

N.H Bingham and [Adam Ostaszewski](#)

Beyond Lebesgue and Baire IV: density topologies and a converse Steinhaus-Weil theorem

**Article (Accepted version)
(Refereed)**

Original citation:

Bingham, N.H and Ostaszewski, Adam (2017). *Beyond Lebesgue and Baire IV: density topologies and a converse Steinhaus-Weil theorem*. [Indagationes Mathematicae](#) ISSN 0019-3577

© 2017 The Authors

This version available at: <http://eprints.lse.ac.uk/85937/>

Available in LSE Research Online: November 2017

LSE has developed LSE Research Online so that users may access research output of the School. Copyright © and Moral Rights for the papers on this site are retained by the individual authors and/or other copyright owners. Users may download and/or print one copy of any article(s) in LSE Research Online to facilitate their private study or for non-commercial research. You may not engage in further distribution of the material or use it for any profit-making activities or any commercial gain. You may freely distribute the URL (<http://eprints.lse.ac.uk>) of the LSE Research Online website.

This document is the author's final accepted version of the journal article. There may be differences between this version and the published version. You are advised to consult the publisher's version if you wish to cite from it.

**Beyond Lebesgue and Baire IV:
Density topologies and a converse Steinhaus-Weil Theorem**
by
N. H. Bingham and A. J. Ostaszewski

*On the centenary of Hausdorff's Mengenlehre (1914) and Denjoy's
Approximate continuity (1915)*

Abstract.

The theme here is category-measure duality, in the context of a topological group. One can often handle the (Baire) category case and the (Lebesgue, or Haar) measure cases together, by working bi-topologically: switching between the original topology and a suitable refinement (a density topology). This prompts a systematic study of such density topologies, and the corresponding σ -ideals of negligibles. Such ideas go back to Weil's classic book, and to Hashimoto's ideal topologies. We make use of *group norms*, which cast light on the interplay between the group and measure structures. The Steinhaus-Weil interior-points theorem ('on AA^{-1} ') plays a crucial role here; so too does its converse, the Simmons-Mospan theorem.

Key words. Steinhaus-Weil property, Weil topology, shift-compact, density topology, Hashimoto ideal topology, group norm

Mathematics Subject Classification (2000): Primary 26A03; 39B62.

1 Introduction

This paper originates from several sources. The first three are our earlier studies 'Beyond Lebesgue and Baire I-III' ([BinO1,2], [Ost3]), the general theme of which is the similarity (indeed, duality) between measure and category, and the primacy of category/topology over measure in many areas. The second three are recent studies by the second author on the Effros Open Mapping Principle ([Ost4,5,6]; §6.4 below).

The *Steinhaus-Weil property* (critical for *regular variation*: see [BinGT, Th. 1.1.1]) of a set S in a topological group G is that 1_G is an interior point of SS^{-1} when S is non-negligible (as in the classic examples in the additive group \mathbb{R} : Baire non-meagre, or measurable non-null). This is implied by the compactness-like property (called *shift-compactness* in [Ost3], cf. [MilO]),

that any *null sequence* z_n (i.e. $z_n \rightarrow 1_G$) has a ‘translator’ $s \in S$ and subsequence $z_{n(m)}$ with $\{s \cdot z_{n(m)} : m \in N\} \subseteq S$. This is not only a stronger property, but also better adapted for use in many proofs.

Results of Steinhaus-Weil type go back to Steinhaus [Ste] on the line and Weil [Wei, p. 50] in a locally compact topological group (see e.g. Grosse-Erdmann [GroE]), and to Kemperman [Kem] (cf [Kuc, Lemma 3.7.2], and [BinO1, Th. K], where this is ‘Kemperman’s Theorem’, [BinO6, Th. 1(iv)]). For present purposes it is natural to call shift-compactness a strong *Steinhaus-Weil-like* property. A first study of the closeness of the two Steinhaus-Weil-like properties appears most recently in [Ost6] by way of the Effros Opening Mapping Principle. Recently, in [BinO8], in work on subadditivity and mid-point convexity, with ‘negligibility’ interpreted via the Christensen Haar-null subsets of a Banach space, we replaced shift-compactness by the already established Steinhaus-Weil property due to Christensen [Chr1,2], or its extension due to Solecki [Sol2]. (Boundedness of a subadditive function on A and B yields its boundedness on AB and hence on an open set, provided AB has the interior-point property – see §6.9.) This new perspective motivates the present return to the question of when the Steinhaus-Weil property implies shift-compactness, and hinges on two themes. The first is the *Lebesgue density theorem*, which Kemperman [Kem] used to reprove the Steinhaus-Weil theorem, very much a local property. The second was our reliance [BinO8] on a localized version of the Steinhaus-Weil property: in S the relative open neighbourhoods of all points were to have the Steinhaus-Weil property. Taken together, these suggested the need to *characterize* in a topological group those analogues of the *Lebesgue density topology* ([HauP], [GofNN], [GofW]) that imply shift-compactness, a matter we turn to in §2. Here we prove Theorems 1 and 2: we study category analogues of the Lebesgue density topology. We turn in §3 to Hashimoto ideal topologies and in §4 to properties of Steinhaus-Weil type and converses (cf. Prop. 2). We establish in §5 the equivalence of weak and strong Steinhaus-Weil-like properties in the presence of three topological restrictions, taking what we call the *Kemperman property* in §2 as a weak form of the Steinhaus-Weil property (see §4). This is reminiscent of the characterization of Borel measures on \mathbb{R} having the Steinhaus-Weil property, for which see [Mos] ([Sim] for the Haar case) and the recent [Dan]. We close in §6 with some complements.

We remind the reader of the tension between two of our themes here: density topologies and topological groups. The real line is not a topological group under the (Lebesgue) density topology (see e.g. [Sch, Prop. 1.9],

[BinO7, § 4]). Instead, what is relevant here is semi-topological group structure [ArhT], in which it is the shift (one argument), rather than multiplication (two arguments) which is continuous.

The Lebesgue Density Theorem, which underlies the density topology (or topologies) crucial here, is already relevant to (though less well known than) the Lebesgue Differentiation Theorem, on the relationship between differentiation and the Lebesgue integral – see Bruckner’s classic survey [Bru]. The first is usually obtained from the second by specializing to indicator functions 1_A . Relevant here are Vitali’s covering lemma and weak L_1 -estimates for the Hardy-Littlewood maximal function; for textbook treatment see e.g. [Rud, Th. 8.8], [Steis, 3.1.2]. Latent here is the relation between a general measure μ and its translation ${}_x\mu$ (defined via ${}_x\mu(B) = \mu(xB)$). That, in turn, is encapsulated in the Radon-Nikodym derivative $d_x\mu/d\mu$ (wherever defined) and is related to the *relative interior-point property*, which arises when studying the difference set $S - S$ relative to the *Cameron-Martin subspace* of a topological vector space; see [Bog, §2.2, 2.4]. This is a matter we hope to return to elsewhere.

In sum: as well as the historical references to Hausdorff in 1914 [Hau] and Denjoy in 1915 [Den], the paper relates to Lebesgue’s approach to the fundamental theorem of calculus. Its roots may thus be traced back to the roots of calculus itself.

2 Density topologies

As usual, we write ZF for Zermelo-Fraenkel and DC for Dependent Choice, which we recall enables standard proofs by induction. All our results here follow from ZF+DC rather than ZFC=ZF+AC (Axiom of Choice), as throughout we deal with *separable* metrizable topological groups, so that there is then a *countable* basis for the topology. This will be understood throughout (though we state it explicitly in Th. 3 for emphasis). In particular, our maximality arguments in the context of such a basis need only DC, rather than AC via Zorn’s Lemma. Our point of view is dictated by the focus here (and in other work elsewhere) on the primary role of category and the equivalence of Baire’s Category Theorem with DC [Bla]; for a wider discussion see [BinO10].

Thus throughout (G, \mathcal{T}) be a separable topological group metrized by a *right*-invariant metric $d = d_R^G$, allowing the topology to be denoted either as

\mathcal{T}_d , or \mathcal{T} . (This is possible by the *Birkhoff-Kakutani metrization theorem*, [HewR, II.8, p. 70 Th., p. 83 Notes], [Ost3].) We make free use of

$$\|t\| := d_R^G(1_G, t),$$

referred to as the *group norm* of G , for which see the textbook account in [ArhT, §3.3] or [BinO4], and denote by $B_\delta(g) := \{h : \|hg^{-1}\| < \delta\} = B_\delta(1_G)g$ the open δ -ball centred at g , briefly the open δ -neighbourhood (δ -*nhd*) of g ; we use B_δ to denote $B_\delta(1_G)$. For G locally compact, we denote (left) Haar measure by η , or context permitting by $|\cdot|$, by analogy with the group norm in view of their close relationship (cf. §6.1). We have in mind, as canonical examples, \mathbb{R} or \mathbb{R}_+ under the usual (Euclidean) topology, denoted \mathcal{E} . For *any* topology τ on G , we write $\mathcal{F}(\tau)$, $\mathcal{F}_\sigma(\tau)$, $\mathcal{G}_\delta(\tau)$ for the corresponding closed sets etc., $\mathcal{B}(\tau)$ for the *Baire* sets, i.e. the sets with the Baire Property (BP), $\mathcal{B}_0(\tau)$ for the corresponding meagre sets, and $\mathcal{B}_+(\tau)$ for the non-meagre members of $\mathcal{B}(\tau)$. If (G, \mathcal{T}) is suppressed, $(\mathbb{R}, \mathcal{E})$ or $(\mathbb{R}_+, \mathcal{E})$ is to be understood. Thus \mathcal{B} denotes the usual Baire sets and \mathcal{B}_0 its negligible sets, the σ -ideal of meagre sets; analogously, \mathcal{L} denotes the Lebesgue (Haar) measurable sets and \mathcal{L}_0 its negligible sets, the σ -ideal of null (measure-zero) sets. We denote by $\mathcal{M}(G)$ the Borel regular σ -finite measures on G , with $\mathcal{P}(G)$ the subfamily of probability measures; here regularity is taken to imply both inner and outer regularity (i.e. compact inner approximation and open outer approximation); these play a significant role in §4. We say that a property holds at *quasi all points* of a set if it holds except on a negligible set (in the category or measure sense).

We will refer to the action of G on itself by $t(x) \mapsto tx$ (or $t(x) = t + x$ in the case of \mathbb{R} – we will feel free to move at will between $(\mathbb{R}, +)$ and (\mathbb{R}_+, \cdot) via the exponential isomorphism). Say that a topology τ on G is (*left*) *shift-invariant* if $tV \in \tau$ for all $t \in G$ and all $V \in \tau$; equivalently: each shift $t : \tau \rightarrow \tau$ is continuous.

A *weak τ -base* for τ is a subfamily \mathcal{W} such that for each non-empty $V \in \tau$ there is $W \in \mathcal{W}$ with $\emptyset \neq W \subseteq V$. When \mathcal{W} above consists of sets *analytic* under \mathcal{T}_d (continuous images of the irrationals – see §5 below), the topology is called in [Ost1] a *generalized Gandy-Harrington* topology, by analogy with its classical antecedent (for a textbook treatment of which see [Gao, Ch. 1]); in such a case the topology τ satisfies the Baire Theorem (see [Oxt2, Ch. 9], [Kec, III.26.18,19, p. 203-4], [Ost1, §2.2]). Here we consider a stronger property generalizing the two observations that

- (a): modulo \mathcal{L}_0 each measurable set is an \mathcal{F}_σ ;
 (b): modulo \mathcal{B}_0 each Baire set is a \mathcal{G}_δ ([Oxt2, Th. 4.4], cf. [Kec, 8.23])
 (these are the forms in which the results are usually stated; it is the similarities, rather than the distinctions, between \mathcal{F}_σ and \mathcal{G}_δ that are relevant here).

Say that τ has the *strong Gandy-Harrington* property if modulo $\mathcal{B}_0(\tau)$ each $\mathcal{B}(\tau)$ set is analytic under \mathcal{T}_d . (Again see the references above.)

Denote by $\mathcal{D}_\mathcal{L}$ the family of all sets M all of whose points are *density points* (i.e. have Lebesgue (Haar) density 1, in the sense of Martin [Mar1,2] or in the more general context Mueller [Mue] – see the more recent development in [Ost3]; cf. [BinO4] for normed groups, [Oxt2] for \mathbb{R}). As noted by Haupt and Pauc [HauP] in \mathbb{R} , $\mathcal{D}_\mathcal{L}$ forms a topology, the (Lebesgue) density topology. It is related to Denjoy approximate continuity. It can be generalized to Haar measure. It is a *fine topology* (refining topology); see [CieLO], [EvaG], [KanK], and [LukMZ], for background on such fine topologies. (For other topologies derived from notions of ‘density point’ see [Wil] and e.g. [FilW]; for aspects of translation invariance see [WilK].)

We list below a number of *qualitative* properties of $\mathcal{D}_\mathcal{L}$, (i)-(viii), all of them classical. We name property (iv) the *Kemperman property*: see §1 for the motivation, and (vii) the *Nikodym property* ([Ost6]; [Rog, §2.9], [Nik]). Property (viii) suggests a category analogue $\mathcal{D}_\mathcal{B}$ of $\mathcal{D}_\mathcal{L}$; we prove the category analogues of (i)-(vii) in Theorems 1 and 2 below.

- (i) $\mathcal{D}_\mathcal{L}$ is a *fine* topology (i.e. refining \mathcal{T}_d) which is shift-invariant:

$$\begin{aligned} \mathcal{S} &\subseteq \mathcal{D}_\mathcal{L} \implies \bigcup \mathcal{S} \in \mathcal{D}_\mathcal{L}, \\ V, V' &\in \mathcal{D}_\mathcal{L} \implies V \cap V' \in \mathcal{D}_\mathcal{L}, \\ \mathcal{T} &\subseteq \mathcal{D}_\mathcal{L}, \\ tV &\in \mathcal{D}_\mathcal{L} \quad (t \in G, V \in \mathcal{D}_\mathcal{L}); \end{aligned}$$

- (ii) the sets in $\mathcal{D}_\mathcal{L}$ are measurable: $\mathcal{D}_\mathcal{L} \subseteq \mathcal{L}$;
 (iii) the $\mathcal{D}_\mathcal{L}$ -boundary of a measurable set is null:

$$M \setminus \text{int}_{\mathcal{D}_\mathcal{L}}(M) \in \mathcal{L}_0 \quad (M \in \mathcal{L});$$

- (iv) the *Kemperman property*, that any $\mathcal{D}_\mathcal{L}$ -open neighbourhood of the identity meets its own small displacements non-meagerly: for $1_G \in U \in \mathcal{D}_\mathcal{L}$ there is $\delta > 0$ with

$$U \cap (tU) \in \mathcal{B}_+(\mathcal{D}_\mathcal{L}) \quad (\|t\| < \delta);$$

(v) $\mathcal{D}_{\mathcal{L}}$ is a *strong generalized Gandy-Harrington topology*: modulo $\mathcal{B}_0(\mathcal{D}_{\mathcal{L}})$ each $\mathcal{B}(\mathcal{D}_{\mathcal{L}})$ set is analytic under \mathcal{T}_d and in fact $\mathcal{F}_{\sigma}(\mathcal{T}_d) \cap \mathcal{D}_{\mathcal{L}}$ is a weak $\mathcal{D}_{\mathcal{L}}$ -base, so that $\mathcal{D}_{\mathcal{L}}$ is a Baire space;

(Proof: Any set $M \in \mathcal{D}_{\mathcal{L}}$ contains a \mathcal{T}_d -closed set F of positive measure. Let H be a $\mathcal{G}(\mathcal{T}_d)_{\delta}$ -null set covering the null set of non-density points; then $F \setminus H \in \mathcal{D}_{\mathcal{L}} \cap \mathcal{F}_{\sigma}(\mathcal{T}_d)$.)

(vi) the $\mathcal{D}_{\mathcal{L}}$ -Baire sets/meagre sets are identical with respectively, the measurable sets and the null sets:

$$\mathcal{B}(\mathcal{D}_{\mathcal{L}}) = \mathcal{L}, \quad \mathcal{B}_0(\mathcal{D}_{\mathcal{L}}) = \mathcal{L}_0;$$

(vii) the *Nikodym property of preservation of category* under displacements (see [Ost2,6] for background and references):

(a) $tU \in \mathcal{B}(\mathcal{D}_{\mathcal{L}})$ ($t \in G, U \in \mathcal{B}(\mathcal{D}_{\mathcal{L}})$), and

(b) $tU \in \mathcal{B}_0(\mathcal{D}_{\mathcal{L}})$ iff $U \in \mathcal{B}_0(\mathcal{B}_{\mathcal{L}})$ ($t \in G$);

(viii) x is a density point of a measurable set M iff $x \in \text{int}_{\mathcal{D}_{\mathcal{L}}}(M)$.

So $\mathcal{D}_{\mathcal{L}}$ yields a topological characterization of local behaviour w.r.t. measure. Property (viii) calls for the Haar generalization of the Lebesgue Density Theorem [Oxt2, Ch. 3]. Below this will be viewed as a special case of the Banach Localization Principle (or Banach Category Theorem, [Oxt2, Ch. 16]; cf. [Ost1]).

We now define a topology $\mathcal{D}_{\mathcal{B}}$ with properties analogous to (i)-(viii) in respect of \mathcal{B} ; here (viii) is a definition of category-density point.

Definitions. 1. Call H τ -locally comeagre at $x \in H$ if there is a τ -open nhd U of x such that $U \setminus H$ is meagre.

2. Say that H is τ -locally comeagre at all of its points if for each $x \in H$ there is a τ -open nhd U of x such that $U \setminus H$ is meagre.

Remarks. If, as in (1), U witnesses the property of H at some point $x \in H$, then each point of $H \cap U$ has this property; note the *monotonicity*: if $H \subseteq H'$ and H has the property at x , then also H' has it at x .

If (2) holds for H , then H is open under the refinement topology generated by the family $\{U \setminus L : U \in \tau, L \in \mathcal{B}_0(\tau)\}$. We consider ‘ideal topology’ refinements such as this, generated by a general σ -ideal in place of $\mathcal{B}_0(\mathcal{T})$, in the next section.

Theorem 1. For $\mathcal{T} = \mathcal{T}_d$ let $\mathcal{D}_{\mathcal{B}}(\mathcal{T})$ be the family of sets which are \mathcal{T} -locally co-meagre at all of their points. Then:

- (i) $t\mathcal{D}_{\mathcal{B}}(\mathcal{T}) \subseteq \mathcal{D}_{\mathcal{B}}(\mathcal{T})$, for all $t \in G$ and $\mathcal{T} \subseteq \mathcal{D}_{\mathcal{B}}(\mathcal{T})$;
- (ii) $\mathcal{D}_{\mathcal{B}}(\mathcal{T}) \subseteq \mathcal{B}(\mathcal{T})$;
- (iii) $H \setminus \text{int}_{\mathcal{D}_{\mathcal{B}}}(\mathcal{H}) \in \mathcal{B}_0(\mathcal{T})$ for $H \in \mathcal{B}(\mathcal{T})$; in fact, if S is Baire, then S has a non-empty $\mathcal{D}_{\mathcal{B}}(\mathcal{T})$ -interior in each nhd on which it is dense and non-meagre;
- (iv) *Kempman property*: for $1_G \in H \in \mathcal{D}_{\mathcal{B}}(\mathcal{T})$ there is $\delta > 0$ with

$$H \cap (tH) \in \mathcal{B}_+(\mathcal{D}_{\mathcal{B}}(\mathcal{T})) \quad (\|t\| < \delta);$$

- (v) $\mathcal{D}_{\mathcal{B}}$ is a strong generalized Gandy-Harrington topology: modulo $\mathcal{B}_0(\mathcal{D}_{\mathcal{L}}(\mathcal{T}))$ each $\mathcal{B}(\mathcal{D}_{\mathcal{L}}(\mathcal{T}))$ set is analytic under \mathcal{T} , and in fact $\mathcal{D}_{\mathcal{B}}(\mathcal{T}) \cap \mathcal{G}_{\delta}(\mathcal{T})$ is a weak $\mathcal{D}_{\mathcal{B}}(\mathcal{T})$ -base, so $(G, \mathcal{D}_{\mathcal{B}}(\mathcal{T}))$ is a Baire space.

In words: parts (i)-(iii) assert that $\mathcal{D}_{\mathcal{B}}(\mathcal{T})$ is a shift-invariant topology refining \mathcal{T} , the sets in $\mathcal{D}_{\mathcal{B}}(\mathcal{T})$ are \mathcal{T} -Baire, and the boundary points of any \mathcal{T} -Baire set H form a \mathcal{T} -meagre set.

Proof. (i) Evidently G and \emptyset are in $\mathcal{D}_{\mathcal{B}}(\mathcal{T})$. For an arbitrary $\mathcal{H} \subseteq \mathcal{D}_{\mathcal{B}}(\mathcal{T})$, the union $H := \bigcup \mathcal{H}$ is \mathcal{T} -locally co-meagre at each element of H , by monotonicity. Next, suppose $x \in H \cap H'$, with $H, H' \in \mathcal{D}_{\mathcal{B}}(\mathcal{T})$. Choose V, V' open nhds of x meeting H, H' in comeagre sets. Then $x \in W = V \cap V'$ is an open nhd of x . As $H \cap W$ and $H' \cap W$ are co-meagre on W , by Baire's Theorem, so is their intersection on W ; so $H \cap H'$ is \mathcal{T} -locally co-meagre at x .

(ii) Suppose H is \mathcal{T} -locally comeagre at all $x \in H$. Being metrizable and separable, G is hereditarily Lindelöf [Dug, VIII §6 and Th. 7.3], so H is Baire; indeed, if $\{U_n\}$ is a countable open cover of H , with each $U_n \setminus H$ meagre, then $H = \bigcup_n (U_n \cap H) = \bigcup_n U_n \setminus (U_n \setminus H)$, which is Baire, since each set $U_n \setminus (U_n \setminus H)$ is Baire.

(iii) If S is non-meagre, it is dense in some open nhd I . Then $T := I \setminus S$ is meagre: for otherwise, T is (Baire and) non-meagre. Then T is dense on some (non-empty) open $J \subseteq I$. Being Baire, both S and T are, modulo meagre sets, $\mathcal{G}_{\delta}(\mathcal{T})$ -sets dense on J . So they meet, by Baire's Theorem – a contradiction.

Let \mathcal{I} be a maximal family of nhds I on which S is co-meagre. Then $\bigcup \{I \cap S : I \in \mathcal{I}\}$ is \mathcal{T} -open. Put $S' = S \setminus \bigcup \{I \cap S : I \in \mathcal{I}\}$. Then S' is meagre; otherwise, as before S' is dense in some nhd I and co-meagre on I , contradicting maximality of \mathcal{I} .

- (iv) If $1_G \in H \in \mathcal{D}_{\mathcal{B}}(\mathcal{T})$, and $H = U \setminus N$, choose $\delta > 0$ such that $B_{\delta}(1_G) \subseteq$

U . Then $\emptyset \neq \{t\} \subseteq B_\delta(1_G) \cap tB_\delta(1_G)$, for $\|t\| < \delta$, so

$$H \cap (tH) \supseteq B_\delta(1_G) \cap tB_\delta(1_G) \setminus (N \cup tN) \in \mathcal{B}_+(\mathcal{D}_\mathcal{B}(\mathcal{T})).$$

(v) If $H \in \mathcal{D}_\mathcal{B}(\mathcal{T})$, and $H = U \setminus N$ with $U \in \mathcal{T}$ and N meagre (in the sense of \mathcal{T}) we may choose a larger meagre $\mathcal{F}_\sigma(\mathcal{T})$ -set $M \supseteq N$, and then $H' = U \setminus M \in \mathcal{G}_\delta(\mathcal{T}) \cap \mathcal{D}_\mathcal{B}(\mathcal{T})$. \square

Recall the observation of Haupt and Pauc [HauP] that (writing *nwd* for nowhere dense)

$$M \text{ is meagre in } \mathcal{D}_\mathcal{L} \text{ iff } M \text{ is nwd in } \mathcal{D}_\mathcal{L} \text{ iff } M \text{ is null} \quad (H-P)$$

[Kec, 17.47] (the left equivalence as $\mathcal{D}_\mathcal{L}$ is a topology). This shows very clearly how changing from the original topology \mathcal{T} (\mathcal{E} in the Euclidean case) to the density topology \mathcal{D} turns qualitative measure considerations into Baire (or topological) considerations. This is the basis for the use of bitopology in [BinO2].

Theorem 2 below extends the list (i)-(v) in Theorem 1 to (vi), (vii) (as noted above (viii) becomes a matter of definition), in particular yielding an abstract form of (H-P).

Theorem 2. *As in Theorem 1, for $\mathcal{T} = \mathcal{T}_d$:*

(vi) Haupt-Pauc property: $\mathcal{B}_0(\mathcal{D}_\mathcal{B}(\mathcal{T})) = \mathcal{B}_0(\mathcal{T})$ and $\mathcal{B}(\mathcal{D}_\mathcal{B}(\mathcal{T})) = \mathcal{B}(\mathcal{T})$;

(vii) Nikodym property: for all $t \in G$

(a) $tU \in \mathcal{B}(\mathcal{D}_\mathcal{B}(\mathcal{T}))$ for $U \in \mathcal{B}(\mathcal{D}_\mathcal{B}(\mathcal{T}))$, and

(b) $tU \in \mathcal{B}_0(\mathcal{D}_\mathcal{B}(\mathcal{T}))$ iff $U \in \mathcal{B}_0(\mathcal{B}_\mathcal{D}(\mathcal{T}))$.

Proof. Let D be $\mathcal{D}_\mathcal{B}(\mathcal{T})$ -nwd. Let \mathcal{W} be a maximal family of pairwise disjoint \mathcal{T} -sets W with $D \cap W$ meagre. By Banach's Category Theorem $\bigcup\{D \cap W : W \in \mathcal{W}\}$ is meagre. Put $D' = D \setminus \bigcup\{D \cap W : W \in \mathcal{W}\}$. Suppose that D' is non-meagre. Then for some open nhd U the set D' is non-meagre on every open subset of U ; but D' is $\mathcal{D}_\mathcal{B}(\mathcal{T})$ -nwd, so there is an open nhd V and meagre M_V with $V \setminus M_V$ disjoint from D' . Then $D' \cap V \subseteq M_V$, a contradiction. So D' is indeed meagre, and so is D .

If D is a countable union of $\mathcal{D}_\mathcal{B}(\mathcal{T})$ -nwd sets, then it is meagre in \mathcal{T} .

The remaining assertions are now clear. \square

3 Hashimoto ideal topologies

As in (*H-P*) above, a key role is played by the family of meagre sets (in \mathcal{E} , or $\mathcal{B}_{\mathcal{L}}$) and (Lebesgue-) null sets. Each forms a σ -ideal of *small sets* and gives rise to a *Hashimoto topology* (= ‘ideal topology’ according to [LukMZ, 1.C]), to which we turn in this section. For an illustration of its use in a Banach space exploiting the σ -ideal \mathcal{HN} of Haar-null subsets (defined in §5 below, cf. §6.6) – see [BinO8]. Also relevant here is the study [CieJ] of topologies τ for which a given σ -ideal is identical with the σ -ideal $B_0(\tau)$ of τ -meagre sets.

Definition. For \mathcal{I} a σ -ideal of subsets of X and τ a topology on X , say that a set S is \mathcal{I} -nearly τ -open if for some τ -open set U and elements M, N in \mathcal{I}

$$S = (U \setminus N) \cup M.$$

We make the blanket assumption that $\{x\} \in \mathcal{I}$ for all $x \in X$ – see [Hay], and also [Sam, especially Ex. 2]).

The following result is an embellishment of Hashimoto’s early insight [Has]; there is a readable account by Janković and Hamlett in [JanH] introducing the topology via a Kuratowski closure operator, rather than via an ideal, as was first done in [Sam], albeit anticipated by [Sch] using \mathcal{L} in \mathbb{R} , as noted in the Introduction.

Theorem 3 (ZF+DC; cf. [Has], [JanH], [Sam, Cor 2]). *For a topology τ with countable basis β and a σ -ideal \mathcal{I} , the (Hashimoto) topology generated by the sets*

$$\mathcal{H} := \{V \setminus M : V \in \tau, M \in \mathcal{I}\}$$

is \mathcal{H} itself, so that, in particular, each $W \in \mathcal{H}$ is \mathcal{I} -nearly τ -open and has the representation

$$W = \left(\bigcup_{B \in \beta_W} B \right) \setminus M, \text{ for some } M \in \mathcal{I},$$

where

$$\beta_W := \{B \in \beta : B \setminus L \subseteq W \text{ for some } L \in \mathcal{I}\}.$$

Furthermore, if no non-empty τ -open set is in \mathcal{I} , then:

- (i) a set is \mathcal{H} -nowhere dense iff it is the union $N \cup M$ of a τ -nowhere dense set and an \mathcal{I} set;
- (ii) the \mathcal{H} -Baire sets are the \mathcal{I} -nearly τ -Baire sets;

(iii) for \mathcal{I} the meagre sets of τ , if τ is a Baire topology, then so is \mathcal{H} .

Proof. For a fixed non-empty set W that is a union of sets in \mathcal{H} , put

$$\beta_W := \{B \in \beta : B \setminus L \subseteq W \text{ for some } L \in \mathcal{I}\}.$$

As this is countable, by DC (see [TomW, Ch. 15]) there is a selector $L(\cdot)$ defined on β_W such that $L(B) \in \mathcal{I}$ for each $B \in \beta_W$; then $B \setminus L(B) \subseteq W$. Set $M(B) := B \cap L(B) \setminus W$, so that $B \setminus M(B) = B \setminus L(B) \cup (B \cap L(B) \cap W) \subseteq W$. Then $M(B) \cap W = \emptyset$. So putting

$$M := \bigcup_{B \in \beta_W} M(B),$$

$M \cap W = \emptyset$. Also $M \in \mathcal{I}$ as $M(B) \subseteq L(B) \in \mathcal{I}$. Then, as $B \setminus M(B) \subseteq W$ for each $B \in \beta_W$,

$$\left(\bigcup_{B \in \beta_W} B \right) \setminus M \subseteq W.$$

In fact, equality holds: indeed, if $x \in W$, then $x \in B \setminus L$ for some $B \in \beta_W$ and $L \in \mathcal{I}$ (and $B \setminus L \subseteq W$). Then $x \notin M$ as $x \in W$, and so $x \in B \setminus M$.

So in particular W is \mathcal{I} -nearly τ -open.

(i) This follows since the \mathcal{I} -nearly τ -open sets are a σ -ideal containing \mathcal{H} .

(ii) If N is \mathcal{H} -nowhere dense, then so is its \mathcal{H} -closure \bar{N} . So $W := X \setminus \bar{N}$ is \mathcal{H} -open and \mathcal{H} -everywhere dense. Write $W = (V \setminus M)$ with $V \in \tau$ and $M \in \mathcal{I}$. It is enough to show now that V is τ -everywhere dense. Indeed, for τ -open U , as U is also \mathcal{H} -open, $U \cap (V \setminus M) = (U \cap V) \setminus M$ is non-empty, as otherwise $U \cap V \subseteq M$, a contradiction.

Conversely, if V is τ -everywhere dense and $M \in \mathcal{I}$, then we are to show that $(X \setminus V) \cup M$ is \mathcal{H} -nowhere dense. For non-empty τ -open U and $M' \in \mathcal{I}$, the set $(U \setminus M') \cap (V \setminus M) = (U \cap V) \setminus (M \cup M')$ is non-empty (otherwise $M \cup M'$ covers the non-empty set $U \cap V$). So $V \setminus M$ is \mathcal{H} -everywhere dense, and so $X \setminus (V \setminus M)$ is \mathcal{H} -nowhere dense and $X \setminus (V \setminus M) = (X \setminus V) \cup M$.

(iii) Straightforward, since if W_n for $n \in \mathbb{N}$ is \mathcal{H} -everywhere dense, then $X \setminus W_n$ is \mathcal{H} -nowhere dense, and so

$$X \setminus W_n = X \setminus G_n \cup M_n$$

for some G_n τ -open τ -everywhere dense, and $M_n \in \mathcal{I}$. That is: the topology is Baire. \square

Cautionary example. For $\mathcal{I} = \mathcal{L}_0$, the Lebesgue null sets, decompose \mathbb{R} into a meagre and a null set [Oxt2, Th. 1.6] to see that starting with τ the usual topology of \mathbb{R} yields a Hashimoto topology that is not Baire, by (i) above.

4 Steinhaus-Weil-like properties

The Kemperman property (iv) of Theorem 1 above, for the \mathcal{D} -open sets U :

$$U \cap (tU) \in \mathcal{B}_+(\mathcal{D}) \quad (||t|| < \delta),$$

yields the Steinhaus-Weil property $B_\delta \subseteq A^{-1}A$ for a set A (see e.g. [BinO6], [BinGT, 1.1.2], [BinO9]), provided the Kemperman property extends to A , since

$$ta = a' \quad \Leftrightarrow \quad t = a'a^{-1} \in AA^{-1}.$$

So we regard the Kemperman property as a weak *Steinhaus-Weil-like* property. The extension beyond \mathcal{D} -sets to $\mathcal{B}_+(\mathcal{D})$ will indeed hold in the presence of the *Nikodym property* (§2 above), when $U \setminus A \in \mathcal{B}_0(\mathcal{D})$ for some $U \in \mathcal{D}$, as then $t(U \setminus A) \in \mathcal{B}_0(\mathcal{D})$. The fact that this is so for A a Haar-measurable set, or a Baire set, is the source of the most direct proofs, recalled below for completeness, of the Steinhaus-Pettis Theorem (see e.g. [BinO4, Th. 6.5]). These generalize (from the Euclidean case) a Lemma first observed by Kemperman ([Kem]; cf. [Kuc, Lemma 3.7.2]).

In Lemma 2 below we take a more direct approach – a streamlined version of the ‘uniformity’ approach in [Hal, Th. 61.A] – inspired by Stromberg [Str], but with *translational subcontinuity* of measure $\mu \in \mathcal{M}(G)$ (its definition below motivated by the upper semicontinuity of the map $x \mapsto \mu(xK)$), in place of *translation-invariance* of measure, and referring to the group norm, more thematic here. This is followed by its Baire-category analogue. Below, for η a left Haar measure of a locally compact group G , $\mathcal{M}_+(\eta)$ denotes the left Haar measurable sets of positive (finite) measure, by analogy with the notation $\mathcal{B}_+ = \mathcal{B}_+(\mathcal{T})$ for the non-meagre Baire sets.

A key tool is provided by a form of the ‘telescope’ or ‘tube’ lemma (cf. [Mun, Lemma 5.8]). Our usage of upper semicontinuity in relation to set-valued maps follows [Rog], cf. [Bor].

Lemma 1. (cf. [BeeV]). *For compact $K \subseteq G$, the map $t \mapsto tK$ is upper*

semicontinuous; in particular, for $\mu \in \mathcal{M}(G)$,

$$m_K : t \mapsto \mu(tK)$$

is upper semicontinuous, hence μ -measurable. In particular, if $m_K(t) = 0$, then m_K is continuous at t .

Proof. For compact K and open $V \supseteq K$, pick for each $k \in K$, an $r(k) > 0$ with $B_{2r(k)}(k) = B_{2r(k)}k \subseteq V$. By compactness, there are k_1, \dots, k_n with $K \subseteq \bigcup_j B_{r(k_j)}k_j \subseteq V$; then for $\delta := \min_j r(k_j) > 0$

$$tK \subseteq \bigcup_j tB_{r(k_j)}k_j \subseteq \bigcup_j B_{2r(k_j)}k_j \subseteq V \quad (\|t\| < \delta).$$

To prove upper semicontinuity of m_K , fix $t \in G$. For $\varepsilon > 0$, as tK is compact, choose by outer regularity an open $U \supseteq tK$ with $\mu(U) < \mu(tK) + \varepsilon$; by the first assertion, there is an open ball B_δ at 1_G with $B_\delta tK \subseteq U$, and then $\mu(B_\delta(t)K) \leq \mu(U) < \mu(K) + \varepsilon$. \square

Of course, in a locally compact group G with left Haar measure μ , m_K is *constant*. By Luzin's theorem, the restriction of m_K (for K compact) to appropriate non-null subsets of G is (relatively) continuous; but of greater significance, as emerges in [BinO9], is a form of subcontinuity *relativized to a fixed sequence* $t_n \rightarrow 1_G$, linking the concept to Solecki's amenability at 1 [Sol2] (see §8.5), and the latter, like *outright continuity at* 1_G , yields the Steinhaus-Weil property (of non left-Haar null universally measurable sets). Regularity of measure also plays a part; it is likewise key in establishing in [Kom] the connection between the (wider) Steinhaus-Weil property concerning AB^{-1} (for which see §6.9) and certain forms of *metric transitivity* of measure (specifically, the co-negligibility/ 'residuality' of AD for any countable dense D cf. [Kuc, Th. 3.6.1], [CichKW, Ch. 7]), known as the *Smítal property* ([KucS], cf. [BarFN]). We begin with a

Cautionary example. In any separable group G , for $D := \{d(n) : n \in \mathbb{N}\}$ a dense subset, define a regular probability measure by $\mu_D := \sum_{n \in \mathbb{N}} 2^{-n} \delta_{d(n)}$, with δ_g the Dirac measure at g (unit point-mass at g); then $\mu_D(U) > 0$ for all non-empty U , as $\mu_D(d(n)) = 2^{-n}$. However, there exist arbitrarily small translations t with $\mu_D(tD) = 0$ (since D is meagre – cf. [MilO]). This is particularly obvious in the case $G = \mathbb{R}$ with $D = \mathbb{Q}$, on taking small

irrational translations; while the situation here is attributable to μ_D having atoms (not just being singular w.r.t. Lebesgue measure – see the Mospan property below), its obverse occurs when $\mu(U) = 0$, for some non-atomic $\mu \in \mathcal{P}(G)$ and U non-empty open, as there is a translation $t = d(n)$ such that $\mu(tU) > 0$ (as $\{d(n)U : n \in \mathbb{N}\}$ covers G). Then ${}_t\mu(U) > 0$ for ${}_t\mu(\cdot) = \mu(t\cdot)$, which is non-atomic.

Motivated by this last comment, and with Proposition 3 below in mind, let us note a non-atomic modification μ of μ_D making all open sets μ -non-null. In $\mathcal{P}(G)$, under its (separable, metrizable) weak topology, the non-atomic measures form a dense \mathcal{G}_δ (see e.g. [Par, II.8]). So for a non-atomic $\tilde{\mu}$ take $\mu := \sum_{n \in \mathbb{N}} 2^{-n} d(n)\tilde{\mu}$, which is non-atomic with $\mu(U) > 0$ for any non-empty open U .

It is inevitable that the significance of small changes in measure relates to amenability (see the Reiter condition in [Pat, Prop. 0.4], cf. §6.5).

Definition. For $\mu \in \mathcal{P}(G)$ and compact $K \subseteq G$, noting that $\mu_\delta(K) := \inf\{\mu(tK) : t \in B_\delta\}$ is weakly decreasing in δ , put

$$\mu_-(K) := \sup_{\delta > 0} \inf\{\mu(tK) : t \in B_\delta\}.$$

Then, as $1_G \in B_\delta$,

$$0 \leq \mu_-(K) \leq \mu(K).$$

We will say that the measure μ is *translation-continuous*, or just *continuous*, if $\mu(K) = \mu_-(K)$ for all compact K ; evidently $m_K(\cdot)$ is continuous if μ is translation continuous, since $m_K(st) = m_{tK}(s)$ and tK is compact whenever K is compact. For G locally compact this occurs for $\mu = \eta$, the left-Haar measure, and also for μ absolutely continuous w.r.t. to η (see below). We call μ *maximally discontinuous* at K if $0 = \mu_-(K) < \mu(K)$. That a measure μ singular w.r.t. Haar measure is just such an example was first discovered by Simmons [Sim] (and independently, much later, by Mospan [Mos]). The intermediate situation when $\mu(K) \geq \mu_-(K) > 0$ for all μ -non-null compact K is of significance; then we call μ *subcontinuous*. The notion of subcontinuity for functions goes back to [Ful] (cf. [Bou]): as applied to the function $m_K(t)$, regarded as a map into the positive reals $(0, \infty)$, subcontinuity at $t = 1_G$ requires that for *every* sequence $t_n \rightarrow 1_G$ there is a subsequence $t_{m(n)}$ with $m_K(t_{m(n)})$ convergent (to a positive value). Thus our usage, applied to a measure μ , is equivalent to demanding, for each μ -non-null compact K and *any* null sequence $\{t_n\}$ (i.e. with $t_n \rightarrow 1_G$, as in §1), that there be a subsequence

$\mu(t_{m(n)}K)$ bounded away from 0. This is already reminiscent of *amenability* at 1_G – again see §6.5. In Lemma 2 below (H-K) is for ‘Haar-Kemperman’, as in Proposition 1 below.

Lemma 2. *Let $\mu \in \mathcal{P}(G)$. For μ -non-null compact $K \subseteq G$, if $\mu_-(K) > 0$ (i.e. μ is subcontinuous at K), then there is $\delta > 0$ with*

$$tK \cap K \in \mathcal{M}_+(\mu) \quad (\|t\| < \delta), \quad (H-K)$$

so in particular,

$$B_\delta \subseteq KK^{-1}$$

(so that B has compact closure), or, equivalently,

$$tK \cap K \neq \emptyset \quad (\|t\| < \delta).$$

Proof. Choose $\Delta > 0$ with $\mu_\Delta(K) > \mu_-(K)/2$. By outer regularity of μ , choose U open with $K \subseteq U$ and $\mu(U) < \mu(K) + \mu_-(K)/2$. By upper semicontinuity of $t \mapsto tK$, w.l.o.g. $B_\delta K \subseteq U$ for some $\delta < \Delta$. Then (H-K) holds for this δ : otherwise, for some $t \in B_\delta$, as $\mu(tK \cap K) = 0$,

$$\mu_\delta(K) + \mu(K) \leq \mu(tK) + \mu(K) = \mu(tK \cup K) \leq \mu(U) < \mu(K) + \mu_-(K)/2,$$

so $\mu_-(K)/2 < \mu_\Delta(K) \leq \mu_\delta(K) < \mu_-(K)/2$, a contradiction. Given $\|t\| < \delta$ and $tK \cap K \in \mathcal{M}_+$, take $s \in tK \cap K \neq \emptyset$; then $s = ta$ for some $a \in K$, so $t = sa^{-1} \in KK^{-1}$. Conversely, $t \in B_\delta \subseteq KK^{-1}$ yields $t = a'a^{-1}$ for some $a, a' \in K$; then $a' = ta \in K \cap tK$. \square

Proposition 1M/1B below, which occurs in two parts (measure and Baire category cases), unifies and extends various previous results due to among others Steinhaus, Weil, Kemperman, Kuczma, Stromberg, Weil, Wilczyński, Simmons [Sim] and Mospan [Mos] – see [BinO6,9] for references.

Proposition 1M (Haar-Kemperman property). *Let $\mu \in \mathcal{P}(G)$ with each map $m_K : t \mapsto \mu(tK)$, for non-null compact $K \subseteq G$, continuous at 1_G . Then for μ -measurable A with $0 < \mu(A) < \infty$ there is $\delta > 0$ with*

$$tA \cap A \in \mathcal{M}_+(\mu) \quad (\|t\| < \delta), \quad (H-K)$$

so in particular,

$$B_\delta \subseteq AA^{-1},$$

or, equivalently,

$$tA \cap A \neq \emptyset \quad (||t|| < \delta).$$

Proof. By inner regularity of μ , there exists a compact $K \subseteq A$ with $0 < \mu(K) \leq \mu(A) < \infty$. Now apply the previous lemma. \square

We can now give a general form of a result of Mospan (sharpened by provision of a converse).

Corollary 1 (Mospan property, [Mos, Th. 2]). *For μ -non-null compact K , if $1_G \notin \text{int}(KK^{-1})$, then $\mu_-(K) = 0$, i.e. μ is maximally discontinuous at K ; equivalently, there is a ‘null sequence’ $t_n \rightarrow 1_G$ with $\lim_n \mu(t_n K) = 0$.*

Conversely, if $\mu(K) > \mu_-(K) = 0$, then there is a null sequence $t_n \rightarrow 1_G$ with $\lim_n \mu(t_n K) = 0$, and there is $C \subseteq K$ with $\mu(K \setminus C) = 0$ with $1_G \notin \text{int}(CC^{-1})$.

Proof. The first assertion follows from Lemma 2. For the converse, as in [Mos]: suppose that $\mu(t_n K) = 0$, for some sequence $t_n \rightarrow 1_G$. By passing to a subsequence, we may assume that $\mu(t_n K) < 2^{-n-1}$ for all $n \in \mathbb{N}$. Put $D_m := K \setminus \bigcap_{n \geq m} t_n K \subseteq K$; then $\mu(K \setminus D_m) \leq \sum_{n \geq m} \mu(t_n K) < 2^{-m}$, so $\mu(D_m) > 0$ provided $2^{-m} < \mu(K)$. Now choose compact $C_m \subseteq D_m$, with $\mu(D_m \setminus C_m) < 2^{-m}$. So $\mu(K \setminus C_m) < 2^{1-m}$. Also $C_m \cap t_n C_m = \emptyset$, for each $n \geq m$, as $C_m \subseteq K$; but $t_n \rightarrow 1_G$, so the compact set $C_m C_m^{-1}$ contains no interior points. Hence, by Baire’s theorem, neither does CC^{-1} for $C = \bigcup_m C_m$ which differs from K by a null set. \square

The significance of Corollary 1 is that in alternative language it asserts a *Converse Steinhaus-Weil Theorem*:

Proposition 2. *A regular Borel measure μ on a topological group G has the Steinhaus-Weil property iff either of the following holds:*

- (i) *for each non-null compact subset K the map $m_K : t \rightarrow \mu(tK)$ is subcontinuous at 1_G ;*
- (ii) *for each non-null compact subset K there is no ‘null’ sequence $t_n \rightarrow 1_G$ with $\mu(t_n K) \rightarrow 0$.*

Remark. In the locally compact case, Simmons and Mospan both prove that this is equivalent to μ being absolutely continuous w.r.t. Haar measure η ; see §6.2 below. For the more general context of a Polish group see [BinO9].

For the Baire-category version, which goes back to Piccard and Pettis (see [BinO6] for references), we recall that the *quasi-interior*, here conveniently denoted \tilde{A} (or A^q) of a set A with the Baire property, is the largest (usual) open set U such that $U \setminus A$ is meagre; it is a regularly-open set (see [Dug, Ch. 3 Problems, Section 4 Q22], §6.3). We note that $(aA)^q = aA^q$. We learn from Theorem 4 below that the counterpart of this ‘quasi-interior’ for measurable sets is provided by the open sets of the density topology.

Proposition 1B (Baire-Kemperman property) ([BinO4, Th. 5.5B/M], [BinO3, §5, Th. K]). *In a normed topological group G , Baire under its norm topology, if A is Baire non-meagre, then there is $\delta > 0$ with*

$$tA \cap A \in \mathcal{B}_+ \quad (\|t\| < \delta);$$

so in particular,

$$B_\delta \subseteq AA^{-1}.$$

Proof. If $a \in A^q$, the quasi-interior of A , then $1_G \in a^{-1}\tilde{A}$, which is open. So w.l.o.g. we may take $a = 1_G$. Choose $\delta > 0$ so that $B := B_\delta \subseteq \tilde{A}$. Then for $t \in B$, since $t \in tB \cap B$,

$$t\tilde{A} \cap \tilde{A} \supseteq tB \cap B \neq \emptyset,$$

so being open, $tB \cap B$ is non-meagre (as G is Baire). But, modulo meagre sets, A and \tilde{A} are identical. For the remaining assertion, argue as in the measure case. \square

Propositions 1M and 1B are both included in Theorem 4 below in the topology refinement context with $\mathcal{B}_+(\mathcal{D}_\mathcal{L}) = \mathcal{M}_+$, and $\mathcal{B}_+(\mathcal{D}_\mathcal{B}) = \mathcal{B}_+(\mathcal{I}_d)$; we apply the result in Theorem 5. (Recall the discussion in §2 of their properties, enlisted below.)

Theorem 4 (Displacements Theorem). *In a topological group under the topology of d_R^G , let \mathcal{D} be a topology refining d_R^G and having the following properties:*

- (i) \mathcal{D} is shift-invariant: $x\mathcal{D} = \mathcal{D}$ for all x ;
- (ii) $\mathcal{B}_0(\mathcal{D})$ is left invariant for all x ;
- (iii) ‘Localization property’: $H \setminus \text{int}_{\mathcal{D}}(H) \in \mathcal{B}_0(\mathcal{D})$ for $H \in \mathcal{B}(\mathcal{D})$;

(iv) the (left) ‘Kemperman property’: for $1_G \in U \in \mathcal{D}$ there is $\delta = \delta_U > 0$ with

$$tU \cap U \in \mathcal{B}_+(\mathcal{D}) \quad (\|t\| < \delta);$$

(v) \mathcal{D} is a strong generalized Gandy-Harrington topology: modulo $\mathcal{B}_0(\mathcal{D})$ each $\mathcal{B}(\mathcal{D})$ set is analytic under d_R^X , so that \mathcal{D} is a Baire space.

Then for $A \in \mathcal{B}_+(\mathcal{D})$ and quasi all $a \in A$, there is $\varepsilon = \varepsilon_A(a) > 0$ with

$$axa^{-1}A \cap A \in \mathcal{B}_+(\mathcal{D}) \quad (\|x\| < \varepsilon);$$

so in particular, with $\gamma_a(x) := axa^{-1}$,

$$\gamma_a(B_\varepsilon) \subseteq AA^{-1} \quad \text{off a } \mathcal{B}_0(\mathcal{D})\text{-set of } a \in A.$$

Proof. For some $U \in \mathcal{D}$ and $N, A' \in \mathcal{B}_0(\mathcal{D})$, $A \setminus A' = U \setminus N$; by (i) and (ii), $U_a = a^{-1}U \in \mathcal{D}$ and $N_a = a^{-1}N \in \mathcal{B}_0(\mathcal{D})$, so $a^{-1}A \supseteq U_a \setminus N_a$ for $a \in A \setminus A'$. By the Kemperman property, for $a \in A \setminus A'$ there is $\varepsilon = \varepsilon_A(a) > 0$ with

$$xU_a \cap U_a \in \mathcal{B}_+(\mathcal{D}) \quad (\|x\| < \varepsilon).$$

Fix $a \in A \setminus A'$. Working modulo $\mathcal{B}_0(\mathcal{D})$ -sets which are left invariant (by the Nikodym property), $xa^{-1}A \supseteq xU_a$ and

$$a^{-1}A \cap xa^{-1}A \supseteq U_a \cap (xU_a), \quad (\|x\| < \varepsilon).$$

So, since also $axa^{-1}A \in \mathcal{B}(\mathcal{D})$, again by the Nikodym property

$$axa^{-1}A \cap A \in \mathcal{B}_+(\mathcal{D}) \quad (\|x\| < \varepsilon).$$

For the remaining assertion, argue as in Prop. 1M above. \square

Remark. The map $\gamma_a : x \mapsto axa^{-1}$ in the preceding theorem is a homeomorphism under the d_R^G topology of the topological group (being continuous, with continuous inverse $\gamma_{a^{-1}}$), so $\gamma_a(B_{\varepsilon_A(a)})$ is open, and also open under the finer topology \mathcal{D} .

Theorem 4 above identifies additional topological properties enabling the Kemperman property to imply the Steinhaus-Weil property. So we regard it as a *weak* Steinhaus-Weil property; the extent to which it is weaker is clarified by Proposition 3 below. For this we need a σ -algebra. Recall that

$E \subseteq G$ is *universally measurable* ($E \in \mathcal{U}(G)$) if E is measurable with respect to every measure $\mu \in \mathcal{P}(G)$ – for background, see e.g. [Kec, §21D], cf. [Fre, 434D, 432]; these form a σ -algebra. Examples are analytic subsets (see e.g. [Rog, Part 1 §2.9], or [Kec, Th. 21.10], [Fre, 434Dc]) and the σ -algebra that they generate. Beyond these are the *provably* Δ_2^1 sets of [FenN], [Kan, Ch. 3 §14.4].

Proposition 3 (cf. BinO8, Cor. 2]). *For \mathcal{H} a left invariant σ -ideal in G , put*

$$\mathcal{U}_+(\mathcal{H}) = \mathcal{U}(G) \setminus \mathcal{H}.$$

If $\mathcal{U}_+(\mathcal{H})$ has the Steinhaus-Weil property, then $\mathcal{U}_+(\mathcal{H})$ has the Kemperman property.

Proof. Suppose otherwise. Then, for some $E \in \mathcal{U}_+(\mathcal{H})$ and some $t_n \rightarrow 1_G$, each of the sets $E \cap t_n E$ is in \mathcal{H} , and so

$$E_0 := E \setminus \bigcup_n t_n E = E \setminus \bigcup_n (E \cap t_n E) \in \mathcal{U}(G),$$

which is in $\mathcal{U}_+(\mathcal{H})$ (as otherwise $E = E_0 \cup \bigcup [E \cap t_n E] \in \mathcal{H}$). So $1_G \in \text{int}(E_0 E_0^{-1})$. So, for ultimately all n , $t_n \in E_0 E_0^{-1}$, and then $E_0 \cap t_n E_0 \neq \emptyset$. But, as $E_0 \subseteq E$, $E_0 \cap t_n E_0 \subseteq E_0 \cap t_n E = \emptyset$, a contradiction. \square

Of interest above is the case \mathcal{H} of *left-Haar-null* sets [Sol2] of a Polish group G (cf. §6.5,6). We close with a result on the density topology \mathcal{D}_μ generated in a Polish group G from an *atomless* measure μ ; this is an immediate corollary of [Mar1]. Unlike the Hashimoto ideal topologies, which need not be Baire topologies, these are Baire. The question of which sets have the Steinhaus-Weil property under μ , hinges on the choice of μ (see the earlier cautionary example of this section), and indeed on further delicate subcontinuity considerations, related to [Sol2], for which see [BinO9]. In this connection see [Oxt1] and [DieS, Ch. 10].

Proposition 4. *For G a Polish group with metric topology \mathcal{T}_d , β a countable basis, and atomless $\mu \in \mathcal{P}(G)$ with $\mu(U) > 0$ for all non-empty $U \in \beta$, then, with density at $g \in G$ computed by reference to $\beta_g := \{B \in \beta : g \in B\}$, the Lebesgue density theorem holds for μ . So the generated density topology \mathcal{D}_μ refines \mathcal{T}_d and is a Baire topology.*

Proof. As β comprises open sets, the Vitali covering lemma applies (see [Bru, §6.3-4]), and implies that the measure μ obeys a density theorem (that μ -almost all points of a measurable set are density points – cf. [Kuc, Th. 3.5.1]). So by results of Martin [Mar1, Cor. 4.4], the family \mathcal{D}_μ of sets all of whose points have (outer) density 1 are μ -measurable, and form a topology on G [Mar1, Th. 4.1]. It is a Baire topology, by [Mar1, Cor. 4.13]. It refines \mathcal{T}_d : the points of any open set have density 1, because the differentiation basis consists of open sets, and these all are μ -non-null. \square

Remark. The assumption of regularity subsumed in $\mu \in \mathcal{P}(G)$ is critical; in its absence the density theorem may fail: see [Kha, Ch.8 Th. 1], where for μ non-regular there is a μ -measurable set with just one density point.

5 A Shift Theorem

Theorem 5 below establishes a compactness-like property of a density topology \mathcal{D} which, according to Corollary 2 below, implies the Steinhaus-Weil property for sets in $A \in \mathcal{B}_+(\mathcal{D})$. So we may regard it as a strong Steinhaus-Weil-like property. Its prototype arises in the relevant infinite combinatorics (the Kestelman-Borwein-Ditor Theorem, KBD: see [Bin06], [Ost3]). The setting for the theorem is that of the Displacements Theorem (Th. 4 above), a key ingredient of which is the Kemperman property, a weak Steinhaus-Weil-like property, as noted in §4. So Theorem 5 establishes the equivalence of a strong and a weak Steinhaus-Weil-like property in the presence of additional topological restrictions on the relevant refinement topology: invariance under shift, localization and some analyticity (namely, a weak base of analytic sets). We raise and leave open the question as to whether the three topological restrictions listed above are minimal here.

Theorem 5 (Fine Topology Shift Theorem). *In a topological group under the topology of d_R^G , let \mathcal{D} be a topology refining d_R^G and having the following properties:*

- (i) \mathcal{D} is shift-invariant: $x\mathcal{D} = \mathcal{D}$ for all x ;
- (ii) ‘Localization property’: $H \setminus \text{int}_{\mathcal{D}}(H) \in \mathcal{B}_0(\mathcal{D})$ for $H \in \mathcal{B}(\mathcal{D})$;
- (iii) the (left) ‘Kemperman property’: for $1_G \in U \in \mathcal{D}$ there is $\delta = \delta_U > 0$ with

$$U \cap (tU) \in \mathcal{B}_+(\mathcal{D}) \quad (||t|| < \delta);$$

(iv) \mathcal{D} is a strong generalized Gandy-Harrington topology: modulo $\mathcal{B}_0(\mathcal{D})$ each $\mathcal{B}(\mathcal{D})$ set is analytic under d_R^G , so that \mathcal{D} is a Baire space.

Then, for $z_n \rightarrow 1_G$ (“null sequence”) and $A \in \mathcal{B}_+(\mathcal{D})$, for quasi all $a \in A$ there is an infinite set \mathbb{M}_a such that

$$\{z_m a : m \in \mathbb{M}_a\} \subseteq A.$$

Proof of Theorem 5. Since the asserted property is monotonic, we may assume by the strong Gandy-Harrington property that A is analytic in the topology of d_R^G . So write $A = K(I)$ with K upper-semicontinuous and single-valued. Below, for greater clarity we write $B(x, r)$ for the open r -ball centered at x . For each $n \in \omega$ we find inductively integers i_n , points x_n, y_n, a_n with $a_n \in A$, numbers $r_n \downarrow 0, s_n \downarrow 0$, analytic subsets A_n of A , and closed nowhere dense sets $\{F_m^n : m \in \omega\}$ w.r.t. \mathcal{D} and $D_n \in \mathcal{D}$ such that:

$$K(i_1, \dots, i_n) \supseteq a_n x_n a_n^{-1} A_n \cap A_n \in \mathcal{B}_+(\mathcal{D}),$$

$$K(i_1, \dots, i_n) \supseteq a_n x_n a_n^{-1} A_n \cap A_n \supseteq D_n \cap B(y_n, s_n) \setminus \bigcup_{m \in \omega} F_m^n,$$

$$y_n \in B(a_n x_n, r_n) \text{ and } D_n \cap B(y_n, s_n) \cap \bigcup_{m, k < n} F_m^k = \emptyset.$$

Assuming this done for n , since $K(i_1, \dots, i_n) = \bigcup_k K(i_1, \dots, i_n, k)$ is non-negligible, there is i_{n+1} with $K(i_1, \dots, i_n, i_{n+1}) \cap a_n x_n a_n^{-1} A_n \cap A_n$ non-negligible. Put $A_{n+1} := K(i_1, \dots, i_n, i_{n+1}) \cap a_n x_n a_n^{-1} A_n \cap A_n \subseteq K(i_1, \dots, i_{n+1})$. As A_{n+1} is non-negligible, we may pick $a_{n+1} \in A_{n+1}$ and $\varepsilon(a_{n+1}, A_{n+1})$ as in Theorem 4 above; also pick $m(n)$ so large that $\|z_m\| < \varepsilon(a_{n+1}, A_{n+1})$ for $m \geq m(n)$ and that for $x_{n+1} := a_{n+1} z_{m(n)}^{-1} a_{n+1}^{-1}$ also $\|x_{n+1}\| \leq 2^{-n}$ (the latter is possible, as G is a topological group in the group-norm topology). Then

$$K(i_1, \dots, i_{n+1}) \supseteq a_{n+1} x_{n+1} a_{n+1}^{-1} A_{n+1} \cap A_{n+1} \in \mathcal{B}_+(\mathcal{D}).$$

So by the Banach Category Theorem, for some $D_{n+1} \in \mathcal{D}$ and some positive $r_{n+1} < r_n/2$

$$K(i_1, \dots, i_{n+1}) \supseteq a_{n+1} x_{n+1} a_{n+1}^{-1} A_{n+1} \cap A_{n+1} \supseteq D_{n+1} \cap B(a_{n+1} x_{n+1}, r_{n+1}) \setminus \bigcup_{m \in \omega} F_m^{n+1},$$

for some closed nowhere dense sets $\{F_m^{n+1} : m \in \omega\}$ of \mathcal{D} . Since the set $\bigcup_{m, k < n+1} F_m^k$ is closed and nowhere dense, there is $y_{n+1} \in D_{n+1} \cap B(a_{n+1} x_{n+1}, r_{n+1})$

and positive $s_{n+1} < s_n/2$ so small that $B(y_{n+1}, s_{n+1}) \subseteq B(a_{n+1}x_{n+1}, r_{n+1})$ and $D_{n+1} \cap B(y_{n+1}, s_{n+1}) \cap \bigcup_{m, k < n+1} F_m^k = \emptyset$. So

$$D_{n+1} \cap B(a_{n+1}x_{n+1}, r_{n+1}) \setminus \bigcup_{m \in \omega} F_m^{n+1} \supseteq D_{n+1} \cap B(y_{n+1}, s_{n+1}) \setminus \bigcup_{m \in \omega} F_m^{n+1},$$

completing the induction.

By the Analytic Cantor Theorem [Ost1, Th. AC, Section 2], there is t with

$$\{t\} = K(i) \cap \bigcap_n B(y_n, s_n) \subseteq \bigcap_n a_n x_n a_n^{-1} A_n \cap A_n.$$

So $t \in A$. Fix n ; then $t \notin \bigcup_{m \in \omega} F_m^n$ (since $D_{m+1} \cap B(y_{m+1}, s_{m+1}) \cap \bigcup_{k < m} F_k^n = \emptyset$ for each m), and so $t \in T$ and

$$t \in D_n \cap B(y_n, s_n) \setminus \bigcup_{m \in \omega} F_m^n \subseteq a_n x_n a_n^{-1} A_n \cap A_n \subseteq K(i_1, \dots, i_n) \subseteq A.$$

As $t \in a_{n+1}x_{n+1}a_{n+1}^{-1}A_{n+1}$, $a_{n+1}x_{n+1}a_{n+1}^{-1}t = z_{m(n)}t \in A_{n+1} \subseteq A$. So $\{z_{m(n)}t : n \in \omega\} \subseteq A$.

Now $d_R^G(a_n x_n, t) \leq d_R^G(a_n x_n, y_n) + d_R^G(y_n, t) \rightarrow 0$, so $a_n \rightarrow t$, since $x_n \rightarrow 1_G$ and so

$$d_R^G(a_n, t) = d_R^G(a_n x_n, t x_n) \leq d_R^G(a_n x_n, t) + d_R^G(1_G, t x_n t^{-1}) \rightarrow 0,$$

again as G is a topological group under the d_R^G -topology. \square

Corollary 2. *In the setting of Theorem 5 above, the sets of $\mathcal{B}_+(\mathcal{D})$ have the Steinhaus-Weil property (§1).*

Proof (cf. [Sol1, Th. 1(ii)], [BinO4, Th. 6.5]). Otherwise, for some set $A \in \mathcal{B}_+(\mathcal{D})$ we may select $z_n \notin AA^{-1}$ with $\|z_n\| < 1/n$. Then there are $a \in A$ and a subsequence $m(n)$ with $z_{m(n)}a \in A$; so $z_{m(n)} \in AA^{-1}$, a contradiction. \square

6 Complements

1. *Weil topology* ([Wei, Ch. VII, §31], cf. [Hal, Ch. XII §62]). We recall that for G a group with a σ -finite left-invariant measure $|\cdot|$ on a σ -ring \mathcal{M} of left-invariant sets and $(x, y) \mapsto (x, xy)$ measurability-preserving, the *Weil topology* is generated by the family of pseudo-norms

$$\|g\|_E := |gE \Delta E|,$$

for $E \in \mathcal{M}_+$ (with \mathcal{M}_+ the family of measurable sets with finite positive measure), so that $\|g\|_E \leq 2|E|$. Provided the pseudo-norms are *separating* (i.e. $\|g\|_E > 0$ for any $g \neq 1_G$ and some $E \in \mathcal{M}_+$ as in (iii) above), G is a topological group under the Weil topology [Hal, 62E]; equivalently, the topology is generated by the neighbourhood base $\mathcal{N}_1 := \{DD^{-1} : 1_G \in D \in \mathcal{M}^+\}$, reminiscent of the Steinhaus-Weil Theorem. The proof relies on a kind of fragmentation lemma (see [BinO9]). That in turn depends on Fubini's Theorem [Hal, 36C], via the average theorem [Hal, 59.F]:

$$\int_G |g^{-1}A \cap B| dg = |A| \cdot |B^{-1}| \quad (A, B \in \mathcal{M}),$$

($g = ab^{-1}$ iff $g^{-1}a = b$), and may be interpreted as demonstrating the continuity at 1_G of $\|\cdot\|_E$ under the density topology.

2. *Steinhaus-Weil property of a Borel measure.* In a locally compact group G , the family $\mathcal{M}_+(\mu)$ of finite non-null measurable sets of a Borel measure μ on G fails to have the Steinhaus-Weil property iff there are a null sequence $z_n \rightarrow 1_G$ and a non-null compact set K with $\lim_n \mu(t_n K) = 0$, as observed by Mospan [Mos] (in \mathbb{R}). Equivalently, this is so iff the measure μ is not absolutely continuous with respect to Haar measure – cf. [Sim] and [BinO9].

3. *Regular open sets.* Recall that U is regular open if $U = \text{int}(\text{cl}U)$, and that $\text{int}(\text{cl}U)$ is itself regular open; for background see e.g. [GivH, Ch. 10], or [Dug, Ch. 3 Problems, Section 4 Q22]. For $\mathcal{D} = \mathcal{D}_{\mathcal{B}}$ the Baire-density topology of a normed topological group, let $\mathcal{D}_{\mathcal{B}}^{RO}$ denote its regular open sets. For $D \in \mathcal{D}_{\mathcal{B}}^{RO}$, put

$$N_D := \{t \in G : tD \cap D \neq \emptyset\} = DD^{-1}, \quad \mathcal{N}_1 := \{N_D : 1_G \in D \in \mathcal{D}_{RO}\};$$

then \mathcal{N}_1 is a base at 1_G (since $1_G \in C \in \mathcal{D}_{RO}$ and $1_G \in D \in \mathcal{D}_{RO}$ yield $1_G \in C \cap D \in \mathcal{D}_{RO}$) comprising \mathcal{T} -neighbourhoods that are $\mathcal{D}_{\mathcal{B}}$ -open (since $DD^{-1} = \bigcup\{Dd^{-1} : d \in D\}$). We raise the (metrizability) question, by analogy with the Weil topology of a measurable group: with $\mathcal{D}_{\mathcal{B}}$ above replaced by a general density topology \mathcal{D} on a group G , when is the topology generated by \mathcal{N}_1 on G a norm topology? Some indications of an answer may be found in [ArhT, §3.3]. We note the following plausible answer: if there exists a separating sequence D_n , i.e. such that for each $g \neq 1_G$ there is n with $\|g\|_{D_n} = 1$, then

$$\|g\| := \sum_n 2^{-n} \|g\|_{D_n}$$

is a norm, since it is separating and, by the Nikodym property, $(D \cap g^{-1}D) = g^{-1}(gD \cap D) \in \mathcal{B}_+$.

4. *The Effros Theorem* asserts that a transitive continuous action of a Polish group G on a space X of second category in itself is necessarily ‘open’, or more accurately is microtransitive (the (continuous) evaluation map $e_x : g \mapsto g(x)$ takes open nhds E of 1_G to open nhds that are the orbit sets $E(x)$ of x). It emerges that this assertion is very close to the shift-compactness property: see [Ost6]. The Effros Theorem reduces to the Open Mapping Theorem when G, X are separable Banach spaces regarded as additive groups, and G acts on X by way of a linear surjection $L : G \rightarrow X$ via $g(x) = L(g) + x$. Indeed, here $e_0(E) = L(E)$. For a neat proof, choose an open neighbourhood U of 0 in G with $E \supseteq U - U$; then $L(U)$ is Baire (being analytic) and non-meagre (since $\{L(nU) : n \in \mathbb{N}\}$ covers X), and so $L(U) - L(U) \subseteq L(E)$ is an open nhd of 0 in X .

5. *Amenability at 1*. Solecki defines G to be amenable at 1 if given $\mu_n \in \mathcal{P}(G)$ for $n \in \mathbb{N}$ with $1_G \in \text{supp}(\mu_n)$ there are ν and ν_n in $\mathcal{P}(G)$ with $\nu_n \ll \mu_n$ and

$$\nu_n * \nu(K) \rightarrow \nu(K) \quad (K \in \mathcal{K}(G)).$$

(The origin of the term may be traced to a localization, via the restriction of supports to contain 1_G , of a *Reiter-like condition* [Pat, Prop. 0.4] characterizing amenability itself.) It is proved in [Sol2, Th. 1(ii)] that, in the class of Polish groups G that are amenable at 1_G , the Steinhaus-Weil property holds for universally measurable sets that are not left-Haar-null; this includes Polish abelian groups [Sol2, Prop. 3.3]. The relativized notion of subcontinuity: *on a compact K along a null sequence $\{t_n\}$* (which requires some subsequence $\mu(t_{m(n)}K)$ to be bounded away from 0, provided $\mu(K) > 0$) yields a connection to amenability at 1_G , which we explore elsewhere [BinO9].

6. *Haar-null and left-Haar-null*. The two families coincide in Polish abelian groups, and in locally compact second countable groups (where they also coincide with the sets of Haar measure zero – by an application of the Fubini theorem). The former family, however, is in general smaller; indeed, (universally measurable) non-Haar-null sets need not have the Steinhaus-Weil property, whereas the (universally measurable) non-left-Haar-null sets do – see [Sol2].

7. *Left-Haar-null Kemperman property*. We note, as this is thematic, that the family of (universally measurable) non-left-Haar-null sets has the left Kemperman property [BinO9, Lemma 1].

8. *Beyond local compactness: Haar category-measure duality.* In the absence of Haar measure, the definition of left-Haar-null subsets of a topological group G requires $\mathcal{U}(G)$, the universally measurable sets – by dint of the role of the totality of (probability) measures on G . The natural dual of $\mathcal{U}(G)$ is the class $\mathcal{U}_{\mathcal{B}}(G)$ of *universally Baire sets*, defined, for G with a Baire topology, as those sets B whose preimages $f^{-1}(B)$ are Baire in any compact Hausdorff space K for any continuous $f : K \rightarrow G$. Initially considered in [FenMW] for $G = \mathbb{R}$, these have attracted continued attention for their role in the investigation of axioms of determinacy and large cardinals – see especially [Woo]; cf. [MarS] and [Kan].

Analogously to the left-Haar-null sets, define in G the family of *left-Haar-meagre* sets, $\mathcal{HM}(G)$, to comprise the sets M coverable by a universally Baire set B for which there are a compact Hausdorff space K and a continuous $f : K \rightarrow G$ with $f^{-1}(gB)$ meagre in K for all $g \in G$. These were introduced, in the *abelian* Polish group setting with K metrizable, by Darji [Dar], cf. [Jab1], and shown there to form a σ -ideal of meagre sets (co-extensive with the meagre sets for G locally compact); as $\mathcal{HM}(G) \subseteq \mathcal{B}_0(G)$, the family is not studied here.

9. *Steinhaus AA^{-1} and AB^{-1} properties.* If the subsets of G lying in a family \mathcal{H} have the property that AA^{-1} for $A \in \mathcal{H}$ has non-empty τ -interior, for τ a translation invariant topology, and furthermore, as in the Haar-Kemperman property, for $A, B \in \mathcal{H}$ there is $g \in G$ such that $C := gA \cap B \in \mathcal{H}$, then of course $g^{-1}CC^{-1} \subseteq AB^{-1}$, and so the latter has non-empty τ -interior. By the Average Theorem (§6.1 above), this is the case for G locally compact with $\tau = \mathcal{T}_d$ and $\mathcal{H} = \mathcal{L}_+$ the Haar-measurable non-null sets [Hal, §59F] (cf. [TomW, §11.3], and [BinO5] for $G = \mathbb{R}$); other examples of families \mathcal{H} are provided by certain refinement topologies τ – see [BinO9] and the extended arXiv version of the current paper arXiv1607.00031v2.

17.12 At end of line add: In a non-abelian setting, note that a left Haar null set need not be right Haar null: for one example see [81], and for more general non-coincidence see Solecki [83, Cor. 6].. However, Matošková and Zelený [MatZ] show that in any non-locally compact abelian Polish group there are closed non (left) Haar null sets A, B such that $A + B$ has empty interior. Recently, Jabłońska [Jab2] has shown that likewise in any non-locally compact abelian Polish group there are closed non-Haar meager sets A, B such that $A + B$ has empty interior.

10. *Non-separability.* The links between the Effros theorem above, the Baire theorem and the Steinhaus-Weil theorem are pursued at length in [Ost6].

There, any separability assumption is avoided. Instead *sequential* methods are used, for example shift-compactness arguments.

11. *Metrizability and Christensen's Theorem.* In connection with the role of analyticity in the generalized Gandy-Harrington property of §2, note that an analytic topological group is metrizable; so if it is also a Baire space, then it is a Polish group [HofT, Th. 2.3.6].

12. *Strong Kemperman property: qualitative versus quantitative measure theory.* We note that property (iv) of Theorem 1 corresponds to the following quantitative, linear Lebesgue-measure property, which we may name the *strong Kemperman property* (see [Kem], [Kuc, Lemma 3.7.2]):

(iv)* for $0 \in U \in \mathcal{D}_{\mathcal{L}}$ there is $\delta > 0$ such that for all $|t| < \delta$

$$|U \cap (t + U)| \geq \varepsilon.$$

This is connected with the continuity of a Weil-like group norm on $(\mathbb{R}, +)$. Indeed, since

$$|U \cap (t + U)| = |U| - |U \Delta (t + U)|/2,$$

the inequality above is equivalent to

$$\|t\|_U := |U \Delta (t + U)| \leq 2(|U| - \varepsilon).$$

The latter holds for any $0 < \varepsilon < |U|$ and for sufficiently small t , by the continuity of the norm $\|t\|_U$.

Acknowledgements. We thank the Referee for a careful and scholarly reading of the paper, and for some very useful presentational suggestions.

References.

- [ArhT] A. Arhangel'skii, M. Tkachenko, *Topological groups and related structures*. Atlantis Studies in Mathematics, 1. Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.
- [BarFN] A. Bartoszewicz, M. Filipczak, T. Natkaniec, On Smital properties. *Topology Appl.* **158** (2011), 2066–2075.
- [BeeV] G. Beer, L. Villar, Borel measures and Hausdorff distance. *Trans. Amer. Math. Soc.* **307** (1988), 763–772.
- [BinGT] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular variation*, 2nd ed., Cambridge University Press, 1989 (1st ed. 1987).
- [BinO1] N. H. Bingham and A. J. Ostaszewski, Beyond Lebesgue and Baire: generic regular variation, *Coll. Math.* **116** (2009), 119–138.
- [BinO2] N. H. Bingham and A. J. Ostaszewski, Beyond Lebesgue and Baire II: Bitopology and measure-category duality, *Coll. Math.*, **121** (2010), 225–238.
- [BinO3] N. H. Bingham and A. J. Ostaszewski, Kingman, category and combinatorics. *Probability and Mathematical Genetics* (Sir John Kingman Festschrift, ed. N. H. Bingham and C. M. Goldie), 135–168, London Math. Soc. Lecture Notes in Mathematics **378**, CUP, 2010.
- [BinO4] N. H. Bingham and A. J. Ostaszewski, Normed groups: Dichotomy and duality. *Dissert. Math.* **472** (2010), 138p.
- [BinO5] N. H. Bingham and A. J. Ostaszewski, Regular variation without limits, *J. Math. Anal. Appl.* **370** (2010), 322–338.
- [BinO6] N. H. Bingham and A. J. Ostaszewski, Dichotomy and infinite combinatorics: the theorems of Steinhaus and Ostrowski. *Math. Proc. Camb. Phil. Soc.* **150** (2011), 1–22.
- [BinO7] N. H. Bingham and A. J. Ostaszewski, The Steinhaus theorem and regular variation: de Bruijn and after, *Indag. Math.* **24** (2013), 679–692.
- [BinO8] N. H. Bingham and A. J. Ostaszewski, Category-measure duality: convexity, mid-point convexity and Berz sublinearity, *Aequationes Math.*, **91.5** (2017), 801–836 (fuller version: arXiv1607.05750).
- [BinO9] N. H. Bingham and A. J. Ostaszewski, The Steinhaus-Weil property: its converse, Solecki amenability and subcontinuity, arXiv1607.00049v3.
- [BinO10] N. H. Bingham and A. J. Ostaszewski, Set Theory and the Analyst, arXiv1801.09149.
- [Bla] C. E. Blair, The Baire category theorem implies the principle of dependent choices. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **25** (1977), 933–934.

- [Bog] V.I. Bogachev, *Gaussian Measures*, Math. Surveys & Monographs **62**, Amer Math Soc., 1998.
- [Bor] K. C. Border, *Fixed point theorems with applications to economics and game theory*. 2nd ed. Cambridge University Press, 1989 (1st ed.1985).
- [Bou] A. Bouziad, Continuity of separately continuous group actions in p-spaces, *Topology Appl.* **71**, (1996),119-124
- [Bru] A. M. Bruckner, Differentiation of integrals. *Amer. Math. Monthly* **78.9** (1971), 1-51
- [Chr1] J. P. R. Christensen, On sets of Haar measure zero in abelian Polish groups. Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972). *Israel J. Math.* **13** (1973), 255–260.
- [Chr2] J. P. R. Christensen, *Topology and Borel structure. Descriptive topology and set theory with applications to functional analysis and measure theory*. North-Holland Mathematics Studies, Vol. 10, 1974.
- [CichKW] J. Cichoń, A. Kharazishvili, B. Węglorz, *Subsets of the real line*, U. Łódzki,1995.
- [CieJ] K. Ciesielski, J. Jasiński, Topologies making a given ideal nowhere dense or meager. *Topology Appl.* **63** (1995), 277–298.
- [CieLO] K. Ciesielski, L. Larson, K. Ostaszewski, \mathcal{I} -density continuous functions, Amer. Math. Soc. Mem. 107, 1994, 133p.
- [Dan] Dang Anh Tuan, A short proof of the converse to a theorem of Steinhaus-Weil, arXiv:1511.05657v1.
- [Dar] U. B. Darji, On Haar meager sets. *Topology Appl.* **160** (2013), 2396–2400.
- [Den] A. Denjoy, Sur les fonctions dérivées sommables, *Bull. Soc. Math. Fr.* **43** (1915), 161-248.
- [DieS] J. Diestel, A. Spalsbury, *The joys of Haar measure*. Grad. Studies in Math. **150**. Amer. Math. Soc., 2014.
- [Dug] J. Dugundji, *Topology*, Allyn & Bacon, 1966.
- [EvaG] L.C. Evans, R.F. Gariepy, *Measure theory and fine properties of functions*, CRC, 1992.
- [FenMW] Q. Feng, M. Magidor, H. Woodin, Universally Baire sets of reals, in H. Judah, W. Just, H. Woodin (eds.), *Set theory of the continuum*, 203–242, Math. Sci. Res. Inst. Publ. **26**, Springer, 1992.
- [FenN] J. E. Fenstad, D. Normann, On absolutely measurable sets. *Fund. Math.* **81** (1973/74), no. 2, 91–98.
- [FilW] M. Filipczak, W. Wilczyński, Strict density topology on the plane.

- Measure case. *Rend. Circ. Mat. Palermo* (2) **60** (2011),113–124.
- [Fre] D. Fremlin, *Measure theory Vol. 3: Measure algebras*. Corrected 2ndsecond printing of the 2002 original. Torres Fremlin, Colchester, 2004.
- [Ful] F. V. Fuller, Relations among continuous and various non-continuous functions. *Pacific J. Math.* **25** (1968), 495–509.
- [Gao] Su Gao, *Invariant descriptive set theory*. Pure and Applied Mathematics **293**, CRC Press, 2009.
- [GivH] S. Givant and P. Halmos, *Introduction to Boolean Algebras*, Springer, 2009 (1st ed., P. Halmos, *Lectures on Boolean Algebras*, Van Nostrand, 1963).
- [GofNN] C. Goffman, C. J. Neugebauer and T. Nishiura, Density topology and approximate continuity, *Duke Math. J.* **28** (1961), 497–505.
- [GofW] C. Goffman, D. Waterman, Approximately continuous transformations, *Proc. Amer. Math. Soc.* **12** (1961), 116–121.
- [GroE] K.-G. Grosse-Erdmann, An extension of the Steinhaus-Weil theorem. *Colloq. Math.* **57.2** (1989), 307–317.
- [Hal] P. R. Halmos, *Measure Theory*, Grad. Texts in Math. **18**, Springer 1974. (1st. ed. Van Nostrand, 1950).
- [Has] H. Hashimoto, On the \ast topology and its application. *Fund. Math.* **91** (1976), no. 1, 5–10.
- [HauP] O. Haupt, C. Pauc, La topologie approximative de Denjoy envisagée comme vraie topologie. *C. R. Acad. Sci. Paris* **234** (1952), 390–392.
- [Hau] F. Hausdorff, *Grundzüge der Mengenlehre*, Veit, Leipzig, 1914 (reprinted, Chelsea, N. York, 1949).
- [Hay] E. Hayashi, Topologies defined by local properties. *Math. Ann.* **156** (1964), 205–215.
- [HewR] E. Hewitt, K. A. Ross, *Abstract Harmonic Analysis*, Vol. I, Grundle. math. Wiss. **115**, Springer 1963 [Vol. II, Grundle. 152, 1970].
- [HofT] J. Hoffmann-Jørgensen, F. Topsøe, Analytic spaces and their application, in [Rog, Part 3].
- [Jab1] E. Jabłońska, Some analogies between Haar meager sets and Haar null sets in abelian Polish groups. *J. Math. Anal. Appl.* **421** (2015),1479–1486.
- [Jab2] E. Jabłońska, A theorem of Piccard’s type in abelian Polish groups. *Anal. Math.* **42** (2016), 159–164.
- [JanH] D. Janković, T. R. Hamlett, New topologies from old via ideals, *Amer. Math. Monthly* **97** (1990), 295-310.
- [Kan] A. Kanamori, *The higher infinity. Large cardinals in set theory from their beginnings*, Springer, 2nd ed. 2003 (1st ed. 1994).
- [KanK] R. Kannan, C.K. Krueger, *Advanced analysis on the real line*, Uni-

- versitext, Springer, 1996.
- [Kec] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics **156**, Springer, 1995.
- [Kem] J. H. B. Kemperman, A general functional equation, *Trans. Amer. Math. Soc.* **86** (1957), 28–56.
- [Kha] A. B. Kharazishvili, *Transformation groups and invariant measures. Set-theoretical aspects*. World Scientific, 1998.
- [Kom] Z. Kominek, On an equivalent form of a Steinhaus theorem, *Math. (Cluj)* **30** (53)(1988), 25-27.
- [Kuc] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*. 2nd ed., Birkhäuser, 2009 [1st ed. PWN, Warszawa, 1985].
- [KucS] M. Kuczma, J. Smítal, On measures connected with the Cauchy equation. *Aequationes Math.* **14** (1976), no. 3, 421–428.
- [LukMZ] J. Lukeš, Jaroslav, J. Malý, L. Zajíček, *Fine topology methods in real analysis and potential theory*. Lecture Notes in Math. **1189**, Springer, 1986.
- [Mar1] N. F. G. Martin, A topology for certain measure spaces, *Trans. Amer. Math. Soc.* **112** (1964), 1-18.
- [Mar2] N. F. G. Martin, Generalized condensation points, *Duke Math. J.*, **28** (1961), 507-514.
- [MarS] D. A. Martin, J. R. Steel, Projective determinacy. *Proc. Nat. Acad. Sci. U.S.A.* **85** (1988), 6582–6586.
- [MatZ] E. Matošková, M. Zelený, A note on intersections of non-Haar null sets, *Colloq. Math.* **96** (2003), 1-4.
- [MilO] H. I. Miller and A. J. Ostaszewski, Group action and shift-compactness, *J. Math. Anal. App.* **392** (2012), 23-39.
- [Mos] Y. V. Mospan, A converse to a theorem of Steinhaus, *Real An. Exch.* **31** (2005), 291-294.
- [Mue] B. J. Mueller, Three results for locally compact groups connected with the Haar measure density theorem, *Proc. Amer. Math. Soc.* **16** (6) (1965), 1414-1416.
- [Mun] J. R. Munkres, *Topology, a first course*, Prentice-Hall, 1975.
- [Nik] O. Nikodym, Sur une propriété de l'opération A , *Fund. Math.* **7** (1925), 149-154.
- [Ost1] A. J. Ostaszewski, Analytically heavy spaces: Analytic Cantor and Analytic Baire Theorems, *Topology and its Applications* **158** (2011), 253-275.

- [Ost2] A. J. Ostaszewski, Analytic Baire spaces, *Fund. Math.* **217** (2012), 189-210.
- [Ost3] A. J. Ostaszewski, Beyond Lebesgue and Baire III: Steinhaus's Theorem and its descendants, *Topology and its Applications* **160** (2013), 1144-1154.
- [Ost4] A. J. Ostaszewski, Almost completeness and the Effros Theorem in normed groups, *Topology Proceedings* **41** (2013), 99-110.
- [Ost5] A. J. Ostaszewski, Shift-compactness in almost analytic submetrizable Baire groups and spaces, *Topology Proceedings* **41** (2013), 123-151.
- [Ost6] A. J. Ostaszewski, Effros, Baire, Steinhaus and non-separability, *Topology and its Applications (Mary Ellen Rudin Memorial Volume)* **195** (2015), 265-274.
- [Oxt1] Oxtoby, John C. Invariant measures in groups which are not locally compact. *Trans. Amer. Math. Soc.* **60**, (1946), 215-237.
- [Oxt2] J. C. Oxtoby: *Measure and category*, 2nd ed. Graduate Texts in Math. **2**, Springer, 1980 (1st ed. 1972).
- [Pat] A. L. T. Paterson, *Amenability*. Math. Surveys and Mon. **29**, Amer. Math. Soc., 1988.
- [Rog] C. A. Rogers, J. Jayne, C. Dellacherie, F. Topsøe, J. Hoffmann-Jørgensen, D. A. Martin, A. S. Kechris, A. H. Stone, *Analytic sets*, Academic Press, 1980.
- [Rud] W. Rudin, *Real and complex analysis*, 3rd ed. McGraw-Hill, 1987.
- [Sam] P. Samuels, A topology formed from a given topology and ideal. *J. London Math. Soc.* (2) **10** (1975), 409-416.
- [Sch] S. Scheinberg, Topologies which generate a complete measure algebra, *Adv. Math.* **7** (1971), 231-239.
- [ShiT] H. Shi, B. S. Thomson, Haar null sets in the space of automorphisms on $[0,1]$. *Real Anal. Exchange* **24** (1998/99), 337-350.
- [Sim] S. M. Simmons, A converse Steinhaus theorem for locally compact groups, *Proc. Amer. Math. Soc.* **49** (1975), 383-386.
- [Sol1] S. Solecki, Size of subsets of groups and Haar null sets. *Geom. Funct. Anal.* **15** (2005), no. 1, 246-273.
- [Sol2] S. Solecki, Amenability, free subgroups, and Haar null sets in non-locally compact groups. *Proc. London Math. Soc.* (3) **93** (2006), 693-722.
- [SteS] E. M. Stein and R. Shakarchi, *Real analysis, Measure theory, integration and Hilbert spaces*, Princeton University press, 2005.
- [Str] K. Stromberg, An elementary proof of Steinhaus-Weil's theorem, *Proc. Amer. Math. Soc.* **36** (6) (1972), 308.

- [TomW] G. Tomkowicz, S. Wagon, *The Banach-Tarski paradox*. Cambridge University Press, 2016 (1st ed., S. Wagon, 1985).
- [Wei] A. Weil, *L'intégration dans les groupes topologiques*, Actualités Scientifiques et Industrielles 1145, Hermann, 1965.
- [Wil] W. Wilczyński, *Density topologies*, Handbook of measure theory, Vol. I, II, 675–702, North-Holland, 2002.
- [WilK] W. Wilczyński, A. B. Kharazishvili, Translations of measurable sets and of sets having the Baire property. (Russian) *Soobshch. Akad. Nauk Gruzii* **145.1** (1992), no. 1, 43–46.
- [Woo] W. H. Woodin, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*. 2nd ed., de Gruyter Series in Logic and its Applications **1**. de Gruyter, 2010.

Mathematics Department, Imperial College, London SW7 2AZ; n.bingham@ic.ac.uk
Mathematics Department, London School of Economics, Houghton Street,
London WC2A 2AE; A.J.Ostaszewski@lse.ac.uk