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On some functionals of the first passage times in models with switching stochastic volatility

Pavel V. Gapeev*† Oliver Brockhaus‡ Mathieu Dubois‡

We compute some functionals related to the joint generalised Laplace transforms of the first times at which two-dimensional diffusion-type Markov processes exit half strips. It is assumed that the state space components are driven by constantly correlated Brownian motions and the dynamics of the coefficients are described by a continuous-time Markov chain. The method of proof is based on the solutions of the equivalent boundary-value problems for systems of elliptic-type partial differential equations for the associated value functions. The results are illustrated on several two-dimensional continuous mean-reverting or diverting models of switching stochastic volatility.

1. Introduction

The main aim of this paper is to derive closed-form expressions for the functionals in (2.14)-(2.15) of the first passage times of the two-dimensional diffusion-type process (S, Q) defined in (2.1)-(2.2). These functionals are related to the joint generalised Laplace transforms in (2.6)-(2.7) and (2.8)-(2.9) of the first times at which the continuous process (S, Q) exits certain regions forming half strips. It is assumed that the stochastic differential equations in (2.1)-(2.2) for (S, Q) are driven by constantly correlated standard Brownian motions and the local drift and diffusion coefficients of Q are switching according to the dynamics of a continuous-time Markov chain. Note that such a model can be used for the description of dynamics of the risky asset prices with stochastic volatility rates which play a central role in the modelling of financial assets (see, e.g. Fouque et al. [12], Kallsen [25] and Gatheral [17] for an overview). For simplicity of presentation, we assume that the process Q solving the equation in (2.2)

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has a linear diffusion coefficient that particularly corresponds to the Black-Karasinski or Cox-Ingersoll-Ross models which are usually used for modelling of dynamics of interest or volatility rates (see, e.g. Shiryaev [35; Chapter III, Section 4a] for an overview). In the latter case, the process (S, Q) forms the Stein-Stein or Heston model of stochastic volatility.

The models of financial markets in which the parameter values are switching according to the dynamics of continuous-time Markov chains have recently found a considerable amount of applications. For instance, the closed-form solutions to the perpetual American lookback and put option pricing problems were obtained by Guo [18] and Guo and Zhang [19] in an extension of such a diffusion model in which both the drift and volatility coefficients of the underlying asset price process are switching between two constant values, according to the change in the state of the observable continuous-time Markov chain. Jobert and Rogers [22] considered the perpetual American put option problem within an extension of that model to the case of several states for the Markov chain and solved numerically the corresponding problem with finite expiry. In the model with a two-state Markov chain and no diffusion part, Dalang and Hongler [6] presented a complete and essentially explicit solution to a similar problem, which exhibited a surprisingly rich structure. These results were further extended by Jiang and Pistorius [21], who studied the perpetual American put option problem within the framework of an exponential jump-diffusion model with observable dynamics of regime-switching behaving parameters. A similar model for the pricing of European options, in which the underlying dividend process is given by a diffusion process with Markov-modulated coefficients, was considered by Di Graziano and Rogers [8] (see also other related references therein).

Mixture models such as the local stochastic volatility model allow to blend features from different models into a single model. In such models the weight of each model is specified initially. For example, in the case of local stochastic volatility models this is achieved through dampening volatility of volatility, (see, e.g. Brockhaus [5; page 85] and the reference therein). In financial markets, one observes regimes. In other words, asset dynamics may be well described by a set of parameters initially and by another set of parameters which prevail after a random time. Regime switching models seem more satisfactory than mixture models since they switch between different models at random times rather than blending two models into a single largely homogenous model with fixed proportion.

The joint distribution law of the first hitting times of constant boundaries for two constantly correlated drifted Brownian motions was obtained by Iyengar [20]. Analytic expressions for the Laplace transforms of the first passage times of compound Poisson processes over linear boundaries were computed in Zacks et al. [36] in the case of positive jumps and in Perry et al. [31]-[30] in certain cases of positive and negative jumps. Kou and Wang [26] and Sepp [34] derived closed-form expressions for the Laplace transforms of the first hitting times over constant boundaries for double-exponential jump-diffusion processes. Other related stopping problems arising from the computation of the Laplace transforms of the first passage times of more complicated spectrally positive and negative Lévy processes over constant levels were recently considered by Mijatović and Pistorius [28]. Monte Carlo schemes for the computation of the distribution of the first exit times of jump-diffusion processes from two-sided intervals in the general size distribution case were developed in Fernandez et al. [11]. The Laplace transforms of the first passage times from intervals for mean-reverting and diverting one-dimensional jump-diffusion processes were recently computed in Gapeev and Stoev [15]. Some functionals related to the generalised Laplace transforms of the first times at which some two-dimensional jump-diffusion processes exit half strips were recently computed in Gapeev and Stoev [16]. In the present paper, we derive closed-form expressions for the functionals related to the generalised joint Laplace transforms of the first passage times as stopping problems for two-dimensional diffusion-type Markov processes with switching coefficients.

It is well known that optimal stopping problems for multi-dimensional continuous-time Markov processes are analytically more difficult than the corresponding problems for the one-dimensional ones and their solutions are very rarely found explicitly. Some necessarily multi-dimensional optimal stopping problems arising mostly from the problems of quickest change-point detection were studied by Bayraktar and Poor [3] and Bayraktar et al. [4] for discontinuous Poisson processes, Dayanik et al. [7] for mixed jump-diffusion processes with mean-reverting components, as well as in Gapeev and Shiryaev [13]-[14] and Johnson and Peskir [23]-[24] for purely continuous diffusion processes. Some analytical results for such optimal stopping problems were recently obtained by Assing et al. [2]. In the present paper, we obtain closed-form solutions to the boundary-value problems for systems of elliptic-type partial differential equations, which are equivalent to the original stopping problems in two-dimensional continuous models of switching stochastic volatility.

The paper is organised as follows. In Section 2, we first introduce the setting and notation of the model with a two-dimensional continuous process which has the price of a risky asset and the (mean-reverting) volatility rate as the state space components. It is assumed that the driving standard Brownian motions are constantly correlated and the local drift and linear diffusion coefficients change according to the dynamics of a continuous-time Markov chain. We define the functionals related to the generalised joint Laplace transforms of the first exit times from half strips of the two-dimensional diffusion-type process and formulate the equivalent boundary-value problem for an elliptic-type partial differential operator. In Section 3, we obtain a closed-form solution to the elliptic partial differential boundary-value problem and show that the value function represents the product of solutions of the associated ordinary problems. We derive explicit expressions for the considered functionals in several classical models of switching stochastic volatility. In Section 4, we show that the solutions to the boundary-value problems provide the original functionals of the first exit times.

2 Preliminaries

In this section, we introduce the setting and notation in the problem of computation of some functionals related to the generalised joint Laplace transforms of the first exit times in diffusion-type models of switching stochastic volatility and formulate the associated boundary-value problems.

2.1 The model. Let us consider a probability space $(\Omega, \mathcal{F}, P)$ supporting two constantly correlated standard Brownian motions $B^i = (B^i_t)_{t \ge 0}$, $i = 1, 2$, such that $\langle B^1, B^2 \rangle_t = \rho t$, for some $\rho \in (-1, 1)$ fixed, and a continuous-time Markov chain $\Theta = (\Theta_t)_{t \ge 0}$ with two states, 0 and 1. Suppose that the processes $B^i = (B^i_t)_{t \ge 0}$, $i = 1, 2$, and Θ are independent. Assume that Θ has the initial distribution $\{1-\pi, \pi\}$, the transition-probability matrix $\{(\lambda_0 e^{-(\lambda_0+\lambda_1)t} + \lambda_1)/(\lambda_0 + \lambda_1), \lambda_0(1 - e^{-(\lambda_0+\lambda_1)t})/(\lambda_0 + \lambda_1); \lambda_1(1 - e^{-(\lambda_0+\lambda_1)t})/(\lambda_0 + \lambda_1), (\lambda_1 e^{-(\lambda_0+\lambda_1)t} + \lambda_0)/(\lambda_0 + \lambda_1)\}$, and the intensity-matrix $\{-\lambda_0, \lambda_0; \lambda_1, -\lambda_1\}$, for all $t \ge 0$, and some $\pi \in [0, 1]$ and $\lambda_i > 0$, $i = 0, 1$, fixed. In other words, the Markov chain Θ changes its state from j to $1-j$ at exponentially distributed times of intensity λ_i, for every $i = 0, 1$, which are independent of the dynamics of the standard Brownian motions B^i, $i = 1, 2$. Such a process Θ is called a *telegraphic signal* in the literature (see, e.g. [27; Chapter IX, Section 4] or [10; Chapter VIII]). Assume that there exists a process $(S, Q) = (S_t, Q_t)_{t \ge 0}$ which provides a (pathwise) unique solution of the system of stochastic differential equations

$$dS_t = S_t \, \delta \sigma^2(Q_t) \, dt + S_t \, \varepsilon \sigma(Q_t) \, dB_t^1 \quad (S_0 = s) \tag{2.1}$$

and

$$dQ_t = \big((1 - \Theta_t) \, \beta_0(Q_t) + \Theta_t \, \beta_1(Q_t) \big) \, dt + \big((1 - \Theta_t) \, \gamma_0 + \Theta_t \, \gamma_1 \big) \, Q_t \, dB_t^2 \quad (Q_0 = q), \tag{2.2}$$

for some $s, q > 0$ fixed, where $\delta \in \mathbb{R}$, $\varepsilon > 0$, and $\gamma_j \ge 0$, $j = 1, 2$, are some constants, and $\sigma(q) > 0$ and $\beta_j(q) \in \mathbb{R}$, $j = 1, 2$, are continuously differentiable functions of at most linear growth on $(0, \infty)$ (see, e.g. [27; Chapter IV, Theorem 4.6] and [29; Chapter V, Theorem 5.2.1] for the existence and uniqueness of solutions of such stochastic differential equations).

Observe that the process S solving the equation in (2.1) admits the representation

$$S_t = s \, \exp \bigg(\int_0^t \Big(\delta - \frac{\varepsilon^2}{2} \Big) \, \sigma^2(Q_u) \, du + \int_0^t \varepsilon \sigma(Q_u) \, dB_u^1 \bigg), \tag{2.3}$$

for all $t \ge 0$. Note that the process (Q, Θ) forms a two-dimensional (strong) Markov jump-diffusion process, while (S, Q, Θ) provides a three-dimensional Markov jump-diffusion process. Without loss of generality and because of the nature of the problems as well as the examples considered below, we can further assume that the state space of the process Q is $(0, \infty)$, so that the state space of the process (S, Q, Θ) is $(0, \infty)^2 \times \{0, 1\}$. In this case, the process S can describe the price of a risky asset on a financial market and Q can represent its volatility rate. Let us finally define the associated with the processes S and Q first passage (stopping) times

$$\tau_a^- = \inf\{t \ge 0 \,|\, S_t \le a\} \quad \text{and} \quad \tau_b^+ = \inf\{t \ge 0 \,|\, S_t \ge b\}, \tag{2.4}$$

as well as

$$\zeta_g^- = \inf\{t \ge 0 \,|\, Q_t \le g\} \quad \text{and} \quad \zeta_h^+ = \inf\{t \ge 0 \,|\, Q_t \ge h\}, \tag{2.5}$$

for some $0 < a < b < \infty$ and $0 \le g < h < \infty$ fixed. The main aim in the present paper is to derive closed form expressions for some functionals related to the generalised joint Laplace transforms of the random times τ_a^-, τ_b^+ and ζ_g^-, ζ_h^+.

2.2 The generalised joint Laplace transforms. Let us first introduce the functionals $\overline{V}_1(a; g, h)$ and $\overline{V}_2(b; g, h)$ given by

$$\begin{aligned} \overline{V}_1(a; g, h) &= E\Big[e^{-\eta A_{\tau_a^-} - \varkappa A_{\zeta_h^+}} \, I(\tau_a^- < \infty, \zeta_h^+ < \zeta_g^-) \Big] \\ &= E\Big[e^{-(\eta+\varkappa) A_{\tau_a^-}} \, e^{-\varkappa (A_{\zeta_h^+} - A_{\tau_a^-})} \, I(\tau_a^- < \zeta_h^+ < \zeta_g^-) \\ &\qquad + e^{-(\eta+\varkappa) A_{\zeta_h^+}} \, e^{-\eta (A_{\tau_a^-} - A_{\zeta_h^+})} \, I(\zeta_h^+ < \zeta_g^- \le \tau_a^- < \infty) \Big] \end{aligned} \tag{2.6}$$

and

$$\overline{V}_2(b;g,h) = E\Big[e^{-\eta A_{\tau_b^+} - \varkappa A_{\zeta_h^+}} \, I(\tau_b^+ < \infty, \zeta_h^+ < \zeta_g^-)\Big] \tag{2.7}$$
$$= E\Big[e^{-(\eta+\varkappa) A_{\tau_b^+}} \, e^{-\varkappa (A_{\zeta_h^+} - A_{\tau_b^+})} \, I(\tau_b^+ < \zeta_h^+ < \zeta_g^-)$$
$$+ e^{-(\eta+\varkappa) A_{\zeta_h^+}} \, e^{-\eta (A_{\tau_b^+} - A_{\zeta_h^+})} \, I(\zeta_h^+ < \zeta_g^- \le \tau_b^+ < \infty)\Big],$$

as well as the functionals $\widehat{V}_1(a;g,h)$ and $\widehat{V}_2(b;g,h)$ given by

$$\widehat{V}_1(a;g,h) = E\Big[e^{-\eta A_{\tau_a^- \wedge \zeta_h^+} - \varkappa A_{\tau_a^- \vee \zeta_h^+}} \, I(\tau_a^- < \infty, \zeta_h^+ < \zeta_g^-)\Big] \tag{2.8}$$
$$= E\Big[e^{-(\eta+\varkappa) A_{\tau_a^-}} \, e^{-\varkappa (A_{\zeta_h^+} - A_{\tau_a^-})} \, I(\tau_a^- < \zeta_h^+ < \zeta_g^-)$$
$$+ e^{-(\eta+\varkappa) A_{\zeta_h^+}} \, e^{-\varkappa (A_{\tau_a^-} - A_{\zeta_h^+})} \, I(\zeta_h^+ < \zeta_g^- \le \tau_a^- < \infty)\Big]$$

and

$$\widehat{V}_2(b;g,h) = E\Big[e^{-\eta A_{\tau_b^+ \wedge \zeta_h^+} - \varkappa A_{\tau_b^+ \vee \zeta_h^+}} \, I(\tau_b^+ < \infty, \zeta_h^+ < \zeta_g^-)\Big] \tag{2.9}$$
$$= E\Big[e^{-(\eta+\varkappa) A_{\tau_b^+}} \, e^{-\varkappa (A_{\zeta_h^+} - A_{\tau_b^+})} \, I(\tau_b^+ < \zeta_h^+ < \zeta_g^-)$$
$$+ e^{-(\eta+\varkappa) A_{\zeta_h^+}} \, e^{-\varkappa (A_{\tau_b^+} - A_{\zeta_h^+})} \, I(\zeta_h^+ < \zeta_g^- \le \tau_b^+ < \infty)\Big],$$

for some $\eta, \varkappa > 0$ fixed, where the process $A = (A_t)_{t \ge 0}$ is defined by

$$A_t = \int_0^t \sigma^2(Q_u)\, du, \tag{2.10}$$

for all $t \ge 0$.

Since the functionals in (2.6)-(2.7) and (2.8)-(2.9) may not generally admit closed-form expressions, we further consider some modified generalised joint Laplace transforms of the random times τ_a^-, τ_b^+ and ζ_g^-, ζ_h^+. For this purpose, let us now assume that the condition

$$E\bigg[\exp\bigg(\frac{1}{2}\int_0^{t \wedge \zeta_g^- \wedge \zeta_h^+} \frac{\rho^2 \varepsilon^2 \alpha_i \sigma^2(Q_u)}{1-\rho^2}\, du\bigg)\bigg] < \infty \tag{2.11}$$

holds, for all $t \ge 0$, where α_i, $i = 1, 2$, are some constants which are specified below. In this case, the processes $(\widetilde{M}^i_{t \wedge \zeta_g^- \wedge \zeta_h^+})_{t \ge 0}$, $i = 1, 2$, with

$$\widetilde{M}^i_t = \exp\bigg(\int_0^t \frac{\rho \varepsilon \alpha_i \sigma(Q_u)}{1-\rho^2}\, dB^2_u - \int_0^t \frac{\rho^2 \varepsilon \alpha_i \sigma(Q_u)}{1-\rho^2}\, dB^1_u - \frac{1}{2}\int_0^t \frac{\rho^2 \varepsilon^2 \alpha_i^2 \sigma^2(Q_u)}{1-\rho^2}\, du\bigg) \tag{2.12}$$

are uniformly integrable martingales. Then, it follows from [32; Chapter VIII, Proposition 1.13] that the probability measures $\widetilde{P}^i$, $i = 1, 2$, defined by

$$\frac{d\widetilde{P}^i}{dP}\bigg|_{\mathcal{F}_{t \wedge \zeta_g^- \wedge \zeta_h^+}} = \widetilde{M}^i_{t \wedge \zeta_g^- \wedge \zeta_h^+}, \tag{2.13}$$

for all $t \geq 0$, are locally equivalent to P on the natural filtration $(\mathcal{F}_{t \wedge \zeta_g^- \wedge \zeta_h^+})_{t \geq 0}$ generated by the driving processes B^i, $i = 1, 2$.

By applying the change-of-measure arguments, let us now define the functionals $V_1^*(a; g, h)$ and $V_2^*(b; g, h)$ by

$$V_1^*(a; g, h) = E\bigg[e^{-(\eta+\varkappa) A_{\tau_a^-}} \, \frac{\widetilde{M}^1_{\zeta_h^+}}{\widetilde{M}^1_{\tau_a^-}} \, e^{-\varkappa (A_{\zeta_h^+} - A_{\tau_a^-})} \, I(\tau_a^- < \zeta_h^+ < \zeta_g^-) \tag{2.14}$$

$$+ \, e^{-(\eta+\varkappa) A_{\zeta_h^+}} \, e^{-\eta (A_{\tau_a^-} - A_{\zeta_h^+})} \, I(\zeta_h^+ < \zeta_g^- \leq \tau_a^- < \infty) \bigg]$$

$$= E\Big[e^{-(\eta+\varkappa) A_{\tau_a^-}} \, \widetilde{E}^1 \Big[e^{-\varkappa (A_{\zeta_h^+} - A_{\tau_a^-})} \, \Big| \, \mathcal{F}_{\tau_a^-} \Big] \, I(\tau_a^- < \zeta_h^+ < \zeta_g^-)$$

$$+ \, e^{-(\eta+\varkappa) A_{\zeta_h^+}} \, E\Big[e^{-\eta (A_{\tau_a^-} - A_{\zeta_h^+})} \, \Big| \, \mathcal{F}_{\zeta_h^+} \Big] \, I(\zeta_h^+ < \zeta_g^- \leq \tau_a^- < \infty) \Big]$$

and

$$V_2^*(b; g, h) = E\bigg[e^{-(\eta+\varkappa) A_{\tau_b^+}} \, \frac{\widetilde{M}^2_{\zeta_h^+}}{\widetilde{M}^2_{\tau_b^+}} \, e^{-\varkappa (A_{\zeta_h^+} - A_{\tau_b^+})} \, I(\tau_b^+ < \zeta_h^+ < \zeta_g^-) \tag{2.15}$$

$$+ \, e^{-(\eta+\varkappa) A_{\zeta_h^+}} \, e^{-\eta (A_{\tau_b^+} - A_{\zeta_h^+})} \, I(\zeta_h^+ < \zeta_g^- \leq \tau_b^+ < \infty) \bigg]$$

$$= E\Big[e^{-(\eta+\varkappa) A_{\tau_b^+}} \, \widetilde{E}^2 \Big[e^{-\varkappa (A_{\zeta_h^+} - A_{\tau_b^+})} \, \Big| \, \mathcal{F}_{\tau_b^+} \Big] \, I(\tau_a^- < \zeta_h^+ < \zeta_g^-)$$

$$+ \, e^{-(\eta+\varkappa) A_{\zeta_h^+}} \, E\Big[e^{-\eta (A_{\tau_b^+} - A_{\zeta_h^+})} \, \Big| \, \mathcal{F}_{\zeta_h^+} \Big] \, I(\zeta_h^+ < \zeta_g^- \leq \tau_b^+ < \infty) \Big],$$

where $\widetilde{E}^i$ denotes the expectation taken under the probability measure $\widetilde{P}^i$, for $i = 1, 2$.

Taking into account the strong Markov property, we observe that the values in (2.14)-(2.15) take the form

$$V_1^*(a; g, h) = E\Big[(1 - \Theta_0) V_{1,0}^*(S_0, Q_0) + \Theta_0 V_{1,1}^*(S_0, Q_0) \Big] = (1 - \pi) V_{1,0}^*(s, q) + \pi V_{1,1}^*(s, q) \tag{2.16}$$

and

$$V_2^*(b; g, h) = E\Big[(1 - \Theta_0) V_{2,0}^*(S_0, Q_0) + \Theta_0 V_{2,1}^*(S_0, Q_0) \Big] = (1 - \pi) V_{2,0}^*(s, q) + \pi V_{2,1}^*(s, q), \tag{2.17}$$

for any $s, q > 0$ and $\pi \in [0, 1]$ fixed. Here, the functions $V_{1,j}^*(s, q) = V_{1,j}^*(s, q; a; g, h)$ and $V_{2,j}^*(s, q) = V_{2,j}^*(s, q; b; g, h)$ are defined by

$$V_{1,j}^*(s, q) = E_{s,q,j} \Big[e^{-(\eta+\varkappa) A_{\tau_a^-}} \, \big((1 - \Theta_{\tau_a^-}) \, U_{1,0}^*(Q_{\tau_a^-}) + \Theta_{\tau_a^-} \, U_{1,1}^*(Q_{\tau_a^-}) \big) \, I(\tau_a^- < \zeta_h^+ < \zeta_g^-) \tag{2.18}$$

$$+ \, e^{-(\eta+\varkappa) A_{\zeta_h^+}} \, \big((1 - \Theta_{\zeta_h^+}) \, W_{1,0}^*(S_{\zeta_h^+}, Q_{\zeta_h^+}) + \Theta_{\zeta_h^+} \, W_{1,1}^*(S_{\zeta_h^+}, Q_{\zeta_h^+}) \big) \, I(\zeta_h^+ < \zeta_g^- \leq \tau_a^- < \infty) \Big]$$

and

$$V_{2,j}^*(s, q) = E_{s,q,j} \Big[e^{-(\eta+\varkappa) A_{\tau_b^+}} \, \big((1 - \Theta_{\tau_b^+}) \, U_{2,0}^*(Q_{\tau_b^+}) + \Theta_{\tau_b^+} \, U_{2,1}^*(Q_{\tau_b^+}) \big) \, I(\tau_b^+ < \zeta_h^+ < \zeta_g^-) \tag{2.19}$$

$$+ \, e^{-(\eta+\varkappa) A_{\zeta_h^+}} \, \big((1 - \Theta_{\zeta_h^+}) \, W_{2,0}^*(S_{\zeta_h^+}, Q_{\zeta_h^+}) + \Theta_{\zeta_h^+} \, W_{2,1}^*(S_{\zeta_h^+}, Q_{\zeta_h^+}) \big) \, I(\zeta_h^+ < \zeta_g^- \leq \tau_b^+ < \infty) \Big],$$

where the functions $W_{1,j}^*(s,q) = W_{1,j}^*(s,q;a)$ and $W_{2,j}^*(s,q) = W_{2,j}^*(s,q;b)$, as well as $U_{i,j}^*(q)$, $i=1,2$, $j=0,1$, are given by

$$W_{1,j}^*(s,q) = E_{s,q,j}\Big[e^{-\eta A_{\tau_a^-}} \, I(\tau_a^- < \infty)\Big] \quad \text{and} \quad W_{2,j}^*(s,q) = E_{s,q,j}\Big[e^{-\eta A_{\tau_b^+}} \, I(\tau_b^+ < \infty)\Big], \tag{2.20}$$

as well as

$$U_{i,j}^*(q) = \widetilde{E}_{q,j}^i\Big[e^{-\varkappa A_{\zeta_h^+}} \, I(\zeta_h^+ < \zeta_g^-)\Big], \tag{2.21}$$

for $\eta, \varkappa > 0$ fixed, and all $s, q > 0$ and every $j = 0,1$. We denote by $\widetilde{E}_{y,j}^i$ and $E_{s,y,j}$ the expectations with respect to the probability measures $\widetilde{P}^i$, $i=1,2$, and P taken under the assumption that the two-dimensional process (Q,Θ) and the three-dimensional process (S,Q,Θ) start at (q,j) and (s,q,j) with some $s,q>0$ and $j=0,1$, respectively. We finally note that the stochastic differential equation in (2.2) for the process Y admits the representation

$$dQ_t = \big((1-\Theta_t)\,\widetilde{\beta}_{i,0}(Q_t) + \Theta_t\,\widetilde{\beta}_{i,1}(Q_t)\big)\,dt + \big((1-\Theta_t)\,\gamma_0 + \Theta_t\,\gamma_1\big)\,Q_t\,d\widetilde{B}_t^2 \quad (Q_0 = q) \tag{2.22}$$

with $\widetilde{\beta}_{i,j}(q) = \beta_j(q) + \rho\varepsilon\alpha_i\sigma(q)\gamma_j q$, for all $q>0$, and every $i=1,2$ and $j=0,1$. Here, by means of Girsanov's theorem for diffusion-type processes (see, e.g. [27; Chapter VII, Theorem 7.19]), the processes $\widetilde{B}^{i,k} = (\widetilde{B}_t^{i,k})_{t\ge 0}$, $k=1,2$, defined by

$$\widetilde{B}_t^{i,1} = B_t^1 \quad \text{and} \quad \widetilde{B}_t^{i,2} = B_t^2 - \rho\varepsilon\alpha_i \int_0^t \sigma(Q_u)\,du \tag{2.23}$$

are standard Brownian motions such that $\langle \widetilde{B}^{i,1}, \widetilde{B}^{i,2}\rangle_t = \rho t$, for all $t \ge 0$ and every $i=1,2$.

2.3 The boundary-value problems. By means of standard arguments based on Itô's formula (see, e.g. [27; Chapter IV, Theorem 4.4] or [32; Chapter IV, Theorem 3.3]), it can be shown that the infinitesimal operator $\mathbb{L}_{(S,Q)}^j$ of the process (S,Q) from (2.1)-(2.2) under the probability measure P acts on a bounded function $V(s,q)$ from the class $C^{2,2}((0,\infty)^2)$ according to the rule

$$\begin{aligned}(\mathbb{L}_{(S,Q)}^j V)(s,q) &= \delta\sigma^2(q)\,s\,\partial_s V(s,q) + \frac{\varepsilon^2\sigma^2(q)}{2}\,s^2\,\partial_{ss}V(s,q) \\ &+ \beta_j(q)\,\partial_q V(s,q) + \frac{\gamma_j^2 q^2}{2}\,\partial_{qq}V(s,q) + \rho\,\varepsilon\sigma(q)s\,\gamma_j q\,\partial_{sq}V(s,q),\end{aligned} \tag{2.24}$$

for all $s,q>0$ and every $j=0,1$. The infinitesimal operator $\widetilde{\mathbb{L}}_Q^{i,j}$ of the process Q from (2.22) under the probability measure $\widetilde{P}^i$ acts on a function $U(q)$ from the class $C^2((0,\infty))$ like

$$(\widetilde{\mathbb{L}}_Q^{i,j}U)(q) = \big(\beta_j(q) + \rho\varepsilon\alpha_i\sigma(q)\gamma_j\,q\big)\,U'(q) + \frac{\gamma_j^2 q^2}{2}\,U''(q), \tag{2.25}$$

for all $q>0$ and every $i=1,2$, $j=0,1$.

In order to find analytic expressions for the unknown value functions from (2.18)-(2.19), let us use the results of general theory of Markov processes (see, e.g. [9; Chapter V]). We reduce the problems of (2.18)-(2.19) for the functions $V^*_{i,j}(s,q)$, $i=1,2$, $j=0,1$, to the equivalent boundary-value problems for the operator $\mathbb{L}^j_{(S,Q)}$ of the form

$$
\begin{aligned}
&(\mathbb{L}^j_{(S,Q)} V_{i,j} - (\eta + \varkappa + \lambda_j)\, \sigma^2(q)\, V_{i,j})(s,q) = -\lambda_j\, V_{i,1-j}(s,q), \quad \text{for} \quad s > a \quad \text{or} \quad s < b, && (2.26)\\
&V_{i,j}(s,q) = U_{i,j}(q), \quad \text{for} \quad s \le a \quad \text{or} \quad s \ge b \quad \text{and} \quad g \le q \le h, && (2.27)\\
&V_{i,j}(s,q) = W_{i,j}(s,q), \quad \text{for} \quad q \le g \quad \text{and} \quad q \ge h \quad \text{and} \quad s \ge a \quad \text{or} \quad s \le b, && (2.28)\\
&V_{1,j}(a+,q) = U_{1,j}(q) \quad \text{or} \quad V_{2,j}(b-,q) = U_{2,j}(q), \quad \text{for} \quad g \le q \le h, && (2.29)\\
&V_{1,j}(s,g+) = W_{1,j}(s,g+) \quad \text{and} \quad V_{2,j}(s,h-) = W_{2,j}(s,h-), \quad \text{for} \quad s \ge a \quad \text{or} \quad s \le b, && (2.30)
\end{aligned}
$$

respectively. Here, the functions $W_{i,j}(s,q)$, $i=1,2$, $j=0,1$, solve the problems for the operator $\mathbb{L}^j_{(S,Q)}$ of the form

$$
\begin{aligned}
&(\mathbb{L}^j_{(S,Q)} W_{i,j} - \eta\, \sigma^2(q)\, W_{i,j})(s,q) = 0, \quad \text{for} \quad s > a \quad \text{or} \quad s < b, && (2.31)\\
&W_{1,j}(s,q) = 1, \quad \text{for} \quad s \le a, \quad \text{and} \quad W_{2,j}(s,q) = 1, \quad \text{for} \quad s \ge b, && (2.32)\\
&W_{1,j}(a+,q) = 1 \quad \text{and} \quad W_{2,j}(b-,q) = 1, && (2.33)
\end{aligned}
$$

for all $q > 0$, while the function $U_{i,j}(q)$ solves the boundary value problem for the operator $\widetilde{\mathbb{L}}^{i,j}_Q$ of the form

$$
\begin{aligned}
&(\widetilde{\mathbb{L}}^{i,j}_Q U_{i,j} - (\varkappa + \lambda_j)\, \sigma^2(q)\, U_{i,j})(q) = -\lambda_j\, U_{i,1-j}(q), \quad \text{for} \quad g < q < h, && (2.34)\\
&U_{i,j}(q) = 0, \quad \text{for} \quad q \le g, \quad \text{and} \quad U_{i,j}(q) = 1, \quad \text{for} \quad q \ge h, && (2.35)\\
&U_{i,j}(g+) = 0 \quad \text{and} \quad U_{i,j}(h-) = 1, && (2.36)
\end{aligned}
$$

for $i = 1,2$ and $j = 0,1$.

Observe that the continuity conditions of (2.29)-(2.30), (2.33), and (2.36) hold in the cases in which the processes S and Q can pass continuously through the points a, b and g, h, respectively. On the other hand, for instance, when $\beta_{i,0}(q) = \gamma_{i,0} = 0$ holds, for all $q > 0$, the stochastic differential equation in (2.30) and (2.2) for Q does not contain the local drift and diffusion parts, so that, the process Q remains constant until the Markov chain Θ changes its state from 0 to 1. In this case, the function $U^*_{i,0}(q)$, $i = 1,2$, may be discontinuous at the boundaries g or h, and thus, the conditions of (2.30) and (2.36) may fail to hold, for $j = 0$.

3 Solutions to the boundary-value problems

In this section, we derive closed-form expressions for the solutions of the boundary-value problems associated with the value functions in (2.18)-(2.21).

3.1 Solutions to the system in (2.34)-(2.36). (i) Let us first assume that $\gamma_j > 0$ and $\lambda_j > 0$ holds, for every $j = 0,1$. In this case, since the coupled system of inhomogeneous second-order linear ordinary differential equations in (2.34) with (2.25) is equivalent to a homogeneous fourth-order linear ordinary differential equation, we may conclude from the general theory that the general solutions take the form

$$
U_{i,j}(q) = C_{i,j,1}\, U_{i,j,1}(q) + C_{i,j,2}\, U_{i,j,2}(q) + C_{i,j,3}\, U_{i,j,3}(q) + C_{i,j,4}\, U_{i,j,4}(q), \tag{3.1}
$$

where $C_{i,j,k}$, $i = 1, 2$, $j = 0, 1$, $k = 1, 2, 3, 4$, are some arbitrary constants. Here, the functions $U_{i,j,k}(q)$, $i = 1, 2$, $j = 0, 1$, $k = 1, 2, 3, 4$, represent the four fundamental positive solutions of the system in (2.34) with (2.25). Without loss of generality, it can be assumed that $U_{i,j,k}(q)$, $i = 1, 2$, $j = 0, 1$, $k = 1, 2, 3, 4$, are (strictly) decreasing and increasing (convex) functions satisfying the properties $U_{i,j,k}(0+) = \infty$ and $U_{i,j,k}(\infty) = +0$, for $k = 1, 2$, as well as $U_{i,j,k}(0+) = +0$ and $U_{i,j,k}(\infty) = \infty$, for $k = 3, 4$, respectively. Moreover, because of the specific structure of the coupled system of second-order linear ordinary differential equations in (2.34) with (2.25), we may also assume that the fundamental solutions $U_{i,j,k}(q)$, $j = 0, 1$, $k = 1, 2, 3, 4$, are chosen such that the equation

$$(\widetilde{\mathbb{L}}_Q^{i,0} U_{i,0,k} - (\varkappa + \lambda_0)\, U_{i,0,k})(q)\, (\widetilde{\mathbb{L}}_Q^{i,1} U_{i,1,k} - (\varkappa + \lambda_1)\, U_{i,1,k})(q) = \lambda_0 \lambda_1\, U_{i,0,k}(q)\, U_{i,1,k}(q) \tag{3.2}$$

holds, for all $g < q < h$, and every $i = 1, 2$ and $k = 1, 2, 3, 4$.

Observe that, due to the fact that $U_{i,j,k}(q)$, $k = 1, 2, 3, 4$, represent a linearly independent system of functions, for each $g < q < h$, and every $i = 1, 2$ and $j = 0, 1$, we may conclude from the structure of the system in (2.34) with (2.25) that the equalities

$$\frac{C_{i,1-j,k}}{C_{i,j,k}} = -\frac{(\widetilde{\mathbb{L}}_Q^{i,j} U_{i,j,k} - (\varkappa + \lambda_j) U_{i,j,k})(q)}{\lambda_j U_{i,1-j,k}(q)} \tag{3.3}$$

hold, for every $i = 1, 2$, $j = 0, 1$, and $k = 1, 2, 3, 4$. Then, by applying the instantaneous-stopping conditions of (2.36) to the function in (3.1), we obtain that the equalities

$$C_{i,j,1}\, U_{i,j,1}(g+) + C_{i,j,2}\, U_{i,j,2}(g+) + C_{i,j,3}\, U_{i,j,3}(g+) + C_{i,j,4}\, U_{i,j,4}(g+) = 0, \tag{3.4}$$

$$C_{i,j,1}\, U_{i,j,1}(h-) + C_{i,j,2}\, U_{i,j,2}(h-) + C_{i,j,3}\, U_{i,j,3}(h-) + C_{i,j,4}\, U_{i,j,4}(h-) = 1 \tag{3.5}$$

hold, for every $i = 1, 2$ and $j = 0, 1$. Hence, we obtain that the candidate solution for $U_{i,j}(q)$ in the system of (2.34)-(2.36) admits the representation

$$U_{i,j}(q; g, h) = C_{i,j,1}(g, h) U_{i,j,1}(q) + C_{i,j,2}(g, h) U_{i,j,2}(q) + C_{i,j,3}(g, h) U_{i,j,3}(q) + C_{i,j,4}(g, h) U_{i,j,4}(q), \tag{3.6}$$

where the constants $C_{i,j,k}(g, h)$, $i = 1, 2$, $j = 0, 1$, $k = 1, 2, 3, 4$, are uniquely determined from the system of linear equations in (3.3) and (3.4)-(3.5). Note that, in the case of $g = 0$, we see that $C_{i,j,k} = 0$ should hold in (3.1), for $i, k = 1, 2$ and $j = 0, 1$, since otherwise $U_{i,j}(q) \to \pm\infty$ as $q \downarrow 0$, that must be excluded, by virtue of the fact that the function $U_{i,j}^*(q)$ in (2.21) is bounded. Therefore, we may conclude that the candidate solution has the form of (3.6), where $C_{i,j,k}(g, h)$, $i = 1, 2$, $j = 0, 1$, $k = 3, 4$, are uniquely determined by the system of (3.3) and (3.4)-(3.5).

(ii) Let us now assume that $\gamma_j > 0$, for $j = 0, 1$, and $\lambda_0 > \lambda_1 = 0$. In this case, in order to solve the system of two inhomogeneous second-order linear ordinary differential equations in (2.34) with (2.25), let us first consider the associated homogeneous equations

$$\frac{\gamma_j^2 q^2}{2}\, H_{i,j}''(q) + \big(\beta_j(q) + \rho\varepsilon\alpha_i\sigma(q)\gamma_j\, q\big)\, H_{i,j}'(q) - (\varkappa + \lambda_j)\, \sigma^2(q)\, H_{i,j}(q) = 0, \tag{3.7}$$

for every $i = 1, 2$ and $j = 0, 1$, the general solution of which takes the form

$$H_{i,j}(q) = \overline{C}_{i,j,1}\, H_{i,j,1}(q) + \overline{C}_{i,j,2}\, H_{i,j,2}(q), \tag{3.8}$$

where $\overline{C}_{i,j,k}$, $i,k=1,2$, are some arbitrary constants. Here, we denote the functions $H_{i,j,k}(q)$, $i,k=1,2$, $j=0,1$, as the two positive fundamental solutions (i.e. nontrivial linearly independent particular solutions) of the second-order ordinary differential equations in (3.7). Without loss of generality, we may assume that $H_{i,j,k}(q)$, $i,k=1,2$, $j=0,1$, are (strictly) decreasing and increasing (convex) functions satisfying the properties $H_{i,j,1}(0+)=\infty$ and $H_{i,j,1}(\infty)=+0$ as well as $H_{i,j,2}(0+)=+0$ and $H_{i,j,2}(\infty)=\infty$, respectively (see, e.g. [33; Chapter V, Section 50] for further details for the diffusion case).

Observe that the expression in (2.34) with (2.25), for $j=1$, becomes a homogeneous second-order ordinary linear differential equation of the form

$$\frac{\gamma_j^2 q^2}{2}\, U''_{i,1}(q) + \big(\beta_j(q) + \rho\varepsilon\alpha_i\sigma(q)\gamma_j\, q\big)\, U'_{i,1}(q) - \varkappa\,\sigma^2(q)\, U_{i,1}(q) = 0, \tag{3.9}$$

for $g<q<h$ and $i=1,2$, and its general solution takes the form

$$\widetilde{U}_{i,1}(q) = \widetilde{C}_{i,1,1}\, \widetilde{H}_{i,1,1}(q) + \widetilde{C}_{i,1,2}\, \widetilde{H}_{i,1,2}(q), \tag{3.10}$$

where $\widetilde{C}_{i,1,k}$, $i,k=1,2$, are some arbitrary constants. Here, we have $\widetilde{H}_{i,1,k}(q)=H_{i,1,k}(q;\varkappa)$, $i,k=1,2$, where we denote by $H_{i,j,k}(q)=H_{i,j,k}(q;\varkappa+\lambda_j)$, $i,k=1,2$, $j=0,1$, the (positive) fundamental solutions of the equation in (3.7). Then, by applying the instantaneous-stopping conditions of (2.36) to the function in (3.10), we obtain that the equalities

$$\widetilde{C}_{i,1,1}\, \widetilde{H}_{i,1,1}(g+) + \widetilde{C}_{i,1,2}\, \widetilde{H}_{i,1,2}(g+) = 0 \quad \text{and} \quad \widetilde{C}_{i,1,1}\, \widetilde{H}_{i,1,1}(h-) + \widetilde{C}_{i,1,2}\, \widetilde{H}_{i,1,2}(h-) = 1 \tag{3.11}$$

hold, for $i=1,2$. Solving the system of linear equations in (3.11), we obtain that the candidate solution for $U_{i,1}(q)$ in the system of (2.34)-(2.36) admits the representation

$$\widetilde{U}_{i,1}(q;g,h) = \frac{\widetilde{H}_{i,1,2}(g+)\widetilde{H}_{i,1,1}(q) - \widetilde{H}_{i,1,1}(g+)\widetilde{H}_{i,1,2}(q)}{\widetilde{H}_{i,1,2}(g+)\widetilde{H}_{i,1,1}(h-) - \widetilde{H}_{i,1,1}(g+)\widetilde{H}_{i,1,2}(h-)}, \tag{3.12}$$

for all $g<q<h$ and every $i=1,2$. Note that, in the case of $g=0$, we see that $\widetilde{C}_{i,0,1} \equiv \widetilde{C}_{i,0,1}(0,h)=0$ should hold in (3.10), for $i=1,2$, since otherwise $U_{i,1}(q)\to\pm\infty$ as $q\downarrow 0$, that must be excluded, by virtue of the fact that the function $U^*_{i,1}(q)$ in (2.21) is bounded. Hence, solving the right-hand equation in (3.11), we conclude that the candidate solution has the form

$$\widetilde{U}_{i,1}(q;0,h) = \widetilde{H}_{i,1,2}(q)/\widetilde{H}_{i,1,2}(h-), \tag{3.13}$$

for all $0<q<h$ and every $i=1,2$.

Observe that the general solution of the inhomogeneous second-order linear ordinary differential equation in (2.34) with (2.25), for $j=0$, admits the general solution

$$\widetilde{U}_{i,0}(q) = \widetilde{C}_{i,0,1}\, H_{i,0,1}(q) + \widetilde{C}_{i,0,2}\, H_{i,0,2}(q) + F_{i,0,}(q;g,h), \tag{3.14}$$

where $\widetilde{C}_{i,1,k}$, $i,k=1,2$, are some arbitrary constants, and we set

$$F_{i,0}(q;g,h) = H_{i,0,1}(q)\int_q \frac{\lambda_0 \widetilde{U}_{i,1}(r;g,h)H_{i,0,2}(r)}{(\gamma_1^2 r^2/2)D_{i,0}(r)}\,dr \tag{3.15}$$
$$+ H_{i,0,2}(q)\int^q \frac{\lambda_0 \widetilde{U}_{i,1}(r;g,h)H_{i,0,1}(r)}{(\gamma_1^2 r^2/2)D_{i,0}(r)}\,dr,$$

and

$$D_{i,0}(q) = H_{i,0,1}(q)H'_{i,0,2}(q) - H'_{i,0,1}(q)H_{i,0,2}(q), \tag{3.16}$$

where the functions $H_{i,j,k}(q)$, $i,k = 1,2$, $j = 0,1$, are the fundamental solutions of the equations in (3.7), and the function $\widetilde{U}_{i,1}(q;g,h)$ is given by (3.12), for $g < q < h$ and $i = 1,2$. Then, by applying the instantaneous-stopping conditions of (2.36) to the function in (3.14), we obtain that the equalities

$$\widetilde{C}_{i,0,1}\, H_{i,0,1}(g+) + \widetilde{C}_{i,0,2}\, H_{i,0,2}(g+) + F_{i,0}(g+;g,h) = 0, \tag{3.17}$$
$$\widetilde{C}_{i,0,1}\, H_{i,0,1}(h-) + \widetilde{C}_{i,0,2}\, H_{i,0,2}(h-) + F_{i,0}(h-;g,h) = 1 \tag{3.18}$$

hold, for $i = 1,2$. Solving the system of linear equations in (3.17)-(3.18), we obtain that the candidate solution for $U_{i,0}(q)$ in the system of (2.34)-(2.36) admits the representation

$$\widetilde{U}_{i,0}(q;g,h) = \widetilde{C}_{i,0,1}(g,h)\, H_{i,0,1}(q) + \widetilde{C}_{i,0,2}(g,h)\, H_{i,0,2}(q) + F_{i,0}(q;g,h) \tag{3.19}$$

with

$$\widetilde{C}_{i,0,1}(g,h) = \frac{F_{i,0}(h-;g,h)H_{i,0,2}(g+) - F_{i,0}(g+;g,h)H_{i,0,2}(h-) - H_{i,0,2}(g+)}{H_{i,0,1}(g+)H_{i,0,2}(h-) - H_{i,0,1}(h-)H_{i,0,2}(g+)}, \tag{3.20}$$
$$\widetilde{C}_{i,0,2}(g,h) = \frac{F_{i,0}(g+;g,h)H_{i,0,1}(h-) - F_{i,0}(h-;g,h)H_{i,0,1}(g+) + H_{i,0,1}(g+)}{H_{i,0,1}(g+)H_{i,0,2}(h-) - H_{i,0,1}(h-)H_{i,0,2}(g+)}, \tag{3.21}$$

for all $g < q < h$ and every $i = 1,2$. Note that, in the case of $g = 0$, we see that the function $\widetilde{U}_{i,1}(q;0,h)$ takes the form of (3.13) with $\widetilde{C}_{i,0,1} \equiv \widetilde{C}_{i,0,1}(0,h) = 0$ should hold in (3.14), for $0 < q < h$ and $i = 1,2$, since otherwise $U_{i,0}(q) \to \pm\infty$ as $q \downarrow 0$, that must be excluded, by virtue of the fact that the function $U^*_{i,0}(q)$ in (2.21) is bounded. In this case, the candidate solution has the form

$$\widetilde{U}_{i,0}(q;0,h) = \widetilde{C}_{i,0,2}(0,h)\, H_{i,0,2}(q) + G_{i,0}(q;0,h) \tag{3.22}$$

with

$$G_{i,0}(q;0,h) = H_{i,0,2}(q) \int^q \frac{D_{i,0}(r)}{H^2_{i,0,2}(r)} \int^r \frac{\lambda_0 \widetilde{U}_{i,1}(p;0,h)H_{i,0,2}(p)}{(\gamma_1^2 p^2/2)D_{i,0}(p)}\, dp\, dr, \tag{3.23}$$

where the function $H_{i,0,2}(q)$, $i = 1,2$, is the fundamental solutions of the equation in (3.7) such that $H_{i,0,2}(0+) = +0$ and $H_{i,0,2}(\infty) = \infty$ holds, the function $\widetilde{U}_{i,1}(q;g,h)$ takes the form of (3.13), and the function $D_{i,0}(q)$ is given by (3.16), for $0 < q < h$ and $i = 1,2$. Hence, solving the linear equation in (3.18), we conclude that the candidate solution for $U_{i,0}(q)$ in the system of (2.34)-(2.36) has the form

$$\widetilde{U}_{i,0}(q;0,h) = (1 - G_{i,0}(h-;0,h))\, H_{i,0,2}(q)/H_{i,0,2}(h-), \tag{3.24}$$

for all $0 < q < h$, where $G_{i,0}(q;0,h)$ is given by (3.23), for every $i = 1,2$.

(iii) Let us finally assume that $\gamma_1 > 0$, $\beta_{i,0}(q) = \gamma_0 = 0$, for all $q > 0$ and every $i = 1,2$, and $\lambda_j > 0$, for $j = 0,1$. In this case, the equation in (2.34) with (2.25), for $j = 0$, takes the form

$$-(\varkappa + \lambda_0)\, U_{i,0}(q) = -\lambda_0\, U_{i,1}(q), \tag{3.25}$$

for $g < q < h$ and $i = 1, 2$. Then, substituting $U_{i,0}(q) = \lambda_0 U_{i,1}(q)/(\varkappa + \lambda_0)$ into the equation of (2.34), for $j = 1$, we get

$$\frac{\gamma_1^2 q^2}{2} \, U''_{i,1}(q) + \big(\beta_1(q) + \rho\varepsilon\alpha_i\sigma(q)\gamma_1 \, q\big) \, U'_{i,1}(q) - \frac{\varkappa(\varkappa + \lambda_0 + \lambda_1)}{\varkappa + \lambda_0} \, U_{i,1}(q) = 0, \tag{3.26}$$

for $g < q < h$ and $i = 1, 2$. Hence, the general solution of the second-order ordinary linear differential equation takes the form

$$\widehat{U}_{i,1}(q) = \widehat{C}_{i,1,1} \, \widehat{H}_{i,1,1}(q) + \widehat{C}_{i,1,2} \, \widehat{H}_{i,1,2}(q), \tag{3.27}$$

where $\widehat{C}_{i,1,k}$, $i, k = 1, 2$, are some arbitrary constants. Here, we have $\widehat{H}_{i,1,k}(q) = H_{i,1,k}(q; (\varkappa(\varkappa + \lambda_0 + \lambda_1))/(\varkappa + \lambda_0))$, $i, k = 1, 2$, where $H_{i,j,k}(q) = H_{i,j,k}(q; \varkappa + \lambda_1)$, $i, k = 1, 2$, $j = 0, 1$, are the fundamental solutions of the equations in (3.7). Then, by applying the instantaneous-stopping conditions of (2.36) to the function in (3.27), we obtain that the equalities

$$\widehat{C}_{i,1,1} \, \widehat{H}_{i,1,1}(g+) + \widehat{C}_{i,1,2} \, \widehat{H}_{i,1,2}(g+) = 0 \quad \text{and} \quad \widehat{C}_{i,1,1} \, \widehat{H}_{i,1,1}(h-) + \widehat{C}_{i,1,2} \, \widehat{H}_{i,1,2}(h-) = 1 \tag{3.28}$$

hold, for $i = 1, 2$. Solving the system of linear equations in (3.11), we obtain that the candidate solution for $U_{i,1}(q)$ in the system of (2.34)-(2.36) admits the representation

$$\widehat{U}_{i,1}(q; g, h) = \frac{\widehat{H}_{i,1,2}(g+)\widehat{H}_{i,1,1}(q) - \widehat{H}_{i,1,1}(g+)\widehat{H}_{i,1,2}(q)}{\widehat{H}_{i,1,2}(g+)\widehat{H}_{i,1,1}(h-) - \widehat{H}_{i,1,1}(g+)\widehat{H}_{i,1,2}(h-)}, \tag{3.29}$$

for all $g < q < h$ and every $i = 1, 2$. Note that, in the case of $g = 0$, we see that $\widehat{C}_{i,1,1} \equiv \widehat{C}_{i,1,1}(0, h) = 0$ should hold in (3.10), for $i = 1, 2$, since otherwise $\widehat{U}_{i,1}(q) \to \pm\infty$ as $q \downarrow 0$, that must be excluded, by virtue of the fact that the function $U^*_{i,1}(q)$ in (2.21) is bounded. Hence, solving the right-hand equation in (3.11), we conclude that the candidate solution has the form

$$\widehat{U}_{i,1}(q; 0, h) = \widehat{H}_{i,1,2}(q)/\widehat{H}_{i,1,2}(h-), \tag{3.30}$$

for all $0 < q < h$ and every $i = 1, 2$. Therefore, we may conclude from (3.25) that the candidate solution for $U_{i,0}(q)$ is given by

$$\widehat{U}_{i,0}(q; g, h) = \lambda_0 \widehat{U}_{i,1}(q; g, h)/(\varkappa + \lambda_0), \tag{3.31}$$

for all $g \le q < h$, as well as $U_{i,0}(h) = 1$, for every $i = 1, 2$. Thus, the function $U_{i,0}(q)$ is discontinuous at h, that can be explained by the fact that the process Q remains constant while the Markov chain Θ is located in the state 0.

3.2 Some examples. Let us further derive explicit expressions for the fundamental system of solutions $H_{i,j,k}(q)$, $i, k = 1, 2$, $j = 0, 1$, from (3.8) for several volatility and drift rates $\sigma(q)$ and $\beta(q)$ in the stochastic differential equations of (2.1)-(2.2).

Example 3.1 Let $\gamma > 0$, $\lambda_j = 0$, $j = 2, 3$, $\sigma(q) = \ln q$, and $\beta(q) = q(\beta_0 - \beta_1 \ln q + \gamma^2/2)$, for all $q > 0$ and some constants $\beta_0, \beta_1 \in \mathbb{R}$, so that Q is an exponential Ornstein-Uhlenbeck process and which represents an extended Black-Karasinski model, and thus, the process $(S, \ln Q)$

constitutes a Stein-Stein model of stochastic volatility (see, e.g. [25; Subsection 4.1]). Then, by performing the change of variable $F(q) = \ln q$, it is seen from [37; Formulas 2.1.31 and 2.1.108] that the fundamental solutions $H_{i,0,k}(q)$, $i, k = 1, 2$, from (3.8) take the form

$$H_{i,0,1}(q) = q^{\upsilon_{i,0} \ln q + \chi_{i,0}} \, \Psi\big(\pi_{i,0}, 1/2; R_{i,0}(q)\big), \quad H_{i,0,2}(q) = q^{\upsilon_{i,0} \ln q + \chi_{i,0}} \, \Phi\big(\pi_{i,0}, 1/2; R_{i,0}(q)\big), \tag{3.32}$$

where we set

$$\upsilon_{i,0} = \frac{\widetilde{\beta}_{i,1} - \sqrt{\widetilde{\beta}_{i,1}^2 + 2\gamma^2 \varkappa}}{2\gamma^2}, \quad \chi_{i,0} = \frac{2\beta_0 \upsilon_{i,0}}{\widetilde{\beta}_{i,1} - 2\gamma^2 \upsilon_{i,0}}, \quad \widetilde{\beta}_{i,1} = \beta_1 - \rho\varepsilon\alpha_i\gamma, \tag{3.33}$$

$$\pi_{i,0} = \frac{\beta_0 \varkappa - \gamma^2 \upsilon_{i,0} (\widetilde{\beta}_{i,1} - 2\gamma^2 \upsilon_{i,0})^2}{2(\widetilde{\beta}_{i,1} - 2\gamma^2 \upsilon_{i,0})^3}, \quad R_{i,0}(q) = \frac{2\beta_0 \upsilon_{i,0}}{\gamma^2 \chi_{i,0}} \left(\ln q - \frac{\beta_0 \widetilde{\beta}_{i,1}}{(\widetilde{\beta}_{i,1} - 2\gamma^2 \upsilon_{i,0})^2} \right)^2, \tag{3.34}$$

for $i = 1, 2$. Here, $\Psi(x, y; z)$ and $\Phi(x, y; z)$ are the Tricomi's and Kummer's confluent hypergeometric functions (see, e.g. [1; Chapter XIII]), respectively, which admit the integral representations

$$\Psi(x, y; z) = \frac{1}{\Gamma(y)} \int_0^\infty e^{-zv} \, v^{x-1} (1 + v)^{y-x-1} \, dv, \tag{3.35}$$

for $y > 0$ and all $z > 0$, and

$$\Phi(x, y; z) = \frac{\Gamma(y)}{\Gamma(x)\Gamma(y - x)} \int_0^1 e^{zv} \, v^{x-1} (1 - v)^{y-x-1} \, dv, \tag{3.36}$$

for $y > x > 0$, and all $z \in \mathbb{R}$. The latter function also has the series expansion

$$\Phi(x, y; z) = 1 + \sum_{k=1}^{\infty} \frac{(x)_k}{(y)_k} \frac{z^k}{k!}, \tag{3.37}$$

for $y \neq 0, -1, -2, \ldots$, and the series converges under all $z > 0$ (see [1; Chapter XIII]), where $(u)_k$ is the Pochhammer symbol defined as $(u)_k = u(u + 1) \cdots (u + k - 1)$, and $(u)_0 = 1$, for $u \in \mathbb{R}$ and $k \in \mathbb{N}$, and $\Gamma(z)$ denotes the Euler's gamma function.

Example 3.2 Let $\gamma > 0$, $\lambda_j = 0$, $j = 2, 3$, $\sigma(q) = \ln q/2$, and $\beta(q) = q(2\beta_0 - \beta_1 \ln^2 q/2 + \gamma^2(\ln q - 1)/2)/\ln q$, for all $q > 0$ and some constants $\beta_0, \beta_1 \in \mathbb{R}$ such that $\beta_0 \geq \gamma^2/2$, so that Q is a diffusion process with state space $(1, \infty)$, and $\ln^2 Q/4$ is a Feller square root process which represents a Cox-Ingersoll-Ross model with state space $(0, \infty)$, and thus, the process $(S, \ln^2 Q/4)$ constitutes a Heston model of stochastic volatility (see, e.g. [25; Subsection 4.2]). Then, by performing the change of variable $F(q) = \ln^2 q/4$, it is seen from [37; Formula 2.1.108] that the fundamental solutions $H_{i,0,k}(q)$, $i, k = 1, 2$, from (3.8) take the form

$$H_{i,0,1}(q) = q^{2\upsilon_{i,0} \ln q} (\ln^2 q/4)^{1-2\beta_0/\gamma^2} \, \Psi(\chi_{i,0}, 2 - 2\beta_0/\gamma^2; R_{i,0}(q)), \tag{3.38}$$

$$H_{i,0,2}(q) = q^{2\upsilon_{i,0} \ln q} (\ln^2 q/4)^{1-2\beta_0/\gamma^2} \, \Phi(\chi_{i,0}, 2 - 2\beta_0/\gamma^2; R_{i,0}(q)), \tag{3.39}$$

where we set

$$\chi_{i,0} = 1 - \frac{\beta_0}{\gamma^2} - \frac{\beta_0 \widetilde{\beta}_{i,1}}{\gamma^2 (\widetilde{\beta}_{i,1} - 2\gamma^2 \upsilon_{i,0})} \quad \text{and} \quad R_{i,0}(q) = \frac{2(\widetilde{\beta}_{i,1} - 2\gamma^2 \upsilon_{i,0})}{\gamma^2} \frac{\ln^2 q}{4} \tag{3.40}$$

with $\upsilon_{i,0}$ and $\widetilde{\beta}_{i,1}$, $i = 1, 2$, from (3.33), and the functions $\Psi(x, y; z)$ and $\Phi(x, y; z)$ are defined as in (3.35)-(3.36).

3.3 Solutions to the system in (2.31)-(2.33). Let us look for a solution of the partial differential equation in (2.24) and (2.31) in the form

$$W_{i,j}(s,q) \equiv W_i(s) = D_{i,1}\, s^{\alpha_1} + D_{i,2}\, s^{\alpha_2}, \tag{3.41}$$

where $D_{i,k}$, $i,k = 1,2$, are some arbitrary constants and α_k, $k = 1,2$, are given by

$$\alpha_k = \frac{1}{2} - \frac{\delta}{\varepsilon^2} + (-1)^k \sqrt{\left(\frac{\delta}{\varepsilon^2} - \frac{1}{2}\right)^2 + \frac{2\eta}{\varepsilon^2}}, \tag{3.42}$$

so that $\alpha_1 < 0 < 1 < \alpha_2$ holds. Note that $D_{1,2} = D_{2,1} = 0$ should hold in (3.41), since otherwise $W_i(s) \to \pm\infty$, $i = 1,2$, as $s \uparrow \infty$ and $s \downarrow 0$, respectively, that must be excluded, by virtue of the fact that the functions $W_i^*(s)$, $i = 1,2$, in (2.20) are bounded. Then, by applying the instantaneous-stopping conditions of (2.33) to the function in (3.41), we obtain that the equalities

$$D_{1,1}\, a^{\alpha_1} = 1 \quad \text{and} \quad D_{2,2}\, b^{\alpha_2} = 1 \tag{3.43}$$

are satisfied. Hence, solving the equations in (3.43), we conclude that the candidate solutions $W_1(s,q;a) = W_1(s;a)$ and $W_2(s,q;b) = W_2(s;b)$ have the form

$$W_1(s;a) = (s/a)^{\alpha_1} \quad \text{and} \quad W_2(s;b) = (s/b)^{\alpha_2}, \tag{3.44}$$

for all $s > a$ and $s < b$, respectively.

4 Main result and proof

In this section, taking into account the facts proved above, we formulate and prove the main results of the paper. We present an analytic solution of the stopping problems of (2.18)-(2.19) in the general case of switching stochastic volatility model of $\gamma_j > 0$ and $\lambda_j > 0$, $j = 0,1$, for the case of a single structural volatility change $\lambda_0 > \lambda_1 = 0$, as well as for the constant volatility case $\beta_0(q) = \gamma_0 = 0$, for all $q > 0$.

Theorem 4.1 *Suppose that the coefficients $\sigma(q) > 0$ and $\beta(q) \in \mathbb{R}$ of the diffusion-type process (S,Q) defined by (2.1)-(2.2) are continuously differentiable functions of at most linear growth, for all $q > 0$. Assume that the condition of (2.11) holds, for all $t \geq 0$, with α_i, $i = 1,2$, given by (3.42). Then, the functionals $V_{1,j}^*(s,q) = V_{1,j}^*(s,q;a;g,h)$ and $V_{2,j}^*(s,q) = V_{2,j}^*(s,q;b;g,h)$ from (2.18)-(2.19) of the associated with (S,Q) random times τ_a^-, τ_b^+ and ζ_g^-, ζ_h^+ from (2.4)-(2.5) admit the representations:*

$$V_{1,j}^*(s,q;b;g,h) = U_{1,j}(q;g,h)\, W_{1,j}(s;a) \quad \text{and} \quad V_{2,j}^*(s,q;b;g,h) = U_{2,j}(q;g,h)\, W_{2,j}(s;b) \tag{4.1}$$

for all $s > a$ or $s < b$ and $g < y < h$, and any $0 < a < b < \infty$ and $0 \leq g < h < \infty$ fixed, where the functions $W_{1,j}(s;a)$ and $W_{2,j}(s;b)$, $j = 0,1$, are given by (3.44), as well as $U_{i,j}(q;g,h)$, $i = 1,2$, are specified as follows:

(i) if $\gamma_j > 0$ and $\lambda_j > 0$, $j = 0,1$, then the functions $U_{i,j}(q;g,h)$, $i = 1,2$, $j = 0,1$, are given by (3.6), with $C_{i,j,k}(g,h)$, $i = 1,2$, $j = 0,1$, $k = 1,2,3,4$, being a unique solution the linear system of algebraic equations in (3.3) and (3.4)-(3.5);

(ii) if $\gamma_j > 0$, $j = 0, 1$, and $\lambda_0 > \lambda_1 = 0$, then the functions $U_{i,j}(q; g, h)$, $i = 1, 2$, $j = 0, 1$, are given by either (3.12)-(3.13) or (3.19)-(3.24), respectively;

(iii) if $\beta_0(q) = \gamma_0 = 0$, for $q > 0$, $i = 1, 2$, and $\lambda_j > 0$, $j = 0, 1$, then the functions $U_{i,j}(q; g, h)$, $i = 1, 2$, $j = 0, 1$, are given by either (3.29)-(3.30) or (3.31), respectively.

Since all the parts of the assertions formulated above are proved using essentially similar arguments, we only give a proof for the two-dimensional stopping problem related to the value function $V^*_{1,j}(s, q; a; g, h)$ in (2.18).

Proof In order to verify the assertion stated above, it remains to show that the functions defined in the right-hand side of (4.1) coincides with the value function in (2.18). For this, let us denote by $V_1(s, q, j) = V_{1,j}(s, q)$ the right-hand side of the first expression in (4.1), as well as the notations $U_1(q, j) = U_{1,j}(q)$ and $W_1(s, j) = W_{1,j}(s)$ for $U_{1,j}(q; g, h)$ and $W_{1,j}(s; a)$, respectively. Then, using the fact that the function $V_1(s, q, j)$ is $C^{2,2,0}((a, \infty) \times (g, h) \times \{0, 1\})$, and taking into account the independence of the processes B^i, $i = 1, 2$, and Θ, by applying Itô's formula to $e^{-(\lambda+\varkappa)A_t} V_1(S_t, Q_t, \Theta_t)$, we obtain that the expression

$$e^{-(\eta+\varkappa)A_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}} V_1(S_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}, Q_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}, \Theta_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}) = V_1(s, q, j) + M_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t} \tag{4.2}$$
$$+ \int_0^{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t} e^{-(\eta+\varkappa)A_u} \big((\mathbb{L}^{\Theta_u}_{(S,Q)} V_1 - (\eta + \varkappa + \lambda_{\Theta_u}) \, \sigma^2(Q_u) V_1)(S_u, Q_u, \Theta_u) + \lambda_{\Theta_u} V_1(S_u, Q_u, 1 - \Theta_u) \big) \, du$$

holds for all $s > a$ and $g < q < h$, and the stopping times τ_a^- and ζ_g^-, ζ_h^+ from (2.4)-(2.5). Here, the process $M = (M_t)_{t \ge 0}$ defined by

$$\begin{aligned} M_t = & \int_0^t e^{-(\eta+\varkappa)A_u} \, \partial_s V_1(S_u, Q_u, \Theta_u) \, S_u \, \sigma(Q_u) \, dB^1_u \\ & + \int_0^t e^{-(\eta+\varkappa)A_u} \, \partial_q V_1(S_u, Q_u, \Theta_u) \, \big((1 - \Theta_u) \, \gamma_0 + \Theta_u \, \gamma_1 \big) \, Q_u \, dB^2_u \\ & + \int_0^t e^{-(\eta+\varkappa)A_u} \, \big(V_1(S_u, Q_u, 1) - V_1(S_u, Q_u, 0) \big) \, dN_u \end{aligned} \tag{4.3}$$

with $N = (N_t)_{t \ge 0}$ given by:

$$N_t = \Theta_t - \int_0^t \big(\lambda_0 \, (1 - \Theta_u) + \lambda_1 \, \Theta_u \big) \, du \tag{4.4}$$

is a local martingale under $P_{s,q,j}$.

By virtue of straightforward calculations and the arguments from the previous section, it is verified that the function $V_1(s, q, j)$ solves the system of elliptic partial differential equations in (2.26) with (2.24) and satisfies the boundary conditions of (2.27)-(2.30). Observe that the process $(M_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t})_{t \ge 0}$ is a uniformly integrable martingale, since the derivatives and the coefficients in (4.3) are bounded functions. Then, taking the expectation with respect to $P_{s,y,j}$ in (4.2), by means of the optional sampling theorem (see, e.g. [27; Chapter III, Theorem 3.6] or [32; Chapter II, Theorem 3.2]), we get

$$\begin{aligned} & E_{s,q,j} \Big[e^{-(\eta+\varkappa)A_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}} \, V_1(S_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}, Q_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}, \Theta_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}) \Big] \\ & = V_1(s, q, j) + E_{s,q,j} \big[M_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t} \big] = V_1(s, q, j) \end{aligned} \tag{4.5}$$

for all $s > a$ and $g < q < h$, and every $j = 0, 1$. Therefore, letting t go to infinity and using the instantaneous-stopping conditions in (2.27)-(2.30) as well as the fact that $V_1(S_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+}, Q_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+}, \Theta_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+}) = 0$ on $\{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ = \infty\}$ ($P_{s,q,j}$-a.s.), we can apply the Lebesgue dominated convergence theorem for (4.5) to obtain the equalities

$$
\begin{aligned}
& E_{s,q,j}\Big[e^{-(\eta+\varkappa)A_{\tau_a^-}} \, U_1(Q_{\tau_a^-}, \Theta_{\tau_a^-}) \, I(\tau_a^- < \zeta_g^- \wedge \zeta_h^+) \\
& \qquad + e^{-(\eta+\varkappa)A_{\zeta_g^- \wedge \zeta_h^+}} \, W_1(S_{\zeta_g^- \wedge \zeta_h^+}, \Theta_{\zeta_g^- \wedge \zeta_h^+}) \, I(\zeta_g^- \wedge \zeta_h^+ \le \tau_a^-)\Big] \\
& = E_{s,q,j}\Big[e^{-(\eta+\varkappa)A_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+}} \, V_1(S_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+}, Q_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+}, \Theta_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+})\Big] = V_1(s, q, j)
\end{aligned} \tag{4.6}
$$

for all $s > a$ and $g < q < h$, and every $j = 0, 1$, which directly implies the desired assertion.

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