Applied Probability Trust (14 April 2017)

RECURSIVE FORMULA FOR THE DOUBLE BARRIER PARISIAN STOPPING TIME

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Abstract

In this paper, we obtain a recursive formula for the density of the double barrier Parisian stopping time. We present a probabilistic proof of the formula for the first few steps of the recursion, and then a formal proof using explicit Laplace inversions. These results provide an efficient computational method for pricing double barrier Parisian options.

Keywords: Brownian excursions, Parisian stopping times, Double-sided Parisian options

2010 Mathematics Subject Classification: Primary 65C50

Secondary 60J65;91G60

1. Introduction

Parisian options are a kind of path dependent option, where the payoff depends not only on the final value of the underlying asset, but also on the path trajectory of the underlying above or below a predetermined barrier L. In particular, the owner of a Parisian down-and-out call loses the option when the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than D, while for a Parisian down-and-in call, the same event gives the owner the right to exercise the option. Parisian options were first introduced in [Chesney et al. (1997)], where the Laplace transforms of the prices of single sided Parisian options were obtained using Azéma martingales. The pricing of Parisian options were also studied

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later in [Haber et al.(1999)], [Schröder (2003)] and [Dassios and Wu (2009)]. There are several motivations for the interest in these options. Parisian options are cheaper than the vanilla option of the same maturity and strike, since the value of the option depends on a barrier event occurring. Parisian options also has the added advantage over a barrier option, as it is not as easily manipulated by an influential agent since the barrier event requires more than just a touch of the barrier. Furthermore, since Parisian options are only triggered when the underlying asset has spent an amount of time beyond the barrier, this smooths the delta and gamma values near the barrier and makes hedging easier. Parisian options also have other practical applications, for instance to real option problems in [Broeders and Chen (2010)], and to insurance in [Dassios and Wu (2009)].

Double barrier Parisian options are a two-barrier version of the standard Parisian options described above. For example, a double barrier Parisian min-in call gives the owner the right to exercise the option if the underlying asset price S either makes an excursion above the upper barrier, or below the lower barrier for a continuous period longer than D, while the owner of a double barrier Parisian min-out call will lose the right to exercise the option when the same event occurs. Pricing of double barrier Parisian options has been studied in [Dassios and Wu (2011)], [Anderluh and van der Weide (2009)] and [Labart and Lelong (2009)]. All these papers have focused on obtaining explicit expressions for the Laplace transforms of the prices, but numerical inversions of these Laplace transforms are sometimes unstable.

In this paper, we derive a recursive formula for the density of the double barrier Parisian stopping time. In [Dassios and Lim (2013)], an explicit solution for the density of the Parisian stopping time with a single barrier was obtained. But here, we consider excursions both above the upper barrier and below the lower barrier. We define the double barrier Parisian stopping time as the first time the Brownian motion remains continuously below the lower barrier b_1 or above the upper barrier b_2 for a fixed amount of time. It turns out that the density is a finite sum of recursive terms, which are convolutions of the previous terms, and hence are fast and easy to compute. This gives us an explicit expression for the price of a double barrier Parisian option, which does not require any numerical inversion of Laplace transforms. Furthermore, our approach is intuitive and easy to understand from a probabilistic viewpoint. Since t is the first time the length of an excursion reaches D, if $kD < t < (k+1)D$, the probability is the same as that of the current excursion starting at time $t - D$, which will be between $(k-1)D < t - D < kD$, and that there are no excursions outside the barriers of length greater than D before this. Hence, we can decompose the Brownian path into each interval of length D, and if there has been no excursions of length greater than D, the density for the stopping time where t is between $kD < t < (k+1)D$ can be computed from the density of the previous step. To illustrate this further, we provide a probabilistic proof for the first few steps of the recursion. This also suggests that the method can be generalised to obtain explicit formulas for densities of the Parisian stopping times of other Markov processes, of which the first and last passage time densities are known. Finally, we use the density to present an efficient computational method for pricing double barrier Parisian options.

This paper will be organised as follows. In Section 2, we define the excursions and the double barrier Parisian stopping time and option. In Section 3, we present the result on the density of the double barrier Parisian stopping time. We first give a heuristic proof for the first few steps of the recursion, and then provide a formal proof of the formula for $t \geq 0$. In Section 4, we derive the pricing formulas for the Parisian double barrier in call options and show how the prices of the double barrier Parisian out call options can be obtained using the in-out parity relationships. In Section 5, we provide numerical examples to demonstrate the accuracy of our results.

2. Definitions

We will use the same definitions for the excursions as in [Chesney et al. (1997)]. Let S be the price process for the underlying asset, and $\mathcal{Q}$ denote the risk neutral probability measure. We assume that S follows a geometric Brownian motion and its dynamics under $\mathcal{Q}$ is

$$dS_t = S_t(rdt + \sigma dW_t), \qquad S_0 = x, \tag{2.1}$$

where W is a standard Brownian motion under $\mathcal{Q}$, and r and σ positive constants. We also introduce the notations

$$m := \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right), \qquad b := \frac{1}{\sigma}\ln\left(\frac{L}{x}\right), \qquad k := \frac{1}{\sigma}\ln\left(\frac{K}{x}\right), \tag{2.2}$$

so that the asset price $S_t = xe^{\sigma(mt+W_t)}$. We define

$$g^S_{L,t} := \sup\{s \le t | S_s = L\}, \qquad d^S_{L,t} := \inf\{s \ge t | S_s = L\}, \tag{2.3}$$

with the usual convention that $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. The trajectory of S between $g^S_{L,t}$ and $d^S_{L,t}$ is the excursion which straddles time t. We are interested here in $t - g^S_{L,t}$, which is the age of the excursion at time t. For $D > 0$, we now define

$$\tau^+_{L,D}(S) := \inf\{t \ge 0 | \mathbf{1}_{S_t > L}(t - g^S_{L,t}) \ge D\}, \tag{2.4}$$

$$\tau^-_{L,D}(S) := \inf\{t \ge 0 | \mathbf{1}_{S_t < L}(t - g^S_{L,t}) \ge D\}. \tag{2.5}$$

We have denoted as $\tau^+_{L,D}(S)$ the first time that the length of an excursion of process S above the barrier L reaches D, while $\tau^-_{L,D}(S)$ is the first time that the length of an excursion of process S below the barrier L reaches D. We also introduce the following notation for the stopping times where we refer to the standard Brownian motion W instead of S. Furthermore, without loss of generality since any time t of interest can be expressed in units of the window length D, we let $D = 1$ from now on.

$$\tau^+_b := \inf\{t \ge 0 | \mathbf{1}_{W_t > b}(t - g^W_{b,t}) \ge 1\}, \tag{2.6}$$

$$\tau^-_b := \inf\{t \ge 0 | \mathbf{1}_{W_t < b}(t - g^W_{b,t}) \ge 1\}. \tag{2.7}$$

We now look at the double barrier Parisian option, which is defined as (for $b_1 < b_2$):

$$\tau^{b_2}_{b_1} := \tau^-_{b_1} \wedge \tau^+_{b_2}. \tag{2.8}$$

This is the first time that for the Brownian motion W, the length of an excursion above b_2, or an excursion below b_1, reaches length 1. We note that we have taken the window length of both sides to be the same (ie. 1 in our case).

The owner of a double-barrier Parisian min-in option receives the payoff only if there is an excursion below the level L_1 or above level L_2 which is of length greater than $D = 1$. This will be the case if $\tau_{L_1}^{L_2}(S) \leq T$. Denoting $C_i^{double}(x, T, L_1, L_2, K)$ as the price of a Parisian min-in call with initial underlying price x, maturity T, strike price K, lower barrier L_1, upper barrier L_2, we have the risk-neutral price of the option

$$C_i^{double}(x, T, L_1, L_2, K) = E_{\mathcal{Q}}\left[e^{-rT}\mathbf{1}_{\{\tau_{L_1}^{L_2}(S)\leq T\}}(xe^{\sigma(mT+W_T)} - K)^+\right]. \tag{2.9}$$

We introduce a new probability measure $\mathcal{P}$, with Radon-Nikodym derivative $\frac{d\mathcal{P}}{d\mathcal{Q}} = e^{-mW_t - \frac{1}{2}m^2 t}$. Applying Girsanov's Theorem and a change of measure from $\mathcal{Q}$ to $\mathcal{P}$, we have

$$C_i^{double}(x, T, L_1, L_2, K) = E_{\mathcal{P}}\left[\frac{d\mathcal{Q}}{d\mathcal{P}}e^{-rT}\mathbf{1}_{\{\tau_{L_1}^{L_2}(S)\leq T\}}(xe^{\sigma(mT+W_T)} - K)^+\right] \tag{2.10}$$

$$= E_{\mathcal{P}}\left[e^{-(r+\frac{1}{2}m^2)T}\mathbf{1}_{\{\tau_{b_1}^{b_2}\leq T\}}e^{mZ_T}\left(xe^{\sigma Z_T} - K\right)^+\right] \tag{2.11}$$

where $Z_t = W_t + mt$ a standard Brownian motion under $\mathcal{P}$. To simplify things, we also let

$$^*C_i^{double}(x, T, L_1, L_2, K) = e^{(r+\frac{1}{2}m^2)T}C_i^{double}(x, T, L_1, L_2, K). \tag{2.12}$$

In the next section, we will first look at the density function of $\tau_{b_1}^{b_2}$, which we will denote by $f_{b_1}^{b_2}(t)$, and then show how it can be used to obtain the prices of a Parisian min-in call option.

3. Density of the double barrier Parisian stopping time

We are interested to derive the density of the double barrier Parisian stopping time $\tau_{b_1}^{b_2}$. We first look at the case when the excursion has not started ($b_1 \leq 0 \leq b_2$) and then discuss results for the case when we are already within an excursion ($b_1 < b_2 \leq 0$ or $0 \leq b_1 < b_2$). We look at two cases, one where the excursion of length 1 occurs above the upper barrier first ($\tau_{b_2}^+ < \tau_{b_1}^-$) and the other where the excursion of length 1 occurs below the lower barrier first ($\tau_{b_1}^- < \tau_{b_2}^+$).

Theorem 3.1. *Let* $f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-)$ *and* $f_{b_1}^{b_2}(t, \tau_{b_1}^- < \tau_{b_2}^+)$ *denote the probability density function of* $\tau_{b_1}^{b_2}$ *on the set* $\tau_{b_2}^+ < \tau_{b_1}^-$ *and* $\tau_{b_1}^- < \tau_{b_2}^+$ *respectively. Then for* $b_1 \leq 0 \leq b_2$, *we have for* $t > 1$, $n < t \leq n+1$, $n = 1, 2, ...$,

$$f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-) = \sum_{k=0}^{n-1} (-1)^k L_k(t-1), \tag{3.1}$$

$$f_{b_1}^{b_2}(t, \tau_{b_1}^- < \tau_{b_2}^+) = \sum_{k=0}^{n-1} (-1)^k \tilde{L}_k(t-1), \tag{3.2}$$

where $L_k(t)$ *and* $\tilde{L}_k(t)$ *are defined recursively as follows for* $t > k+1$*:*

$$L_{k+1}(t) = \int_1^{t-k} \Big(L_k(t-s)\varphi_1(s) + \tilde{L}_k(t-s)\varphi_2(s, b_2 - b_1) \Big)\, ds, \tag{3.3}$$

$$\tilde{L}_{k+1}(t) = \int_1^{t-k} \Big(\tilde{L}_k(t-s)\varphi_1(s) + L_k(t-s)\varphi_2(s, b_2 - b_1) \Big)\, ds, \tag{3.4}$$

with initial conditions

$$L_0(t) = \frac{1}{2\pi\sqrt{t}} e^{-\frac{b_2^2}{2t}}, \qquad \tilde{L}_0(t) = \frac{1}{2\pi\sqrt{t}} e^{-\frac{b_1^2}{2t}}, \qquad \text{for } t > 0, \tag{3.5}$$

and the functions $\varphi_1(s)$ *and* $\varphi_2(s, b)$ *are defined as*

$$\varphi_1(s) := \frac{\sqrt{s-1}}{2\pi s}, \tag{3.6}$$

$$\varphi_2(s, b) := \frac{\sqrt{s-1}}{2\pi s} e^{-\frac{b^2}{2(s-1)}} - \frac{b}{\sqrt{2\pi} s^{3/2}} e^{-\frac{b^2}{2s}} \mathcal{N}\left(-\frac{b}{\sqrt{s(s-1)}} \right). \tag{3.7}$$

Before we begin the formal proof of Theorem 3.1, we will first give, in the following subsection, an intuitive proof for $1 < t < 3$.

3.1. A probabilistic explanation for the recursion

Here, we explain the above result using excursions. We will prove the result for small values of t by using a path decomposition of the Brownian motion around time $t = 1$. The general result will then follow by induction. We only look at $f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-)$, the case when the excursion above b_2 occurs before the excursion below b_1, but the same idea applies to $f_{b_1}^{b_2}(t, \tau_{b_1}^- < \tau_{b_2}^+)$, the case when the excursion below b_1 occurs before the excursion above b_2. We denote by P^x the law of a Brownian motion starting at x at time 0, $p_{x,y}(t)$ the transition density of a Brownian motion from x to y in time t, and T_x the first hitting time of level x of the Brownian motion. Recall the notation that $g_{b_1,t}$ is the last time the Brownian motion hits level b_1 before time t. We want to find the density of $\tau_{b_1}^{b_2}$ when $\tau_{b_2}^{+} < \tau_{b_1}^{-}$. First, we note that there is no density for $\tau_{b_1}^{b_2}$ when $t < 1$. For $1 < t < 2$, if $\{\tau_{b_1}^{b_2} \in dt\}$, the excursion must start at $t-1$, where $0 < t-1 < 1$, and it must be the first excursion. Hence, we need to find $\nu_{b_2}(t) := P^0(\tau_{b_1}^{b_2} - 1 \in dt, \tau_{b_2}^{+} < \tau_{b_1}^{-})$, the probability of t being the start of the excursion above b_2 greater than length 1 for a Brownian motion starting at 0, by decomposing it into the part of the excursion between $g_{b_2,1}$ and 1, and between 1 and $g_{b_2,1}+1$. We have for $0 < t < 1$,

$$P(\tau_{b_1}^{b_2} - 1 \in dt, \tau_{b_2}^{+} < \tau_{b_1}^{-}) = \int_{b_2}^{\infty} P^0(g_{b_2,1} \in dt, W_1 \in dx) P^x(T_{b_2} \geq 1 - (1-t)). \quad (3.8)$$

Using the time reversal property of Brownian motion, we have

$$\begin{aligned} P^0(g_{b_2,1} \in dt, W_1 \in dx) &= P^x(T_{b_2} \in 1-t) p_{b_2,0}(t) & (3.9) \\ &= \frac{x-b_2}{\sqrt{2\pi(1-t)^3}} e^{-\frac{(x-b_2)^2}{2(1-t)}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{b_2^2}{2t}} dxdt. & (3.10) \end{aligned}$$

Hence,

$$\begin{aligned} & P(\tau_{b_1}^{b_2} - 1 \in dt, \tau_{b_2}^{+} < \tau_{b_1}^{-}) & (3.11) \\ = & \int_{b_2}^{\infty} \frac{x-b_2}{\sqrt{2\pi(1-t)^3}} e^{-\frac{(x-b_2)^2}{2(1-t)}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{b_2^2}{2t}} \int_t^{\infty} \frac{x-b_2}{\sqrt{2\pi u^3}} e^{-\frac{(x-b_2)^2}{2u}} dudxdt & (3.12) \\ = & \int_t^{\infty} \frac{1}{2\pi\sqrt{t(1-t)^3}} \frac{1}{\sqrt{2\pi u^3}} e^{-\frac{b_2^2}{2t}} \int_0^{\infty} x^2 e^{-\frac{x^2}{2(1-t)}} e^{-\frac{x^2}{2u}} dxdudt & (3.13) \\ = & \int_t^{\infty} \frac{1}{2\pi\sqrt{t(1-t)^3}} \frac{1}{\sqrt{2\pi u^3}} e^{-\frac{b_2^2}{2t}} \sqrt{\frac{\pi}{2}} \left(\frac{u(1-t)}{1-t+u}\right)^{3/2} dudt & (3.14) \\ = & \frac{1}{2\pi\sqrt{t}} e^{-\frac{b_2^2}{2t}} dt. & (3.15) \end{aligned}$$

This is the first term in the recursion, which we have denoted by $L_0(t)$. Note that due to the symmetry of Brownian motion, this probability is the same for the excursion below b_1, and only depends on the difference between the barrier and the starting point. Hence, it follows that the probability of t being the start of the excursion below b_1 greater than length 1, for $0 < t < 1$, is

$$P^0(\tau_{b_1}^{b_2} - 1 \in dt, \tau_{b_1}^- < \tau_{b_2}^+) = \nu_{b_1}(t) = \frac{1}{2\pi\sqrt{t}} e^{-\frac{b_1^2}{2t}} dt, \tag{3.16}$$

which we denote by $\tilde{L}_0(t)$, and corresponds to the first term of the second recursion (3.5). This proves the equations (3.1) and (3.2) for $1 < t < 2$.

Now, for $2 < t < 3$, the same interpretation for $L_0(t-1)$ as the start of the excursion greater than length 1 for $1 < t-1 < 2$ still applies, but now there can be up to 2 excursions greater than length 1. Hence, we need to subtract the probability of t being the start of the second excursion greater than length 1. We denote by $\varphi_1(t-s+1)$ the probability that we will start another excursion above b_2 greater than length 1 at time t, given that at time s, we are already length 1 into an excursion above b_2. We have for $1 < s < t$,

$$\varphi_1(t-s+1) = \int_{b_2}^{\infty} P^{b_2}(W_s \in dx | g_{b_2,s} = s-1) \int_s^t P^x(T_{b_2} \in du) \nu_0(t-u), \tag{3.17}$$

where we have decomposed the excursion, conditioning on the value of the Brownian motion at time s, and the time when it comes back to level b_2, and $\nu_0(t-u)$ is the probability that the Brownian motion will start another excursion above b_2 of length 1 at time t, given that it is at level b_2 at time u. Now, the Brownian motion conditioned to stay above b_2 up to time 1 is a Brownian meander, which has density

$$P^{b_2}(W_s \in dx | g_{b_2,s} = s-1) = (x - b_2) e^{-\frac{(x-b_2)^2}{2}} \mathbf{1}_{\{x > b_2\}} dx. \tag{3.18}$$

Hence, we have

$$\begin{aligned} \varphi_1(t-s+1) &= \int_{b_2}^{\infty} (x-b_2) e^{-\frac{(x-b_2)^2}{2}} \int_s^t \frac{x-b_2}{\sqrt{2\pi(u-s)^3}} e^{-\frac{(x-b_2)^2}{2(u-s)}} \frac{1}{2\pi\sqrt{t-u}} du\,dx \qquad (3.19) \\ &= \frac{1}{2\pi} \frac{\sqrt{t-s}}{t-s+1}. \qquad (3.20) \end{aligned}$$

Next, we denote by $\varphi_2(t-s+1, b_2-b_1)$ the probability that we will start an excursion above b_2 of length at least 1 at time t, given that at time s, we are already length 1 into an excursion below b_1. We have for $1 < s < t$,

$$\varphi_2(t-s+1, b_2-b_1) = \int_{-\infty}^{b_1} P^{b_1}(W_s \in dx | g_{b_1,s} = s-1) \int_s^t P^x(T_{b_1} \in du) \nu_{b_2-b_1}(t-u), \tag{3.21}$$

where we have decomposed the excursion, conditioning on the value of the Brownian motion at time s, and the time when it comes back to level b_1. Then $\nu_{b_2-b_1}(t-u)$ is the probability that the Brownian motion will start an excursion above b_2 of length 1 at time t, given that it is at level b_1 at time u. Computations lead to

$$\varphi_2(t-s+1, b) \tag{3.22}$$

$$= \int_{-\infty}^{b_1} (b_1-x) e^{-\frac{(b_1-x)^2}{2}} \int_s^t \frac{b_1-x}{\sqrt{2\pi(u-s)^3}} e^{-\frac{(b_1-x)^2}{2(u-s)}} \frac{1}{2\pi\sqrt{t-u}} e^{-\frac{b^2}{2(t-u)}} du dx \tag{3.23}$$

$$= \frac{1}{4\pi} \int_s^t \frac{1}{\sqrt{t-u}(1+u-s)^{3/2}} e^{-\frac{b^2}{2(t-u)}} du \tag{3.24}$$

$$= \frac{1}{4\pi} \left[-\frac{2\sqrt{t-u}}{(1+t-s)\sqrt{1+u-s}} e^{-\frac{b^2}{2(t-u)}} \right. \tag{3.25}$$

$$\left. -\frac{\sqrt{2\pi} b}{(1+t-s)^{3/2}} e^{-\frac{b^2}{2(1+t-s)}} \left(1 - 2\mathcal{N}\left(-\frac{b\sqrt{1+u-s}}{\sqrt{(t-u)(1+t-s)}} \right) \right) \right]_s^t \tag{3.26}$$

$$= \frac{1}{2\pi} \frac{\sqrt{t-s}}{1+t-s} e^{-\frac{b^2}{2(t-s)}} + \frac{b}{\sqrt{2\pi}(1+t-s)^{3/2}} e^{-\frac{b^2}{2(1+t-s)}} \mathcal{N}\left(-\frac{b}{\sqrt{(t-s)(1+t-s)}} \right) \tag{3.27}$$

Note that due to the symmetry of Brownian motion, $\varphi_1(t-s+1)$ is also the probability that t is the start of another excursion below b_1 greater than length 1, given that at time s, we are already in an excursion of length 1 below b_1. Likewise, $\varphi_2(t-s+1, b_2-b_1)$ is also the probability that t is the start of an excursion below b_1 greater than length 1, given that at time s, we are already in an excursion of length 1 above b_2.

Since the first excursion can either be above b_2 or below b_1, there are two scenarios. The probability that t is the start of the second excursion above b_2 greater than length 1 is the sum of the two cases:

$$\int_1^t L_0(s-1)\varphi_1(t-s+1) ds + \int_1^t \tilde{L}_0(s-1)\varphi_2(t-s+1, b_2-b_1) ds \tag{3.28}$$

$$= \int_1^t L_0(t-s)\varphi_1(s)ds + \int_1^t \tilde{L}_0(t-s)\varphi_2(s, b_2-b_1)ds, \tag{3.29}$$

which is $L_1(t-1)$ in the recursion equation (3.3). Similarly, the probability that t is the start of the second excursion below b_1 greater than length 1 is

$$\int_1^t \tilde{L}_0(t-s)\varphi_1(s)ds + \int_1^t L_0(t-s)\varphi_2(s, b_2-b_1)ds, \tag{3.30}$$

which is $\tilde{L}_1(t-1)$ in the recursion equation (3.4). Hence for $2<t<3$, the density of $\tau_{b_1}^{b_2}$ for the cases $\tau_{b_2}^+ < \tau_{b_1}^-$ and $\tau_{b_1}^- < \tau_{b_2}^+$ are $L_0(t-1) - L_1(t-1)$ and $\tilde{L}_0(t-1) - \tilde{L}_1(t-1)$ respectively, and we proved the equations (3.1) and (3.2) for $2 < t < 3$. The same argument would follow by induction for $t > 3$ and thus we obtain the recursion.

3.2. Formal proof

In this section, we give a formal proof of the recursive formula based on Laplace transforms. We define the Laplace transform $\hat{h}(\beta)$ of a function $h(t)$ on the positive real line as

$$\mathcal{L}(h(t)) = \hat{h}(\beta) := \int_0^\infty e^{-\beta t} h(t)dt, \tag{3.31}$$

and the inverse Laplace transform operator is denoted by $\mathcal{L}^{-1}(.)$. Furthermore, for ease of notation, we define as in previous papers the following function

$$\Psi(x) := 1 + x\sqrt{2\pi}e^{\frac{x^2}{2}}\mathcal{N}(x), \tag{3.32}$$

where $\mathcal{N}(x)$ is the cumulative distribution function for the standard normal distribution.

Proof. We only show the calculations for the case when $\{\tau_{b_2}^+ < \tau_{b_1}^-\}$, but the case when $\{\tau_{b_1}^- < \tau_{b_2}^+\}$ can be proved in the same way. The Laplace transform of the double barrier Parisian stopping time was studied in [Anderluh and van der Weide (2009)] and we use here the result in Theorem 3.2 of the paper. The Laplace transform for $\tau_{b_1}^{b_2}$ on the set $\{\tau_{b_2}^+ < \tau_{b_1}^-\}$ is

$$E\left(e^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}_{\{\tau_{b_2}^+ < \tau_{b_1}^-\}}\right) = \frac{e^{-\sqrt{2\beta}b_1}\Psi\left(\sqrt{2\beta}\right) - e^{\sqrt{2\beta}b_1}\Psi\left(-\sqrt{2\beta}\right)}{e^{\sqrt{2\beta}(b_2-b_1)}\Psi\left(\sqrt{2\beta}\right)^2 - e^{\sqrt{2\beta}(b_1-b_2)}\Psi\left(-\sqrt{2\beta}\right)^2}. \tag{3.33}$$

Factorising, we obtain

$$E\left(e^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}_{\{\tau_{b_2}^+<\tau_{b_1}^-\}}\right) = \frac{\frac{1}{2}\left(e^{\sqrt{2\beta}(b_1+b_2)/2}+e^{-\sqrt{2\beta}(b_1+b_2)/2}\right)}{e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right)+e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right)} \quad (3.34)$$

$$-\frac{\frac{1}{2}\left(e^{\sqrt{2\beta}(b_1+b_2)/2}-e^{-\sqrt{2\beta}(b_1+b_2)/2}\right)}{e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right)-e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right)} \quad (3.35)$$

Now we refer to Dassios and Lim [Dassios and Lim (2013)] for the derivation of the following equality

$$e^{-\beta}\frac{1}{\beta}\Psi\left(\sqrt{2\beta}\right) = 2\sqrt{\frac{\pi}{\beta}}\left(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right). \quad (3.36)$$

Similarly, we derive

$$e^{-\beta}\frac{1}{\beta}\Psi\left(-\sqrt{2\beta}\right) = \frac{e^{-\beta}}{\beta}-2\sqrt{\frac{\pi}{\beta}}\mathcal{N}\left(-\sqrt{2\beta}\right) \quad (3.37)$$

$$= \int_1^\infty e^{-\beta s}ds-2\sqrt{\frac{\pi}{\beta}}\int_{-\infty}^{-\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \quad (3.38)$$

$$= \int_1^\infty e^{-\beta s}ds-\int_1^\infty\frac{e^{-\beta s}}{\sqrt{s}}ds \quad (3.39)$$

$$= 2\sqrt{\frac{\pi}{\beta}}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right). \quad (3.40)$$

Adding the two, we obtain an expression for the denominator in the expression on the RHS of (3.34),

$$e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right)+e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right) \quad (3.41)$$

$$= e^{\beta}e^{\sqrt{2\beta}(b_2-b_1)/2}2\sqrt{\pi\beta}\left(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds+e^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right), \quad (3.42)$$

and (3.35),

$$e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right)-e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right) \quad (3.43)$$

$$= e^{\beta}e^{\sqrt{2\beta}(b_2-b_1)/2}2\sqrt{\pi\beta}\left(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds-e^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right). \quad (3.44)$$

Denoting by $\hat{g}_0(\beta)$, $\hat{\hat{g}}_0(\beta)$, $\hat{g}_k(\beta)$ and $\hat{\hat{g}}_k(\beta)$ the following expressions,

$$\hat{g}_0(\beta) = \frac{e^{\sqrt{2\beta}b_1} + e^{-\sqrt{2\beta}b_2}}{4\sqrt{\pi\beta}}, \tag{3.45}$$

$$\hat{\hat{g}}_0(\beta) = \frac{e^{\sqrt{2\beta}b_1} - e^{-\sqrt{2\beta}b_2}}{4\sqrt{\pi\beta}}, \tag{3.46}$$

$$\hat{g}_1(\beta) = \frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds + e^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds, \tag{3.47}$$

$$\hat{\hat{g}}_1(\beta) = \frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds - e^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds, \tag{3.48}$$

we can write the expression on the RHS of (3.34) as

$$\frac{\frac{1}{2}\left(e^{\sqrt{2\beta}(b_1+b_2)/2} + e^{-\sqrt{2\beta}(b_1+b_2)/2}\right)}{e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right) + e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right)} \tag{3.49}$$

$$= \frac{\frac{1}{2}\left(e^{\sqrt{2\beta}(b_1+b_2)/2} + e^{-\sqrt{2\beta}(b_1+b_2)/2}\right)}{e^{\beta}e^{\sqrt{2\beta}(b_2-b_1)/2}2\sqrt{\pi\beta}\left(1 + \frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds + e^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right)} \tag{3.50}$$

$$= e^{-\beta}\frac{\hat{g}_0(\beta)}{1+\hat{g}_1(\beta)}, \tag{3.51}$$

and (3.35) as

$$-\frac{\frac{1}{2}\left(e^{\sqrt{2\beta}(b_1+b_2)/2} - e^{-\sqrt{2\beta}(b_1+b_2)/2}\right)}{e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right) - e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right)} = -e^{-\beta}\frac{\hat{\hat{g}}_0(\beta)}{1+\hat{\hat{g}}_1(\beta)}. \tag{3.52}$$

Since $\hat{g}_1(\beta)$ is a continuous and decreasing function of β, it goes to 0 when $\beta \to \infty$. Hence, there exists some $\beta > 0$ such that $|\hat{g}_1(\beta)| < 1$, and so (3.51) can be written as the sum of a convergent geometric series with first term $\hat{g}_0(\beta)$ and common ratio $-\hat{g}_1(\beta)$. Similarly, since

$$|\hat{\hat{g}}_1(\beta)| \leq \frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds < 1, \tag{3.53}$$

(3.52) can be written as the sum of a convergent geometric series with first term $\hat{\hat{g}}_0(\beta)$ and common ratio $-\hat{\hat{g}}_1(\beta)$. We obtain

$$E\left(e^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}_{\{\tau_{b_2}^+<\tau_{b_1}^-\}}\right) = e^{-\beta}\left(\hat{g}_0(\beta)\sum_{i=0}^{\infty}(-1)^k\left(\hat{g}_1(\beta)\right)^k - \hat{\hat{g}}_0(\beta)\sum_{i=0}^{\infty}(-1)^k\left(\hat{\hat{g}}_1(\beta)\right)^k\right) \tag{3.54}$$

Now, we invert the Laplace transform (3.54). If we denote the Laplace inversions of $\hat{g}_0(\beta)$, $\hat{\tilde{g}}_0(\beta)$, $\hat{g}_1(\beta)$, and $\hat{\tilde{g}}_1(\beta)$ by $g_0(t)$, $\tilde{g}_0(t)$, $g_1(t)$ and $\tilde{g}_1(t)$, we have for $t > 1$,

$$f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-) = \sum_{k=0}^{\infty} (-1)^k \left(g_k(t-1) - \tilde{g}_k(t-1) \right), \tag{3.55}$$

where $g_k(t)$ is the convolution of $g_0(t)$ with $k-1$ times of $g_1(t)$, and $\tilde{g}_k(t)$ is the convolution of $\tilde{g}_0(t)$ with $k-1$ times of $\tilde{g}_1(t)$. Next, we have the following explicit Laplace inversions:

$$\mathcal{L}^{-1}\left(\frac{e^{\sqrt{2\beta}b_1}}{4\sqrt{\pi\beta}} \right) = \frac{1}{4\pi\sqrt{t}} e^{-\frac{b_1^2}{2t}}, \tag{3.56}$$

$$\mathcal{L}^{-1}\left(\frac{e^{-\sqrt{2\beta}b_2}}{4\sqrt{\pi\beta}} \right) = \frac{1}{4\pi\sqrt{t}} e^{-\frac{b_2^2}{2t}}, \tag{3.57}$$

$$\mathcal{L}^{-1}\left(\frac{1}{2\sqrt{\pi\beta}} \int_1^{\infty} \frac{e^{-\beta s}}{2s^{3/2}} ds \right) = \frac{\sqrt{t-1}}{2\pi t} \mathbf{1}_{\{t>1\}}, \tag{3.58}$$

and

$$\mathcal{L}^{-1}\left(e^{\sqrt{2\beta}(b_1-b_2)} \frac{1}{2\sqrt{\pi\beta}} \int_1^{\infty} \frac{e^{-\beta s}}{2s^{3/2}} ds \right) \tag{3.59}$$

$$= \left(\frac{\sqrt{t-1}}{2\pi t} e^{-\frac{(b_2-b_1)^2}{2(t-1)}} - \frac{b_2-b_1}{\sqrt{2\pi} t^{3/2}} e^{-\frac{(b_1-b_2)^2}{2t}} \mathcal{N}\left(-\frac{b_2-b_1}{\sqrt{t(t-1)}} \right) \right) \mathbf{1}_{\{t>1\}}. \tag{3.60}$$

The first three inversions were computed in [Dassios and Lim (2013)], and the last one can be derived as the convolution of the following two functions:

$$\mathcal{L}^{-1}\left(e^{\sqrt{2\beta}(b_1-b_2)} \frac{1}{2\sqrt{\pi\beta}} \right) = \frac{1}{2\pi\sqrt{t}} e^{-\frac{(b_1-b_2)^2}{2t}} \tag{3.61}$$

$$\mathcal{L}^{-1}\left(\int_1^{\infty} \frac{e^{-\beta s}}{2s^{3/2}} ds \right) = \frac{1}{2t^{3/2}} \mathbf{1}_{\{t>1\}}, \tag{3.62}$$

so that

$$\mathcal{L}^{-1}\left(e^{\sqrt{2\beta}(b_1-b_2)} \frac{1}{2\sqrt{\pi\beta}} \int_1^{\infty} \frac{e^{-\beta s}}{2s^{3/2}} ds \right) \tag{3.63}$$

$$= \mathbf{1}_{\{t>1\}} \int_0^{t-1} \frac{1}{2\pi\sqrt{s}} \frac{1}{2(t-s)^{3/2}} e^{-\frac{(b_1-b_2)^2}{2s}} ds \tag{3.64}$$

$$= \mathbf{1}_{\{t>1\}} \int_{-\infty}^{\frac{b_1-b_2}{\sqrt{t-1}}} \frac{-x(b_2-b_1)}{2\pi\left(tx^2-(b_1-b_2)^2\right)^{3/2}} e^{-\frac{x^2}{2}} dx \tag{3.65}$$

$$= \mathbf{1}_{\{t>1\}} \left(\frac{\sqrt{t-1}}{2\pi t} e^{-\frac{(b_1-b_2)^2}{2(t-1)}} - \int_{-\infty}^{\frac{b_1-b_2}{\sqrt{t-1}}} \frac{-x(b_2-b_1)}{2\pi t\sqrt{tx^2-(b_1-b_2)^2}} e^{-\frac{x^2}{2}} dx \right) \tag{3.66}$$

$$= \mathbf{1}_{\{t>1\}} \left(\frac{\sqrt{t-1}}{2\pi t} e^{-\frac{(b_1-b_2)^2}{2(t-1)}} - \frac{b_2-b_1}{\sqrt{2\pi} t^{3/2}} e^{-\frac{(b_1-b_2)^2}{2t}} \mathcal{N}\left(-\frac{b_2-b_1}{\sqrt{t(t-1)}} \right) \right). \tag{3.67}$$

Thus, adding the appropriate terms, we have

$$g_0(t) = \frac{e^{-\frac{b_1^2}{2t}} - e^{-\frac{b_2^2}{2t}}}{4\pi\sqrt{t}}, \tag{3.68}$$

$$\tilde{g}_0(t) = \frac{e^{-\frac{b_1^2}{2t}} + e^{-\frac{b_2^2}{2t}}}{4\pi\sqrt{t}}, \tag{3.69}$$

$$g_1(t) = \mathbf{1}_{\{t>1\}} \left(\varphi_1(t) - \varphi_2(t, b_2-b_1) \right), \tag{3.70}$$

$$\tilde{g}_1(t) = \mathbf{1}_{\{t>1\}} \left(\varphi_1(t) + \varphi_2(t, b_2-b_1) \right), \tag{3.71}$$

where $\varphi_1(t)$ and $\varphi_2(t,b)$ are as defined in (3.6) and (3.7), and for $k \geq 1$,

$$g_{k+1}(t) = \int_1^{t-k} g_k(t-s) \left(\varphi_1(s) - \varphi_2(s, b_2-b_1) \right) ds, \qquad \text{for } t > k+1, \tag{3.72}$$

$$\tilde{g}_{k+1}(t) = \int_1^{t-k} \tilde{g}_k(t-s) \left(\varphi_1(s) + \varphi_2(s, b_2-b_1) \right) ds, \qquad \text{for } t > k+1. \tag{3.73}$$

We also note that for $n < t \leq n+1$, $g_k(t)$ and $\tilde{g}_k(t)$ are zero for $k > n$, and thus $g_k(t-1)$ and $\tilde{g}_k(t-1)$ are zero for $k > n-1$, so we only need a finite sum up to $n-1$. Finally, we let $L_k(t) = g_k(t) - \tilde{g}_k(t)$, and $\tilde{L}_k(t) = g_k(t) + \tilde{g}_k(t)$, to obtain the result. For $n < t \leq n+1$, $n = 1, 2, ...$,

$$f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-) = \sum_{k=0}^{n-1} (-1)^k \left(g_k(t-1) - \tilde{g}_k(t-1) \right) \tag{3.74}$$

$$= \sum_{k=0}^{n-1} (-1)^k L_k(t-1), \tag{3.75}$$

where

$$L_0(t) = g_0(t) - \tilde{g}_0(t) = \frac{1}{2\pi\sqrt{t}} e^{-\frac{b_2^2}{2t}}, \tag{3.76}$$

$$\tilde{L}_0(t) \quad = \quad g_0(t) + \tilde{g}_0(t) = \frac{1}{2\pi\sqrt{t}} e^{-\frac{b_1^2}{2t}}, \tag{3.77}$$

for $t > 0$, and

$$L_{k+1}(t) \quad = \quad \int_1^{t-k} \Big(L_k(t-s)\varphi_1(s) + \tilde{L}_k(t-s)\varphi_2(s, b_2 - b_1) \Big)\, ds, \tag{3.78}$$

$$\tilde{L}_{k+1}(t) \quad = \quad \int_1^{t-k} \Big(\tilde{L}_k(t-s)\varphi_1(s) + L_k(t-s)\varphi_2(s, b_2 - b_1) \Big)\, ds, \tag{3.79}$$

for $t > k+1$, which completes the proof. □

Remark 1. When $b_1 = b_2$, the above formula reduces to the formula for the two-sided Parisian stopping time for $b = b_1 = b_2$.

3.3. Starting above ($b_1 < b_2 \leq 0$) or below ($0 \leq b_1 < b_2$) both barriers

We denote by T_b the first hitting time of level b of a standard Brownian motion W. For the case when we start above both barriers ($b_1 < b_2 \leq 0$), we consider only the case when $T_{b_2} < D = 1$, because if this is not the case, then we would have $\tau_{b_1}^{b_2} = 1$ since we are already above the upper barrier.

Theorem 3.2. *For $b_1 < b_2 \leq 0$, we have for $T_{b_2} < 1$, for the two cases where $\tau_{b_2}^{+} < \tau_{b_1}^{-}$ and $\tau_{b_1}^{-} < \tau_{b_2}^{+}$, we have the following formulas for the probability density function of $\tau_{b_1}^{b_2}$, for $t > 1$, $n < t \leq n+1$ and $n = 1, 2, ...$:*

$$f_{b_1}^{b_2}(t, \tau_{b_2}^{+} < \tau_{b_1}^{-}, T_{b_2} < 1) \quad = \quad \sum_{k=0}^{n-1} (-1)^k L_k(t-1), \tag{3.80}$$

$$f_{b_1}^{b_2}(t, \tau_{b_1}^{-} < \tau_{b_2}^{+}, T_{b_2} < 1) \quad = \quad \sum_{k=0}^{n-1} (-1)^k \tilde{L}_k(t-1), \tag{3.81}$$

where $L_k(t)$ and $\tilde{L}_k(t)$ are defined recursively as follows for $t > k+1$:

$$L_{k+1}(t) \quad = \quad \int_1^{t-k} \Big(L_k(t-s)\varphi_1(s) + \tilde{L}_k(t-s)\varphi_2(s, b_2 - b_1) \Big)\, ds, \tag{3.82}$$

$$\tilde{L}_{k+1}(t) \quad = \quad \int_1^{t-k} \Big(\tilde{L}_k(t-s)\varphi_1(s) + L_k(t-s)\varphi_2(s, b_2 - b_1) \Big)\, ds, \tag{3.83}$$

with the initial conditions

$$L_0(t) = \mathbf{1}_{\{0\le t\le 1\}}\left(\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_1^2}{2t}}+\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_2^2}{2t}}\right) \tag{3.84}$$

$$+\mathbf{1}_{\{t>1\}}\left(\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_1^2}{2t}}\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}}+\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\right)\right. \tag{3.85}$$

$$\left.+\frac{1}{4\pi\sqrt{t}}e^{-\frac{(2b_2-b_1)^2}{2t}}\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}}-\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\right)+\frac{1}{2\pi\sqrt{t}}e^{-\frac{b_2^2}{2t}}\mathcal{N}\left(-b_2\sqrt{\frac{t-1}{t}}\right)\right), \tag{3.86}$$

$$\tilde{L}_0(t) = \mathbf{1}_{\{0\le t\le 1\}}\left(\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_1^2}{2t}}-\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_2^2}{2t}}\right) \tag{3.87}$$

$$+\mathbf{1}_{\{t>1\}}\left(\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_1^2}{2t}}\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}}+\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\right)\right. \tag{3.88}$$

$$\left.+\frac{1}{4\pi\sqrt{t}}e^{-\frac{(2b_2-b_1)^2}{2t}}\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}}-\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\right)-\frac{1}{2\pi\sqrt{t}}e^{-\frac{b_2^2}{2t}}\mathcal{N}\left(-b_2\sqrt{\frac{t-1}{t}}\right)\right), \tag{3.89}$$

where $\varphi_1(s)$ and $\varphi_2(s,b)$ are as defined in Theorem 3.1.

Proof. For $b_1 < b_2 \le 0$, when $T_{b_2} < 1$, using the strong Markov property of the Brownian motion, we can restart it the first time it hits b_2. Then $\tau_{b_1}^{b_2}$ can be decomposed into the sum of T_{b_2} and $\tau^0_{b_1-b_2}$, which are independent of each other, and furthermore, due to the symmetry of Brownian motion, $\tau^0_{b_1-b_2} = \tau_0^{b_2-b_1}$. The Laplace transform of the stopping time on the sets we are interested in is thus:

$$E\left(e^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}_{\{\tau_{b_2}^+<\tau_{b_1}^-\}}\mathbf{1}_{\{T_{b_2}<1\}}\right) \tag{3.90}$$

$$= E\left(e^{-\beta(T_{b_2}+\tau^0_{b_1-b_2})}\mathbf{1}_{\{\tau_0^+<\tau_{b_1-b_2}^-\}}\mathbf{1}_{\{T_{b_2}<1\}}\right) \tag{3.91}$$

$$= E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)E\left(e^{-\beta\tau^0_{b_1-b_2}}\mathbf{1}_{\{\tau_0^+<\tau_{b_1-b_2}^-\}}\right) \tag{3.92}$$

$$= E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)E\left(e^{-\beta\tau_0^{b_2-b_1}}\mathbf{1}_{\{\tau_{b_2-b_1}^+<\tau_0^-\}}\right). \tag{3.93}$$

Using now equation (3.54), but with the two barriers now being 0 and b_2-b_1, we have

$$E\left(e^{-\beta\tau_0^{b_2-b_1}}\mathbf{1}_{\{\tau_{b_2-b_1}^+<\tau_0^-\}}\right) = e^{-\beta}\left(\hat{g}_0(\beta)\sum_{i=0}^{\infty}(-1)^k\left(\hat{g}_1(\beta)\right)^k-\hat{\hat{g}}_0(\beta)\sum_{i=0}^{\infty}(-1)^k\left(\hat{\hat{g}}_1(\beta)\right)^k\right), \tag{3.94}$$

with $\hat{g}_1(\beta)$ and $\hat{\hat{g}}_1(\beta)$ the same as in (3.47) and (3.48), but $\hat{g}_0(\beta)$ and $\hat{\hat{g}}_0(\beta)$ becomes

$$\hat{g}_0(\beta) = \frac{e^{\sqrt{2\beta}(b_1-b_2)}+1}{4\sqrt{\pi\beta}}, \tag{3.95}$$

$$\hat{\hat{g}}_0(\beta) = \frac{e^{\sqrt{2\beta}(b_1-b_2)}-1}{4\sqrt{\pi\beta}}. \tag{3.96}$$

Hence,

$$E\left(e^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}_{\{\tau_{b_2}^+<\tau_{b_1}^-\}}\mathbf{1}_{\{T_{b_2}<1\}}\right) \tag{3.97}$$

$$= e^{-\beta}\left(E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)\hat{g}_0(\beta)\sum_{i=0}^{\infty}(-1)^k\left(\hat{g}_1(\beta)\right)^k\right. \tag{3.98}$$

$$\left.-E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)\hat{\hat{g}}_0(\beta)\sum_{i=0}^{\infty}(-1)^k\left(\hat{\hat{g}}_1(\beta)\right)^k\right). \tag{3.99}$$

This can be inverted the same way as before, and each convolution term $L_1(t)$ and $\tilde{L}_1(t)$ is the same as before, but the initial conditions become

$$L_0(t) = \mathcal{L}^{-1}\left(E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)\hat{g}_0(t)\right), \tag{3.100}$$

$$\tilde{L}_0(t) = \mathcal{L}^{-1}\left(E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)\hat{\hat{g}}_0(t)\right). \tag{3.101}$$

To find $L_0(t)$ and $\tilde{L}_0(t)$, we invert the following Laplace transforms:

$$\mathcal{L}^{-1}\left(E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)\frac{e^{\sqrt{2\beta}(b_1-b_2)}}{4\sqrt{\pi\beta}}\right) \tag{3.102}$$

$$= \mathbf{1}_{\{0\le t<1\}}\int_0^t \frac{-b_2}{\sqrt{2\pi s^3}}e^{-\frac{b_2^2}{2s}}\frac{1}{4\pi\sqrt{t-s}}e^{-\frac{(b_2-b_1)^2}{2(t-s)}}ds \tag{3.103}$$

$$+\mathbf{1}_{\{t>1\}}\int_0^1 \frac{-b_2}{\sqrt{2\pi s^3}}e^{-\frac{b_2^2}{2s}}\frac{1}{4\pi\sqrt{t-s}}e^{-\frac{(b_2-b_1)^2}{2(t-s)}}ds \tag{3.104}$$

$$= \mathbf{1}_{\{0\le t<1\}}\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_1^2}{2t}} \tag{3.105}$$

$$+\mathbf{1}_{\{t>1\}}\left(e^{-\frac{b_1^2}{2t}}\frac{1}{4\pi\sqrt{t}}\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}}+\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\right)\right. \tag{3.106}$$

$$\left.+e^{-\frac{(2b_2-b_1)^2}{2t}}\frac{1}{4\pi\sqrt{t}}\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}}-\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\right)\right). \tag{3.107}$$

where the derivation from (3.103) to (3.105) is:

$$\begin{aligned}
& \mathbf{1}_{\{0\le t<1\}}\int_0^t \frac{-b_2}{\sqrt{2\pi s^3}}e^{-\frac{b_2^2}{2s}}\frac{1}{4\pi\sqrt{t-s}}e^{-\frac{(b_2-b_1)^2}{2(t-s)}}ds && (3.108)\\
= & \mathbf{1}_{\{0\le t<1\}}\int_{-\infty}^{\frac{b_2}{\sqrt{t}}}\frac{1}{2\pi\sqrt{2\pi}}\frac{-x}{\sqrt{x^2t-b_2^2}}e^{-\frac{x^2}{2}}e^{-\frac{(b_2-b_1)^2x^2}{2(tx^2-b_2^2)}}dx && (3.109)\\
= & \mathbf{1}_{\{0\le t<1\}}e^{-\frac{b_2^2}{2t}}e^{-\frac{(b_2-b_1)^2}{2t}}\int_0^\infty \frac{1}{2\pi\sqrt{2\pi t}}e^{-\frac{1}{2t}\left(x^2+\frac{(b_2-b_1)^2b_2^2}{x^2}\right)}dx && (3.110)\\
= & \mathbf{1}_{\{0\le t<1\}}\frac{1}{2}e^{-\frac{b_2^2}{2t}}e^{-\frac{(b_2-b_1)^2}{2t}}\left(\int_0^\infty \frac{1+\frac{(b_2-b_1)b_2}{x^2}}{2\pi\sqrt{2\pi t}}e^{-\frac{1}{2t}\left(x-\frac{(b_2-b_1)b_2}{x}\right)^2}e^{-\frac{1}{2t}(2(b_2-b_1)b_2)}dx\right. && (3.111)\\
& \left.+\int_0^\infty \frac{1-\frac{(b_2-b_1)b_2}{x^2}}{2\pi\sqrt{2\pi t}}e^{-\frac{1}{2t}\left(x+\frac{(b_2-b_1)b_2}{x}\right)^2}e^{\frac{1}{2t}(2(b_2-b_1)b_2)}dx\right) && (3.112)\\
= & \mathbf{1}_{\{0\le t<1\}}\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_1^2}{2t}}. && (3.113)
\end{aligned}$$

Likewise, we also have

$$\mathcal{L}^{-1}\left(E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)\frac{1}{4\sqrt{\pi\beta}}\right)=\mathbf{1}_{\{0\le t\le 1\}}\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_2^2}{2t}}+\mathbf{1}_{\{t>1\}}\frac{1}{2\pi\sqrt{t}}e^{-\frac{b_2^2}{2t}}\mathcal{N}\left(-b_2\sqrt{\frac{t-1}{t}}\right). \tag{3.114}$$

Finally, $L_0(t)$ is the sum of (3.102) and (3.114), and $\tilde{L}_0(t)$ is the difference of (3.102) and (3.114), so this gives us the result. □

Corollary 1. *For $0\le b_1<b_2$, on the set $T_{b_1}<1$, we have*

$$\begin{aligned}
f_{b_1}^{b_2}(t,\tau_{b_2}^+<\tau_{b_1}^-,T_{b_1}<1) &= f_{-b_2}^{-b_1}(t,\tau_{-b_2}^-<\tau_{-b_1}^+,T_{-b_1}<1) && (3.115)\\
f_{b_1}^{b_2}(t,\tau_{b_1}^-<\tau_{b_2}^+,T_{b_1}<1) &= f_{-b_2}^{-b_1}(t,\tau_{-b_1}^+<\tau_{-b_2}^-,T_{-b_1}<1). && (3.116)
\end{aligned}$$

Proof. The results are due to the symmetry of Brownian motion. The positive barriers can be reflected to give the same result as in the case with negative barriers. □

4. Pricing a Double barrier Parisian call option

In the previous section, we obtained a recursive formula for the density of the double barrier Parisian stopping time, for each of the three cases where we start in between the barrier ($b_1 \leq 0 \leq b_2$), above both barriers ($b_1 < b_2 \leq 0$), and below both barriers ($0 \leq b_1 < b_2$). In this section, we will show how we can use the densities to compute the price of a double barrier Parisian min-in call option.

4.1. Double barrier Parisian min-in call

A double barrier Parisian min-in call is a call option that gets knocked in if $\tau_{b_1}^{b_2} \leq T$. We denote by $C_i^{double}(x, T, L_1, L_2, K)$ the price of such an option with strike price K, barrier level L_1 and L_2, where $L_1 < L_2$, window length $D = 1$, initial underlying price x and maturity T. The payoff at maturity of such an option is $\mathbf{1}_{\{\tau_{b_1}^{b_2} \leq T\}}(S_T - K)^+$. When the underlying asset price follows a Geometric Brownian motion, and when it is in between the two barriers L_1 and L_2, we have the following pricing formula.

Theorem 4.1. *For $L_1 \leq S_0 \leq L_2$ ($b_1 \leq 0 \leq b_2$), the risk neutral price of a double barrier Parisian min-in call with maturity $T > 1$ is given by*

$$
\begin{aligned}
& {}^*C_i^{double}(x, T, L_1, L_2, K) && (4.1)\\
= \quad & \sqrt{2\pi}\left(\int_0^T f_{b_1}^{b_2}(t, \tau_{b_1}^- < \tau_{b_2}^+)(x\psi(\sigma + m, h_{b_1}, b_1, \rho, t) - K\psi(m, h'_{b_1}, b_1, \rho, t))dt\right. && (4.2)\\
& \left. + \int_0^T f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-)(x\psi(-(\sigma + m), h_{b_2}, -b_2, -\rho, t) - K\psi(-m, h'_{b_2}, -b_2, -\rho, t))dt\right), && (4.3)
\end{aligned}
$$

where we have used the following functions in order to simplify notations:

$$
\begin{aligned}
\psi(x, y, b, \rho, t) \quad := \quad & e^{\frac{x^2(1+T-t)+2bx}{2}}\left(Z(-x)\mathcal{N}\left(\frac{-x\rho - y}{\sqrt{1-\rho^2}}\right) - \rho Z(y)\mathcal{N}\left(\frac{-x - \rho y}{\sqrt{1-\rho^2}}\right)\right. && (4.4)\\
& \left. -x\left(\mathcal{N}(-x) - \mathcal{N}_\rho(-x, y)\right)\right), && (4.5)
\end{aligned}
$$

with $\mathcal{N}_\rho(\cdot, \cdot)$ denoting the joint cumulative distribution of a pair of bivariate normal random variables with correlation coefficient ρ, and

$$
\begin{aligned}
h_b \quad &:= \quad \frac{1}{\sqrt{1+T-t}}\left(k - b - (\sigma + m)(1 + T - t)\right), && (4.6)\\
h'_b \quad &:= \quad \frac{1}{\sqrt{1+T-t}}\left(k - b - m(1 + T - t)\right), && (4.7)\\
\rho \quad &:= \quad \frac{1}{\sqrt{1+T-t}}. && (4.8)
\end{aligned}
$$

Proof. As discussed in (2.7) and (2.8), we have

$$ {}^*C_i^{double}(x,T,L_1,L_2,K) = E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_{b_1}^{b_2}\le T\}} e^{mZ_T}(xe^{\sigma Z_T}-K)^+\right]. \quad (4.9) $$

We denote by $\mathcal{F}_t=\sigma(Z_s, s\le t)$ the natural filtration of the Brownian motion $(Z_t, t\ge 0)$. For ease of notation, since there is no ambiguity here, we refer to the double barrier stopping time $\tau_{b_1}^{b_2}$ as just τ. Then τ is an $\mathcal{F}_t$-stopping time, and by the strong Markov property of Brownian motion,

$$ \begin{aligned} {}^*C_i^{double}(x,T,L_1,L_2,K) &= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau\le T\}} E\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right] & (4.10)\\ &= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau\le T\}}\mathbf{1}_{\{\tau_{b_1}^-<\tau_{b_2}^+\}} E_{\mathcal{P}}\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right] & (4.11)\\ &\quad + E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau\le T\}}\mathbf{1}_{\{\tau_{b_2}^+<\tau_{b_1}^-\}} E_{\mathcal{P}}\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right] & (4.12) \end{aligned} $$

We have split into the two cases, $\tau_{b_1}^- < \tau_{b_2}^+$ and $\tau_{b_2}^+ < \tau_{b_1}^-$. In the first case, (4.11) is equal to

$$ \begin{aligned} & E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau\le T\}}\mathbf{1}_{\{\tau_{b_1}^-<\tau_{b_2}^+\}} E_{\mathcal{P}}\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right] & (4.13)\\ = & E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau\le T\}}\mathbf{1}_{\{\tau_{b_1}^-<\tau_{b_2}^+\}} \int_{-\infty}^{\infty} e^{my}(xe^{\sigma y}-K)^+ \frac{1}{\sqrt{2\pi(T-\tau)}} e^{-\frac{(y-Z_\tau)^2}{2(T-\tau)}}dy\right] & (4.14) \end{aligned} $$

Since the stopping time $\tau\mathbf{1}_{\{\tau_{b_1}^-<\tau_{b_2}^+\}}$ is independent of the Brownian meander $Z_{\tau_{b_1}^-}$, and the density of $Z_{\tau_{b_1}^-}$ is

$$ v(dz) = P(Z_{\tau_{b_1}^-}\in dz) = (b_1-z)e^{-\frac{(z-b_1)^2}{2}}\mathbf{1}_{\{z<b_1\}}dz, \quad (4.15) $$

(4.14) is equal to

$$ \int_0^T\int_{-\infty}^{b_1} f_{b_1}^{b_2}(t;\tau_{b_1}^-<\tau_{b_2}^+)\nu(dz)\int_k^\infty e^{my}(xe^{\sigma y}-K)\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(y-z)^2}{2(T-t)}}dydzdt, \quad (4.16) $$

for $k=\frac{1}{\sigma}\ln\left(\frac{K}{x}\right)$. Making use of the calculations in [Dassios and Lim (2013)], we have

$$ x\int_{-\infty}^{b_1}\int_k^\infty \frac{1}{2\pi\sqrt{T-t}}e^{(\sigma+m)y}(b_1-z)e^{-\frac{(z-b_1)^2}{2}}e^{-\frac{(y-z)^2}{2(T-t)}}dzdy \quad (4.17) $$

$$= xe^{\frac{(\sigma+m)^2(1+T-t)+2b_1(\sigma+m)}{2}} \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{-(\sigma+m)} \int_{h_{b_1}}^{\infty} (-v-(\sigma+m))e^{-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}} dudv \quad (4.18)$$

$$= xe^{\frac{x^2(1+T-t)+2b_1x}{2}} \left(Z(-(\sigma+m))\mathcal{N}\left(\frac{-(\sigma+m)\rho - h_{b_1}}{\sqrt{1-\rho^2}}\right) + \rho Z(h_{b_1})\mathcal{N}\left(\frac{-(\sigma+m)-\rho}{\sqrt{1-\rho^2}}\right)\right) \quad (4.19)$$

$$-xe^{\frac{x^2(1+T-t)+2b_1x}{2}}(\sigma+m)\left(\mathcal{N}(-(\sigma+m)) - \mathcal{N}_\rho(-(\sigma+m), h_{b_1})\right) \quad (4.20)$$

$$= x\psi(\sigma+m, h_{b_1}, b_1, \rho, t) - K\psi(m, h'_{b_1}, b_1, \rho, t), \quad (4.21)$$

where we have used the transformation $u = \frac{y-(b_1+(\sigma+m)(1+T-t)}{\sqrt{1+T-t}}$ and $v = z-(b_1+(\sigma+m))$, and denoted the function $\psi(x,y,b,\rho,t)$, h_b, h'_b, and ρ as in (4.4)-(4.8). Similar calculations for the expectation (4.12) leads to the price of the option given in (4.2) and (4.3). □

Theorem 4.2. *For $S_0 \leq L_1 < L_2$ ($0 \leq b_1 < b_2$), the price of a double barrier Parisian in call with maturity $T > 1$ is given by*

$$^*C_i^{double}(x,T,L_1,L_2,K) \quad (4.22)$$

$$= x\phi(\sigma+m) - K\phi(m) \quad (4.23)$$

$$+\sqrt{2\pi}\left(\int_0^T f_{b_1}^{b_2}(t, \tau_{b_1}^- < \tau_{b_2}^+, T_{b_1} < 1)(x\psi(\sigma+m, h_{b_1}, b_1, \rho, t) - K\psi(m, h'_{b_1}, b_1, \rho, t))dt \right. \quad (4.24)$$

$$\left. + \int_0^T f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-, T_{b_1} < 1)(x\psi(-(\sigma+m), h_{b_2}, -b_2, -\rho, t) - K\psi(-m, h'_{b_2}, -b_2, -\rho, t))dt\right), \quad (4.25)$$

and for $L_1 < L_2 \leq S_0$ ($b_1 < b_2 \leq 0$), the price of a double barrier Parisian in call with maturity $T > 1$ is given by

$$^*C_i^{double}(x,T,L_1,L_2,K) \quad (4.26)$$

$$= x\phi'(\sigma+m) - K\phi'(m) \quad (4.27)$$

$$+\sqrt{2\pi}\left(\int_0^T f_{b_1}^{b_2}(t, \tau_{b_1}^- < \tau_{b_2}^+, T_{b_2} < 1)(x\psi(\sigma+m, h_{b_1}, b_1, \rho, t) - K\psi(m, h'_{b_1}, b_1, \rho, t))dt \right. \quad (4.28)$$

$$\left. + \int_0^T f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-, T_{b_2} < 1)(x\psi(-(\sigma+m), h_{b_2}, -b_2, -\rho, t) - K\psi(-m, h'_{b_2}, -b_2, -\rho, t))dt\right), \quad (4.29)$$

In the above, we have defined the functions $\phi(x)$ and $\phi'(x)$ to be:

$$\phi(x) := e^{\frac{x^2T}{2}}\left(\mathcal{N}(b-x)-\mathcal{N}_{\frac{1}{\sqrt{T}}}\left(b-x,\frac{k-xT}{\sqrt{T}}\right)\right) \tag{4.30}$$

$$-e^{\frac{x^2T+4bx}{2}}\left(\mathcal{N}(-b-x)-\mathcal{N}_{\frac{1}{\sqrt{T}}}\left(-b-x,\frac{k-2b-xT}{\sqrt{T}}\right)\right), \tag{4.31}$$

$$\phi'(x) := e^{\frac{x^2T}{2}}\left(\bar{\mathcal{N}}_\rho\left(b-x,\frac{k-xT}{\sqrt{T}}\right)-e^{\frac{x^2T+4bx}{2}}\bar{\mathcal{N}}_\rho\left(-b-x,\frac{k-(2b+xT)}{\sqrt{T}}\right)\right) \tag{4.32}$$

Proof. For $S_0 \le L_1 < L_2$ $(0 \le b_1 < b_2)$,

$$^*C_i^{double}(x,T,L_1,L_2,K) = E_{\mathcal{P}}\left[\mathbf{1}_{\{T_{b_1}\ge 1\}}E\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right] \tag{4.33}$$

$$+E_{\mathcal{P}}\left[\mathbf{1}_{\{T_{b_1}<1\}}\mathbf{1}_{\{\tau\le T\}}E\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right] \tag{4.34}$$

If $T_{b_1} \ge 1$, we have $\tau = 1$. Furthermore, the law of Z_1 on the set $\{T_{b_1} \ge 1\}$ is

$$P(Z_1 \in dz, T_{b_1} \ge 1) = P(Z_1 \in dz) - P(Z_1 \in dz, T_{b_1} < 1) \tag{4.35}$$

$$= \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{z^2}{2}}-e^{-\frac{(z-2b_1)^2}{2}}\right)dz. \tag{4.36}$$

Hence, we have

$$E_{\mathcal{P}}\left[\mathbf{1}_{\{T_{b_1}\ge 1\}}E\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right] \tag{4.37}$$

$$= \frac{1}{\sqrt{2\pi(T-1)}}\int_{-\infty}^{b_1}\int_k^\infty e^{my}(xe^{\sigma y}-K)e^{-\frac{(y-z)^2}{2(T-1)}}\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{z^2}{2}}-e^{-\frac{(z-2b_1)^2}{2}}\right)dydz \tag{4.38}$$

$$= x\phi(\sigma+m)-K\phi(m), \tag{4.39}$$

for $\phi(x)$ defined in (4.30)-(4.31). (4.34) is the same as before with the density of the stopping time now being restricted to the set $\{T_{b_1} < 1\}$, and the pricing formula in (4.23)-(4.25) thus follows. For $L_1 < L_2 \le S_0$ $(b_1 < b_2 \le 0)$, we have

$$^*C_i^{double}(x,T,L_1,L_2,K) = E_{\mathcal{P}}\left[\mathbf{1}_{\{T_{b_2}\ge 1\}}E\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right] \tag{4.40}$$

$$+E_{\mathcal{P}}\left[\mathbf{1}_{\{T_{b_2}<1\}}\mathbf{1}_{\{\tau\le T\}}E\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right] \tag{4.41}$$

The law of Z_1 on the set $\{T_{b_2} \geq 1\}$ is

$$P(Z_1 \in dz, T_{b_2} \geq 1) = \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{z^2}{2}} - e^{-\frac{(z+2b_2)^2}{2}} \right) dz. \tag{4.42}$$

Expression (4.40) is thus

$$\begin{aligned} & E_{\mathcal{P}} \left[\mathbf{1}_{\{T_{b_2} \geq 1\}} E \left[e^{mZ_T} (xe^{\sigma Z_T} - K)^+ | \mathcal{F}_\tau \right] \right] && (4.43) \\ = & \frac{1}{\sqrt{2\pi(T-1)}} \int_{b_2}^{\infty} \int_{k}^{\infty} e^{my} (xe^{\sigma y} - K) e^{-\frac{(y-z)^2}{2(T-1)}} \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{z^2}{2}} - e^{-\frac{(z+2b_2)^2}{2}} \right) dydz && (4.44) \\ = & x\phi'(\sigma + m) - K\phi'(m), && (4.45) \end{aligned}$$

where $\phi'(x)$ is as defined in (4.32). Since (4.41) is the same as before with the density of the stopping time being restricted to the set $\{T_{b_2} < 1\}$, we have the pricing formula (4.27)-(4.29). □

4.2. Double barrier Parisian out call

The double barrier Parisian out call is a call option which gets knocked out when the price of the underlying asset goes beyond the barriers. Hence it has payoff $(S_T - K)^+ \mathbf{1}_{\{\tau \leq T\}}$ at time T. We denote by $C_o^{double}(x, T, L_1, L_2, K)$ the price of such an option with initial price x and time to maturity T. Then since

$$E_{\mathcal{P}} \left[e^{-rT} (S_T - K)^+ \right] = E_{\mathcal{P}} \left[e^{-rT} (S_T - K)^+ \mathbf{1}_{\{\tau \leq T\}} \right] + E_{\mathcal{P}} \left[e^{-rT} (S_T - K)^+ \mathbf{1}_{\{\tau > T\}} \right], \tag{4.46}$$

we have for $L_1 \leq S_0 \leq L_2$ $(b_1 \leq 0 \leq b_2)$,

$$C_o^{double}(x, T, L_1, L_2, K) = C_{BS}(x, T) - C_i^{double}(x, T), \tag{4.47}$$

where $C_{BS}(x, T)$ denotes the price of the vanilla call option with payoff $(S_T - K)^+$ at maturity T. For $S_0 \leq L_1 < L_2$ $(0 \leq b_1 < b_2)$, the Parisian out call becomes useless if $T_{b_1} > 1$, hence we have

$$\begin{aligned} & C_o^{double}(x, T, L_1, L_2, K) = E_{\mathcal{P}} \left[e^{-rT} \mathbf{1}_{\{T_{b_1} < 1\}} \mathbf{1}_{\{\tau > T\}} (S_T - K)^+ \right] && (4.48) \\ = & E_{\mathcal{P}} \left[\mathbf{1}_{\{T_{b_1} < 1\}} e^{-rT} (S_T - K)^+ \right] - E_{\mathcal{P}} \left[\mathbf{1}_{\{T_{b_1} < 1\}} e^{-rT} \mathbf{1}_{\{\tau \leq T\}} (S_T - K)^+ \right] && (4.49) \end{aligned}$$

$$= \int_0^1 \frac{b_1}{\sqrt{2\pi t^3}} e^{-\frac{b_1^2}{2t}} C_{BS}(L_1, T-t)dt - \left(C_i^{double}(x, T, L_1, L_2, K) - (x\phi(\sigma+m) - K\phi(m))\right) \tag{4.50}$$

For $L_1 < L_2 \leq S_0$ ($b_1 < b_2 \leq 0$), the Parisian out call becomes useless if $T_{b_2} > 1$, hence we have

$$C_o^{double}(x, T, L_1, L_2, K) = E_{\mathcal{P}}\left[e^{-rT}\mathbf{1}_{\{T_{b_2}<1\}}\mathbf{1}_{\{\tau>T\}}(S_T - K)^+\right] \tag{4.51}$$

$$= E_{\mathcal{P}}\left[\mathbf{1}_{\{T_{b_2}<1\}}e^{-rT}(S_T - K)^+\right] - E_{\mathcal{P}}\left[\mathbf{1}_{\{T_{b_2}<1\}}e^{-rT}\mathbf{1}_{\{\tau\leq T\}}(S_T - K)^+\right] \tag{4.52}$$

$$= \int_0^1 \frac{-b_2}{\sqrt{2\pi t^3}} e^{-\frac{b_2^2}{2t}} C_{BS}(L_2, T-t)dt - \left(C_i^{double}(x, T, L_1, L_2, K) - (x\phi'(\sigma+m) - K\phi'(m))\right) \tag{4.53}$$

5. Numerical Results

In this section, we compute the prices of the two-sided Parisian in call options and compare the prices across different initial asset price S_0, and different window length D. We demonstrate the correctness of these results by comparing them with the numerical prices obtained in [Anderluh and van der Weide (2009)] using Fourier transform inversion. We note that since we have chosen the window length D as the unit of time, all parameters (r, σ) are correspondingly normalised depending on the window length.

The table below is a comparison of the prices of double barrier Parisian in calls across different S_0 between 80 and 100, and D between 2 weeks and 3 months, fixing the parameters $\sigma = 0.2$, $r = 0.05$, $T = 1$ year, $K = 90$, and $L_1 = 80$ and $L_2 = 100$.

TABLE 1: Price of Parisian min-in call for $L_1 = 80$, $L_2 = 100$

S_0	$D = 1/25$	$D = 1/12$	$D = 1/6$	$D = 1/4$
80	3.894968	3.540956	2.844183	2.210231
82	4.650542	4.245481	3.478998	2.782763
84	5.526743	5.082531	4.256382	3.498445
86	6.525658	6.055637	5.182448	4.366187
88	7.644386	7.161490	6.255264	5.387473
90	8.877623	8.394327	7.470518	6.561364
92	10.21816	9.746352	8.821621	7.884263
94	11.65736	11.20818	10.29992	9.349889
96	13.18567	12.76930	11.89502	10.94943
98	14.79303	14.41847	13.59514	12.67189
100	16.39683	16.03719	15.23193	14.31076

The prices decrease with longer window lengths, as it becomes more difficult to knock in the option. For comparison, we have computed the same call prices for the case when the barriers are widened to $L_1 = 70$ and $L_2 = 110$. As can be expected, the options become cheaper as it is now more difficult for the option to be knocked in. This is shown in the following table.

TABLE 2: Price of Parisian min-in call $L_1 = 70$, $L_2 = 110$

S_0	$D = 1/25$	$D = 1/12$	$D = 1/6$	$D = 1/4$
80	2.192441	1.780915	1.240433	0.857615
82	2.814304	2.332982	1.679245	1.199394
84	3.563176	3.007037	2.229091	1.639840
86	4.444798	3.811887	2.902650	2.194032
88	5.462882	4.754442	3.711202	2.876336
90	6.618767	5.839241	4.663993	3.699677
92	7.911251	7.068144	5.767730	4.674874
94	9.336592	8.440194	7.026212	5.810075
96	10.88864	9.951636	8.440122	7.110332
98	12.55912	11.59608	10.00698	8.577325
100	14.33796	13.36478	11.72125	10.20927

The convolutions are evaluated using the `convolve` function in R. Due to the recursions, computation time decreases with the window length and are recorded in the following table:

Table 3: Computation time in seconds

	$D = 1/25$	$D = 1/12$	$D = 1/6$	$D = 1/4$
Elapsed time	38.23	6.24	2.42	1.75

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