + 12q}, \\ \lim\nolimits_{\delta \to 1} V_b(N) = \lim\nolimits_{\delta \to 1} V_e(N) &= \frac{7 + 3y}{12}, \\ \lim\nolimits_{\delta \to 1} V_c(N) = \lim\nolimits_{\delta \to 1} V_d(N) &= 1. \end{aligned}$$

All values are strictly positive for any $y \in [2, 3]$ and any $q \in [0, 1]$. Moreover, $\partial V_a/\partial q > 0$ for $y < 3$, but $\partial V_a/\partial q = 0$ for $y = 3$. Thus, player a and f do not delay and set $q = 1$ for $y < 3$. But, there might be equilibrium delay for $y = 3$. In fact, an equilibrium exists in which $q = 0$ when $y = 3$. This discontinuity arises because multiple matches are efficient when $y = 3$. Although agent a delays, there is an efficient match in which he is unmatched. With heterogeneities, instances of multiple efficient matches are non-generic. When the core match is unique, delay occurs only because of fundamental strategic reasons, as we documented in Examples 3 and 4.

3 Relationship to Okada (2011)

There are some similarities between Okada (2011) and our paper. Both papers relate the existence of an efficient MPE in a non-cooperative bargaining game to whether different statistics belong to the core of an associated cooperative game. Nevertheless, the models are significantly different in a crucial dimension. Okada models coalitional bargaining, while we allow only pairs of players to bargain. The models are geared towards different applications (legislative bargaining for Okada, while decentralized markets in our case) and, in this section, we argue that applying Okada's model to decentralized markets may lead to strange predictions.

Consider the 4-player example shown in Figure 10. In terms of Okada's notation, this is coalitional game with $N = \{a, b, c, d\}$ and

(1) $v(N) = s_{ab} + s_{cd}$,

(2) $v(a, b) = v(a, b, c) = v(a, b, d) = s_{ab}$,

(3) $v(c, d) = v(a, c, d) = v(b, c, d) = s_{cd}$,

(4) $v(S) = 0$ for any other coalition $S \subset N$.

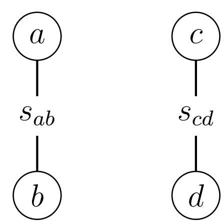

Figure 10: A Four Player Economy

For an efficient equilibrium as defined by Okada, each agent must make an acceptable proposal to the grand coalition with probability 1 if selected as proposer. Unlike in our model, this option is available to agents and by offering to grand coalition all players can reach an outcome immediately eliminating any losses from agents' limited patience. By Okada's Proposition 3.1, in an efficient stationary equilibrium, the expected payoffs are given by the solution to the following system of value equations

$$V_i = p_i \left[v(N) - \delta \textstyle\sum_{j \in N \setminus i} V_j \right] + \delta V_i \textstyle\sum_{j \in N \setminus i} p_j \quad \text{for all } i \in N.$$

In the limit as $\delta \to 1$, this yields expected payoffs

$$\lim_{\delta \to 1} V_i = \left(\frac{p_i}{p_a + p_b + p_c + p_d} \right) v(N) \quad \text{for all } i \in N.$$

Moreover, Okada shows that an efficient equilibrium only exists if these payoffs belong to the core of the associated cooperative game. In the limit, two necessary conditions for an efficient equilibrium are

$$V_i + V_{\eta(i)} \geq v(i, \eta(i)) \geq s_{i\eta(i)} \text{ for all } i \in N.$$

Substituting the efficient payoff characterization and rearranging the conditions simplify to

$$(p_a + p_b)s_{cd} \geq (p_c + p_d)s_{ab} \text{ and } (p_c + p_d)s_{ab} \geq (p_a + p_b)s_{cd}.$$

But if so, an efficient MPE only exists if $(p_a + p_b)s_{cd} = (p_c + p_d)s_{ab}$. This condition is a knife-edge. Indeed, even if the condition was satisfied, any perturbation to the surpluses by some small independent noise terms (drawn from continuous distributions) would lead to the condition being violated with probability 1. The knife-edge nature of the condition is not an artifact of the of the example, but a general feature of Okada's setting in the context of assignment economies which implies that efficient outcomes are very unlikely to occur with multilateral negotiations. Intuitively, having to agree with all players, imposes further constraints on agreeable outcomes and restrict the scope for efficient negotiations. In contrast in our setting, a strongly (and thus weakly) efficient MPE would exist for any values of $(p_a, p_b, p_c, p_d, s_{ab}, s_{cd})$, as Rubinstein payoffs would belong to the core for any such parameter values. Intuitively with bilateral negotiations, non-core partners cannot affect bargaining outcomes and constrain efficiency when they generate no surplus with their alternative partners.

The example highlights the differences in the approach and the conclusions relative to Okada (2011). His model most suitable for situations in which coalitions can jointly bargain. In contrast, ours is intended to capture decentralized markets in which buyer-seller pairs bargain in solitude. When this is the case, decentralized negotiations may actually lead to more efficient and arguably more plausible outcomes.

4 Omitted Proofs

Proof of Remark 1. First, establish part (a). By assumption there is a unique preferred match at any active player set. Thus, for all $i \in A$ and all $A \subseteq N$, if $\max_{j\in A} s_{ij} > 0$ then $\text{argmax}_{j\in A} s_{ij}$ is a singleton. Moreover, i's continuation value when selected as the proposer satisfies

$$\lim_{\delta\to 0} v_i(A) = \lim_{\delta\to 0} \max\{\delta V_i(A), \max_{j\in A\setminus i} \{s_{ij} - \delta V_j(A)\}\} = \max_{j\in A\setminus i} s_{ij},$$

as $V_j(A) < \max_{k\in A} s_{jk} < \infty$ for all players $j \in A$ and active player sets $A \subseteq N$. Hence, in all MPE for all δ close to 0, $\pi_{ij}(A) = 1$ if and only if $j = \text{argmax}_{j\in A} s_{ij}$. If we have that $\max_{k\in A} s_{ik} > 0$ at some active player set $A \subseteq N$, then there is a unique player $j =$

$\text{argmax}_{k\in A}\, s_{ik}$ and for all δ close to zero

$$\max_{k\in A}(s_{ij} - s_{ik}) > \delta \sum_{k\in A} s_{k\eta(k)}.$$

Thus, independently of the constraints imposed on subsequent matching, expected social surplus is maximized by matching agent i to agent j, if agent i is selected as the proposer. Maximizing utilitarian welfare for δ all close to 0 simply amounts to setting $\pi_{ij}(A) = 1$ if and only if $j = \text{argmax}_{j\in A}\, s_{ij}$. So, all MPE maximize utilitarian welfare for all δ close to zero.[1]

Now, establish part (b). Payoffs in any subgame $A \in C(N)$ of an efficient MPE are pinned down by Proposition 2 for any $\delta \in (0, 1)$. We show that, if $s_{i\eta(i)} > s_{ij}$ for all $i \neq j$, complying with efficient strategies is an equilibrium when δ is sufficiently low. Recall that any player $j \in A$ accepts any offer that is worth at least $\delta V_j(A)$. Suppose, by contradiction, that some player $i \in A$ at some subgame $A \in C(N)$ has a profitable deviation which entails agreeing with $j \neq \eta(i)$ when all are playing efficient strategies. For such an offer to be profitable for player i, it must be that

$$s_{ij} - \delta V_j(A) \geq s_{i\eta(i)} - \delta V_{\eta(i)}(A). \tag{19}$$

By taking limits on both sides of the inequality as δ converges to 0, we obtain

$$s_{ij} \geq s_{i\eta(i)}.$$

But this cannot be as players strictly prefer their core match by assumption, $s_{ij} < s_{i\eta(i)}$. Thus, any player $i \in A$ at any subgame $A \in C(N)$ does not have a profitable deviation when the discount factor is sufficiently low, which implies the existence of an efficient MPE for any δ close to 0.

Next, establish part (c). By contradiction, assume that an efficient MPE exists for all δ close to 0, but that $s_{ij} > s_{i\eta(i)}$ for some $j \neq \eta(i)$. If so, player i has a strictly profitable deviation from an efficient equilibrium if condition (19) holds strictly. But, since $\delta V_i(A) \to 0$ and $\delta V_j(A) \to 0$, condition (19) must be strict for δ sufficiently low, and thus player i must have a profitable deviation. ■

Proof of Remark 2. To establish part (a), let u be a vector of core payoffs associated to the core match η. Consider two players $i, j \in N$ such that $\eta(i) = j$, and set

$$\frac{p_i}{p_j} = \frac{u_i}{s_{ij} - u_i}.$$

[1] If the preferred match is not unique, then a planner maximizing welfare may have preferences over preferred partners that differ from those of the proposer. Thus, for all δ close 0 there may be no welfare maximizing MPE.

This condition ensures that i and j receive their core payoffs, u_i and u_j, if everyone plays the strategies characterized in the proof of Proposition 2. This removes at most $N/2$ degrees of freedom from the vector p. Thus, it is straightforward to find a probability vector p that satisfies the above condition for all $i \in N$.

Part (b) is a trivial consequence of the Rubinstein payoffs not being affected by proportional changes in probabilities. Part (c) is also straightforward. Let $U(S)$ denote the set of core payoffs when the surplus matrix is S. Observe that if the surplus changes from S to S', it must be that $s_{i\eta(i)} = s'_{i\eta(i)}$ for any $i \in N$. This is because the core match cannot change when S changes to S', and because $s_{i\eta(i)} \neq s'_{i\eta(i)}$ implies that any core payoff in S would not belong to S' (since $u_i + u_{\eta(i)} = s_{i\eta(i)}$ for any $u \in U(S)$). Thus, Rubinstein payoffs in the two markets must coincide,

$$\sigma = (\sigma_1, ..., \sigma_n) = (\sigma'_1, ..., \sigma'_n) = \sigma'.$$

The conclusion then follows immediately from these observations, since $\sigma \in U(S) \subseteq U(S')$.

To prove part (d), it is useful to introduce the notions of an offer graph and a cyclical offer graph. For any subgame with active player set $A \subseteq N$ and any MPE, the *offer graph* (A, G) consists of a directed graph with vertices in A and with edges satisfying

$$ij \in G \quad \Leftrightarrow \quad i \in A \;\text{ and }\; j \in \{k \mid \pi_{ik}(A) > 0\} \cup \eta(i).$$

We say that an offer graph is cyclical whenever there exists a subset of active players choosing to make offers so as to exchange their respective core partners with one another. Formally, an offer graph is *cyclical* if there exists a map $\varphi : N \to N$ and a set of players $F \subseteq P_k \cap A$ for $k \in \{1, 2\}$ such that

$$\begin{aligned}
&(1)\ \varphi(i) = j \quad \Rightarrow \quad ij \in G, \\
&(2)\ \varphi(i) \neq \eta(i) \;\text{ for some } i \in F, \\
&(3)\ \{\varphi(i) | i \in F\} = \{\eta(i) | i \in F\}.
\end{aligned}$$

Next, we establishes that MPE offer graphs are never cyclical. If offers were cyclical, a subset of players who prefer offering to one another's core matches instead of their own core match would exist. These players would have to achieve a higher aggregate surplus by matching with non-core partners, thereby violating the efficiency properties of the core. Formally, suppose the offer graph is cyclical. By revealed preferences for any player $i \in F$ and $\varphi(i)$ such that $\pi_{i\varphi(i)}(A) > 0$, subgame perfection requires that

$$s_{i\varphi(i)} - \delta V_{\varphi(i)} \geq s_{i\eta(i)} - \delta V_{\eta(i)}.$$

Furthermore, because of cyclicality, by summing over all players in F we would have that

$$\textstyle\sum_{i\in F}(s_{i\varphi(i)} - \delta V_{\varphi(i)}) \geq \sum_{i\in F}(s_{i\eta(i)} - \delta V_{\eta(i)}) \quad \Leftrightarrow \quad \sum_{i\in F} s_{i\varphi(i)} \geq \sum_{i\in F} s_{i\eta(i)}.$$

However, this leads to a contradiction as core match was assumed to be be unique.

Next, we establish that the core match always obtains with positive probability in an MPE without delay. The uniqueness of the core match and the non-negativity of surpluses imply that all players on one side of the market are matched at the unique core allocation.[2] Fix an MPE without delay. No delay implies that every player with a positive value agrees with probability 1 when selected to propose in every possible subgame. Without loss of generality, suppose that $P_1 \cap A \geq P_2 \cap A$. If for any A there exists $i \in P_1 \cap A$ such that $\pi_{i\eta(i)}(A) > 0$, the conclusion obviously holds. Thus assume that this is not the case. Then for some A, $\pi_{i\eta(i)}(A) = 0$ for all $i \in P_1 \cap A$. Next, we show that this leads to a contradiction, as the offer graph would necessarily be cyclical. Pick any match φ satisfying $\varphi(i) = j$ for $\pi_{ij}(A) > 0$, and $\varphi(i) \neq \eta(i)$ for any $i \in P_1 \cap A$. Such a match exists because players in $P_1 \cap A$ do not delay, and because $\pi_{i\eta(i)}(A) = 0$. Observe that, since the core match is unique, $P_2 = \{\eta(i)|i \in P_1 \cap A\} \cap P_2$. Furthermore, by construction it must be that $P_2 \supseteq \{\varphi(i)|i \in P_1 \cap A\} \cap P_2$. Since $\eta(i) \neq \eta(k)$ for any $i, k \in P_1 \cap A$, there must exist a set $F \subseteq P_2$ such that

$$\{\varphi(i)|i \in F\} = \{\eta(i)|i \in F\},$$

as otherwise a player $i \in P_1 \cap A$ would exist such that $\varphi(i) = \eta(i)$. This in turn implies the desired contradiction to the first part of the proposition, as the offer graph would necessarily be cyclical. ■

Proof of Remark 3. For convenience, when $A = N$, value functions and proposal probabilities omit the dependence on the active player set A. First observe that players on one of the two core matches never delay in any weakly efficient LMPE for all δ close to 1. Delay on both core matches would require

$$\delta V_a + \delta V_b \geq s_{ab} \quad \text{and} \quad \delta V_c + \delta V_d \geq s_{cd}, \tag{20}$$

which violates feasibility as $\sum_{i\in N} V_i > s_{ab} + s_{cd}$. Thus, in any weakly efficient LMPE there exists a core match in which no player delays. Call such a match $i\eta(i)$ so that $\pi_{ii} + \pi_{\eta(i)\eta(i)} = 0$. Next observe that players agree at most on one of the two non-core with positive probability in any weakly efficient LMPE for all δ close to 1. Agreement on both non-core matches would

[2] This is the only result in which the assumption on non-negativity of the surplus is substantive.

require

$$\delta V_a + \delta V_d = s_{ad} \text{ and } \delta V_b + \delta V_c = s_{bc}.$$

But this would violate the weak efficiency of the limiting equilibrium as

$$\lim_{\delta\to 1} \sum\nolimits_{k\in N} V_k = s_{ad} + s_{bc} < s_{ab} + s_{cd}.$$

Thus, in any weakly efficient LMPE there exists a non-core match with disagreement. As this link must involve either i or $\eta(i)$ it is without loss of generality to call such a match ij, so that $\pi_{ij} = 0$. This establishes that $\pi_{i\eta(i)} = 1$ and that $\pi_{jj} = 1$. Furthermore, there must be agreement in match $\eta(i)\eta(j)$. If instead we had that $\pi_{\eta(i)\eta(j)} + \pi_{\eta(j)\eta(i)} = 0$, value equation for a player $k \in \{j, \eta(j)\}$ would simplify to

$$V_k = (1-2p)\delta V_k + 2p\delta V_k(\{j,\eta(j)\}) = (1-2p)\delta V_k + 2p\delta\sigma_k.$$

Thus, $\delta V_j + \delta V_{\eta(j)} < s_{i\eta(j)}$ and the equilibrium would be strongly efficient and not sequential. Thus, $\pi_{\eta(i)\eta(j)} + \pi_{\eta(j)\eta(i)} > 0$. Finally, observe that $\pi_{\eta(i)\eta(j)} = 0$. Otherwise,

$$s_{i\eta(i)} - \delta V_i = s_{\eta(j)\eta(i)} - \delta V_{\eta(j)} = \delta V_{\eta(i)},$$

where the first equality would hold by player $\eta(i)$'s indifference, while the latter by player $\eta(j)$'s indifference. This implies that $\delta V_i + \delta V_{\eta(i)} \geq s_{i\eta(i)}$. But, as j and $\eta(j)$ delay, the condition (20) would be satisfied and the values would be infeasible. Thus, we must have that $\pi_{\eta(i)i} = 1$ for all δ close to 1. This completely pins down the acceptance probabilities up to relabelling, and consequently, for $\pi_{\eta(j)\eta(i)} = q$, the value equations reduce to

$$\begin{aligned}
s_{\eta(i)\eta(j)} &= \delta V_{\eta(i)} + \delta V_{\eta(j)} \\
V_{\eta(i)} &= (1-p)\delta V_{\eta(i)} + p(s_{i\eta(i)} - \delta V_i) \\
V_{\eta(j)} &= (1-2p)\delta V_{\eta(j)} + 2p\delta V_{\eta(j)}(\{j,\eta(j)\}) \\
V_i &= (1-p-pq)\delta V_i + pq\delta V_i(\{i,j\}) + p(s_{i\eta(i)} - \delta V_{\eta(i)}) \\
V_j &= (1-2p-pq)\delta V_j + pq\delta V_j(\{i,j\}) + 2p\delta V_j(\{j,\eta(j)\})
\end{aligned} \tag{21}$$

where obviously for any $k, l \in N$, we have that

$$V_k(kl) = \frac{p}{1-\delta+2p\delta} s_{kl}.$$

First observe that $\eta(j)$'s value equation trivially implies that $V_{\eta(j)} \leq V_{\eta(j)}(\{j,\eta(j)\})$ for all $\delta \leq 1$. As j delays when $A = N$, $s_{j\eta(j)} - \delta V_j \leq \delta V_{\eta(j)}$. Thus $s_{j\eta(j)} \leq \delta V_j + \delta V_{\eta(j)}(\{j,\eta(j)\})$.

Towards a contradiction, suppose that $\delta V_j < \delta V_j(\{j, \eta(j)\})$. Then $s_{j\eta(j)} < \delta V_j(\{j, \eta(j)\}) + \delta V_{\eta(j)}(\{j, \eta(j)\})$ and $\eta(j)$ would have a profitable deviation delaying instead of offering to j in the subgame where only j and $\eta(j)$ are active. We therefore conclude that $\delta V_j \geq \delta V_j(\{j, \eta(j)\})$. From j's value function, this implies that $V_j(\{i, j\}) \geq V_j(\{j, \eta(j)\})$. Moreover, with equal proposal probabilities, this is equivalent to $s_{ij} \geq s_{j\eta(j)}$. By adding this inequality to the inequality defining the core match, $s_{i\eta(i)} + s_{j\eta(j)} > s_{\eta(i)\eta(j)} + s_{ij}$, we further obtain that $s_{i\eta(i)} > s_{\eta(i)\eta(j)}$.

In any sequential LMPE $\lim_{\delta\to 1} q = 0$. Taking limits of value equations (21) as $\delta \to 1$, immediately delivers that

$$\begin{aligned} &\lim_{\delta\to 1} V_{\eta(i)} = s_{\eta(i)\eta(j)} - \sigma_{\eta(j)} && \lim_{\delta\to 1} V_{\eta(j)} = \sigma_{\eta(j)} \\ &\lim_{\delta\to 1} V_i = s_{i\eta(i)} - s_{\eta(i)\eta(j)} + \sigma_{\eta(j)} && \lim_{\delta\to 1} V_j = \sigma_j \end{aligned}$$

Now observe that player $\eta(i)$ always possesses a deviation that sets $q = 0$ (namely rejecting any offer from $\eta(j)$ when $A = N$). If so, i's and $\eta(i)$'s value functions reduce to

$$\hat{V}_i = (1-p)\delta\hat{V}_i + p(s_{i\eta(i)} - \delta\hat{V}_{\eta(i)})$$
$$\hat{V}_{\eta(i)} = (1-p)\delta\hat{V}_{\eta(i)} + p(s_{i\eta(i)} - \delta\hat{V}_i)$$

and $\eta(i)$ secures a payoff $\hat{V}_{\eta(i)} = \frac{p}{1-\delta+2p\delta} s_{i\eta(i)} \to \sigma_{\eta(i)}$. For $q > 0$ to be an equilibrium for all δ close to 1 such a deviation cannot be profitable. Thus, $V_{\eta(i)} \geq \hat{V}_{\eta(i)}$ for all δ close to 1, and

$$\lim_{\delta\to 1} V_{\eta(i)} = s_{\eta(i)\eta(j)} - \sigma_{\eta(j)} = s_{\eta(i)\eta(j)} - (s_{j\eta(j)}/2) \geq s_{i\eta(i)}/2 = \sigma_{\eta(i)} = \lim_{\delta\to 1} \hat{V}_{\eta(i)}.$$

This implies that $2s_{\eta(i)\eta(j)} \geq s_{j\eta(j)} + s_{i\eta(i)}$, which by efficiency and uniqueness of the core immediately implies that $s_{\eta(i)\eta(j)} > s_{ij}$. We thus conclude that

$$s_{i\eta(i)} > s_{\eta(i)\eta(j)} > s_{ij} \geq s_{j\eta(j)},$$

and, invoking our labelling conventions, that $\eta(i) = a$, $i = b$, $j = c$, and $\eta(j) = d$.

To establish the final part of the result, we first find necessary conditions for the existence of a sequential LMPE, and then show that these conditions are also sufficient. Recall that the previous part of the proof establishes that a sequential LMPE exists only if

$$s_{ab} > s_{ad} > s_{bc} \geq s_{cd}. \tag{22}$$

For the proposed strategy profile to be an equilibrium c and d weakly prefer to delay instead of offering to each other and so $\delta V_c + \delta V_d \geq s_{cd}$ for all δ sufficiently close to 1. Moreover,

$\lim_{\delta\to 1} \delta(V_c + V_d) = s_{cd}$. Thus the strategy is consistent with equilibrium behavior only if $\delta(V_c + V_d)$ to converges to s_{cd} from above. By solving value equations (21) it is possible to show that

$$\lim_{\delta\to 1} \frac{\delta(V_c + V_d) - s_{cd}}{1-\delta} = \frac{s_{cd}(s_{bc} - s_{cd}) + 2s_{ad}(s_{bc} + s_{cd}) - s_{ab}(s_{bc} + 3s_{cd})}{2p[2(s_{ab} - s_{ad}) - (s_{bc} - s_{cd})]}. \tag{23}$$

If $s_{bc} = s_{cd}$ then the right hand side of equation (23) reduces to $-s_{cd}/p < 0$ which is not consistent with equilibrium behavior. Thus, $s_{bc} > s_{cd}$. Next observe that the denominator in equation (23) must be positive since $s_{ab} - s_{ad} > 0$ by (22) and since $s_{ab} - s_{ad} > s_{bc} - s_{cd}$ by definition of the core. Thus, as the denominator is always positive, equation (23) is satisfied if and only if the numerator is also positive. This requires that

$$\frac{s_{bc} - s_{cd}}{s_{ab} - s_{ad}} \geq 2\frac{s_{bc} + s_{cd}}{s_{ab} + s_{cd}}. \tag{24}$$

The first part of the proof also establishes that a strategy is consistent with weak efficiency only if $2s_{ad} \geq s_{ab} + s_{cd}$. However, if $s_{ad} = (s_{ab} + s_{cd})/2$, by substituting s_{ad} in (24) one obtains

$$\frac{s_{bc} - s_{cd}}{s_{ab} - s_{cd}} \geq 4\frac{s_{bc} + s_{cd}}{s_{ab} + s_{cd}},$$

which with some rearrangement in turn implies that

$$0 \geq 3\left(s_{ab} - s_{cd}\right)\left(s_{bc} + s_{cd}\right) + 2s_{cd}\left(s_{ab} - s_{bc}\right),$$

which cannot be by (22). Thus, $2s_{ad} > s_{ab} + s_{cd}$. Combining the above inequalities establishes that

$$s_{ab} > s_{ad} > (s_{ab} + s_{cd})/2 > s_{bc} > s_{cd}.$$

This establishes why the above condition is necessary for the existence of a sequential LMPE.

To show this condition is sufficient we verify that no player can have a profitable deviation given the agreement probabilities pinned down in the first part of the proof. First observe that c and b prefer delaying to offering to each other as

$$\lim_{\delta\to 1} \delta(V_b + V_c) = s_{ab} - s_{ad} + \sigma_d + \sigma_c = s_{ab} + s_{cd} - s_{ad} > s_{bc},$$

where the last inequality holds by the uniqueness of the efficient match. By construction, d is indifferent about offering to a or delaying. Players c and d weakly prefer delaying to offering to each other as argued earlier in the proof. Players a and b weakly prefer offering to each other than delaying as

$$\lim_{\delta\to 1}\frac{\delta(V_a+V_b)-s_{ab}}{1-\delta}=\frac{s_{cd}-2s_{ad}}{2p}<0,$$

which implies that $\delta V_a+\delta V_b\leq s_{ab}$ for all δ close to 1. Thus, for sufficiently high δ, a and b prefer offering to each other than delaying. As we have already established that b prefers delaying to offering to c, b's optimal offer strategy is to offer to a with probability 1 for all δ close to 1. Player a prefers offering to b than offering to d as

$$\lim_{\delta\to 1}\frac{s_{ab}-\delta V_b-s_{ad}-\delta V_d}{1-\delta}=\frac{2s_{ad}-s_{cd}}{2p}>0.$$

Thus, it is optimal for a to offer to d with probability 1. Finally, mixing probabilities are consistent with a weakly efficient LMPE as the probability that d and a agree converges to zero from above by

$$\lim_{\delta\to 1}\frac{q}{1-\delta}=\frac{2(2s_{ad}-s_{ab}-s_{cd})}{p(2s_{ab}-2s_{ad}-s_{bc}+s_{cd})}>0,$$

where the inequality holds as the numerator is positive by $2s_{ad}>s_{ab}+s_{cd}$, while the denominator is positive by $s_{ab}-s_{ad}>0$ and $s_{ab}-s_{ad}>s_{bc}-s_{cd}$. All players thus best respond for δ close to 1, and so the condition we needed to show is sufficient for the existence of a sequential LMPE is indeed sufficient. ■

Proof of Proposition 7. Any simple assignment economy S can be represented as an unweighted bipartite network $L\subseteq P_1\times P_2$ in which links capture the opportunity to generate a unit surplus. For any component of the network $\hat{L}\subseteq L$, let $\hat{L}_k\subseteq P_k$ denote the projection of $\hat{L}$ on P_k. The components of any such network must be of two types: (i) balanced components with the same number of players on both sides, $\left|\hat{L}_1\right|=\left|\hat{L}_2\right|$; (ii) unbalanced components with more players on one side $k\in\{1,2\}$, $\left|\hat{L}_k\right|>\left|\hat{L}_r\right|$. We begin by invoking a result implied by conclusions from Corominas-Bosch (2004).

Remark (Corominas-Bosh 2004): Any unweighted bipartite network $L\subseteq P_1\times P_2$ possesses a sub-network $L'\subseteq L$ such that:
(a) any efficient match in L belongs to L';
(b) in unbalanced components of L', the unique core payoff of all players on the long side is 0 and that of all players on the short side is 1;
(c) in balanced components of L', all players one side receiving payoff $\beta\in[0,1]$ and all the remaining players receiving payoff $1-\beta$ is a core outcome.

By the assumptions on the economy immediately observe that Rubinstein payoffs are 1/2 on any efficient match. Thus, by the Corominas-Bosh Remark, these payoffs are in the core if and only if all players who have a neighbor in L belong to balanced components in the resulting sub-network L'. If so, pick any efficient match η. Suppose that any player $i \in N$ agrees with $\eta(i)$ with probability 1 when proposing in any equilibrium path active player set $A \in C^\eta(N)$. As in the proof of Proposition 2, these strategies, imply that, in any equilibrium-path subgames $A \in C^\eta(N)$, the continuation value of any player $i \in A$ satisfies

$$V_i(A) = \frac{p}{1-\delta+2\delta p},$$

where p denotes the proposal probability of the representative player. If so, player i has no strictly profitable deviation when proposing, since all other players have the same continuation value as $\eta(i)$ and since

$$2V_i(A) = \frac{2p}{1-\delta(1-2p)} < 1 \text{ for all } \delta < 1.$$

Thus, the constructed strategies are a MPE. ■

Proof of Remark 4. By Proposition 3 in the main document, a sufficient condition for the existence of an efficient MPE is that there exist no worker i and firm j who have a weakly profitable pairwise deviation when receiving their Rubinstein payoffs. As the efficient match is assortative, the core match of worker i is firm i. Thus, there is an assortative MPE if, for all $i \neq j$,

$$\frac{q_i}{p_i+q_i}S(i,i) + \frac{p_j}{p_j+q_j}S(j,j) > S(i,j),$$

where $S(k,k) = 0$ for all $k > \min w, f$.

If $w = f$, no agent is unmatched in the efficient match. Along with the condition that $p_i = q_i = p$, the above expression then simplifies to

$$S(i,i) + S(j,j) > 2S(i,j) = S(i,j) + S(j,i), \tag{25}$$

where the equality follows from the condition that $S(i,j) = S(j,i)$. The existence of an efficient MPE then follows, as condition (25) holds by the increasing differences assumption C3.[3]

To prove the second part of the remark, we show that there exist vertically differentiated markets for which there is no weakly efficiently LMPE whenever we relax one of three conditions in the statement of the result: (i) $w = f$; (ii) $p_i = q_i = p$ for all i; and (iii) $S(i,j) = S(j,i)$ for all $i, j \leq \min\{w, f\}$. We do so by relying on the earlier results as well as the following two lemmas (which are proven below).

[3] This also follows by applying results from Eeckhaut (2006).

Lemma 1 *There is no weakly efficient LMPE in any market $S \in \bar{\mathcal{S}}$ satisfying (ii) and (iii) if $w = 2$, $f = 3$, and*

$$\max\{S(1,1)/2, S(1,3)\} + S(2,2) - S(2,3) < S(1,2), \quad S(1,2)/2 < S(2,3).$$

Lemma 2 *There is no weakly efficient LMPE, if $w = f = 2$, $S(1,1) = 9$, $S(1,2) = S(2,1) = 6$, $S(2,2) = 4$, $p_1 = q_2 = 1/16$ and $p_2 = q_1 = 7/16$.*

Unbalanced Market: Lemma 1 identifies conditions on market $S \in \bar{\mathcal{S}}$ for the non-existence of weakly efficient LMPE in markets satisfying (ii) and (iii), but violating (i). What remains to be shown is that conditions are not vacuous and can be satisfied for some $S \in \bar{\mathcal{S}}$. Consider the economy

$$S(1,1) = 25;\ S(2,1) = S(1,2) = 20;\ S(2,2) = 16;\ S(1,3) = 12.$$

The economy trivially fulfills C1, C2 and C3; and thus $S \in \bar{\mathcal{S}}$. Moreover, we have that $S(1,2)/2 = 10 < S(2,3) = 12$ and

$$\max\{S(1,1)/2, S(1,3)\} + S(2,2) - S(2,3) = 19 < S(1,2) = 20.$$

Thus the economy S satisfies the conditions of Lemma 1 and no weakly efficient LMPE exists.

Heterogeneous Probabilities: Lemma 2 provides an example in which conditions (i) and (iii) are satisfied, but condition (ii) is violated and there does not exist a weakly efficient LMPE.

Asymmetric Surpluses: Finally, consider the case in which $S(i,j) \neq S(j,i)$. Setting $w = f = 2$, we appeal directly to Remark 3 for a characterization of when there is no weakly efficient LMPE. To do so, it suffices to observe that the surpluses can satisfy C1-C3 (and thus belong to $\bar{\mathcal{S}}$), while violating conditions for weak efficient LMPE existence identified in this result.

For convenience, in the proof of the next two lemmas, whenever $A = N$, we omit the dependence on the active player set A from value functions and proposal probabilities. ■

Proof of Lemma 1. By Proposition 3, an efficient MPE exists only if, for no worker-firm pair such that $i \neq j$, we have that

$$\frac{q_i}{p_i + q_i} S(i,i) + \frac{p_j}{p_j + q_j} S(j,j) < S(i,j),$$

where $S(k,k)=0$ for $k>\min\{f,w\}$. But (ii) implies that this condition must be violated as

$$S(2,2)/2 < S(1,2)/2 < S(2,3).$$

The first inequality holds by C1 and the second by assumption. By Proposition 4 there is no strongly efficient LMPE if shifted Rubinstein payoffs are not in the core. In this case, the profile of shifted Rubinstein payoffs are

$$\bar{\sigma}_1^w = \max\{S(1,1)/2, S(1,3)\} \text{ and } \bar{\sigma}_2^w = S(2,3);$$
$$\bar{\sigma}_1^f = S(1,1) - \bar{\sigma}_1^w, \ \bar{\sigma}_2^f = S(2,2) - \bar{\sigma}_2^w \text{ and } \bar{\sigma}_3^f = 0.$$

Thus, no strongly efficient LMPE exists as $S(1,2) > \bar{\sigma}_1^w + \bar{\sigma}_2^f$.

By Proposition 5, any weakly efficient LMPE that is not strongly efficient must be sequential LMPE. Next, we focus on ruling out the existence of a sequential LMPE. Recall that by the proof of Proposition 2 we have that, at any active player set $A \in C(N)$, a player $i \in A \backslash E$ agrees with positive probability in any weakly efficient LMPE for all sufficiently high δ. So, if only one worker is active at A, any weakly efficient LMPE is strongly efficient. So for the LMPE to be sequential one core match must delay in th limit when $A = N$.

Let $A = N$. Suppose that worker 1 and firm 1 delay with probability 1 in the limit, $\beta_1^w = \beta_1^f = 0$. If so, with probability 1, worker 1 and firm 1 end up in the subgame $B_1 \subset N$ in which worker 2 and firm 2 exit. In this subgame there is a unique MPE with limit payoffs

$$\bar{V}_1^w(B_1) = \max\{S(1,3), S(1,1)/2\} \text{ and } \bar{V}_1^f(B_1) = S(1,1) - \bar{V}_1^w(B_1).$$

As this subgame is reached with probability 1, $\bar{V}_1^w = \bar{V}_1^w(B_1)$ and $\bar{V}_1^f = \bar{V}_1^f(B_1)$. In a weakly efficient LMPE, the probability that worker 2 and firm 3 agree must converge zero. For worker 2 not to benefit by offering to firm 3, requires $\bar{V}_2^w \geq S(2,3)$. By Proposition 2, we also know that a weak efficient LMPE would further require $\bar{V}_2^w + \bar{V}_2^f = S(2,2)$. But for these conditions to hold at once we would have that

$$S(2,2) - S(2,3) \geq \bar{V}_2^f.$$

Finally, in a weakly efficient LMPE, worker 1 must prefer delaying than offering to firm 2 for δ sufficiently high, which requires

$$\bar{V}_1^w + \bar{V}_2^f \geq S(1,2).$$

Combining these observations we find that

$$\max\{S(1,3), S(1,1)/2\} + S(2,2) - S(2,3) \geq \bar{V}_1^w + \bar{V}_2^f \geq S(1,2),$$

which contradicts the assumption in the statement of our result. Thus, there is no sequential LMPE in which worker and firm 1 delay.

Next suppose instead that worker 2 and firm 2 delay with probability 1 in the limit, $\beta_2^w = \beta_2^f = 0$. If so, with probability 1, worker 2 and firm 2 end up in the subgame $B_2 \subset N$ in which worker 1 and firm 1 exit. In this subgame there is a unique MPE with limit payoffs

$$\bar{V}_2^w(B_2) = S(2,3) \text{ and } \bar{V}_2^f(B_2) = S(2,2) - \bar{V}_2^w(B_2).$$

As firm 3 delays with positive probability, $V_3^f(B_2) = 0$ for all δ sufficiently high. As this subgame is reached with probability 1, $\bar{V}_2^w = \bar{V}_2^w(B_2)$, $\bar{V}_2^f = \bar{V}_2^f(B_2)$, and $\bar{V}_3^f = \bar{V}_3^f(B_2)$. Suppose that there is a weakly efficient LMPE in which firm 3 and worker 1 agree with positive probability for all sufficiently high $\delta < 1$. If so, $S(1,3) \geq \delta(V_1^w + V_3^f)$. But in the limit, this implies that $\bar{V}_1^w = S(1,3)$. If so however, worker 1 would benefit by offering to firm 2 with strictly positive probability in the limit by C3 as

$$S(1,2) - \bar{V}_2^f = S(1,2) - S(2,2) + S(2,3) > S(1,3) = \bar{V}_1^w.$$

This contradicts the premise that this is a weakly efficient LMPE. Thus, firm 3 and worker 1 must reach agreement with probability 0 when all players are active for all sufficiently high δ.

Consider now the subgame in which worker 1 and firm 2 are not active. If so, for all δ sufficiently high, we have that worker 2's continuation value is $S(2,3)/\delta$. The latter follows because by assumption we have that $S(2,3) > S(2,1)/2$ and because in the unique MPE of this subgame firm 3 must mix between delaying and agreeing with worker 2 for all sufficiently high δ (as in Example 2 in the main document). Similarly, in the subgame in which worker 1 and firm 1 are not active, worker 2's continuation value is $S(2,3)/\delta$. Next consider the value equation of worker 2 when all players are active, and recall that $\pi_{13}^w = \pi_{31}^f = 0$ for all sufficiently high δ. As we have characterized the value equation of worker 2 in every other subgame, the value equation at N simplifies to

$$V_2^w = p(\pi_{11}^f + \pi_{21}^f + \pi_{11}^w + \pi_{12}^w)S(2,3) + (1 - p(\pi_{11}^f + \pi_{21}^f + \pi_{11}^w + \pi_{12}^w))\delta V_2^w$$

From this, we conclude that for all sufficiently high $\delta < 1$, $\bar{V}_2^w < S(2,3)$. However, for worker

2 to delay with positive probability in the limit, it must be that for all sufficiently high $\delta < 1$

$$\delta(\bar{V}_3^f + \bar{V}_2^w) \geq \delta \bar{V}_2^w \geq S(2,3)$$

which is a contradiction. So, there can be no sequential LMPE. ■

Proof of Lemma 2. As $w = f = 2$, shifted Rubinstein payoffs are the same as their Rubinstein payoffs. Moreover, these payoffs do not belong to the core as, by the previous inequalities, $pS(1,1) + pS(2,2) < S(1,2)$. So, there are no efficient MPE by Propositions 3, and no strongly efficient LMPE by Proposition 4. If so, any weakly efficient LMPE must be sequential by Proposition 5. Thus, we establish there is no sequential LMPE. For notational ease, denote unconditional link-agreement probabilities as

$$\upsilon_{ij} = p_i \pi_{ij}^w + p_j \pi_{ji}^f \text{ for any } (i,j) \in W \times F.$$

First observe that in any weakly efficient LMPE that is not a (weakly) efficient MPE, we must have $\max\{\upsilon_{21}, \upsilon_{12}\} > 0$ for all $\delta < 1$ sufficiently high. Moreover, $\min\{\upsilon_{21}, \upsilon_{12}\} = 0$ for all $\delta < 1$ sufficiently high. If $\upsilon_{21} > 0$, then one of the following two conditions would hold:

$$\text{(a1)} \;\; S(1,2) - \delta V_2^w \geq S(1,1) - \delta V_1^w; \text{ (a2)} \;\; S(1,2) - \delta V_1^f \geq S(2,2) - V_2^w.$$

If $\upsilon_{12} > 0$, then one of the following two conditions would hold:

$$\text{(b1)} \;\; S(1,2) - \delta V_1^w \geq S(2,2) - \delta V_2^w; \text{ (b2)} \;\; S(1,2) - \delta V_2^f \geq S(1,1) - \delta V_1^f.$$

If (a1) and (b1) held at once, then summing inequalities would yield a contradiction, as

$$2S(1,2) \geq S(1,1) + S(2,2).$$

But, as the same argument applies when (a1) and (b2) hold, or when (a2) and (b1) hold, or when (a2) and (b2) hold, it must be that $\min\{\upsilon_{21}, \upsilon_{12}\} = 0$.

Define the following two-player active player sets, where the first entry denotes the active buyer and the second the active seller:

$$B_1 = \{2,2\}, \;\; B_2 = \{2,1\}, \;\; B_3 = \{1,1\}, \;\; B_4 = \{1,2\}.$$

In the unique MPE of each of these subgames we have that

$$
\begin{aligned}
V_2^w(B_1) &= \frac{S(2,2)}{8(2-\delta)} \to \frac{1}{2} = \bar{V}_2^w(B_1) \ \ \& \ \ V_2^f(B_1) = \frac{7S(2,2)}{8(2-\delta)} \to \frac{7}{2} = \bar{V}_2^f(B_1);\\
V_2^w(B_2) &= \frac{S(1,2)}{2(8-7\delta)} \to 3 = \bar{V}_2^w(B_2) \ \ \& \ \ V_1^f(B_2) = \frac{S(1,2)}{2(8-7\delta)} \to 3 = \bar{V}_1^f(B_2);\\
V_1^w(B_3) &= \frac{7S(1,1)}{8(2-\delta)} \to \frac{63}{8} = \bar{V}_1^w(B_3) \ \ \& \ \ V_1^f(B_3) = \frac{S(1,1)}{8(2-\delta)} \to \frac{9}{8} = \bar{V}_1^f(B_3);\\
V_1^w(B_4) &= \frac{7S(2,1)}{2(8-\delta)} \to 3 = \bar{V}_1^w(B_4) \ \ \& \ \ V_2^f(B_4) = \frac{7S(2,1)}{2(8-\delta)} \to 3 = \bar{V}_2^f(B_4).
\end{aligned}
$$

In any sequential LMPE one of the two core pairs agrees in the limit, while the other does not by Propositions 2 and 5. Thus, four possible cases must be considered for δ sufficiently high:

(A) $\pi_{21}^f + \pi_{22}^f < 1$, $\pi_{21}^w + \pi_{22}^w < 1$, and $\pi_{21}^f + \pi_{12}^w = 0$.

(B) $\pi_{11}^f + \pi_{12}^f < 1$, $\pi_{11}^w + \pi_{12}^w < 1$, and $\pi_{21}^f + \pi_{12}^w = 0$.

(C) $\pi_{21}^f + \pi_{22}^f < 1$, $\pi_{21}^w + \pi_{22}^w < 1$, and $\pi_{12}^f + \pi_{21}^w = 0$.

(D) $\pi_{11}^f + \pi_{12}^f < 1$, $\pi_{11}^w + \pi_{12}^w < 1$, and $\pi_{12}^f + \pi_{21}^w = 0$.

Case A: For sufficiently high $\delta < 1$, the value equations of worker 2 and firm 2 amount to

$$
\begin{aligned}
V_2^w &= (1-\upsilon_{11})\,\delta V_2^w + \upsilon_{11}\delta V_2^w(B_1),\\
V_2^f &= (1-\upsilon_{11}-\upsilon_{21})\,\delta V_2^f + \upsilon_{11}\delta V_2^w(B_1) + \upsilon_{21}\delta V_2^w(B_4).
\end{aligned}
$$

Rearranging the first of these equations implies that $V_2^w < \delta V_2^w(B_1)$, while the second equation implies that $V_2^f < \delta V_2^f(B_1)$ where the latter holds for δ high enough as $\bar{V}_2^f(B_1) > \bar{V}_2^f(B_4)$. We therefore have that for all sufficiently high $\delta < 1$,

$$
V_2^w + V_2^f < \delta(V_2^w(B_1) + V_2^f(B_1)) < S(2,2).
$$

But, as worker 2 and firm 2 delay in the limit, we must have $\delta(V_2^w + V_2^f) \geq S(2,2)$ for all sufficiently high $\delta < 1$ which is a contradiction. So, there is no such weakly efficient LMPE.
Case B: The argument here is identical to that for Case A. Writing out the value equations for firm 1 and worker 1 and rearranging them, shows that $V_1^f < \delta V_1^f(B_3)$ and $V_1^w < \delta V_1^w(B_3)$. Combining these equations yields

$$
V_1^f + V_1^w < \delta(V_1^f(B_3) + V_1^w(B_3)) < S(1,1).
$$

But as firm 1 and worker 1 delay in the limit, $\delta(V_1^f + V_1^w) \geq S(1,1)$, which is a contradiction. So, there is no such weakly efficient LMPE.

Case C: Writing out the value equations we get:

$$V_1^w = q_1(S(1,1) - \delta V_1^f) + \upsilon_{22}\delta V_1^w(B_3) + (1 - q_1 - \upsilon_{22})\delta V_1^w, \tag{26}$$

$$V_1^f = p_1\left(S(1,1) - \delta V_1^w\right) + \upsilon_{22}\delta V_1^f(B_3) + \upsilon_{12}\delta V_1^f(B_2) + \left(1 - p_1 - \upsilon_{22} - \upsilon_{12}\right)\delta V_1^f, \tag{27}$$

$$V_2^w = \upsilon_{12}\delta V_2^w(B_2) + \upsilon_{11}\delta V_2^w(B_1) + \left(1 - \upsilon_{11} - \upsilon_{12}\right)\delta V_2^w, \tag{28}$$

$$V_2^f = \upsilon_{11}\delta V_2^f(B_1) + (1 - \upsilon_{11})\delta V_2^f, \tag{29}$$

In any sequential LMPE, it must be that either $\upsilon_{22} = 0$ for all $\delta < 1$ sufficiently large or $\lim_{\delta\to 1}\upsilon_{22} = 0$. Allowing for these possibilities, there are three subcases to be considered. For all $\delta < 1$ sufficiently high we could have (a) $\pi_{11}^w < 1$ and $\pi_{21}^f > 0$; (b) $\pi_{11}^w < 1$ and $\pi_{21}^f = 0$; or (c) $\pi_{11}^w = 1$ and $\pi_{21}^f > 0$. Further, if $\pi_{11}^w < 1$ then

$$S(1,2) - \delta V_2^f = S(1,1) - \delta V_1^f, \tag{30}$$

while if $\pi_{21}^f > 0$ then

$$S(1,2) - \delta V_1^w = \delta V_2^f. \tag{31}$$

In subcase (a), both worker 1 and firm 2 play a mixed strategy. If so, the conjectured equilibrium is pinned down by a system of value equations that includes (26-31). Substituting equations (28) and (29) into (30) and (31) eliminates V_2^f and V_2^w. Rearranging these new equations creates expressions for V_1^w and V_1^f in terms of just the mixing probability π_{11}^w. Substituting these expressions into equations (26) and (27) to eliminate V_1^w and V_1^f then gives a system of two equations in just the mixing probabilities (π_{11}^w, π_{21}^f, π_{22}^f and π_{22}^w). Using these equations to eliminate π_{21}^f, we get an expression for π_{11}^w in terms of just parameters, δ, π_{22}^f and π_{22}^w. Taking limits, and using that in a sequential LMPE we must have $\bar{\pi}_{22}^f + \bar{\pi}_{22}^w = 0$, we get that $\bar{\pi}_{11}^w = -\frac{1}{7}$. As mixing probabilities cannot be negative, this implies that we cannot have both worker 1 and firm 2 both offering to each other with positive probability.

In subcase (b), worker 1 plays a mixed strategy but firm 2 does not. If so, the conjectured equilibrium is pinned down by a system of value equations that includes (26-30). Equation (29) identifies V_2^f in terms of just parameters and $\bar{V}_2^f = 7/2$. Manipulating equations (28), (26) and (27), we get an expression for V_1^f in terms of just the mixing probabilities π_{11}^w, π_{22}^f and π_{22}^w. Moreover, using that $\bar{\pi}_{22}^f + \bar{\pi}_{22}^w = 0$ and $\bar{\pi}_{11}^w = 1$, we get that $\bar{V}_1^f = 3$. Combining these values with equation (30) yields a contradiction. So, we cannot have that worker 1 offers to firm 2, but firm 2 does not offer to worker 1.

In subcase (c), firm 2 plays a mixed strategy, but worker 1 does not. If so, the conjectured equilibrium is pinned down by a system of value equations that includes (26-29) and (31). Combining equations (26) and (27) to eliminate V_1^f gives an expression for V_1^w just in terms of just mixing probabilities. Substituting equation (29) into equation (31) to eliminate V_2^f gives a second expression for V_1^w just in terms of mixing probabilities. Using these two expression to eliminate V_1^w gives

$$\pi_{21}^f = -\frac{(7\delta-8)(\delta(83\delta-318)+192)(\delta(7\pi_{22}^f+\pi_{22}^w-16)+16)(\delta(7\pi_{22}^f+\pi_{22}^w-8)+16)}{7\delta(\delta(\delta(\delta\Phi+16(1701\pi_{22}^f+243\pi_{22}^w-5056))-768(14\pi_{22}^f+2\pi_{22}^w-99))-24576)}, \quad (32)$$

for $\Phi = 7\delta(581\pi_{22}^f + 83\pi_{22}^w - 708) - 20230\pi_{22}^f - 2890\pi_{22}^w + 34592$.

If so, there are two further possibilities to consider: $\upsilon_{22} > 0$ and $\upsilon_{22} = 0$. If $\upsilon_{22} > 0$, we have that $\delta(V_2^w + V_2^f) = S(2,2)$ as worker 2 and firm 2 must delay with positive probability for all sufficiently high $\delta < 1$ in any sequential LMPE. Substituting equations (29) and (28) into this expression to eliminate V_2^f and V_2^w, we get a second expression for π_{21}^f in terms of just parameters. Eliminating π_{21}^f by combining this equation with equation (32) yields an expression for π_{22}^w which is linear in π_{22}^f,

$$\pi_{22}^w = \Psi(\delta) - 7\pi_{22}^f.$$

Clearly, $\pi_{22}^w \leq \Psi(\delta)$ as $\pi_{22}^f \geq 0$. Minor manipulations then establish that $\lim_{\delta\to 1}\Psi(\delta) = 0$ and

$$\lim_{\delta\to 1} \partial\Psi(\delta)/\partial\delta > 0.$$

This implies that $\Psi(\delta) < 0$ for all sufficiently high $\delta < 1$, and thus that $\pi_{22}^w < 0$ for all sufficiently high $\delta < 1$. But, this is a contradiction, and so $\upsilon_{22} = 0$.

Setting $\upsilon_{22} = 0$, equation (32) simplifies to

$$\pi_{21}^f = \frac{32(\delta-2)(\delta-1)(7\delta-8)(\delta(83\delta-318)+192)}{7\delta(\delta(\delta(\delta(1239\delta-8648)+20224)-19008)+6144)}.$$

This implies that $\bar{\pi}_{21}^f = 0$ and that

$$\lim_{\delta\to 1} \partial\pi_{21}^f/\partial\delta > 0$$

Thus, for all sufficiently high values of $\delta < 1$, we would have $\pi_{21}^f < 0$. But again this is a contradiction as π_{21}^f is a probability, and so there is not sequential LMPE consistent with the proposal probabilities.

Case D: The argument is identical to that for Case C, but with worker 1 swapping roles with firm 2 and worker 2 swapping roles with firm 1. ■