Exact Simulation for a Class of Tempered Stable Distributions*

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Abstract

In this paper, we develop a new scheme of exact simulation for a class of tempered stable (TS) and other related distributions with similar Laplace transforms. We discover some interesting integral representations for the underlying density functions that imply a unique simulation framework based on a backward recursive procedure. Therefore, the foundation of this simulation design is very different from existing schemes in the literature. It works pretty efficiently for some subclasses of TS distributions, where even the conventional acceptance-rejection mechanism can be avoided. It can also generate some other distributions beyond the TS family. For applications, this scheme could be easily adopted to generate a variety of TS-constructed random variables and TS-driven stochastic processes for modelling observational series in practice. Numerical experiments and tests are performed to demonstrate the accuracy and effectiveness of our scheme.

Keywords: Monte Carlo simulation; Exact simulation; Backward recursive scheme; Stable distribution; Tempered stable distribution; Exponentially tilted stable distribution; Lévy process; Lévy subordinator; Leptokurtosis

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1 Introduction

From the empirical analysis, it is well known that asset returns in financial markets are not normally distributed, and the observations often present skewness and heavy tails, especially during crises. Stable distributions, first introduced by Lévy (1925), is a very important class that often provides an alternative model for price changes observed in financial markets. More precisely, the distributions form a four-parameter family of *infinitely divisible distributions* (Sato, 1999, p.31), which can be defined by the logarithm of the characteristic function,

$$\ln \phi(v) = \begin{cases} \mathrm{i}\ell v - \gamma^\alpha |v|^\alpha \Big(1 - \mathrm{i}\zeta \tan\left(\frac{\alpha}{2}\pi\right) \operatorname{sign}(v)\Big), & \alpha \neq 1, \\ \mathrm{i}\ell v - \gamma |v| \Big(1 + \mathrm{i}\zeta \frac{2}{\pi} \ln |v|\Big), & \alpha = 1, \end{cases} \qquad v \in \mathbb{R},$$

where $\phi(v)$ is the characteristic function of stable distributions, i is the imaginary unit, i.e. $\mathrm{i}^2 = -1$, $\alpha \in (0, 2]$ is the *stability index*, $\zeta \in [-1, 1]$ is the *skewness parameter*, $\gamma \in \mathbb{R}^+$ is the *scale parameter* and $\ell \in \mathbb{R}$ is the *location parameter*. Their great flexibility attracted many scholars in economics and finance at least since the 1960s, and the study was initialised by the seminal work of Mandelbrot (1960, 1961, 1963a,b) and followed by Fama (1965) and Fama and Roll (1968, 1971). Tremendous applications of stable distributions and their various extensions can be found in the literature (e.g. option pricing and portfolio management); see a detailed survey by McCulloch (1996).

However, stable distributions could involve infinite moments, and this property limits its applications. For instance, this model may lead to an infinite price in the option pricing; see Carr and Wu (2003). Therefore, the distribution, in particular its tail, needs to be adjusted. *Tempered stable* (TS), or, *exponentially tilted stable* (ETS) distribution[1] provides a very important class that attempts to overcome this limitation of the original stable distribution. It was initially proposed by Tweedie (1984); see also Hougaard (1986) and Jorgensen (1987). An additional parameter of *tilt* is included so that their moments exist. The TS law is a natural generalisation, as it inherits desirable features from the stable distribution, such as infinite divisibility, skewness and leptokurtosis. The resulting tails are lighter than those of the stable but heavier than those of the Gaussian, so it is more suitable for modelling financial returns. Moreover, for those distributions that are obtained by tilting *one-sided stable distributions*[2] with $\zeta = 1$, namely the *positive TS distributions*, they can be further adopted as the building blocks to construct various stochastic processes (such as Lévy TS subordinators and processes) that facilitate numerous applications in finance and many other

[1]Here, we adopt the naming for the TS distribution similarly as Küchler and Tappe (2013) and Kim and Kim (2016). In fact, the class of TS distributions has been recently enlarged by Rosiński (2007) and Grabchak et al. (2012) in a variety of specifications, so we consider them as *generalised TS distributions*.

[2]The stable distributions are one-sided when $\alpha \in (0, 1)$ and $\zeta = \pm 1$.

fields, e.g. for modelling the dynamics of stock prices, stochastic volatilities or interest rates just to name a few; see Barndorff-Nielsen and Shephard (2001a, 2002), Schoutens (2003), Cont and Tankov (2004), Kyprianou (2006), and more recently Barndorff-Nielsen et al. (2012).

Still, the challenging issue is that, with only certain exceptions, density functions of stable and TS distributions have no explicit expressions (Zolotarev, 1986, p.2). Hence, simulation-based methods, in particular exact simulation schemes without bias, become extremely important in practice, especially for the purposes of statistical inference, numerical validation, risk analysis and pricing exotic financial derivatives.

Let us first have a brief review of some important simulation schemes for stable, TS and their related distributions in the literature. For a stable distribution in general, the standard algorithm for exact simulation is the classical method of Chambers et al. (1976) (CMS)[3] which is the extension of the work by Kanter (1975). This state-of-the-art algorithm is remarkably elegant, accurate and fast, since the procedure is *direct* which means that even the acceptance-rejection (A/R) procedure is not required. However, for the TS distribution, it is usually hard to find such a *direct* scheme for exact simulation. The very popular algorithm is the classical *simple stable rejection* (SSR) scheme (with the algorithm provided in Appendix A). It is straightforward to be implemented, as it is developed by simply combining the classical CMS method with the A/R scheme; see Brix (1999) and Baeumer and Meerschaert (2010) for more details. More recently, Devroye (2009) developed an alternative algorithm of *double rejection* method such that the complexity is uniformly bounded. Hofert (2011b,a) suggested a *fast rejection* algorithm to enhance the original SSR scheme. All of the three algorithms are *exact*[4] and based on the A/R scheme, which can be applied to the class of positive TS distributions with stability index $\alpha \in (0, 1)$[5].

For the existing algorithms to simulate a TS random variable (r.v.), each of them has its own advantages but also have some limitations for some choices of parameters. It is therefore very useful to develop some alternative simulation schemes to sample the TS family. In this paper, we discover a multiple integral representation for the density functions of TS distributions with stability index in a general *dyadic form* $\frac{q}{2^n} \in (0, 1)$ for $q, n \in \mathbb{N}^+$, the integral representation allows us to develop an alternative simulation framework for the TS distributions with stability index $\frac{q}{2^n} \in (0, 1)$. In particular, when q is a binary number, an extremely efficient *direct* scheme can be established,

[3]The detailed proof is given by Weron (1996).

[4]On the other hand, some approximation-based algorithms have also been proposed in the literature, see Bondesson (1982), Rosiński (2001) and Ridout (2009), however, they obviously involve numerical errors.

[5]For the class of $\alpha \in (1, 2)$, a non-exact simulation algorithm based on the A/R scheme with Gaussian approximation is developed in Kawai and Masuda (2011).

which means that the algorithm only involves the explicit inverse transformation without the A/R mechanism. On one hand, this alternative algorithm could be very helpful for numerical validating for TS-based stochastic models, because a variety of choices of q, n combined with the different values of parameters β and θ would provide enough flexibility for the purpose of numerical testing, and at the same time, the corresponding computational cost of implementing TS-based models could be substantially reduced. On the other hand, since the foundation of our simulation design is indeed very different from existing schemes in the literature, this representation therefore allows us to design the simulation algorithms for a wider range of distributions with Lévy measures beyond the general tempered stable family.

The paper is organised as follows: In Section 2, we first present the fundamental results on the multiple integral representation for a special recursive type of Laplace transform functions and provide an simulation framework for some distributions beyond the TS $\frac{q}{2^n}$-family. In Section 3, we derive a new simulation scheme, namely the *backward recursive (BR) scheme*, for exact simulation based on the multiple integral representation for the density of $\frac{q}{2^n}$-TS distributions. In addition, we outline the improved algorithms tailored for two important subclasses, one is a *direct* scheme for the $\frac{1}{2^n}$-family, and the other is an enhanced scheme for the $\frac{3}{2^n}$-family. In Section 4, numerical experiments for our algorithms as well as the associated comparisons with other schemes have been carried out and reported. Section 5 makes a conclusion of this paper, and proposes some issues for possible further extensions and future research.

2 Fundamental Results

Before investigating the TS distributions, let us first introduce a general integral representation for a special type of Laplace transform functions based on the explicit formula for the inverse Gaussian (IG) density. This special type of Laplace transforms has an interesting recursive structure, which characterises a variety of different distributions even beyond the TS family. The integral representation leads to an recursive simulation scheme if the initial distribution is given. The theoretical foundation in general is illustrated in Theorem 2.1 below.

Theorem 2.1. *If a series of non-negative functions $\hat{f}_1(v)$, ..., $\hat{f}_n(v)$ for any $n \in \mathbb{N}^+$ possess a recursive structure of*

$$\hat{f}_k(v) = \begin{cases} \hat{f}_1(v), & k = 1, \\ a_k\sqrt{b_k + c_k v + \hat{f}_{k-1}(v)} - a_k\sqrt{b_k}, & k = 2, ..., n, \end{cases} \tag{2.1}$$

where $a_k > 0$ and $b_k, c_k \geq 0$, then, $e^{-\hat{f}_n(v)}$ can be represented by

$$e^{-\hat{f}_n(v)} = \begin{cases} \int_0^\infty e^{-vs_1} f_1(s_1)\mathrm{d}s_1, & n=1, \\ \int_0^\infty \dots \int_0^\infty e^{-s_1 v - \sum_{k=2}^{n} c_k s_k v} f_1(s_1 \mid s_2) \frac{a_2 s_3 e^{-\frac{\left(\sqrt{2b_2}s_2 - \frac{a_2 s_3}{\sqrt{2}}\right)^2}{2s_2}}}{2\sqrt{\pi s_2^3}} \times \dots \times \frac{a_n e^{-\frac{\left(\sqrt{2b_n}s_n - \frac{a_n}{\sqrt{2}}\right)^2}{2s_n}}}{2\sqrt{\pi s_n^3}} \mathrm{d}s_n \dots \mathrm{d}s_1, & n>1, \end{cases} \tag{2.2}$$

where $f_1(s_1) = \mathcal{L}_{s_1}^{-1}\left\{e^{-\hat{f}_1(v)}\right\}$ and $f_1(s_1 \mid s_2) = \mathcal{L}_{s_1}^{-1}\left\{e^{-s_2\hat{f}_1(v)}\right\}$.

Proof. We use $IG\left(\frac{a}{b}, a^2\right)$ to denote an IG distribution[6] with mean $\frac{a}{b}$ and the shape parameter a^2 where $a > 0$, $b \geq 0$. It is well known that its Laplace transform is given by

$$\mathbb{E}\left[e^{-vIG\left(\frac{a}{b},a^2\right)}\right] = e^{-a\left(\sqrt{b^2+2v}-b\right)}.$$

Since (2.2) holds for $n = 2$. The proof for the general case can be conducted by the mathematical induction: Assuming (2.2) also holds for an arbitrary integer $n = j > 2$, given $s_{j+1} \in (0, \infty)$, we have

$$\begin{aligned} e^{-\hat{f}_j(v)s_{j+1}} &= \exp\left(-a_j s_{j+1}\left[\sqrt{b_j + c_j v + \hat{f}_{j-1}(v)} - \sqrt{b_j}\right]\right) \\ &= \int_0^\infty \dots \int_0^\infty e^{-\left(s_1 + \sum_{k=2}^{j} c_k s_k\right)v} f_1(s_1 \mid s_2) \frac{a_2 s_3 e^{-\frac{\left(\sqrt{2b_2}s_2 - \frac{a_2 s_3}{\sqrt{2}}\right)^2}{2s_2}}}{2\sqrt{\pi s_2^3}} \times \dots \times \frac{a_j s_{j+1} e^{-\frac{\left(\sqrt{2b_j}s_j - \frac{a_j s_{j+1}}{\sqrt{2}}\right)^2}{2s_j}}}{2\sqrt{\pi s_j^3}} \mathrm{d}s_j \dots \mathrm{d}s_1. \end{aligned}$$

Then, for $n = j + 1$, we have

$$\begin{aligned} e^{-\hat{f}_{j+1}(v)} &= \int_0^\infty e^{-c_{j+1}s_{j+1}v} e^{-\hat{f}_j(v)s_{j+1}} \frac{a_{j+1}}{2\sqrt{\pi s_{j+1}^3}} e^{-\frac{\left(\sqrt{2b_{j+1}}s_{j+1} - \frac{a_{j+1}}{\sqrt{2}}\right)^2}{2s_{j+1}}} \mathrm{d}s_{j+1} \\ &= \int_0^\infty \dots \int_0^\infty e^{-\left(s_1 + \sum_{k=2}^{j+1} c_k s_k\right)v} f_1(s_1 \mid s_2) \frac{a_2 s_3 e^{-\frac{\left(\sqrt{2b_2}s_2 - \frac{a_2 s_3}{\sqrt{2}}\right)^2}{2s_2}}}{2\sqrt{\pi s_2^3}} \times \dots \times \frac{a_{j+1} e^{-\frac{\left(\sqrt{2b_{j+1}}s_{j+1} - \frac{a_{j+1}}{\sqrt{2}}\right)^2}{2s_{j+1}}}}{2\sqrt{\pi s_{j+1}^3}} \mathrm{d}s_{j+1} \dots \mathrm{d}s_1. \end{aligned}$$

Hence, (2.2) also holds for $n = j + 1$, which means that this statement holds for any $n \in \mathbb{N}^+$. □

With the integral representation (2.2), we can develop an associate simulation scheme for a variety class of random variables by setting different values for the parameters a_k, b_k, and c_k. For instance, if we set $c_k = 0$ for all k, we can develop a general simulation scheme to sample a r.v. S, where the Laplace transform is of the following form

$$\mathbb{E}\left[e^{-vS}\right] = a_n\sqrt{\dots\sqrt{\dots\sqrt{b_2 + \hat{f}_1(v)} - a_2\sqrt{b_2}} - a_{n-1}\sqrt{b_{n-1}}} - a_n\sqrt{b_n}, \tag{2.3}$$

[6]The distributional properties of IG distributions have been well documented in Chhikara and Folks (1989).

when the initial simulation scheme for the r.v. with Laplace transform $\hat{f}_1(v)$ is known. Let f_1 be the density of r.v. with Laplace transform $\hat{f}_1(v)$, then, the full simulation structure in general is summarised in Algorithm 2.1.

Algorithm 2.1 General Framework for Simulating S with Laplace Transform (2.3)

1. **if**$(n == 1)\{$
2. **sample** S_1 with distribution $f_1(s_1)$
3. $\}$
4. **else** $\{$
5. **for** $i \in \{n-1, ..., 2\}\{$
6. **sample** $S_n \sim IG\left(a_n/2\sqrt{b_n},\ a_n^2/2\right)$
7. **set** $a_i = a_k$ and $b_i = b_k$
8. **sample** $S_i \sim IG\left(S_{i+1}/2\sqrt{b_i},\ a_i^2 S_{i+1}^2/2\right)$ and **sample** S_1 with distribution $f_1(s_1|S_2)$
9. $\}$
10. **return** S_1

Alternatively, if we set $b_k = c_k = 0$ for all k, this integral representation then lead to an exact simulation scheme for a class of random variables with Laplace transform

$$a_n\sqrt{...\sqrt{...a_2\sqrt{\hat{f}_1(v)}}}. \tag{2.4}$$

More specifically, if we set $a_k = 1$, for all k and let $\hat{f}_1(v) = v^q$, then, the Laplace transform (2.1) corresponds to a class of stable distributions with stability index $\frac{q}{2^n}$. Indeed, this representation for the stable distributions is of great interests, as it provides us an alternative simulation framework to generate the associated TS class. In the rest of this paper, we concentrate on the exact simulation for the TS class with stability index $\frac{q}{2^n}$, which is mainly demonstrated in Section 3.

3 Tempered Stable distributions and Exact Simulation

Positive TS distribution is a one-sided tilted stable distribution with stability index $\alpha \in (0,1)$. It is an *infinitely divisible* distribution that combines both of the stable distribution and Gaussian trends by tilting the *Lévy measure* with an exponential function. More precisely, it has a very flexible structure with three parameters, and the Lévy measure ν is specified by

$$\nu(\mathrm{d}s) = e^{-\beta s}\frac{\theta}{s^{\alpha+1}}\mathbf{1}_{\{s>0\}}\mathrm{d}s, \qquad \alpha \in (0,1), \quad \beta, \theta \in (0,\infty), \tag{3.1}$$

where α is the *stability index*, θ is the *intensity parameter* and β is the *tilting parameter*. In particular, if $\alpha = \frac{1}{2}$, it reduces to a very important distribution, the inverse Gaussian (IG) distribution.

In the sequel, for simplicity, this TS distribution uniquely determined by (3.1) is denoted by $TS(\alpha, \beta, \theta)$, and its probability density function (PDF) is denoted by $f(s; \alpha, \beta, \theta)$. Obviously, if $\beta = 0$, it then returns to a standard positive stable distribution $S(\alpha, \theta)$ with Lévy measure

$$\nu(\mathrm{d}s) = \frac{\theta}{s^{\alpha+1}} \mathbf{1}_{\{s>0\}} \mathrm{d}s,$$

and density $f(s; \alpha, 0, \theta)$. Proposition 3.1 below illustrates an important and well known connection between these two distributions.

Proposition 3.1. *The Laplace transform of $TS(\alpha, \beta, \theta)$ is given by*

$$\mathbb{E}\left[e^{-vTS(\alpha,\beta,\theta)}\right] = \exp\left(-\frac{\theta\Gamma(1-\alpha)}{\alpha}\left[(\beta+v)^{\alpha} - \beta^{\alpha}\right]\right), \qquad v \in \mathbb{R}^{+}, \tag{3.2}$$

with the density

$$f(s; \alpha, \beta, \theta) = \exp\left(\frac{\theta\Gamma(1-\alpha)}{\alpha}\beta^{\alpha} - \beta s\right) \times f(s; \alpha, 0, \theta), \tag{3.3}$$

where $\Gamma(u)$ is the gamma function, i.e. $\Gamma(u) := \int_0^\infty s^{u-1} e^{-s} \mathrm{d}s$.

As the name *stable* implies in stable distribution, stability index α is the most crucial parameter, since this fundamentally determines its distributional property of the so-called *stable laws* (or α-*stable*): a sum of any two independent stable r.v.s with the same index α is still a stable r.v. with index α, and this invariance property does not hold across different α's (Borak et al., 2005, p.22). It has been commonly recognised that, the density functions for general specifications on α are hard to be obtained analytically, and this indeed poses the greatest challenge to the further study of their distributional properties and statistical inference. Nevertheless, a different specification for α leads to a different family of stable distributions, and some are particularly attractive in respect of their distinctive distributional properties and potentials for applications. Many of these interesting special families have been extensively investigated in the literature, see Brown and Tukey (1946), Mitra (1981, 1982), Montroll and Bendler (1984) and Penson and Górska (2010), and also see surveys in Holt and Crow (1973), Devroye (1986), Zolotarev (1986), Samoradnitsky and Taqqu (1994) and Uchaikin and Zolotarev (1999). For example, the $\frac{1}{2^n}$-family of stable distributions for $n = -1, 0, 1, ...$ is particularly prestigious, and the first three members are all famous: they are Gaussian, Cauchy and Lévy, respectively, and each one has an explicit density function with a tailored algorithm available for exact simulation (Uchaikin and Zolotarev, 1999, p.214–216). However, explicit densities beyond these three members rely on heavy use of the hypergeometric function, which are not ideally suited to our goal for exact simulation.

In the following parts of this section, we first set up our proposed framework of exact simulation for the TS families with stability index that can be expressed in a general dyadic form of

$$\alpha = \frac{q}{2^n} \in (0,1), \qquad q \in \mathbb{N}^+, \quad n \in \left\{ k \in \mathbb{N}^+ \mid 2^k > q \right\}. \tag{3.4}$$

We mainly consider the case where q is an odd number, as obviously an even numerator will reduce to an odd numerator by moving down the degree n. Meanwhile, we focus on some of its important subclasses where further enhanced algorithms (such as *direct* simulation procedure) are available. The associated numerical examples will be presented later in Section 4.

3.1 Tempered Stable Distributions with Stability Index $\alpha = \frac{q}{2^n}$

Based on Theorem 2.1, if $a_k = 1$ and $b_k = c_k = 0$ for any $k = 2, ..., n$ and $\hat{f}_1(v) = v^q$, then, (2.1) represents the Laplace transform of a stable distribution with stability index $\frac{q}{2^n}$. Therefore, we can represent the density of a TS distribution with stability index $\frac{q}{2^n} \in (0,1)$ based on (3.3) and (2.2) in Theorem 3.1 as follows.

Theorem 3.1. *For $q, n \in \mathbb{N}^+$ specified by (3.4) and $p := \min\limits_{1 \le i \le n} \left\{ i \in \mathbb{N}^+ : \ 2^i \ge q \right\}$, the density of $TS\left(\frac{q}{2^n}, \beta, \theta\right)$ has an integral representation of*

$$f(s_1) = \begin{cases} \exp\left(-\beta s_1 + A_p \beta^{\frac{q}{2^p}}\right) f(s_1; \frac{q}{2^p}, 0, \theta), & n = p, \\ \displaystyle\int_0^\infty \cdots \int_0^\infty \frac{\frac{s_2}{\sqrt{2}} e^{-\frac{\left(\sqrt{2\beta} s_1 - \frac{s_2}{\sqrt{2}}\right)^2}{2 s_1}}}{\sqrt{2\pi s_1{}^3}} \times \ldots \times \frac{\frac{s}{\sqrt{2}} e^{-\frac{\left(\sqrt{2}\beta^{\frac{1}{2^{n-p}}} s_{n-p} - \frac{s}{\sqrt{2}}\right)^2}{2 s_{n-p}}}}{\sqrt{2\pi s_{n-p}^3}} f_S(s) e^{-\beta^{\frac{1}{2^{n-p}}} s + A_n \beta^{\frac{q}{2^n}}} \mathrm{d}s \ldots \mathrm{d}s_2, & n > p, \end{cases} \tag{3.5}$$

where $A_n := \frac{2^n}{q}\theta\Gamma\left(1 - \frac{q}{2^n}\right)$, and $f_S(s)$ is the density of a stable distribution with stability index $\frac{q}{2^p}$ such that its Laplace transform is $\exp\left(-A_n v^{\frac{q}{2^p}}\right)$.

When $n = p$, i.e. n is the smallest integer for $\frac{q}{2^n} < 1$, the density is equivalent to (3.3), and the simulation scheme degenerates to the SSR scheme (with the algorithm in Appendix A). Whereas when $n > p$, from the multiple integral representation (3.5) for the density function, we can observe that $TS\left(\frac{q}{2^n}, \beta, \theta\right)$ is closely related to an IG distribution which can be directly simulated by Michael et al. (1976)[7]. We therefore develop an alternative simulation framework based on the

[7]Note that, the exact simulation for IG distribution designed by Michael et al. (1976) is *direct* and very efficient, as

integral representation for $TS\left(\frac{q}{2^n}, \beta, \theta\right)$ in Algorithm 3.1, namely the *backward recursive (BR) scheme*. The recursion first needs to be initialised by generating a r.v. (named as *"seed"* throughout this paper) $\tilde{S}$ via the A/R scheme, the general BR scheme in Algorithm 3.1 as below to sample $TS\left(\frac{q}{2^n}, \beta, \theta\right)$.

Algorithm 3.1 Backward Recursive Scheme for Simulating $TS\left(\frac{q}{2^n}, \beta, \theta\right)$

1. **set** $\tilde{\theta} := 2^{n-p}\theta\Gamma\left(1 - \frac{q}{2^n}\right)/\Gamma\left(1 - \frac{q}{2^p}\right)$
2. **repeat** {
3. **sample** $\tilde{S} \sim S\left(q/2^p, \tilde{\theta}\right)$ and $V \sim \mathcal{U}[0,1]$
4. **if** ($V \leq \exp\left(-\beta^{1/2^{n-p}}\tilde{S}\right)$) **break**
5. }
6. **set** $\tilde{S}_{n-p+1} = \tilde{S}$
7. **for** $i \in \{n-p, ..., 2, 1\}${
8. **sample** $\tilde{S}_i \sim IG\left(\tilde{S}_{i+1}/2\beta^{1/2^i},\ \tilde{S}_{i+1}^2/2\right)$
9. }
10. **return** $\tilde{S}_1$

This alternative scheme turns out to be useful for a large n, in the way that it avoids the low acceptance rate of sampling a TS distribution with a lower stability index, by generating a TS distribution with a relative higher stability index and using a recursive procedure to produce the rest. To illustrate the key idea of recursion, we could adopt the SSR scheme to generate the seed, therefore, the corresponding expected complexity of the algorithm based on this SSR scheme is $n - p + 2\exp\left(A_p\beta^{\frac{q}{2^p}}\right)$. Since the complexity is exponentially increasing with respect to the parameter β, to accelerate the algorithms, one should replace the SSR algorithm by the double rejection method (Devroye, 2009), or, the fast rejection algorithm (Hofert, 2011b), to speed up the simulation of the associated seed.

3.2 Tempered Stable Distributions with Stability Index $\alpha = \frac{1}{2^n}$

Mathematically elegant schemes of *direct* exact simulation without the A/R mechanism, like the classical CMS method (Chambers et al., 1976), are indeed very rare in the families of TS distributions but extremely desirable. In this section, we present our first discovery of such unique family that allows *direct* simulation, that is, the family of TS distributions with the stability index of *binary*

it is developed based on the explicit inverse transformation of multiple roots without the A/R scheme.

fractions $\frac{1}{2^n}$, $n = 1, 2, ...$ In fact, it inherits from the prestigious family of stable distributions with the same style as we mentioned at the beginning of this section. The first member of this $\frac{1}{2^n}$-family of TS distributions is familiar: $TS\left(\frac{1}{2}, \beta, \theta\right)$ follows an IG distribution, since the associated stable distribution $S\left(\frac{1}{2}, \theta\right)$ follows a Lévy distribution. However, it is almost unknown for the remaining members.

This family is very unique, which allows us to develop an extremely fast algorithm for exactly sampling any member based on recursively calling the IG generator developed by Michael et al. (1976). It is accurate and very straightforward to implement. Amazingly, it barely depends on the choices of parameters, as the recursion is only driven by the efficient IG generator and no A/R mechanism is involved. This *direct* procedure is summarised in Algorithm 3.2, and it is derived immediately from a multiple integral representation of its density function in Corollary 3.1.

Corollary 3.1. *The density of* $TS\left(\frac{1}{2^n}, \beta, \theta\right)$ *has an integral representation of*

$$f\left(s_1; \frac{1}{2^n}, \beta, \theta\right) = \int_0^\infty ... \int_0^\infty \frac{\frac{s_2}{\sqrt{2}}}{\sqrt{2\pi s_1{}^3}} e^{-\frac{\left(\sqrt{2\beta}s_1 - \frac{s_2}{\sqrt{2}}\right)^2}{2s_1}} \times ... \times \frac{\frac{A_n}{\sqrt{2}}}{\sqrt{2\pi s_n^3}} e^{-\frac{\left(\sqrt{2}\beta^{\frac{1}{2^n}} s_n - \frac{A_n}{\sqrt{2}}\right)^2}{2s_n}} \mathrm{d}s_n ... \mathrm{d}s_2, \tag{3.6}$$

where $A_n = 2^n \theta \Gamma\left(1 - \frac{1}{2^n}\right)$.

Proof for Corollary 3.1 immediately follows Theorem 2.1 by setting $\hat{f}_1(v) = \sqrt{v}$. Hence, based on (3.6), we can directly generate a $TS\left(\frac{1}{2^n}, \beta, \theta\right)$ r.v. for $n > 1$ via the following Algorithm 3.2.

Algorithm 3.2 Direct Scheme for Simulating $TS\left(\frac{1}{2^n}, \beta, \theta\right)$

1. **sample** $\tilde{S}_{n+1} \sim IG\left(2^n\theta\Gamma(1 - \frac{1}{2^n})/2\beta^{1/2^n},\ [2^n\theta\Gamma(1 - 1/2^n)]^2/2\right)$
2. **for** $i \in \{n, ..., 2, 1\}\{$
3. **sample** $\tilde{S}_i \sim IG\left(\tilde{S}_{i+1}/2\beta^{1/2^{i-1}},\ \tilde{S}_{i+1}^2/2\right)$
4. $\}$
5. **return** $\tilde{S}_1$

To illustrate how Algorithm 3.2 works, let us consider the simplest case when $n = 2$ as an example: Since $TS\left(\frac{1}{4}, \beta, \theta\right)$ is equal in distribution to $\tilde{S}_1$, it can be simulated by first generating an IG r.v. of

$$\tilde{S}_2 \sim IG\left(\frac{2\theta\Gamma\left(\frac{3}{4}\right)}{\beta^{\frac{1}{4}}},\ 8\theta^2\Gamma^2\left(\frac{3}{4}\right)\right);$$

then, conditional on one realisation of $\tilde{S}_2$, we can generate $TS\left(\frac{1}{4}, \beta, \theta\right)$ i.e. $\tilde{S}_1$ via another IG r.v.

of

$$\tilde{S}_1 \mid \tilde{S}_2 \sim IG\left(\frac{\tilde{S}_2}{2\sqrt{\beta}}, \frac{\tilde{S}_2^2}{2}\right).$$

3.3 Tempered Stable Distributions with Stability Index $\alpha = \frac{3}{2^n}$

To sample TS r.v.s beyond the $\frac{1}{2^n}$-family, we find that it is hard to obtain the associated *direct* algorithms and eventually the A/R scheme is unavoidable. Similar to the stable distribution, there is no explicit formula for the density of TS distribution in general. However, for some special choices of q, we find an alternative way to generate the stable distribution $S\left(\frac{q}{2^p}, \tilde{\theta}\right)$ of the seed without using the *Zolotarev's integral representation*. In this section, we take a further inside of the density of stable distribution of $S(\frac{3}{4}, \theta)$ based on some very interesting distributional results we discovered in Lemma (3.1) for a special family of r.v.s with Laplace transform (3.7). Then, we design an enhanced algorithm to generate the $\frac{3}{2^n}$-TS family via Algorithm for $n = 2$ and Algorithm for $n > 2$, respectively.

Let us first introduce a class of r.v.s with Laplace transform (3.7), which is a simplified version of (2.1). This class of r.v.s has a closed-form distribution function which lead to a direct simulation scheme. Simply setting $\gamma = 0$, (3.7) becomes the Laplace transform of a $S\left(\frac{3}{4}, \theta\right)$ r.v.. Details of the distribution property and simulation scheme for this special family are presented in Lemma 3.1 and Algorithm 3.3.

Lemma 3.1. *If the random variable S has the Laplace transform*

$$\mathbb{E}\left[e^{-vS}\right] = \exp\left(-\zeta\sqrt{v^{\frac{3}{2}} + \gamma v}\right), \qquad \zeta, \gamma \in \mathbb{R}^+, \tag{3.7}$$

then, the corresponding CDF of S can be represented by

$$F_S(s) = \int_0^\infty \frac{1}{\pi(w+1)\sqrt{w}} \exp\left(-\frac{\zeta^4(w+1)^3}{64w}\frac{1}{s^3} - \frac{\gamma\zeta^2(w+1)}{4}\frac{1}{s}\right) \mathrm{d}w. \tag{3.8}$$

Proof. Let $f_S(s)$ and $F_S(s)$ denote the PDF and the cumulative distribution function (CDF) of S, respectively. According to (3.7), the Laplace transform of $F_S(s)$ can be derived

$$\begin{aligned}\int_0^\infty e^{-vs}F_S(s)\mathrm{d}s &= \int_0^\infty\int_u^\infty e^{-vs}f_S(u)\mathrm{d}s\mathrm{d}u \\ &= \frac{1}{v}\int_0^\infty e^{-vu}f_S(u)\mathrm{d}u \\ &= \frac{1}{v}\exp\left(-\zeta\sqrt{v^{\frac{3}{2}} + \gamma v}\right), \qquad v \in \mathbb{R}^+. \end{aligned} \tag{3.9}$$

Note that, by change of variable $u = vx$, (3.9) can be rewritten as

$$\begin{aligned}\frac{1}{v}\exp\left(-\zeta\sqrt{v^{\frac{3}{2}}+\gamma v}\right) &= \frac{1}{v}\int_0^\infty e^{-(\sqrt{v}-\gamma)vx}\frac{\zeta}{2\sqrt{\pi x^3}}\exp\left(-\frac{\zeta^2}{2x}\right)\mathrm{d}x \\ &= \frac{1}{\sqrt{v}}\int_0^\infty e^{-\sqrt{v}u}\frac{\zeta}{2\sqrt{\pi u^3}}\exp\left(-\frac{\zeta^2 v}{4u}-\gamma u\right)\mathrm{d}u.\end{aligned}$$

From the tables of Laplace transforms listed in Bateman (1954, p.246), we recognise that

$$\frac{1}{\sqrt{v}}e^{-\sqrt{v}u} = \int_0^\infty e^{-vy}\frac{1}{\sqrt{\pi y}}\exp\left(-\frac{u^2}{4y}\right)\mathrm{d}y.$$

Finally, (3.9) has an integral representation of

$$\begin{aligned}\frac{1}{v}\exp\left(-\zeta\sqrt{v^{\frac{3}{2}}+\gamma v}\right) &= \int_0^\infty\int_0^\infty e^{-vy}\frac{1}{\sqrt{\pi y}}\exp\left(-\frac{u^2}{4y}\right)\frac{\zeta}{2\sqrt{\pi u^3}}\exp\left(-\frac{\zeta^2 v}{4u}-\gamma u\right)\mathrm{d}u\mathrm{d}y \\ &= \int_0^\infty\int_0^\infty e^{-vy}e^{vz}\frac{1}{\pi\sqrt{yz}}\exp\left(-\frac{\zeta^4}{64yz^2}-\frac{\gamma\zeta^2}{4z}\right)\mathrm{d}z\mathrm{d}y. \qquad (3.10)\end{aligned}$$

Applying transformations $s = y + z$ and $w = \frac{y}{z}$ to (3.10), we have $z = \frac{s}{w+1}$, $y = \frac{sw}{w+1}$ with the Jacobian of the transformation

$$\begin{vmatrix} \frac{1}{w+1} & \frac{w}{w+1} \\ -\frac{s}{(w+1)^2} & \frac{s}{(w+1)^2} \end{vmatrix} = \frac{s}{(w+1)^2}.$$

Hence, (3.9) can be finally rewritten as

$$\frac{1}{v}\exp\left(-\zeta\sqrt{v^{\frac{3}{2}}+\gamma v}\right) = \int_0^\infty\int_0^\infty e^{-vs}\frac{1}{\pi(1+w)\sqrt{w}}\exp\left(-\frac{\zeta^4(1+w)^3}{64w}\frac{1}{s^3}-\frac{\gamma\zeta^2(1+w)}{4}\frac{1}{s}\right)\mathrm{d}s,$$

which corresponds to the CDF of (3.8). □

According to the closed-form CDF (3.8) for S with Laplace transform (3.7), we can directly generate S using an explicit inverse transformation. The simulation scheme for sampling S is provided in Algorithm 3.3 as follows.

Proof. We observe that, the CDF of S in (3.8) of Lemma 3.1 can be represented by

$$F_S(s) = \int_0^\infty f_W(w)F_{S|W}(s|w)\mathrm{d}w,$$

Algorithm 3.3 Exact Scheme for Simulating S with Laplace Transform (3.7)

1. **sample** $U^{(1)} \sim \mathcal{U}[0,1]$, and $U^{(2)} \sim \mathcal{U}[0,1]$
2. **set** $W = \tan^2\left(\frac{\pi}{2}U^{(1)}\right)$, $A = \frac{\zeta^4(1+W)^3}{64W}$, $B = \frac{\gamma\zeta^2(1+W)}{4}$, $C = -\ln(U_2)$, and
$$D = 3\sqrt{3}\sqrt{27A^2C^4 + 4AB^3C^2} + 27AC^2 + 2B^3$$
3. **return**
$$\frac{1}{3C}\left(D^{\frac{1}{3}}/2^{\frac{1}{3}} + 2^{\frac{1}{3}}B^2/D^{\frac{1}{3}} + B\right) \tag{3.11}$$

where W is a well-defined r.v. with the density

$$f_W(w) = \frac{1}{\pi\sqrt{w}(w+1)}.$$

The corresponding CDF is

$$F_W(w) = \int_0^w \frac{1}{\pi\sqrt{x}(1+x)}\mathrm{d}x = \frac{2}{\pi}\tan^{-1}\left(\sqrt{w}\right),$$

which allows W to be directly sampled via an explicit inverse transform by setting

$$W \overset{\mathcal{D}}{=} \tan^2\left(\frac{\pi}{2}U^{(1)}\right), \qquad U^{(1)} \sim \mathcal{U}[0,1].$$

Conditional on W, the CDF of S is therefore given by

$$F_{S|W}(s|w) = \exp\left(-\frac{\zeta^4(1+w)^3}{64w}\frac{1}{s^3} - \frac{\gamma\zeta^2(1+w)}{4}\frac{1}{s}\right).$$

To sample S given W, we then need to solve the following cubic equation

$$-\ln(U_2)\times S^3 - \frac{\gamma\zeta^2(1+W)}{4}\times S^2 - \frac{\zeta^4(1+W)^3}{64W} = 0, \qquad U^{(2)} \sim \mathcal{U}[0,1]. \tag{3.12}$$

The solution for this cubic equation immediately follows (3.11) using the parameter setting in Algorithm 3.3 for the coefficients in (3.12). □

For a stable r.v. $S\left(\frac{3}{4},\theta\right)$, its Laplace transform satisfies (3.7) with $\gamma = 0$. The corresponding distribution function for $S\left(\frac{3}{4},\theta\right)$ satisfies (3.13).

Corollary 3.2. *The CDF of stable distribution $S\left(\frac{3}{4},\theta\right)$ for any $\theta \in \mathbb{R}^+$ can be represented by*

$$F_{S(\frac{3}{4},\theta)}(s) = \int_0^\infty \frac{1}{\pi(w+1)\sqrt{w}}\exp\left(-\frac{\kappa^4(w+1)^3}{64s^3w}\right)\mathrm{d}w, \tag{3.13}$$

and the simulation scheme for the stable r.v. $S\left(\frac{3}{4},\theta\right)$ *directly follows Algorithm 3.3 by setting* $B=0$.

Proof. The Laplace transform of $S\left(\frac{3}{4},\theta\right)$ is given by

$$\mathbb{E}\left[e^{-vS\left(\frac{3}{4},\theta\right)}\right]=e^{-\kappa v^{\frac{3}{4}}},\qquad \kappa=\frac{4}{3}\theta\Gamma\left(\frac{1}{4}\right).$$

Based on Lemma 3.1, setting $\gamma=0$ and $\zeta=\kappa$ in (3.8), the CDF of $S\left(\frac{3}{4},\theta\right)$ follows (3.13). □

To generate the associated TS distribution $TS\left(\frac{3}{4},\beta,\theta\right)$, instead of using the A/R based on the Zolotarev's integral representation, we redesign an alternative A/R scheme using a gamma distributed envelop based on the distribution function we obtained for the stable distribution $S\left(\frac{3}{4},\theta\right)$. The detail is given in Algorithm 3.4.

Algorithm 3.4 A/R Scheme for Simulating $TS\left(\frac{3}{4},\beta,\theta\right)$

1. **set** $\kappa=4\theta\beta^{\frac{3}{4}}\Gamma\left(\frac{1}{4}\right)/3$
2. **repeat** {
3. **sample** $W=\tan^2\left(\frac{\pi}{2}U\right)$ with $U\sim\mathcal{U}[0,1]$
4. **sample** $E\sim\Gamma(m,1)$ and $V\sim\mathcal{U}[0,1]$
5. **if** $(V\leq 3^{m+1}6^{\frac{m}{3}}\kappa^{\frac{4(m+3)}{3}}e^{\frac{m+3}{3}}(1+W)^3e^{-\frac{\kappa^4(1+W)^3}{64WE^3}}/\left(2^{3(m+2)}(m+3)^{\frac{m+3}{3}}WE^{m+3}\right))$ **break**
6. }
7. **return** E/β

Proof. Let $X\sim TS\left(\frac{3}{4},\beta,\theta\right)$ and set $S=\beta X$. According to Proposition 3.1, the Laplace transform of S is given by

$$\begin{aligned}
\mathbb{E}\left[e^{-vS}\right] &= \mathbb{E}\left[e^{-v\beta X}\right]\\
&= \exp\left(-\frac{4\theta\Gamma\left(\frac{1}{4}\right)}{3}\left[(\beta+\beta v)^{\frac{3}{4}}-\beta^{\frac{3}{4}}\right]\right)\\
&= \exp\left(-\frac{4\theta\beta^{\frac{3}{4}}\Gamma\left(\frac{1}{4}\right)}{3}\left[(1+v)^{\frac{3}{4}}-1\right]\right)\\
&= \exp\left(-\kappa\left[(1+v)^{\frac{3}{4}}-1\right]\right),
\end{aligned}$$

where $\kappa=\frac{4}{3}\theta\beta^{\frac{3}{4}}\Gamma\left(\frac{1}{4}\right)$. Using (3.3) and (3.13), the density of S can be written as

$$f_S(s)=\int_0^\infty\frac{1}{\pi(w+1)\sqrt{w}}\frac{3\kappa^4(1+w)^3}{64ws^4}\exp\left(-\frac{\kappa^4(1+w)^3}{64ws^3}\right)e^{-s+\kappa}\mathrm{d}w. \tag{3.14}$$

The joint density of S and W is

$$f(s,w) = \frac{1}{\pi(w+1)\sqrt{w}} \frac{3\kappa^4(1+w)^3}{64ws^4} \exp\left(-\frac{\kappa^4(1+w)^3}{64ws^3}\right) e^{-s+\kappa}, \qquad w,s>0.$$

To generate S, we consider a Gamma distributed envelop with rate parameter $m > 0$ and scale parameter 1 for S, and this envelop appears to be the most natural choice to be adopted according to its density (3.14). The joint density we have is

$$g(s,w) = \frac{1}{\pi(w+1)\sqrt{w}} \frac{1}{\Gamma(m)} s^{m-1} e^{-s}, \qquad w,s>0.$$

We have

$$\begin{aligned}
\frac{f(s,w)}{g(s,w)} &= \frac{3\kappa^4(w+1)^3\Gamma(m)e^{\kappa}}{64w} \frac{1}{s^{3+m}} \exp\left(-\frac{\kappa^4(1+w)^3}{64ws^3}\right) \\
&\leq \frac{\Gamma(m)e^{\kappa}e^{-\frac{m+3}{3}}\left(\frac{64}{3\kappa^4}\right)^{\frac{m}{3}}(m+3)^{\frac{m+3}{3}}w^{\frac{m}{3}}}{(w+1)^m} \\
&\leq \Gamma(m)e^{\kappa}e^{-\frac{m+3}{3}}\left(\frac{8}{3}\right)^m\left(\frac{1}{6\kappa^4}\right)^{\frac{m}{3}}(m+3)^{\frac{m+3}{3}} = C(m,\kappa),
\end{aligned}$$

where $C(m,\kappa)$ can be minimised over m. It is equivalent to find the m that minimises

$$\log C(m,\kappa) = \frac{1}{3}\Big[\log\Gamma(m)+3\kappa-(m+3)+8m\log(2)-4m\log(3)-4m\log(\kappa)+(m+3)\log(m+3)\Big].$$

The optimal value m^* satisfies the following equation

$$\frac{1}{4}\left[\log\left(m^*+3\right)+3\psi^{(0)}\left(m^*\right)\right] = \log\left(\frac{3\kappa}{4}\right), \qquad \text{for} \quad \psi^{(0)}(m) = \frac{\mathrm{d}\Gamma(m)}{\mathrm{d}m},$$

and we have

$$m^* = \frac{3\kappa}{4} - \frac{3}{8} + \frac{143}{128}\frac{1}{m^*} - \frac{1857}{1024}\frac{1}{m^{*2}} + \mathcal{O}\left(\frac{1}{m^{*3}}\right) \approx \frac{3\kappa}{4} + c.$$

When we numerically fit this linear setting, the optimal value c obtained is -0.14. Hence, given κ, the optimal rate m^* for the gamma distributed envelop is chosen by setting

$$m^* = -0.14 + \frac{3}{4}\kappa.$$

To generate S, we first sample W using the explicit inverse transformation, and then generate the

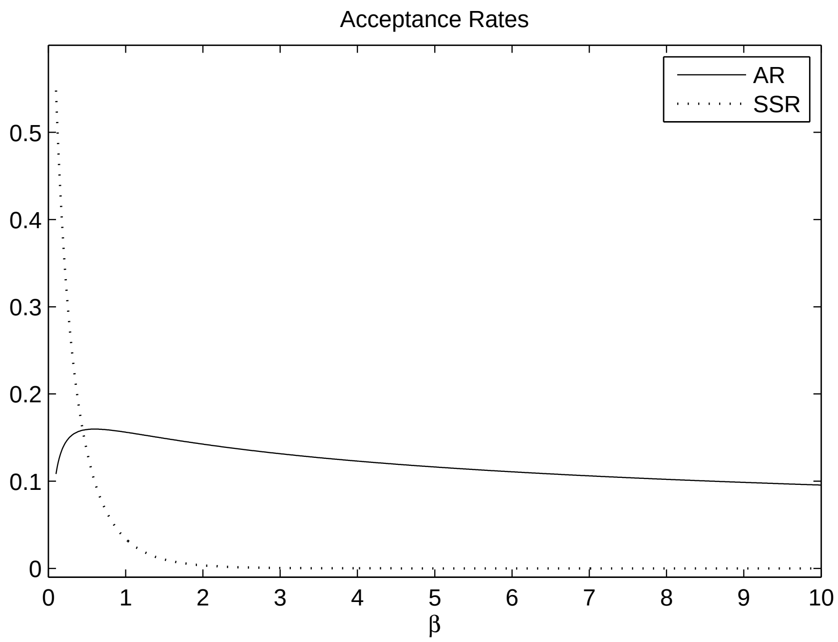

Figure 1: Acceptance rates of Algorithm 3.4 and the SSR scheme for sampling $TS\left(\frac{3}{4}, \beta, \theta\right)$ based on the parameter setting $\theta = 0.7$ and $\beta \in (0, 10)$

gamma distribution envelop $E \sim \Gamma(m^*, 1)$. The A/R decision follows

$$V \le \frac{f(E,W)}{C(m^*,\kappa)g(E,W)} = \frac{3^{m^*+1}6^{\frac{m^*}{3}}\kappa^{\frac{4(m^*+3)}{3}}e^{\frac{m^*+3}{3}}}{2^{3(m^*+2)}(m^*+3)^{\frac{m^*+3}{3}}}\frac{(1+W)^3}{W}\frac{e^{-\frac{\kappa^4(1+W)^3}{64WE^3}}}{E^{m^*+3}},$$

and the corresponding acceptance rate is $\frac{1}{C(m^*,\kappa)}$. Finally, we obtain $TS\left(\frac{3}{4}, \beta, \theta\right)$ by setting $X = \frac{S}{\beta}$. □

Comparing with the acceptance rate $\exp\left(-\frac{4}{3}\theta\Gamma\left(\frac{1}{4}\right)\beta^{\frac{3}{4}}\right)$ using the SSR scheme to generate $TS\left(\frac{3}{4}, \beta, \theta\right)$, Algorithm 3.4 leads to a higher acceptance rate, in particular for a large value of β. The difference of the acceptance rates to generate one $TS\left(\frac{3}{4}, \beta, \theta\right)$ r.v. for these two schemes with respect to β is plotted in Figure 1. The SSR scheme is only desirable when the tilting parameter is small, whereas our Algorithm 3.4 is more competitive when the tilting parameter is large. Hence, instead of using the SSR scheme to generate the seeds for the $TS\left(\frac{3}{2^n}, \beta, \theta\right)$ family, we adopt Algorithm 3.4 to sample seeds. The enhanced algorithm for $TS\left(\frac{3}{2^n}, \beta, \theta\right)$ is provided in Algorithm 3.5.

Proof. Letting $X \sim TS\left(\frac{3}{2^n}, \beta, \theta\right)$ and $S = \beta X$, we have

$$\begin{aligned}\mathbb{E}\left[e^{-vS}\right] &= \mathbb{E}\left[e^{-v\beta X}\right] \\ &= \exp\left(-\frac{2^n\theta\Gamma\left(1-\frac{3}{2^n}\right)}{3}\left[(\beta+\beta v)^{\frac{3}{2^n}} - \beta^{\frac{3}{2^n}}\right]\right)\end{aligned}$$

Algorithm 3.5 Backward Recursive Scheme for Simulating $TS\left(\frac{3}{2^n},\beta,\theta\right)$

1. **set** $\kappa = 2^n\theta\beta^{\frac{3}{2^n}}\Gamma\left(1-3/2^n\right)/3$
2. **repeat** {
3. **sample** $W=\tan^2\left(\frac{\pi}{2}U\right)$ with $U\sim\mathcal{U}[0,1]$
4. **sample** $E\sim\Gamma(m,1)$ and $V\sim\mathcal{U}[0,1]$
5. **if** $(V\leq 3^{m+1}6^{\frac{m}{3}}\kappa^{\frac{4(m+3)}{3}}e^{\frac{m+3}{3}}(1+W)^3e^{-\frac{\kappa^4(1+W)^3}{64WE^3}}\big/\left(2^{3(m+2)}(m+3)^{\frac{m+3}{3}}WE^{m+3}\right))$**break**
6. }
7. **set** $\tilde{S}_{n-1}=E$
8. **for** $i\in\{n-2,...,2,1\}\{$
9. **sample** $\tilde{S}_i\sim IG\left(\tilde{S}_{i+1}/2,\ \tilde{S}_{i+1}^2/2\right)$
10. }
11. **return** $\tilde{S}_1/\beta$

$$= \exp\left(-\frac{2^n\theta\beta^{\frac{3}{2^n}}\Gamma\left(1-\frac{3}{2^n}\right)}{3}\left[(1+v)^{\frac{3}{2^n}}-1\right]\right).$$

Based on Theorem 3.1 and (3.14), the density of $S\sim TS\left(\frac{3}{2^n},1,\theta\beta^{\frac{3}{2^n}}\right)$ has the integral representation

$$f\left(s_1;\frac{3}{2^n},1,\theta\beta^{\frac{3}{2^n}}\right) = \int_0^\infty\cdots\int_0^\infty \frac{\frac{s_2}{\sqrt{2}}}{\sqrt{2\pi s_1{}^3}}e^{-\frac{\left(\sqrt{2}s_1-\frac{x_2}{\sqrt{2}}\right)^2}{2s_1}}\times\cdots\times\frac{\frac{s}{\sqrt{2}}}{\sqrt{2\pi s_{n-2}^3}}e^{-\frac{\left(\sqrt{2}s_{n-2}-\frac{s}{\sqrt{2}}\right)^2}{2s_{n-2}}} \times\frac{3\kappa^4(w+1)^2}{64\pi w^{\frac{3}{2}}s^4}\exp\left(-\frac{\kappa^4(w+1)^3}{64s^3w}-s+\kappa\right)\mathrm{d}w...\mathrm{d}s_2,$$

where

$$\kappa=\frac{2^n}{3}\theta\beta^{\frac{3}{2^n}}\Gamma\left(1-\frac{3}{2^n}\right). \tag{3.15}$$

The seed, denoted $\tilde{S}$, follows a distribution of $TS\left(\frac{3}{4},1,\theta\beta^{\frac{3}{2^n}}\right)$, and the density satisfies (3.14) with the new κ defined in (3.15). We directly follow Algorithm 3.4 to generate seeds and follow the general BR scheme to generate X. □

The total number of random variables needed to generate one $TS\left(\frac{3}{2^n},\beta,\theta\right)$ for our BR scheme would be $n-2+3C(m^*,\kappa)$, which is the sum of the expected number of iterations of the A/R algorithm required to generate a seed and the number of r.v.s generated via the recursion. The total number of r.v.s needed for the classical SSR scheme is $2\exp\left(\frac{2^n\theta}{3}\Gamma\left(1-\frac{3}{2^n}\right)\beta^{\frac{3}{2^n}}\right)$, so our BR

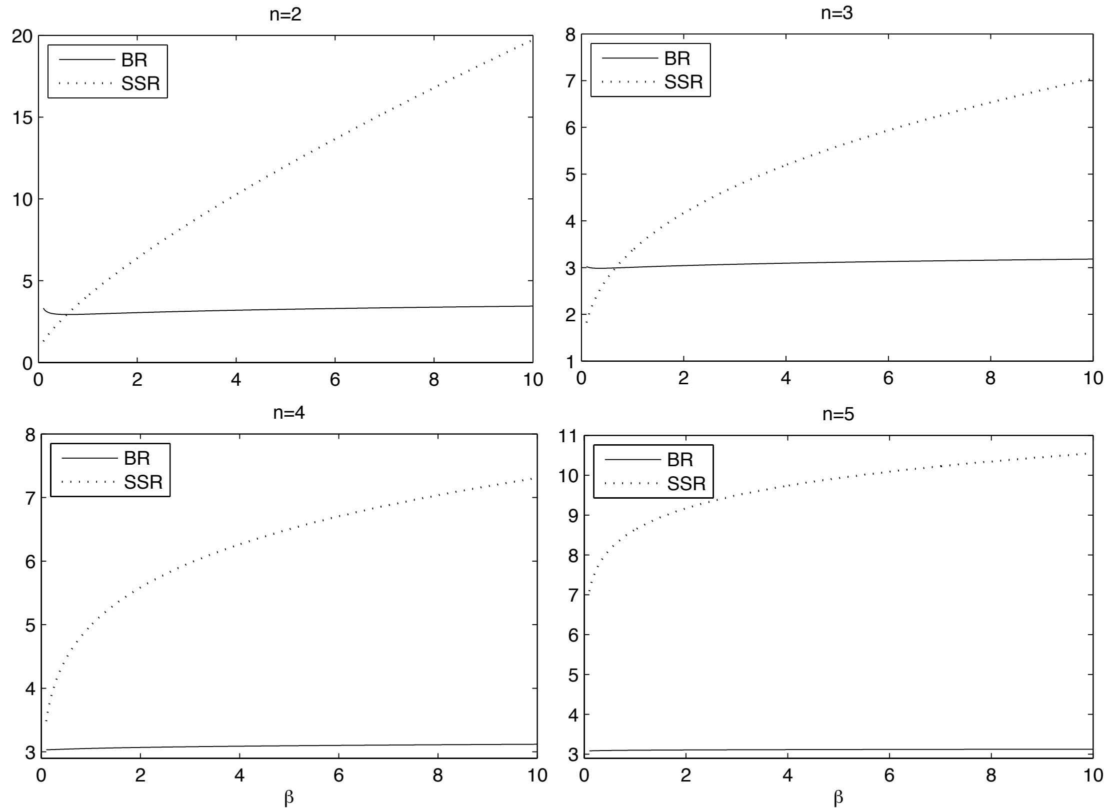

Figure 2: Logarithm of expected number of random variables needed for our *backward recursive* (BR) scheme (via Algorithm 3.4) v.s. the *simple stable rejection* (SSR) via Algorithm A.1 based on the parameter setting $\theta = 0.7$, $n = 2, 3, 4, 5$, respectively for the tilting parameter $\beta \in (0, 10)$

scheme shall have a better performance in general. In Figure 2, we compare the logarithm of the expected number of iterations for these two algorithms. The computation cost for the BR scheme is less than the SSR scheme, especially, when the tilting parameter β becomes larger. In general, different from the SSR scheme, the expected complexity of the BR scheme does not increase exponentially with respect to the β.

4 Numerical Examples

In this section, we provide numerical examples for these three families of TS distributions with stability index $\alpha = \frac{1}{2^n}, \frac{3}{2^n}, \frac{q}{2^n}$, respectively. The simulation experiments are all conducted on a normal laptop with the Intel Core i7-6500U CPU@2.50GHz processor, 8.00GB RAM, Windows 10 Home and 64-bit Operating System. The algorithms are coded and performed in MatLab (R2016b), and the computing time is measured by the *elapsed CPU time* in seconds. Numerical validation and tests for our simulation algorithms are based on the PDF and CDF of $TS(\alpha, \beta, \theta)$, which can be calculated by inverting the Laplace transform (3.2) numerically[8]. In particular, we

[8] A variety of methods are available for numerically inverting Laplace transforms with high accuracy, such as Gaver (1966), Stehfest (1970), Abate and Whitt (1992, 1995, 2006). Here, we use the Euler scheme in Abate and Whitt (2006).

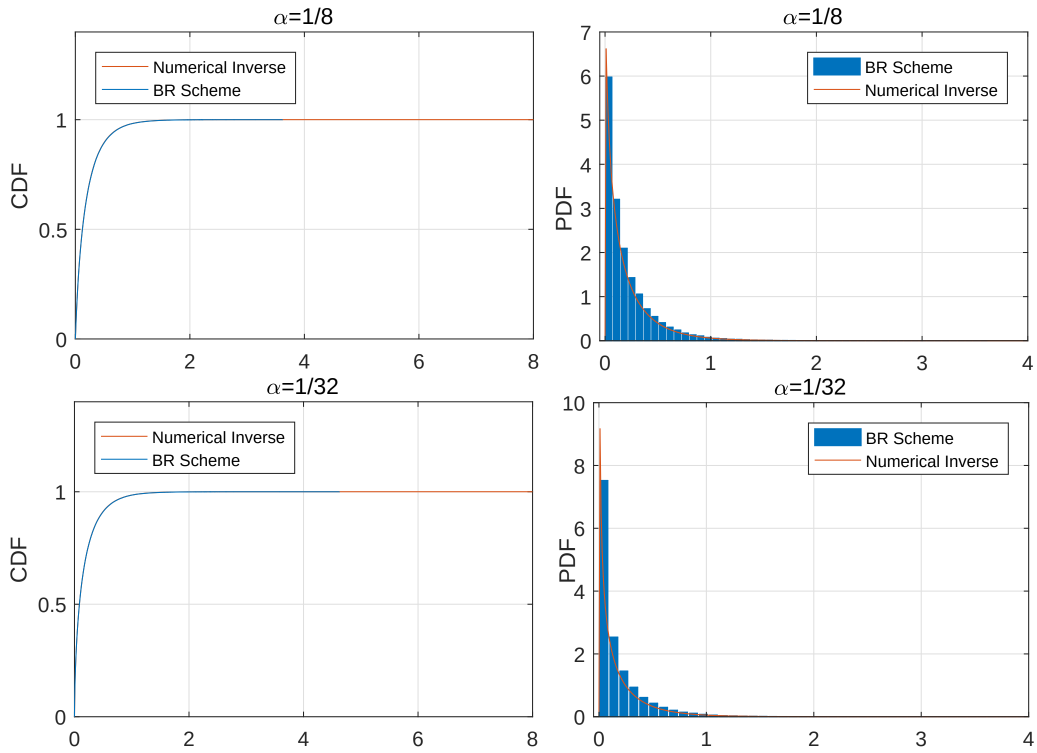

Figure 3: Comparison of the empirical CDF/PDF for our backward recursive (BR) scheme (via Algorithm 3.2) with the CDF/PDF obtained via numerical inverse under the parameter setting $\alpha = 1/2^2, 1/2^5$, $\theta = 0.5, \beta = 3.0$, respectively

assess the goodness-of-fit by comparing the CDFs and PDFs obtained by our BR scheme and by the numerical inverse of Laplace transform. The associated plots of CDFs, PDFs and histograms under different parameter settings are illustrated in Figure 3, 4, and 5, respectively. Overall, we can observe that each of these algorithms can achieve a very high level of accuracy, and the simulated CDF and PDF are fitted well to the associated numerical inversion.

We carry out a comparison of CPU time for our backward recursive (BR) scheme against the simple stable rejection (SSR) scheme and the double rejection (DR) scheme for three families of TS distributions with stability index $\alpha = \frac{1}{2^n}, \frac{3}{2^n}, \frac{q}{2^n}$. The associated numerical results in detail is reported in Table 1. Comparing with the SSR scheme and DR scheme, our direct scheme of Algorithm 3.2 for the first family of $\alpha = \frac{1}{2^n}$ performances extraordinarily fast, since no A/R procedure is involved as clearly demonstrated in Section 3.2. For example, it is nearly $1,000$ times faster than the SSR scheme and 10 times faster than the DR scheme for simulating $100,000$ replications based on parameter setting $\alpha = 1/32, \theta = 0.5, \beta = 1$. The out-performance of our algorithm would even become much more substantial when θ or β increases or n increases. More remarkably, the speed of simulation is barely effected by the changes of parameters θ and β, and the CPU time

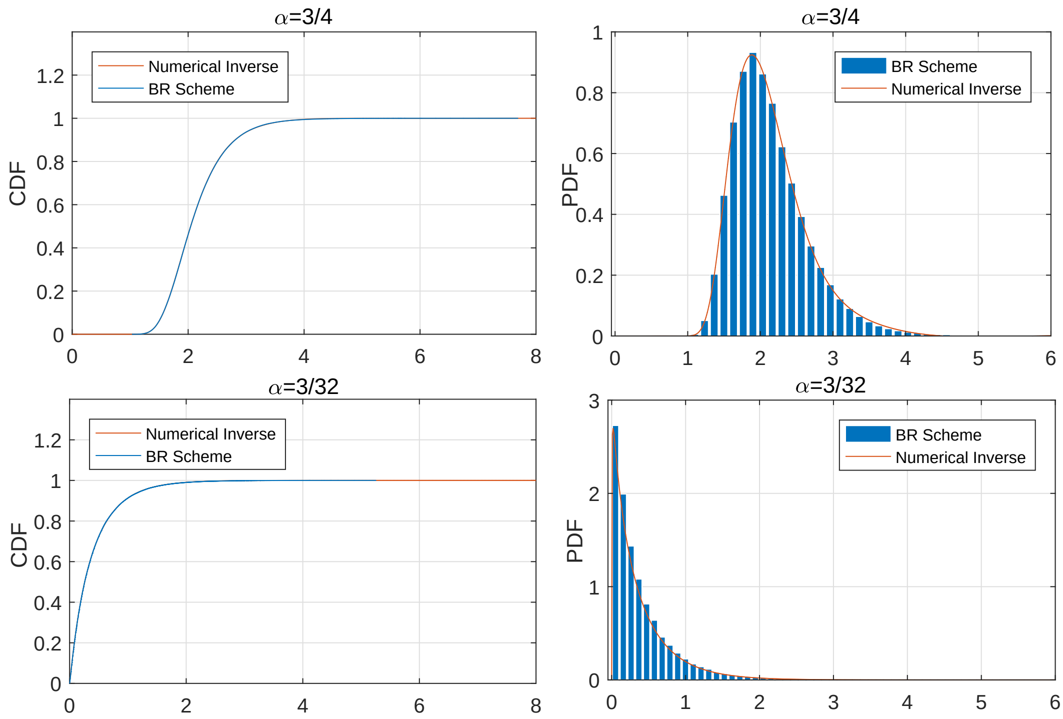

Figure 4: Comparison of the empirical CDF/PDF for our backward recursive (BR) scheme (via Algorithm 3.4,3.5) with the CDF/PDF obtained via numerical inverse under the parameter setting $\alpha = 3/2^2, 3/2^5, \theta = 0.7, \beta = 2.0$, respectively

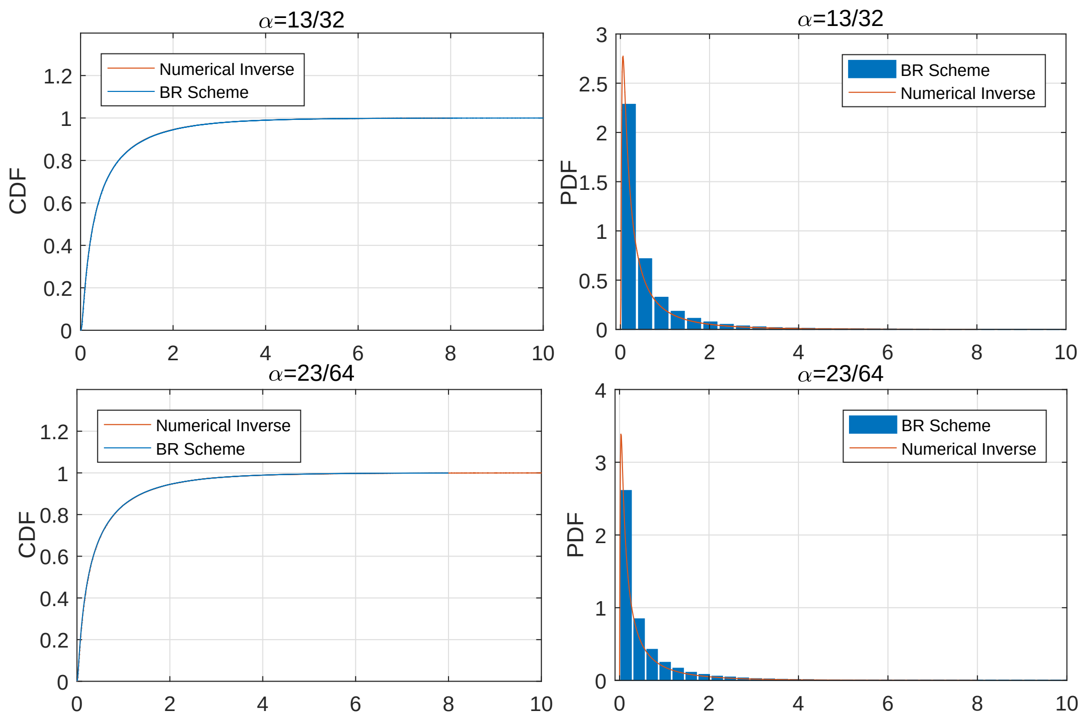

Figure 5: Comparison of the empirical CDF/PDF for our backward recursive (BR) scheme (via Algorithm 3.1) with the CDF/PDF obtained via numerical inverse under the parameter setting $\alpha = 13/2^5, 23/2^6, \theta = 0.25, \beta = 0.5$, respectively

only slightly increases when n increases. The associated numerical results in detail for this sensitivity analysis are reported in Table 2. Our Algorithm 3.5 is also very fast and can easily beat the SSR scheme, as the generator for the stable law of seeds is enhanced by our own design of Algorithm 3.4. For example, it is about 400 times faster than the SSR scheme for exactly simulating $100,000$ replications with parameter setting of $\alpha = 3/32, \theta = 0.7, \beta = 2.0$. Moreover, it is about four times faster than the DR scheme. However, if we generate the seed using the SSR scheme for our Algorithm 3.1 in general for the third family of $\alpha = \frac{q}{2^n}$, then, the performance is at a similar level as the SSR scheme. In general, increasing n would increase the computing time, as more IG variables are needed but the computing time mainly depends on the time of generating the seed. Hence, when the acceptance rate of SSR becomes smaller for a larger tilting parameter β, we can replace the simulation scheme of the seed by DR scheme to improve the simulation speed.

Table 1: Comparison of CPU time for backward recursive (BR), simple stable rejection (SSR), double rejection (DR) for $TS\left(\frac{q}{2^n}, \theta, \beta\right)$ based on parameter setting $q = 1, 3, 23, 27, 35$, $n = 2, 3, 5, 6, 7$, $\theta = 0.50, 0.70$, $\beta = 1.0, 2.0$; each value in the tables is produced from 100, 000 replications

α	$1/2^2$	$1/2^3$	$1/2^5$	$3/2^2$	$3/2^3$	$3/2^5$	$23/2^6$	$27/2^6$	$35/2^7$
				$\theta = 0.50$	$\beta = 1.0$				
BR	1.58	2.20	2.34	1.25	7.62	7.50	4.45	4.36	8.85
SSR	7.45	47.25	2152.53	7.53	6.52	178.39	4.97	4.47	6.15
DR	25.31	23.85	24.89	27.94	23.67	23.48	26.68	25.34	31.52
				$\theta = 0.70$	$\beta = 2.0$				
BR	1.42	2.14	2.87	1.29	8.21	8.29	30.78	45.45	32.53
SSR	54.41	581.38	3272.53	257.76	32.28	3681.59	35.25	39.65	28.93
DR	41.23	37.72	32.89	34.24	30.78	31.45	31.12	32.65	35.12

5 Concluding Remarks

In this paper, we provide a new framework for the exact simulation of a class of TS and related distributions based on the multiple integral representation. The principle of our approach, which is based on the backward recursion, is clearly distinguishing from the existing algorithms in the literature. It works pretty efficiently for some subclasses of TS distributions, and is also applicable to some other classes beyond the TS family. This scheme could lead to many prospective applications in practice with improved numerical efficiency: It could be used as the basis to exactly generate TS-constructed r.v.s. For instance, it is well known that, one two-sided TS r.v. on the real line $\mathbb{R}$ can be decomposed as the difference of two independent one-sided TS r.v.s on the positive half line $\mathbb{R}^+$(Küchler and Tappe, 2013, p.4262). Moreover, it can also be used to generate paths of various TS-driven stochastic processes, such as *Lévy TS subordinators*; see an abundance of their applications for modelling the stochastic volatility with the associated econometric methods

Table 2: Sensitivity analysis of the CPU time against the varying parameters $(\theta, \beta) \in (0, 1] \times (0, 1]$ for our *backward recursive* (BR) scheme (via Algorithm 3.2) for $\alpha = 1/2^n$, $n = 2, 3, 4, 5$, respectively; each value in the tables is produced from $100,000$ replications

β \ θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
					$\alpha = 1/4$					
0.2	1.34	1.17	1.23	1.28	1.42	1.19	1.30	1.34	1.30	1.39
0.6	1.16	1.20	1.22	1.14	1.22	1.13	1.13	1.22	1.16	1.30
1.0	1.11	1.14	1.17	1.19	1.13	1.13	1.22	1.14	1.16	1.25
					$\alpha = 1/8$					
0.2	2.52	2.19	2.17	2.59	2.33	2.19	2.22	2.30	2.14	2.20
0.6	2.45	2.20	2.19	2.42	2.16	2.13	2.16	2.19	2.25	2.22
1.0	2.50	2.19	2.17	2.41	2.23	2.16	2.25	2.11	2.09	2.11
					$\alpha = 1/16$					
0.2	2.91	3.08	2.55	2.47	2.92	2.92	2.58	2.44	2.84	2.48
0.6	2.91	2.91	2.45	2.52	2.91	3.19	2.47	2.52	2.89	2.53
1.0	2.91	2.88	2.42	2.52	2.88	2.88	2.48	2.50	2.89	2.64
					$\alpha = 1/32$					
0.2	2.88	2.81	2.88	3.33	2.94	2.80	2.81	2.81	3.38	2.91
0.6	2.88	3.05	2.92	3.45	2.94	2.81	2.84	2.80	3.27	2.89
1.0	2.88	2.80	2.81	3.28	2.84	2.73	2.80	2.73	3.31	2.84

in Barndorff-Nielsen and Shephard (2001b, 2002, 2003), Barndorff-Nielsen et al. (2002), Gander and Stephens (2007a,b), Andrieu et al. (2010) and Todorov et al. (2015). These r.v.s and stochastic processes could further lead to numerous applications in many other fields. In particular, it would be extremely useful for numerical validating and testing the newly developed statistical inference or econometric methods for the TS-based models, where the parameters are set up for Monte Carlo studies. Our Algorithm 3.2 for the $\frac{1}{2^n}$-family is strongly recommended to be adopted for generating the required data, and a variety of choices for the parameters n, β and θ would provide enough flexibility for the purpose of numerical testing and validation. Since it is a bit of an art to find appropriate envelops in the development of A/R schemes, the further enhancement for these A/R schemes adopted in this paper could be a meaningful topic for future research.

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Appendices

A Simple Stable Rejection (SSR) Scheme

The Simple Stable Rejection (SSR) scheme introduced by Brix (1999) is illustrated in Algorithm A.1 as below.

Algorithm A.1 Simple Stable Rejection (SSR) Scheme for $TS(\alpha, \beta, \theta)$

1. **repeat**{
2. **sample** $U \sim \mathcal{U}[0, \pi]$, $E_s \sim Exp(1)$ and $V \sim \mathcal{U}[0, 1]$
3. **set** $S = (-\theta\Gamma(-\alpha))^{-\frac{1}{\alpha}} \sin(\alpha U + \pi\alpha/2) \cos(U)^{-\frac{1}{\alpha}} \cos\big((1 - \alpha)U - \pi\alpha/2\big)^{\frac{1-\alpha}{\alpha}} E_s^{-\frac{1-\alpha}{\alpha}}$
4. **if** $(V \leq \exp(-\beta S))$ **break**
5. }
6. **return** S

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