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Asset pricing under optimal contracts *

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Abstract. We consider the problem of finding equilibrium asset prices in a financial market in which a portfolio manager (Agent) invests on behalf of an investor (Principal), who compensates the manager with an optimal contract. We extend a model from Buffa, Vayanos and Woolley (2014) by allowing general contracts, and by allowing the portfolio manager to invest privately in individual risky assets or the index. To alleviate the effect of moral hazard, Agent is optimally compensated by benchmarking to the index, which, however, may incentivize him to be too much of a "closet indexer". To counter those incentives, the optimal contract rewards Agent for taking specific risk of individual assets in excess of the systematic risk of the index, by rewarding the deviation between the portfolio return and the return of an index portfolio, and the deviation's quadratic variation.

Keywords: asset-management, equilibrium asset pricing, optimal contracts, principal–agent problem.
2000 Mathematics Subject Classification: 91B40, 93E20
JEL classification: C61, C73, D82, J33, M52

1 Introduction

We consider the problem of asset pricing with delegated portfolio management, that is, of finding asset prices so that the financial market is in equilibrium when the portfolio managers are offered optimal compensation contracts. The fact that an increasing percentage of investment funds is run by investment managers underlines the importance of studying the effect of managerial actions

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on asset prices. Thus, the problem is important, however, it is also difficult. There are extensive studies that consider various equilibrium models of asset prices, but, partly due to technical difficulties, there are almost no results where asset pricing is combined with optimal contracting between portfolio managers and investors. A notable exception is [Buffa et al., 2014], henceforth BVW (2014), which inspired the current paper.[1]

BVW (2014) considers a market with three types of participants: portfolio managers who decide on the investment strategy, but who can also get benefit from (non contractible) shirking that reduces the managed return; rational investors who can hire managers to invest on investors behalf in individual assets, while investors invest privately only in the index; and buy-and-hold investors. Portfolio managers have expert knowledge about individual assets, which is why investors can benefit from contracting managers to get access to individual assets. Both the investor and the portfolio manager have CARA utility functions. BVW (2014) considers two models: one in which the dividends have square-root dynamics, and the other in which they have OU (Orstein-Uhlenbeck) dynamics. The representative CARA investor chooses optimally the contract to pay the representative manager, but is allowed to do so only in a subfamily of all possible contracts – those that are linear in the investor's portfolio value and the stock index. This would, indeed, be optimal in the classical moral hazard continuous-time models of [Holmstrom and Milgrom, 1987] and [Sannikov, 2008], in which the manager can only affect the return of the output process. However, when the manager can also affect the volatility of the output, as is the case in portfolio management, it was shown in [Cvitanić et al., 2016a] and [Cvitanić et al., 2016b], henceforth CPT (2016ab), that the optimal contract makes use also of the quadratic (co)variations of the contractible factors. We use that insight to extend the family of admissible contracts in this paper.

The differences between this paper and CPT (2016ab) are as follows. In the latter, the manager is paid only once, at the final time, and the model is one of partial equilibrium, that is, the asset prices are exogenous. In contrast, in this paper the manager is paid at a continuous rate on an infinite horizon, and the asset prices are determined endogenously in equilibrium, as in BVW (2014). We use a mathematical methodology similar to that of CPT (2016ab), but adapted to the infinite horizon and continuous payments. In our setting, as in CPT (2016ab), the optimal contract uses quadratic (co)variations of contractible variables. More precisely, in the OU model, the optimal contract is linear in the investor's portfolio value, the index, and the quadratic variation of the deviation of the portfolio return from the return of an index portfolio. We find that the contract sensitivity to the quadratic variation of the deviation is positive, meaning that the contract rewards the agent for taking specific risk of individual risky assets beyond the systematic risk of the index. We show in a numerical example that the contract with the quadratic variation component can substantially increase investor's optimal value.

To the best of our knowledge, this, together with Leung (2016), is the first general equilib-

[1]Other related literature is discussed below.

rium model in which such a contract is shown to be optimal. The use of the quadratic variation, which, in practice, would correspond to using the sample variance, is, as noted in CPT (2016a), in the spirit of using the sample Sharpe ratio when compensating portfolio managers. However, in our model, in equilibrium, the principal rewards the agent for higher values of the quadratic variation, rather than penalizing him, to provide proper incentives for risk-taking beyond solely taking the risk of the index. The helpfulness of the second order variations is in agreement with [Admati and Pfleiderer, 1997] who emphasized that linear benchmark contracts may be inefficient for providing the right incentives. It is also consistent with [Bhattacharaya and Pfleiderer, 1985] who pointed out that quadratic contracts could be helpful in improving the incentives (albeit in the context of adverse selection rather than moral hazard).[2]

Let us elaborate further on the need for a contract which benchmarks to an index fund. In the first best risk sharing between two CARA agents with risk aversions ρ and $\bar{\rho}$, the fraction $\frac{\rho}{\rho+\bar{\rho}}$ of the output is paid to the agent with risk aversion $\bar{\rho}$. However, in the model of BVW (2014), the portfolio manager can apply a non-contractible shirking action, resulting in agency friction. To prevent the shirking action, the investor has to offer the pay-per-performance fraction Z that is higher than the shirking benefit b. Thus, when $b > \frac{\rho}{\rho+\bar{\rho}}$, the investor cannot offer the first best compensation. However, the higher than the first best fraction, $Z \geq b$, exposes the manager to higher risk. It then becomes profitable for the investor to reduce that exposure by benchmarking the output to the index. In turn, the benchmarking may make the manager invest too much in the index risk, and not enough in the specific risks of individual assets. This makes the investor use quadratic variation terms in the contract to reward the portfolio manager for taking specific risks.

When the contract depends only on the fraction of the output, Merton's theory of optimal portfolio selection would imply that the optimal investment into risky assets would be implemented via one fund only, call it Merton's fund, which is the fund with the vector of holdings equal to the risk premium vector. However, with a term depending on the index returns also in the contract, the manager would also like to invest in a second fund, the index. The Merton's fund is not equal to the index fund in our model because we assume that some shares are not available for trading. When the agency friction is more severe, the manager puts more weight on the index fund. Thus, the managed fund in this case is a "closet indexer". Our finding is consistent with the empirical analysis of [Cremers and Petajisto, 2009] which argued that the "closet indexers" are disproportionately expensive, relative to their performance.[3]

We also extend the model in BVW(2014) by allowing the portfolio manager to invest privately in individual risky assets. When the manager can trade in all the individual assets privately, the managed fund under the optimal contract is simply the index fund. Given that the investor can invest in the index directly, in this case the investor does not benefit from contracting. When the manager is allowed to trade privately only in the index, the second best can still be obtained with

[2]We thank an anonymous referee for pointing out to us the relevance of these two references.

[3]We are in debt to an anonymous referee for pointing out this connection to us.

a contract that incentivizes the manager to report his private portfolio value truthfully.

We leave the square-root model for dividends for future research. The difficulty with the square-root model is that the linear contracts with constant coefficients (the admissible contracts in BVW (2016)) are time-inconsistent – the investor would optimally want to change the coefficients as the time goes by. This makes the problem non-stationary and difficult.

Our asset pricing results are similar to those of the OU case in BVW (2014): the stocks in large supply have high risk premia, and the stocks in low supply have low risk premia, and this effect is stronger as agency friction increases. However, by using the contract that provides optimal risk-taking incentives, the sensitivity of the price distortion to agency frictions is of an order of magnitude smaller compared to the price distortion in BVW (2014). In other words, by including the risk-incentive terms in the compensation, the investor mitigates somewhat the effect of agency frictions in equilibrium.

Other than BVW (2014), Leung (2016), and the current paper, most of the existing literature either looks at the case of a fixed contract and then finds asset prices in equilibrium, or the case of fixed asset prices and then finds the optimal contract. In the first strand of the literature with fixed contracts, none of the papers, other than the current one and Leung (2016), allows for quadratic variation and co-variation components in the contract. That literature includes the following papers (a more thorough literature review can be found in BVW 2014): [Brennan, 1993] considers a static model with preferences based on a benchmark, resulting in a two-factor equilibrium model; [Basak and Pavlova, 2013] consider a similar set-up, but in a dynamic model; [Cuoco and Kaniel, 2011] have a dynamic setting with two risky assets, and the contract is a piece-wise affine function of the portfolio return and the return relative to a benchmark; [Malamud and Petrov, 2014] consider two types of managers, less and more informed.

The second strand of the literature with fixed asset prices includes the following papers: [Ou-Yang, 2003] has a dynamic model in which the portfolio value is only observable at the terminal time, and in which there is no moral hazard due to shirking, so that the optimal contract does not have quadratic variation/covariation components; [Cadenillas et al., 2007] extend some of [Ou-Yang, 2003] results to non-CARA utility functions, still with no moral hazard; [Lioui and Poncet, 2013] assume that the agent has enough bargaining power to require that the contract be linear in the output and in a benchmark factor; [Leung, 2014] studies a model with a single risky asset, in which moral hazard arises because there is an exogenous factor multiplying the volatility choice of the agent, and that factor is not observed by the principal; CPT (2016ab) find the optimal contract when the primary source of moral hazard is not due to shirking, but to the volatility vector being unobserved and the agent's cost of modifying it. Their model has finite horizon T and the agent is paid with a lump-sum contract payment at T only, unlike the present paper in which the payments are continuous over an infinite horizon.

Papers that do combine asset pricing and contracting include the following. [Ou-Yang, 2005] studies the interaction of asset pricing and moral hazard. However, the managers in his model are not portfolio managers, they are managers of firms, and they affect only the return rate of the firms' cash flows and not their volatilities. Moreover, the compensation payment is not continuous, but a lump sum at the end of the horizon. [Sung and Wan, 2015] consider an economy with N principals who hire N agents, extending the [Holmstrom and Milgrom, 1987] framework; as in the latter, the agents again only affect the drift of the cash-flows and the optimal contract depends only on the returns. In [Kaniel and Kondor, 2013] the authors consider a market with a fraction of investors delegating their capital for management, while the others invest directly at a cost. They do not look for the optimal contract; instead, they assume that a flow of capital to the managers is a convex function of performance, as empirically documented. [Leung, 2016] is an interesting recent paper that also considers an interplay between the asset price equilibrium and optimal contracting, but in a different model from ours, in which the manager affects with his effort the dividend growth rate and not the shirking level (as in our model); more importantly, the returns from the managed portfolio go to the manager, while the investors receive only the dividends. Despite the model differences, in his model the optimal contract has a similar form as ours, a linear term, and a term depending on the quadratic variation.

The rest of the paper is organized as follows: Section 2 sets up the model and the optimization problems, Section 3 describes the main results, Section 4 extends the result to the case when Agent can invest privately, Section 5 concludes, and Section 6 provides the proofs.

Some notational conventions. Let $(\Omega, \mathbf{F} = \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ denote a filtrated probability space, whose filtration $\mathbf{F}$ is the augmented filtration generated by independent Brownian motions $B^p, (B^e)_{i=1,\ldots,N}$, and satisfies the usual conditions of completeness and right-continuity. For a $\mathbf{F}$-adapted process X, $\mathbf{F}^X$ denotes the filtration generated by X and satisfies the usual conditions.

2 Model

2.1 Assets

The market consists of a risk-free asset with an exogenous constant risk-free rate r, and N risky assets whose prices $(S_{it})_{i=1,\ldots,N}$ will be determined in equilibrium. We work with the following model considered by [Buffa et al., 2014], henceforth BVW (2014). Assume that the dividend process of asset $i = 1, \ldots, N$ is given by

$$D_{it} = a_i p_t + e_{it}, \tag{2.1}$$

where p and e_i follow Ornstein-Uhlenbeck processes

$$\begin{aligned} dp_t &= \kappa^p(\bar{p} - p_t)dt + \sigma_p dB_t^p, \\ de_{it} &= \kappa_i^e(\bar{e}_i - e_{it})dt + \sigma_{ei} dB_{it}^e. \end{aligned} \tag{2.2}$$

Here, B^p and $(B_i^e)_{i=1,\dots,N}$ are independent Brownian motions, and model coefficients $a_i, \bar{p}, \bar{e}_i, \kappa^p, \kappa_i^e$, σ_p, σ_{ei}, for $i = 1,\dots,N$, are all positive constants. The filtration $\mathbf{F}^{B^p,B^e}$, denoted by $\mathbf{F}$, represents the full information in the model. We introduce the following vector and matrix notation for future use:

$$e = \mathrm{diag}\{e_1,\dots,e_N\}, \quad \bar{e} = \mathrm{diag}\{\bar{e}_1,\dots,\bar{e}_N\}, \quad \sigma_e = \mathrm{diag}\{\sigma_{e1},\dots,\sigma_{eN}\},$$
$$D = (D_1,\dots,D_N)', \quad S = (S_1,\dots,S_N)', \quad \kappa^e = (\kappa_1^e,\dots,\kappa_N^e)', \quad \text{and} \quad B^e = (B_1^e,\dots,B_N^e)',$$

where $\mathrm{diag}\{a_1,\dots,a_N\}$ is a diagonal matrix with diagonal elements $a_1,\dots,a_N$ and $(\cdot)'$ represents the vector transpose.

The vector of assets' return per share in excess of the riskless rate follows

$$dR_t = D_t dt + dS_t - rS_t dt. \tag{2.3}$$

The excess return of the market portfolio, or index, is given by

$$I_t = \eta' R_t, \tag{2.4}$$

where $\eta = (\eta_1,\dots,\eta_N)'$ is a constant vector, with η_i equal to the number of shares of asset i in the market. However, we assume that not all the shares of assets are available for trade. A constant vector $\theta = (\theta_1,\dots,\theta_N)'$, with entries equal to the number of shares available to trade is called the *residual supply*. The difference $\eta_i - \theta_i$ equals the number of shares of asset i held by buy-and-hold investors who do not trade. We assume that each component of θ is strictly positive.

2.2 Agent and Principal

In addition to buy-and-hold investors, there are two market participants in the model: Agent (portfolio manager) and Principal (investor). They can be considered as representatives of identical agents and principals. Both Agent and Principal are price-takers, that is, they take prices as given, without taking into account the feedback effects in equilibrium.

Principal can hire Agent to manage a portfolio of assets on Principal's behalf. Agent, if hired by Principal, receives compensation (fee) paid by Principal, manages a portfolio of assets, and he can also undertake a "shirking" action that has a detrimental effect on the portfolio, but it provides Agent with a private benefit.

In the benchmark model, Agent can only invest in the riskless asset in his private account, and he can also consume from it. Thus, Agent is exposed to the risky assets only via the compensation paid by Principal. We consider two additional cases in Section 4, the one in which Agent can invest privately in all the risky assets, and the one in which he can invest privately only in the index.

In the benchmark model, Agent's wealth process is given by

$$d\bar{W}_t = r\bar{W}_t dt + (bm_t - \bar{c}_t)dt + dF_t, \tag{2.5}$$

where

- $\bar{c}_t$ is Agent's consumption rate;

- bm_t is the private benefit from his rate m_t of shirking with the benefit rate $b \in [0,1]$;

- F_t is the cumulative compensation paid by Principal.

Principal can trade in the index, but not in the individual risky assets. The only way she can access individual risky assets is by hiring Agent. As noted in BVW (2014), one way to interpret this participation constraint is that Principal is unable to identify assets that have good return/risk characteristics relative to the index. This happens when Principal does not observe assets return vector R. As a result, Principal must employ Agent for non-index investing, because Agent has private information on individual asset characteristics, and, thus, contracting might be beneficial to Principal. As for Agent, because he can privately invest only in the riskless asset, the only way for him to have exposure to risky assets is through the fee paid by Principal. To achieve that exposure, Agent takes specific risk of individual risky assets in Principal's portfolio. This in turn provides benefits to Principal, and both Agent and Principal gain from contracting with each other. When Agent can invest privately in all the risky assets, we show in Section 4.1 that Agent gets the benefit of individual risky assets fully through his private investment, and would only invest in the index for Principal; as a result, there are no gains from contracting. [4]

Recall also that in the CARA-normal framework of this paper the investor would split her capital between the risk-free asset and the fund corresponding to the risk premium vector. However, due to the presence of buy-and-hold investors, not all the shares are available for trade, and, therefore, if vector θ of the numbers of available shares in the residue supply is not co-linear with vector η of the total number of shares, the risk premium vector is not equal to the index fund in equilibrium. This is why it is suboptimal for Principal to trade only in the index and the risk-free asset – she has a need for Agent, to gain access to individual risky assets.

Principal's wealth process follows

$$dW_t = rW_t dt + dG_t + y_t dI_t - c_t dt - dF_t, \tag{2.6}$$

where:

- $G_t = \int_0^t [Y_s' dR_s - m_s ds]$ is the *reported* cumulative fund return process, where Y is the vector of the number of shares of the risky assets held by Agent in the managed portfolio;

- y_t is the number of shares of the index held by Principal;

- c_t is Principal's consumption rate.

Agent's rate m_t of shirking action m_t is assumed to be nonnegative. It reduces Principal's wealth; in addition to shirking, it can also be interpreted as diverting money from the portfolio for expenses that do not contribute to the performance of the fund. More generally, it may be thought of as a measure of (lack of) Agent's efficiency when running the portfolio; see [DeMarzo and Sannikov, 2006].

[4] We thank the AE for pointing this out.

Agent maximizes utility over intertemporal consumption:

$$\bar{V} = \max_{\Xi \text{ admissible}} \mathbb{E}\Big[\int_0^\infty e^{-\bar{\delta}t} u_A(\bar{c}_t)dt\Big],$$

where u_A is exponential utility with constant absolute risk aversion $\bar{\rho}$, i.e., $u_A(c) = -\frac{1}{\bar{\rho}}e^{-\bar{\rho}c}$, and $\bar{\delta} > 0$ is Agent's discounting rate. Given Agent's utility function u_A, we can assume, without loss of generality, that the initial wealth of Agent is zero, i.e., $\bar{W}_0 = 0$.

Given Principal's strategy $\Theta = (c, F, y)$, Agent's strategy $\Xi = (\bar{c}, m, Y)$ is *admissible* if the following conditions are satisfied

- $(\bar{c}, m, Y)$ is adapted to $\mathbf{F}$;
- Y is predictable, $\int_0^t |Y_s|^2 ds < \infty$ for all $t > 0$;
- $m \geq 0$;
- $\bar{c}$ is financed by wealth process $\bar{W}$ satisfying (2.5).
- The following transversality condition is satisfied:

$$\lim_{T\to\infty}\lim_{n\to\infty} \mathbb{E}\Big[e^{-\bar{\delta}(T\wedge\tau_n)}e^{-r\bar{\rho}\bar{W}^u_{T\wedge\tau_n}}\Big] = 0, \tag{2.7}$$

for any sequence of stopping time $\{\tau_n\}_n$ with $\lim_n \tau_n = \infty$.

If Agent is not employed by Principal, he chooses his private portfolio Y^u and consumption rate $\bar{c}^u$ to maximize his utility over consumption

$$\bar{V}^u = \max_{(Y^u, \bar{c}^u) \text{ admissible}} \mathbb{E}\Big[\int_0^\infty e^{-\bar{\delta}t} u_A(\bar{c}^u_t)dt\Big],$$

subject to the budget constraint

$$d\bar{W}^u_t = r\bar{W}^u_t dt + Y^u_t dR_t - \bar{c}^u_t dt. \tag{2.8}$$

Agent's private investment and consumption strategy $(Y^u, \bar{c}^u)$ is *admissible* if

- $(Y^u, \bar{c}^u)$ is adapted to $\mathbf{F}$;
- Y^u is predictable, $\int_0^t |Y^u_s|^2 ds < \infty$ for all $t > 0$;
- $\bar{c}^u$ is financed by wealth process $\bar{W}^u$ satisfying (2.8).
- The transversality condition (2.7) is satisfied.

Agent takes the contract offered by Principal if and only if the following *participation constraint* is satisfied:

$$\bar{V} \geq \bar{V}^u. \tag{2.9}$$

When this inequality is an equality, Agent is indifferent with respect to taking the contract or not. In this case, as is standard in contract theory, we assume that Agent chooses to work for Principal.

Principal maximizes utility over intertemporal consumption:

$$V = \max_{\Theta \text{ admissible}} \mathbb{E}\Big[\int_0^\infty e^{-\delta t} u_P(c_t)dt\Big],$$

where u_P is an exponential utility with constant absolute risk aversion ρ, i.e., $u_P(c) = -\frac{1}{\rho}e^{-\rho c}$, and $\delta > 0$ is Principal's discounting rate. Principal's strategy $\Theta = (c, F, y)$ is *admissible* if

- Agent's optimization problem admits at least one admissible optimal strategy $\Xi^* = (\bar{c}^*, m^*, Y^*)$;

- (c, F, y) is adapted to $\mathbf{F}^{G^*, I}$, where $G_t^* = \int_0^t (Y_s^*)' dR_s - m_s^* ds$ is the reported cumulative fund return when Agent employs his optimal strategy Ξ^*;

- y is predictable, $\int_0^t y_s^2 ds < \infty$ for all $t \geq 0$;

- The consumption stream c is financed by the wealth process W satisfying

$$dW_t = rW_t dt + dG_t^* + y_t dI_t - c_t dt - dF_t;$$

- The following transversality condition is satisfied:

$$\lim_{T\to\infty} \lim_{n\to\infty} \mathbb{E}\left[e^{-\delta(T\wedge\tau_n)} e^{-r\rho W^u_{T\wedge\tau_n}}\right] = 0, \tag{2.10}$$

for any sequence of stopping time $\{\tau_n\}_n$ with $\lim_n \tau_n = \infty$.

If Principal does not hire Agent, she chooses investment y^u in the index and consumption rate c^u to maximize her utility over consumption

$$V^u = \max_{(y^u, c^u) \text{ admissible}} \mathbb{E}\left[\int_0^\infty e^{-\delta t} u_p(c_t^u) dt\right],$$

subject to the budget constraint

$$dW_t^u = rW_t^u dt + y_t^u dI_t - c_t^u dt. \tag{2.11}$$

Principal's private investment and consumption strategy (y^u, c^u) is *admissible* if

- (y^u, c^u) is adapted to $\mathbf{F}^I$;

- y^u is predictable, $\int_0^t |y_s^u|^2 ds < \infty$ for all $t > 0$;

- c^u is financed by wealth process W^u satisfying (2.11).

- The transversality condition (2.10) is satisfied.

Principal hires Agent if and only if

$$V \geq V^u. \tag{2.12}$$

When the inequality above is an equality, Principal is indifferent to hiring Agent or not. In this case, we assume that Principal chooses to hire Agent.

2.3 Equilibrium

We will look for equilibria in which Principal hires Agent. The notion of equilibrium is similar to BVW (2014). The only difference is that the class of the contracts which Principal is allowed to optimize over is incorporated in our notion of equilibrium.

Definition 2.1 *A price process S, a contract F in a class of contracts $\mathscr{F}$, and an index investment y form an equilibrium if:*

(i) Given S, $(F, \mathscr{F})$ and y, Agent takes the contract, and $Y = \theta - y\eta$ solves Agent's optimization problem.

(ii) Given S, Principal hires Agent, contract F is optimal for Principal in the class $\mathscr{F}$ and y is her optimal index investment strategy.

We will look for the equilibrium in which the price of asset i is of the form

$$S_{it} = a_{0i} + a_{pi} p_t + a_{ei} e_{it}, \tag{2.13}$$

where (a_{0i}, a_{pi}, a_{ei}) are constants that will be determined in equilibrium. The form of (2.13), combined with (2.1), (2.2), and (2.3), imply that the excess return for asset i follows

$$\begin{aligned} dR_{it} = &[(a_i - a_{pi}(r+\kappa^p))p_t + (1 - a_{ei}(r+\kappa_i^e))e_{it} + \kappa^p a_{pi}\bar{p} + \kappa_i^e a_{ei}\bar{e}_i - r a_{0i}]\, dt \\ &+ a_{pi}\sigma_p dB_t^p + a_{ei}\sigma_i dB_{it}^e \\ =:&\ [A_{1i} p_t + A_{2i} e_{it} + A_{3i}]\, dt + a_{pi}\sigma_p dB_t^p + a_{ei}\sigma_{ei} dB_{it}^e. \end{aligned} \tag{2.14}$$

Denote also

$$\gamma = (a_{p1}, \ldots, a_{pN})'\sigma_p, \quad \sigma = \mathrm{diag}\{a_{e1}, \ldots, a_{eN}\}\sigma_e, \quad A_\ell = (A_{\ell 1}, \ldots, A_{\ell N})', \quad \ell = 1,2,3,$$

$$\mu_t - r = p_t A_1 + e_t A_2 + A_3, \quad \text{and} \quad \Sigma_R = \gamma\gamma' + \sigma^2.$$

Then, the vector of asset returns follows

$$dR_t = (\mu_t - r)dt + \gamma dB_t^p + \sigma dB_t^e, \tag{2.15}$$

with (instantaneous) covariance matrix Σ_R.

3 Optimal strategies and equilibrium

Since the main new effects in this paper arise from the optimal contract including terms which depend on quadratic (co)variations of the contractible variables, we first discuss the need for these terms in the contract, in a market with fixed asset prices.

3.1 Incentive effects of quadratic variations and covariations

Let us first recall that the first best risk sharing between two CARA agents is obtained by paying the fraction $\frac{\rho}{\rho+\bar{\rho}}$ of the output G to the agent with risk aversion $\bar{\rho}$. This is still the case in our framework when there is no agency friction. However, when agency friction is present, i.e., Agent receives benefit at rate b from non-contractible shirking action, if the fraction of the output in Agent's compensation is less than b, Agent would apply an infinite level of shirking (i.e., $m =$

∞), which is clearly suboptimal for Principal. Therefore, the pay-per-performance fraction Z in Agent's compensation should be no smaller than b. As a result, if $b > \frac{\rho}{\rho+\bar{\rho}}$, Principal cannot offer the first best compensation. However, high pay-per-performance fraction exposes Agent to more risk. It then becomes profitable for Principal to benchmark the output to the index by offerring a term UI in the contract, where U is the contract sensitivity of the index.

With only the term ZG in the contract, standard Merton's theory of optimal portfolio selection would imply that the optimal investment into risky assets would be implemented via one fund only, call it Merton's fund, which is the fund with the vector of holdings equal to the risk premium vector $\Sigma_R^{-1}(\mu - r)$. However, with the term UI also in the contract, Agent would also like to invest in a second fund, that is, the index, as we illustrate in what follows. Note that Merton's fund is the fund Principal would invest into in absence of agency frictions, but it is not equal to the index fund in our model, due to some shares not being available for trading. This is why Principal needs access to individual assets via delegation.

One of the main points of this paper is that Principal also should include terms depending on quadratic (co)variations of the contractible variables in the contract. A reason for this is that including the term UI in the contract may make Agent invest too much in the index risk, and not enough in the specific risk of individual assets, which Principal benefits from. To better understand the intuition behind the effects of those terms, we first use a simplified framework to compare simple linear contracts, that do not include those (co)variations, with linear contracts that do include them.

Assume, for simplicity, that (2.15) holds with constant $\mu_t \equiv \mu$. Suppose first that the contract is linear only in the portfolio returns and the index returns, that is, of the form,

$$F_t = ZG_t + UI_t, \tag{3.1}$$

for some constants U and Z.

As mentioned above, Principal needs to choose $Z \geq b$, and we will show that this will make Agent choose $m \equiv 0$. Then, from (2.5), the optimization term involving the choice of strategy Y in the HJB optimality equation (the dynamic programming equation) for Agent's value function $\bar{V} = \bar{V}(\bar{W})$ is

$$\sup_Y \left\{ \bar{V}_{\bar{W}} Z Y'(\mu - r) + \tfrac{1}{2}\bar{V}_{\bar{W}\bar{W}}(ZY + U\eta)'\Sigma_R(ZY + U\eta) \right\}, \tag{3.2}$$

where $\bar{V}_{\bar{W}}$ and $\bar{V}_{\bar{W}\bar{W}}$ denote the appropriate partial derivatives. This reflects the familiar mean-variance tradeoff, with the first term representing the mean of the part of the compensation driven by the choice of the portfolio strategy Y, and the second term driven by the variance of the compensation. We conjecture that Agent's value function is of the form $\bar{V}(\bar{W}) = \bar{V}_0 e^{-r\bar{\rho}\bar{W}}$ for some constant $\bar{V}_0$ (which will be verified later). If that is the case, we see that Agent will optimally choose

$$Y^* = \frac{1}{r\bar{\rho}Z}\Sigma_R^{-1}(\mu - r) - \frac{U}{Z}\eta. \tag{3.3}$$

Thus, Agent's strategy is a linear combination of the risk premium vector $\Sigma_R^{-1}(\mu - r)$ and the index, that is, his strategy is a weighted average of two funds. There is no investment in the index fund if the index fund is not used in the contract, that is, if $U = 0$. However, note that since Z is restricted to be not less than b, Principal can place at most weight $\frac{1}{r\bar{\rho}b}$ on Merton's fund. This restricts the range of strategies attainable under the contracts of the above form.

Suppose now we consider contracts that include the quadratic (co)variations, that is, of the form, with constants $C, Z, U, \gamma^G, \gamma^I$, and γ^{GI},

$$F_t = Ct + ZG_t + UI_t + \tfrac{1}{2}\gamma^G \langle G \rangle_t + \tfrac{1}{2}\gamma^I \langle I \rangle_t + +\gamma^{GI} \langle G, I \rangle_t, \text{[5]} \tag{3.4}$$

where the quadratic (co)variations are defined as

$$d\langle G \rangle_t = Y'\Sigma_R Y dt, \quad d\langle I \rangle_t = \eta'\Sigma_R \eta dt, \quad \text{and} \quad d\langle G, I \rangle_t = Y'\Sigma_R \eta dt.$$

The maximization term in Agent's optimality equation contains now an extra term $\bar{V}_{\bar{W}}(\frac{1}{2}\gamma^G Y'\Sigma_R Y + \gamma^I Y'\Sigma_R \eta)$ and becomes

$$\sup_Y \left\{ \bar{V}_{\bar{W}} ZY'(\mu - r) + \bar{V}_{\bar{W}}(\tfrac{1}{2}\gamma^G Y'\Sigma_R Y + \gamma^{GI} Y'\Sigma_R \eta) + \tfrac{1}{2}\bar{V}_{\bar{W}\bar{W}}(ZY + U\eta)'\Sigma_R(ZY + U\eta) \right\}. \tag{3.5}$$

Using the conjectured form of Agent's value function $\bar{V}(\bar{W}) = \bar{V}_0 e^{-r\bar{\rho}\bar{W}}$, we see that Agent will optimally choose

$$Y^* = \frac{Z}{r\bar{\rho}Z^2 - \gamma^G}\Sigma_R^{-1}(\mu - r) - \frac{r\bar{\rho}ZU - \gamma^{GI}}{r\bar{\rho}Z^2 - \gamma^G}\eta.$$

To make Y^* above more transparent, we reparameterize (3.4) to

$$F_t = Ct + ZG_t + UI_t + \tfrac{1}{2}\Gamma^G \langle G \rangle_t + \tfrac{1}{2}\Gamma^I \langle I \rangle_t + \Gamma^{GI} \langle G, I \rangle_t + \tfrac{1}{2} r\bar{\rho} \langle ZG + UI \rangle_t. \tag{3.6}$$

Then, the additional term $\frac{1}{2} r\bar{\rho} \langle ZG + UI \rangle_t$ cancels the quadratic term $\frac{1}{2}\bar{V}_{\bar{W}\bar{W}}(ZY + U\eta)'\Sigma_R(ZY + U\eta)$ in (3.5), and replaces it with a quadratic term $\bar{V}_{\bar{W}}(\frac{1}{2}\Gamma^G Y'\Sigma_R Y + \Gamma^{GI} Y'\Sigma_R \eta)$, with arbitrary weights Γ^G and Γ^{GI}. Under the new parametrization, Agent will optimally choose

$$Y^* = -\frac{Z}{\Gamma^G}\Sigma_R^{-1}(\mu - r) - \frac{\Gamma^{GI}}{\Gamma^G}\eta. \tag{3.7}$$

Thus, even though Z is no smaller than b, because Γ^G is not constrained Principal can achieve arbitrary weights on Merton's fund.

Let us conclude this informal discussion by reiterating in which case it would actually be beneficial for Principal to have the Y^* from (3.7) different from the one in (3.3): this is the case if and only if Principal wants to have a weight of more than $1/(r\bar{\rho}b)$ on Merton's portfolio, which is the case if an only if $b > \rho/(\rho + \bar{\rho})$, as argued in the beginning of this section. In other words, if the agency friction, as measured by the shirking benefit b, is high, it is in Principal's interest to include quadratic variation and covariation in the contract. In the remainder of the paper, we develop a rigorous approach for justifying the intuition given in this section, and for placing it in an equilibrium framework.

[5] Since Agent cannot control the returns of the index, the term $\frac{1}{2}\gamma_t^I \langle I \rangle_t$ can be incorporated into the term Ct.

3.2 Family of viable contracts

A priori, we do not know that it is sufficient to consider the contracts of the form (3.6). In particular, it is not clear whether it is beneficial for Principal to offer non-linear contracts or even non-Markovian contracts. In order to include more general contracts that Principal can choose, we follow the approach of Cvitanić, Possamai and Touzi (2016ab), henceforth CPT (2016ab). The approach consists of *admitting only such contracts for which Agent's problem satisfies the martingale principle of optimality (dynamic programming principle).* We call such contracts viable. This makes Agent's problem straightforward to solve. More importantly, CPT (2016ab) prove that in the finite horizon model this approach represents no loss of generality, that is, Principal's maximal utility when optimizing over the restricted family of contracts satisfying the martingale principle is the same as when maximizing over arbitrary contracts satisfying mild integrability conditions. While we have not proved similar result in our infinite horizon framework, we conjecture that this is still true in our model.[6] If the conjecture is true, then our optimal contract would be optimal in an essentially as large as possible family of contracts (including non-linear and non-Markovian).

We remark that Lemma 3.3 below shows that any viable contract can be represented by a contract of type (3.6). Thus, the intuition of Section 3.1 is based on the most general form of the contracts we consider.

In order to state the definition of viability, we need the following notation. For $t \geq 0$ and a given Agent's admissible strategy $\Xi = (\bar{c}, Y, m)$, consider the following class of admissible strategies that agree with Ξ on $[0,t]$:

$$\Xi^t = \{\hat{\Xi} \text{ admissible } | \hat{\Xi}_s = \Xi_s, s \in [0,t]\}.$$

Define Agent's *continuation value process* $\bar{\mathscr{V}}(\Xi)$ as

$$\bar{\mathscr{V}}_t(\Xi) = \text{ess sup}_{\Xi^t} \mathbb{E}_t\left[\int_t^\infty e^{-\bar{\delta}(s-t)} u_A(\bar{c}_s) ds\right], \quad t \geq 0.$$

That is, $\bar{\mathscr{V}}_t(\Xi)$ is Agent's optimal value at time t if he employs the strategy Ξ before time t and acts optimally from time t onward. The continuation value process is expected to satisfy the *martingale principle of optimality* (which can be viewed as the dynamic programming principle in non-Markovian settings): process $\tilde{\mathscr{V}}(\Xi)$, defined as

$$\tilde{\mathscr{V}}_t(\Xi) = e^{-\bar{\delta} t} \bar{\mathscr{V}}_t(\Xi) + \int_0^t e^{-\bar{\delta} s} u_A(\bar{c}_s) ds,$$

is a supermartingale for arbitrary admissible strategy Ξ, and is a martingale for the optimal strategy Ξ^*. The definition of viability essentially requires that there exists such a process, and it identifies the sensitivities of the process with respect to the contractible variables and their quadratic variations and co-variations. To do this properly, we need to introduce some additional notation.

[6] A working paper Lin, Ren and Touzi (2017) is developing the mathematical theory needed to prove such a conjecture.

For real numbers $X > 0, Z \geq b, U, \Gamma^G < 0, \Gamma^I, \Gamma^{GI}$, interpreted later below as the (normalized) sensitivities of the Agent's value to the observable factors, define the Hamiltonian H by

$$H(X,Z,U,\Gamma^G,\Gamma^I,\Gamma^{GI}) = \sup_{(\bar{c},m\geq 0,Y)} \Big\{ u_A(\bar{c}) + X\Big[bm - \bar{c} - Zm + ZY'(\mu - r) + U\eta'(\mu - r) + \tfrac{1}{2}\Gamma^G Y'\Sigma_R Y + \tfrac{1}{2}\Gamma^I \eta'\Sigma_T \eta + \Gamma^{GI} Y'\Sigma_R \eta\Big]\Big\}. \tag{3.8}$$

It is also convenient to introduce process P via

$$dP_t = (bm_t - \bar{c}_t)dt, \quad P_0 = 0,$$

which records the impact of Agent's private action on his wealth.

Definition 3.1 *Principal's admissible strategy $\Theta = (c, F, y)$ is* viable *if there exist*

- *a constant $\bar{V}_0$ and*
- *a class of Agent's admissible strategies $\Xi(\Theta)$,*

such that

(a) for any Agent's strategy $\Xi \in \Xi(\Theta)$, there exist $\mathbf{F}^{G,I}$ adapted processes $Z, U, \Gamma^G, \Gamma^I, \Gamma^{GI}$, satisfying $\int_0^t Z_s^2 ds < \infty, \int_0^t U_s^2 ds < \infty$, for all $t > 0$, and $Z \geq b$, $\Gamma^G < 0$ such that the process $\bar{V}(\Xi)$ defined by

$$d\bar{V}_t(\Xi) = X_t\Big[dP_t + Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma_t^I \langle I\rangle_t + \Gamma_t^{GI} d\langle G, I\rangle_t\Big] + \bar{\delta}\bar{V}_t(\Xi)dt - H(Z_t, U_t, \Gamma_t^G, \Gamma_t^I, \Gamma_t^{GI})dt, \quad \bar{V}_0(\Xi) = \bar{V}_0, \tag{3.9}$$

where $X_t = -r\bar{\rho}\bar{V}_t(\Xi)$, satisfies the transversality condition

$$\lim_{T\to\infty}\lim_{n\to\infty} \mathbb{E}\Big[e^{-\bar{\delta}T\wedge\tau_n}\bar{V}_{T\wedge\tau_n}(\Xi)\Big] = 0, \tag{3.10}$$

for any sequence of stopping times $\{\tau_n\}_n$ with $\lim_n \tau_n = \infty$.

(b) the class $\Xi(\Theta)$ contains a strategy $\Xi^* = (\bar{c}^*, m^*, Y^*)$ that maximizes the Hamiltonian, that is, the strategy with

$$\bar{c}^* = (u_A')^{-1}(-r\bar{\rho}\bar{V}(\Xi^*)), \quad m^* = 0, \quad Y^* = -\frac{Z}{\Gamma^G}\Sigma_R^{-1}(\mu - r) - \frac{\Gamma^{GI}}{\Gamma^G}\eta; \tag{3.11}$$

(c) Denoting the reported portfolio value and the contract value by G^* and F^*, respectively, when Agent employs strategy Ξ^*, then, Principal's wealth process, following the dynamics

$$dW_t = rW_t dt + dG_t^* + y dI_t - c_t dt - dF_t^*, \tag{3.12}$$

satisfies the transversality condition

$$\lim_{T\to\infty}\lim_{n\to\infty}\mathbb{E}\left[e^{-r\rho W_{T\wedge\tau_n}}\right]=0, \tag{3.13}$$

for any sequence of stopping times $\{\tau_n\}_n$ with $\lim_n \tau_n = \infty$.

Remark 3.1 *As we will argue below, process $\bar{V}_t(\Xi)$ is Agent's continuation value process, and process $\tilde{V}_t(\Xi) = e^{-\bar{\delta}t}\bar{V}_t(\Xi) + \int_0^t e^{-\bar{\delta}s}u_A(\bar{c}_s)ds$ is, indeed, by the definition of H, a supermartingale for an arbitrary Agent's admissible action, and a martingale for the strategy in Ξ^* in* (3.11)*. Moreover,* (3.9) *gives a representation for Agent's continuation value process with sensitivities with respect to P, $G, I, \langle G\rangle$, $\langle I\rangle$, $\langle G,I\rangle$ given by processes $X, XZ, XU, \frac{1}{2}X\Gamma^G$, $\frac{1}{2}X\Gamma^I$, and $X\Gamma^{GI}$, respectively. Since $U, Z, \Gamma^G, \Gamma^I, \Gamma^{GI}$ can be arbitrary $\mathbf{F}^{G,I}$-adapted processes (with $Z \ge b$ and $\Gamma^G < 0$), a viable strategy allows all possible sensitivities with respect to G, I and quadratic variation and covariations of G and I. Note also that the sensitivity of Agent's continuation value with respect to P is the same as the sensitivity with respect to $\bar{W}$. Lemma 6.1 below shows that this sensitivity is equal to $-r\bar{\rho}\bar{V}(\Xi)$, which is why we set sensitivity X equal to that value.*

The reason we require $Z \ge b$ is that, when $Z_t < b$ for t, Hamiltonian H is maximized for $m = \infty$. This would lead to Principal's wealth being equal to $-\infty$, hence not optimal for Principal. When $Z = b$, all nonnegative values of m maximize the Hamiltonian, and Agent is indifferent which m to choose. In this case, we follow the usual convention in contract theory and assume that Agent will choose the best value for Principal, i.e., $m = 0$.

The following lemma shows why the above definition of viability is useful and natural in our context.

Lemma 3.2 *Consider any Principal's viable strategy $\Theta = (c, F, y)$. Assume that Σ_R is invertible. Then, the strategy $\Xi^* = (\bar{c}^*, m^*, Y^*)$ in* (3.11) *is Agent's optimal strategy in the class $\Xi(\Theta)$, and $\bar{V}_0$ is Agent's optimal value at time* 0*. Moreover, $\bar{V}(\Xi)$ is equal to Agent's continuation value process $\bar{\mathcal{V}}(\Xi)$.*

The following result provides a representation for any viable contract.

Lemma 3.3 *Contract F in any viable strategy satisfies*

$$\begin{aligned} dF_t =& Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma_t^I d\langle I\rangle_t + \Gamma_t^{GI} d\langle G,I\rangle_t + \tfrac{1}{2}r\bar{\rho}\, d\langle Z\cdot G + U\cdot I\rangle_t \\ & - \bar{H}_t dt, \end{aligned} \tag{3.14}$$

where $Z\cdot G = \int_0^\cdot Z_s dG_s$, and

$$\bar{H}_t = \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) - \tfrac{1}{\bar{\rho}} + \tfrac{\bar{\delta}}{r\bar{\rho}} + (Z_tY_t^* + U_t\eta)'(\mu_t - r) + \tfrac{1}{2}\Gamma_t^G (Y_t^*)'\Sigma_R Y_t^* + \tfrac{1}{2}\Gamma_t^I \eta'\Sigma_R\eta + \Gamma_t^{GI}(Y_t^*)'\Sigma_R\eta. \tag{3.15}$$

In particular, when $\mu - r$ is a constant vector, F is adapted to $\mathbf{F}^{G,I}$.

Remark 3.4 *The lemma shows that a viable contract is linear, in the integration sense, with respect to G,I, their quadratic variation and covariations, and the quadratic variation $\langle Z\cdot G+U\cdot I\rangle$ of Agent's wealth $\bar{W}$. In particular, the linear contracts considered in BVW (2014) of the form, for some constants ϕ,χ, and ψ,*

$$dF_t^{BVW} = \phi dG_t - \chi dI_t + \psi dt \tag{3.16}$$

are viable, but our family of viable contracts include many more possible contracts.

Let us also note that we could have simply started by requiring that a viable contract is of the form (3.14) *for an arbitrary adapted process $\bar{H}_t$ (or* (3.4) *similarily). However, then, it wouldn't have been clear how to solve Agent's problem for arbitrary adapted processes Z,U,Γ^G, Γ^I, and Γ^{GI} satisfying the above conditions, and, more importantly, our approach shows why the contracts of the form* (3.14) *are as general as can be expected if Agent's problem can be solved by the martingale principle.*

3.3 Main results

Let us introduce some notation before stating the main results:

- The instantaneous variance of the index portfolio:

$$Var^\eta = \eta'\Sigma_R\eta.$$

- The instantaneous variance of the fund portfolio for the fund that invests Y in risky assets:

$$Var^Y = Y'\Sigma_R Y.$$

-The instantaneous covariance between the fund portfolio and the index portfolio:

$$Covar^{Y,\eta} = \eta'\Sigma_R Y.$$

- The CAPM beta of the fund portfolio:

$$\beta^Y = \tfrac{Covar^{Y,\eta}}{Var^\eta}.$$

3.3.1 Optimal contract and strategies

Given fixed asset prices, not necessarily in asset pricing equilibrium, we first state the results on the optimal contract and optimal strategies. As we indicated above, if the shirking benefit is high enough, the optimal contract will have a term depending on the index returns, because the first best contract, which does not depend on I_t, would not prevent Agent from shirking. Moreover, since Principal can invest in the index, she cares about the portfolio's performance relative to the index. However, the index term, while alleviating the effect of moral hazard, may provide incentives to Agent to be too much exposed to the index risk. To counter those incentives, Principal will reward

Agent for returns in excess of a specific index fund, and will also reward him for the high variation of deviation from that index fund. As for the optimal asset holdings, Agent will invest not only in the usual Merton's fund, rather, he will add the index fund to the mixture. The two funds are not the same in our model, because the Merton's fund is not the market portfolio, due to some shares not being available for trading.

We state these results precisely in the following

Theorem 3.5 *Consider a financial market in which the vector of asset returns (per share) has a constant drift vector μ and constant covariance matrix Σ_R such that Σ_R is invertible and $\eta'\Sigma_R\eta > 0$. Assume that Principal can attain a higher value than V^u by hiring Agent, and Agent can attain a higher value than $\bar{V}^u$ by working for Principal. Then, one optimal strategy for Principal is not to invest in the index, and*

(a) The optimal contract in the viable class is given by

$$dF_t = Cdt + \tfrac{\rho}{\rho+\bar{\rho}}dG_t + \xi(dG_t - \beta^{Y^*}dI_t) + \tfrac{r}{2}\zeta d\langle G - \beta^{Y^*}I\rangle_t, \tag{3.17}$$

where G is the reported portfolio return process,

$$\begin{aligned}
\xi =& (b - \tfrac{\rho}{\rho+\bar{\rho}})_+,\\
\zeta =& (\rho+\bar{\rho})Z(1-Z)(b - \tfrac{\rho}{\rho+\bar{\rho}})_+,\\
C =& \tfrac{1}{2r}\tfrac{1}{\bar{\rho}}(\mu - r)'\Sigma_R^{-1}(\mu - r) - (ZY^* + U\eta)'(\mu - r)\\
& - \tfrac{r}{2}\zeta(Y^* - \beta^{Y^*}\eta)'\Sigma_R(Y^* - \beta^{Y^*}\eta) + \tfrac{r}{2}\bar{\rho}(ZY^* + U\eta)'\Sigma_R(ZY^* + U\eta),\\
Z =& \max\{b, \tfrac{\rho}{\rho+\bar{\rho}}\} = \tfrac{\rho}{\rho+\bar{\rho}} + (b - \tfrac{\rho}{\rho+\bar{\rho}})_+,\\
U =& -(b - \tfrac{\rho}{\rho+\bar{\rho}})_+\beta^{Y^*}.
\end{aligned} \tag{3.18}$$

(b) Agent's vector of optimal holdings is given by

$$Y^* = \frac{1}{r}\frac{1}{\mathscr{C}_b}\Sigma_R^{-1}(\mu - r) + \frac{1}{r}\Big(\frac{\rho+\bar{\rho}}{\rho\bar{\rho}}\frac{\mathscr{D}_b}{\mathscr{C}_b}\Big)\frac{\eta'(\mu - r)}{Var^{\eta}}\eta, \tag{3.19}$$

where

$$\begin{aligned}
\mathscr{D}_b &= (\rho+\bar{\rho})\big(b - \tfrac{\rho}{\rho+\bar{\rho}}\big)_+^2,\\
\mathscr{C}_b &= \tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}} + \mathscr{D}_b.
\end{aligned} \tag{3.20}$$

(c) Principal's value process V_t is of the form

$$V_t = V(W_t) = Ke^{-r\rho W_t}, \tag{3.21}$$

for an appropriate constant K.

(d) Agent's value process satisfies the linear SDE

$$d\bar{V}_t = \bar{V}_t\big[-r\bar{\rho}(ZY + U\eta)'(\gamma dB_t^p + \sigma dB_t^{\sigma}) + (\bar{\delta} - r)dt\big]. \tag{3.22}$$

3.4 Contract properties

We now further discuss the optimal contract properties.

1. **First best.** If $b \leq \frac{\rho}{\rho+\bar{\rho}}$, Principal can attain the first best utility, the one she would get if she was the one choosing portfolio holdings Y rather than Agent choosing them. In this case $\xi = \zeta = 0$ in (3.17), and Agent receives the fraction $\frac{\rho}{\rho+\bar{\rho}}$ of the reported portfolio return. That is, the optimal contract does not have quadratic variation term and is equal to

$$dF_t = \frac{\rho}{\rho+\bar{\rho}} dG_t.^7$$

The fraction $\frac{\rho}{\rho+\bar{\rho}}$ is the classical risk-sharing fraction of the wealth between two agents with CARA utilities. Moreover, in this case $\mathscr{D}_b = 0$, and there is no agency friction.

2. **Second best.** As mentioned above, in the optimal contract (3.17) the term $\frac{\rho}{\rho+\bar{\rho}} dG_t$ is the risk-sharing term that is the only incentive part of the contract in the first best case of no agency friction. The term $\xi(dG_t - \beta^{Y^*} dI_t)$ benchmarks the reported portfolio return against the portfolio that invests β^{Y^*} in the index. Note that ξ, when strictly positive, corresponds exactly to the difference between the minimal pay-per-performance sensitivity b that would prevent shirking and the first best sensitivity, We can think of $\beta^{Y^*} I$ as the index-based approximation of the portfolio with strategy Y^* which is optimal (in L_2 sense) among the approximations of the form cI_t for some constant c. We call this portfolio the *optimal benchmark portfolio*. Thus, $\xi(dG_t - \beta^{Y^*} dI_t)$ rewards Agent when the portfolio return is above the return of the optimal benchmark portfolio, and penalizes Agent when the portfolio return is below the return of the optimal benchmark portfolio. We emphasize that we endogenously obtain this benchmarking term, unlike much of the literature that assumes it exogenously.

   When $b \leq \frac{\rho}{\rho+\bar{\rho}}$, the quadratic variation and covariation parts of the contract are zero. However, when $b > \frac{\rho}{\rho+\bar{\rho}}$, there is a new quadratic variation term compared to BVW (2014). This new term provides additional incentives for aligning Agent's risk taking with Principal's objectives by rewarding the quadratic variation of the deviation $G - \beta^{Y^*} I$ from the optimal benchmark portfolio. Note that when $b > \frac{\rho}{\rho+\bar{\rho}}$ the sensitivity with respect to $\langle G - \beta^{\theta} I \rangle_t$ is $\frac{r}{2}\zeta$, which is positive, thus rewarding Agent for deviating from the optimal benchmark portfolio. Thus, the quadratic variation term rewards Agent for taking the specific risk of individual stocks in a sufficient amount, and not to have too high a bias for the systematic risk of the index in his portfolio. Thus, it is beneficial for investors to provide incentives for deviating from the index, as in our contract.

   When agency friction b increases, ξ increases, so as to make Agent to not employ the shirking action. As a result, the portfolio is benchmarked more heavily to the optimal benchmark

[7] When $\xi = \zeta = 0$, using, from (3.3), $Z = \frac{\rho}{\rho+\bar{\rho}}$ and $ZY^* + U\eta = \frac{1}{r\bar{\rho}} \Sigma_R^{-1}(\mu - r)$, we obtain $C = 0$.

portfolio. Dependence of ζ on the agency friction is demonstrated in Figure 1. When agency friction is small, ζ increases with respect to agency friction, so that Agent is increasingly awarded by taking specific risks. However, when agency friction is large, the benefits of taking that specific risk are lower, therefore ζ decreases with respect to agency friction, so that Agent is incentivized not to take as much specific risk.

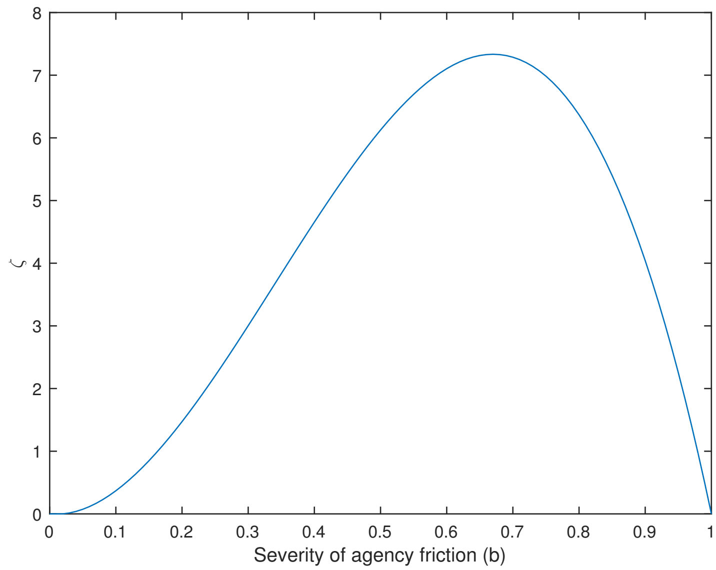

Figure 1: Sensitivity to quadratic variation

Finally, the quadratic variation term depends on the interest rate, but the profit sharing and benchmarking terms do not.

3. **Optimal fund holdings.** Note that $\Sigma_R^{-1}(\mu - r)$ is the vector of risk premia of the individual risky assets, and $\frac{\eta'(\mu-r)}{Var^{\eta}}$ is the risk premium of the index. Therefore, item (c) in Theorem 3.8 shows that Agent's optimal holding in asset i is a linear combination of the risk premium of asset i and the portion of the risk premium of the index corresponding to asset i. Put differently, as mentioned above, a two-fund theorem holds here, with Agent diversifying between the usual fund based on the risk-premia and an index fund (plus the risk-free asset). Moreover, when agency friction increases, the weight on the usual (first best) fund decreases, while the weight on the index fund increases, in agreement with the incentives against shirking being driven by the index fund. Thus, the model predicts that if the agency frictions are high, the fund managed by Agent will be more of a "closet indexer". In our model, though, this is not due to Agent having outside motives to follow the index more closely, rather, it's due to the contract with Principal providing him with incentives to do so, in order to prevent him from shirking.

3.4.1 Equilibrium prices

We will need the following assumption for the equilibrium result.

Assumption 3.6

(i) θ *and* η *are not linearly dependent.*

(ii) *Denote* $a_p = (a_{p1}, \dots, a_{pN})'$ *and* $a_e = diag\{a_{e1}, \dots, a_{eN}\}$*. For the values*

$$a_{pi} = \tfrac{a_i}{r+\kappa^p} \quad a_{ei} = \tfrac{1}{r+\kappa_i^e}, \quad i = 1, \dots, N, \tag{3.23}$$

the matrix

$$\Sigma_R = a_p \sigma_p^2 a_p' + a_e' \sigma_e^2 a_e \tag{3.24}$$

is invertible.

Remark 3.7 *Since Principal can invest in the index directly, if* $\theta = \alpha\eta$ *for some* $\alpha \in \mathbb{R}$*, then there exists an equilibrium in which Principal invests in* α *units of index directly without hiring Agent. Item (i) in the above assumption excludes this trivial case. Item (ii) ensures that the equilibrium we characterize is endogenously complete.*

The following is the main equilibrium result of the paper.

Theorem 3.8 *Suppose that Assumption 3.6 holds. Then, there exists an equilibrium in which Principal is indifferent with respect to the amount y invested by herself in the index (in particular, she may choose not to invest in the index directly, i.e., to set* $y = 0$*), in which asset prices are as in* (2.13)*, the volatility matrix is* Σ_R *in* (3.24)*, and:*[8]

(a) *Vectors* a_p *and* a_e *are given by* (3.23) *and vector* $a_0 = (a_{01}, \dots, a_{0N})'$ *is given by, with* $\mathscr{D}_b$ *given in* (3.20)*,*

$$a_0 = \tfrac{1}{r}\kappa^p \bar{p} a_p + \tfrac{1}{r}(\kappa^e)' \bar{e} a_e - \tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}} \Sigma_R \theta - \mathscr{D}_b \Sigma_R (\theta - \beta^\theta \eta), \tag{3.25}$$

(b) *The vector of asset excess returns is given by*

$$\mu - r = r\tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}} \Sigma_R \theta + r\mathscr{D}_b \Sigma_R (\theta - \beta^\theta \eta). \tag{3.26}$$

The index excess return is

$$\eta'(\mu - r) = r\tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}} Covar^{\theta,\eta}. \tag{3.27}$$

The excess return of Agent's portfolio is

$$\theta'(\mu - r) = r\tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}} Var^\theta + r\mathscr{D}_b \big(Var^\theta - \tfrac{(Covar^{\theta,\eta})^2}{Var^\eta}\big). \tag{3.28}$$

[8] The intuition why the equilibrium quantities do not depend on y is that, with the appropriate choice of the contract, Principal can get Agent to invest for her in the index in the exact amount that she would have, and this has no impact on asset prices.

(c) Principal offers optimally the contract that assigns to Agent the value

$$\bar{V}_0 = \bar{V}_0^u = -\exp\Big(1 - \frac{\bar{\delta}}{r} - \log(r\bar{\rho}) - \frac{1}{2r}(\mu - r)'\Sigma_R^{-1}(\mu - r)\Big),$$

that is the minimal value Agent would accept. With this choice, Principal is always willing to offer the contract. Moreover, Principal's value process is given by $V(W_t) = Ke^{-r\rho W_t}$ *where*

$$K = -\exp\Big(1 - \frac{\delta}{r} - \log(r\rho) + \frac{r}{2}\frac{\rho}{\bar{\rho}}(\mathscr{C}_b^2 - \bar{\rho}\mathscr{C}_b)Var^\theta + \frac{r}{2}\frac{\rho}{\bar{\rho}}(\mathscr{D}_b^2 - 2\mathscr{C}_b\mathscr{D}_b + \bar{\rho}\mathscr{D}_b)\frac{(Covar^{\theta,\eta})^2}{Var^\eta}\Big). \tag{3.29}$$

In this model with Gaussian dividends, the equilibrium volatility matrix is constant, and, in fact, the same as in Theorem 6.1 in BVW (2014). That is, the extra incentives coming from the quadratic (co)variations affect only Agent's impact on the excess returns, and not on the variance-covariance structure. We discuss in Conclusions the possibility of having stochastic equilibrium volatility in other models.

3.5 Equilibrium properties

1. **Price and returns distortion.** Note that $\mathscr{D}_b$ increases with b. We see then, from (3.26), that the risk premium of asset i increases (resp. decreases) with b when $\theta_i/\eta_i > \beta^\theta$ (resp. $\theta_i/\eta_i < \beta^\theta$). That is, whether the risk premium goes up or down with agency frictions depends on how large is the fund's relative holding θ_i/η_i of asset i compared to the CAPM beta of the fund. Thus, the stocks in large supply have high risk premia, and the stocks in low supply have low risk premia, and this effect is stronger as agency friction increases. As noted in BVW (2014), this is because the assets in high supply, for example, have to offer high premium for Agent to be willing to give them weight higher than their weight in the index. The price is distorted reversely. We see from (3.25) that the price of asset i decreases (resp. increases) with b when $\theta_i/\eta_i > \beta^\theta$ (resp. $\theta_i/\eta_i < \beta^\theta$). Therefore assets in large supply have lower prices and assets in low supply have higher price, and the effect is stronger as agency friction increases.

   The above is the same qualitative behavior as in BVW (2014). However, there is a quantitative difference. In BVW (2014), $\mathscr{D}_b$ is replaced by

   $$\mathscr{D}_b^{BVW} = \bar{\rho}(b - \frac{\rho}{\rho+\bar{\rho}})_+.$$

   Note that $\mathscr{D}_b < \mathscr{D}_b^{BVW}$ for any $b \in (0,1)$. Therefore, our price and returns distortions are less sensitive to agency friction than those in BVW (2014). Moreover, when agency friction is small, our sensitivities are of second order magnitude compared to the first order magnitude in BVW (2014). However, when $b = 1$, $\mathscr{D}_1$ and $\mathscr{D}_1^{BVW}$ are the same.

Let us now take the same parameters as in BVW (2014): $\rho = 1, \bar{\rho} = 50, r = 4\%, \kappa^p = \kappa_i^e = 10\%, N = 6, \eta_i = 1, \theta_1 = \theta_2 = \theta_3 = 0.7, \theta_4 = \theta_5 = \theta_6 = 0.3, a_i = 1, \bar{p} = 0.65, \bar{e}_i = 0.4, \sigma_p = 1, \frac{\sigma_p^2}{\bar{p}} = \frac{\sigma_{ei}^2}{\bar{e}_i}$, for $i = 1, \ldots, 6$. Figure 2 compares distortion of excess return in our equilibrium with the one in BVW (2014).

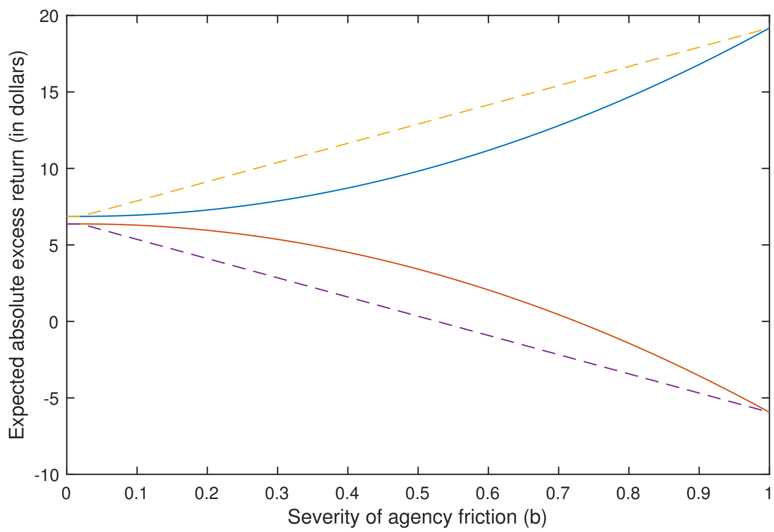

Figure 2: Expected excess return of the two groups of assets. Assets with large supply are in the top half, assets with low supply are in the bottom half. The results of this paper are presented in solid lines, the results in BVW (2014) are presented in dashed lines.

2. **Portfolio returns.** As we see from (3.27), agency friction does not have impact on index excess return. This is because Principal can trade the index privately. However, (3.28) indicates that excess return of Agent's portfolio depends on agency friction. Since $Var^{\theta} > \frac{(Covar^{\theta,\eta})^2}{Var^{\theta}}$ by Cauchy-Schwarz inequality, the excess return of Agent's portfolio increases with agency friction. This means the increase in return of large supply assets dominates the decrease in return of low supply assets. To give an intuition for this, note that the value-weighted market share of the assets in large supply becomes very high relative the share of the assets in low supply, because the former are not only in large supply to start with, but also more expensive in equilibrium. This asymmetry in the relative share of the two classes of assets makes it harder for Agent to trade against the high value of the larger class. Figure 3 demonstrates the excess return of Agent's portfolio in our equilibrium in comparison with BVW (2014). As with price distortions, the increase in excess return due to agency frictions is lower in our model, due to a more efficient contract.

   Since the excess return of index does not change, the increase in Agent's portfolio return implies that there is a decrease in the return of the portfolio held by buy-and-hold investors.

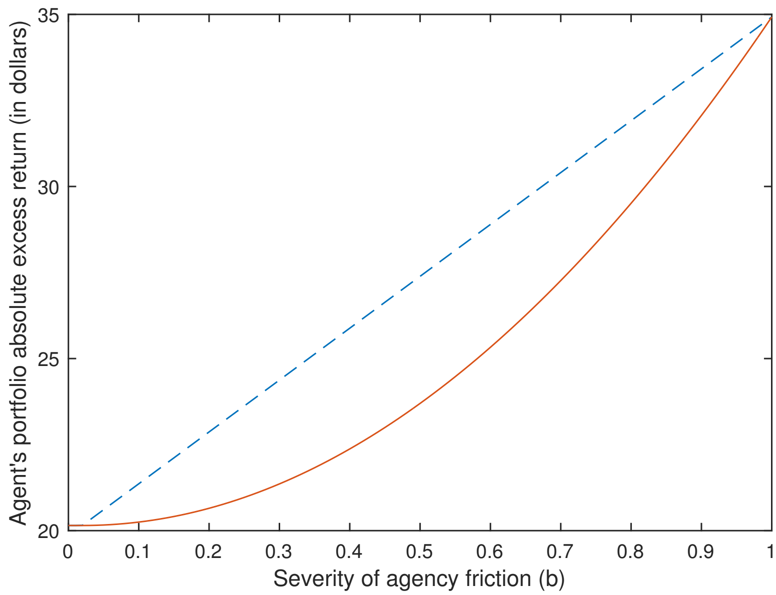

Figure 3: Expected excess return of Agent's portfolio. The result of this paper is presented as a solid line, the result in BVW is presented in dashed lines.

However, since $\mathscr{D}_b$ is lower in our model, this means that buy-and-hold investors lose less compared to BVW (2014).

3. **Contract.** In equilibrium $Y^* = \theta$, so that from (3.27) we get

$$\beta^\theta = \tfrac{1}{r}\tfrac{\rho+\bar{\rho}}{\rho\bar{\rho}}\tfrac{\eta'(\mu-r)}{Var^\eta}$$

This is recognized as the optimal portfolio holding in the index in the case in which Agent and Principal can invest only in the index and they share the risk in the first best situation. We call this portfolio the *index-sharing portfolio*. Thus, $\xi(dG_t - \beta^\theta dI_t)$ rewards Agent when the portfolio return is above the return of the index-sharing portfolio, and penalizes Agent when the portfolio return is below the return of the index-sharing portfolio; the term $\zeta d\langle G - \beta^\theta I\rangle_t$ rewards Agent for taking more of the specific risk of individual stocks, relative to the risk of the index-sharing portfolio. Note that this term makes the cumulative compensation F_t history dependent; in contrast to BVW (2014) or to other literature on the subject, with an exception of Leung (2016), the portfolio manager's current cumulative compensation depends on historical returns, and more precisely (in the natural discretized implementation of the model) on the sample variance of the portfolio's deviations from the index sharing portfolio; and we get this term endogenously, as a part of the optimal contract.

4. **Principal's optimal value.** Given the excess return in BVW (2014) equal to

$$\mu - r = r\bar{\rho}\Sigma_R(Z\theta + U\eta),$$

and the same parameters as in BVW (2014) (see Figure 2), Figure 4 demonstrates that Principal's certainty equivalence could be improved substantially when the contract (3.17) is employed compared to (3.16). However, under this new contract, holding the residual demand to clear the market is no longer optimal for Agent. Therefore, the equilibrium in BVW (2014) fails to be an equilibrium when Principal is allowed to choose contracts from our viable class.

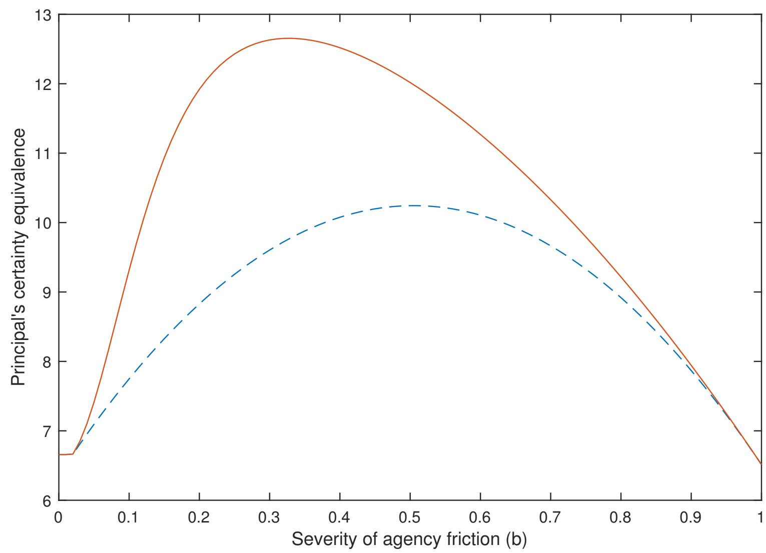

Figure 4: Certainty equivalence of Principal when asset prices are as in BVW (2014). Solid line represents Principal's certainty equivalence when contract (3.17) is implemented, dashed line represents Principal's certainty equivalence when the contract in BVW (2014) is implemented.

4 Extensions: Agent can invest privately

In this section, we extend the baseline model in Section 2 to two cases in which Agent is allowed to invest privately.

4.1 Private investment in individual risky assets

In this section, Agent is allowed to trade any individual risky assets. However, neither his investment strategy nor his private wealth are observable by Principal. Therefore, similar to the baseline model, Principal can only use the contract variables G and I.

Given contract F, Agent's private wealth process follows

$$d\bar{W}_t = (r\bar{W}_t + bm_t - \bar{c}_t)dt + \bar{Y}_t' dR_t + dF_t, \tag{4.1}$$

where the vector valued process $\bar{Y}$ represents the number of shares that Agent invests in each risky asset privately. This investment process is assumed to be adapted to $\mathbf{F}$ and to satisfy $\int_0^t |\bar{Y}_s|^2 ds < \infty$ for all $t \geq 0$. Principal's optimization problem is the same as before. The notion of equilibrium in Definition 2.1 is modified accordingly, so that in item (i) Agent's optimal investment strategy Y and $\bar{Y}$ satisfy $Y + \bar{Y} = \theta - y\eta$.

Similar to the baseline model, we restrict Principal's contracts to those for which Agent's optimization problem satisfies the martingale principle Rather than starting from the general viable class in Definition 3.1, we give the explicit representation F directly. The viable class of contracts Θ together with Agent's admissible strategies $\Xi(\Theta)$ can be introduced similarly.

Given $\mathbf{F}^{G,I}$-adapted processes $Z, U, \Gamma^G, \Gamma^I, \Gamma^{GI}$ such that $Z \geq b$, $\Gamma^G \neq 0$, and $\Gamma^G - \frac{1}{2} r\bar{\rho} Z^2 < 0$, consider a contract F whose dynamics follow

$$dF_t = Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma_t^I d\langle I\rangle_t + \Gamma_t^{GI} d\langle G, I\rangle_t - \bar{H}(Z_t, U_t, \Gamma_t^G, \Gamma_t^I, \Gamma^{GI})_t dt, \quad (4.2)$$

where

$$\begin{aligned} &\bar{H}(Z, U, \Gamma^G, \Gamma^I, \Gamma^{GI}) \\ &= \tfrac{\bar{\delta}}{r\bar{\rho}} + \tfrac{1}{\bar{\rho}} \log(-r\bar{\rho}\bar{V}_0) - \tfrac{1}{\bar{\rho}} \\ &\quad + \sup_{Y, \bar{Y}} \Big\{ (ZY + U\eta + \bar{Y})'(\mu - r) + \tfrac{1}{2}\Gamma^G Y'\Sigma_R Y + \tfrac{1}{2}\Gamma^I \eta'\Sigma_R \eta + \Gamma^{GI} Y'\Sigma_R \eta \\ &\qquad\qquad - \tfrac{1}{2} r\bar{\rho}(ZY + U\eta + \bar{Y})'\Sigma_R(ZY + U\eta + \bar{Y}) \Big\}. \end{aligned} \quad (4.3)$$

The optimizers of $\bar{H}$ are

$$Y^* = -\frac{\Gamma^{GI}}{\Gamma^G}\eta \quad \text{and} \quad \bar{Y}^* = \frac{1}{r\bar{\rho}}\Sigma_R^{-1}(\mu - r) + \frac{Z\Gamma^{GI} - U\Gamma^G}{\Gamma^G}\eta. \quad (4.4)$$

Therefore, Agent's candidate optimal strategy is given by $\Xi^* = (\bar{c}^*, m^*, Y^*, \bar{Y}^*)$, where $m^* = 0$ and $\bar{c}^* = \left(u_A'\right)^{-1}(-r\bar{\rho}\bar{V}_0 e^{-r\bar{\rho}\bar{W}})$ for some constant $\bar{V}_0 < 0$. Similarly to Lemmas 3.2 and 3.3, the following result holds.

Lemma 4.1 *Suppose that Σ_R is invertible, and the transversality condition* (2.7) *is satisfied when Agent employs the strategy Ξ^*. Then Ξ^* is Agent's optimal strategy in $\Xi(\Theta)$ and $\bar{V}_0$ is Agent's optimal value at time* 0.

Agent's optimal strategy Y^* in (4.4) is such that Principal gets access only to index investment through contracting. Since Principal can invest in the index by herself, Principal is indifferent with respect to not hiring Agent or hiring Agent with any contract of type (4.2). In this case, there exists an equilibrium in which both Agent and Principal invest only privately, as stated next.

Theorem 4.2 *Suppose that Assumption 3.6 holds. Then Principal's index investment and the optimal contract in the viable class are given by y^* and (4.2) for any $\mathbf{F}^{GI}$-adapted processes $y^*, Z, U, \Gamma^G, \Gamma^I, \Gamma^{GI}$ satisfying $Z \geq b$, $\Gamma^G \neq 0$, $\Gamma^G - \frac{1}{2}r\bar{\rho}Z^2 < 0$, and*

$$(Z-1)\frac{\Gamma^{GI}}{\Gamma^G} - U + y^* = \frac{1}{r\rho}\frac{\eta'(\mu-r)}{\eta'\Sigma_R\eta}. \tag{4.5}$$

In particular, Principal can choose $\Gamma^{GI} = U = 0$, $Z = 1$, and $y^* = \frac{1}{r\rho}\frac{\eta'(\mu-r)}{\eta'\Sigma_R\eta}$. Then, Agent does not invest for Principal, y^* and $Y^* = \frac{1}{r\bar{\rho}}\Sigma_R^{-1}(\mu-r)$ are the optimal investment strategies corresponding to Principal's and Agent's outside options. Moreover, any of the above contracts are optimal in the equilibrium in which

$$\frac{1}{r\bar{\rho}}\Sigma_R^{-1}(\mu-r) + \frac{1}{r\rho}\frac{\eta'(\mu-r)}{\eta'\Sigma_R\eta}\eta = \theta \quad \text{and} \quad \mu - r = r\bar{\rho}\Sigma_R\Big(\theta - \frac{\bar{\rho}}{\rho+\bar{\rho}}\beta^\theta\eta\Big).$$

Intuitively, unlike in our benchmark model, here Agent can benefit fully from the individual risky assets by investing on his own, and there are no gains from contracting.

4.2 Private investment in the index

Theorem 4.2 shows that there is no benefit in contracting for Principal when Agent is allowed to invest in individual assets privately. We now restrict Agent's private investment opportunities to the index. Moreover, Principal also requires Agent to report his private investment value. Agent may misreport it, but we show that Principal can offer a contract to incentivize truth-telling and restore the second best case of the baseline model in Section 2.

When Agent is only allowed to invest in the index, his investment strategy can be described by a $\mathbf{F}$-adapted scalar process $\bar{y}$. We introduce Agent's reported portfolio value by

$$\bar{G}_t = \int_0^t [\bar{y}_s\eta' dR_s - \bar{m}_s ds],$$

where $\bar{m}$ denotes Agent's misreporting. Unlike the shirking action affecting Principal's portfolio, misreporting action $\bar{m}$ is allowed to be either negative or positive. When $\bar{m}$ is negative (resp. positive), Agent's reported portfolio value is less (resp. more) than the true value. The under-reporting (resp. over-reporting) benefits (hurts) Agent's private wealth process that follows

$$d\bar{W}_t = (r\bar{W}_t + bm_t + \bar{m}_t - \bar{c}_t)dt + d\bar{G}_t + dF_t.$$

We assume that Principal can contract on $\bar{G}$.

Similar to the previous section, we will introduce a class of contracts which are derived from the martingale principle. The corresponding viable class can be defined similarly. Given $\mathbf{F}^{G,\bar{G},I}$-adapted processes $Z, U, \Gamma's$ such that $Z \geq b$, $\Gamma^G < 0$, and $\Gamma^G\Gamma^{\bar{G}} - (\Gamma^{G\bar{G}}) > 0$, consider the contract

$$\begin{aligned} dF_t =& Z_t dG_t + U_t dI_t \\ &+ \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma_t^{\bar{G}} d\langle \bar{G}\rangle_t + \tfrac{1}{2}\Gamma_t^I d\langle I\rangle_t + \Gamma_t^{GI} d\langle G, I\rangle_t + \Gamma_t^{G\bar{G}} d\langle G, \bar{G}\rangle_t + \Gamma_t^{\bar{G}I} d\langle \bar{G}, I\rangle_t \\ &+ \tfrac{1}{2}r\bar{\rho}\, d\langle Z\cdot G + U\cdot I + \bar{G}\rangle_t - \bar{H}_t dt, \end{aligned} \tag{4.6}$$

where

$$
\begin{aligned}
\bar{H} =& \frac{\bar{\delta}}{r\bar{\rho}} + \frac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) - \frac{1}{\bar{\rho}} \\
&+ \sup_{Y,\bar{y}} \Big\{ (ZY + U\eta + \bar{y}\eta)'(\mu - r) + \tfrac{1}{2}\Gamma^G Y'\Sigma_R Y + \tfrac{1}{2}\Gamma^{\bar{G}}\bar{y}^2\eta'\Sigma_R\eta + \tfrac{1}{2}\Gamma^I\eta'\Sigma_R\eta \\
&\quad + \Gamma^{GI}Y'\Sigma_R\eta + \Gamma^{G\bar{G}}\bar{y}Y'\Sigma_R\eta + \Gamma^{\bar{G}I}\bar{y}\eta'\Sigma_R\eta \Big\}. \qquad (4.7)
\end{aligned}
$$

Observe that F in (4.6) is invariant under different $\bar{m}$. Therefore, Agent is indifferent between truth-telling and misreporting. Therefore, we assume Agent reports his portfolio value truthfully, i.e., $\bar{m}^* = 0$. The optimizers of $\bar{H}$ are

$$
Y^* = -\frac{Z}{\Gamma^G}\Sigma_R^{-1}(\mu - r) - \frac{\Gamma^{GI}}{\Gamma^G}\eta - \frac{\Gamma^{G\bar{G}}}{\Gamma^G}\bar{y}^*\eta, \qquad (4.8)
$$

$$
\bar{y}^* = \frac{Z\Gamma^{G\bar{G}} - \Gamma^G}{\Gamma^{\bar{G}}\Gamma^G - (\Gamma^{G\bar{G}})^2}\frac{\eta'(\mu - r)}{\eta'\Sigma_R\eta} + \frac{\Gamma^{GI}\Gamma^{G\bar{G}} - \Gamma^G\Gamma^{\bar{G}I}}{\Gamma^G\Gamma^{\bar{G}} - (\Gamma^{G\bar{G}})^2}. \qquad (4.9)
$$

Agent's candidate optimal strategy is then $\Xi^* = (\bar{c}^*, m^*, \bar{m}^*, Y^*, \bar{y}^*)$ with $\bar{c}^*$ and $\bar{m}^*$ the same as in the previous section. Similar to Lemmas 3.2, 3.3, the same statements of 4.1 hold in this case, and in particular, Ξ^* is Agent's optimal strategy.

For the contract in (4.6), there exists a particular choice of sensitivity processes Γ's such that it is optimal for Agent and Principal not to invest in the index. Therefore, the equilibrium is the same as in the baseline case.

Theorem 4.3 *Suppose that Assumption 3.6 holds. Then, there exists an equilibrium in which statements (a)-(c), (e), and (f) in Theorem 3.8 hold. Moreover, both Agent and Principal do not invest in the index. The optimal contract in the viable class is*

$$
\begin{aligned}
dF_t =& Cdt + \tfrac{\rho}{\rho+\bar{\rho}}dG_t + \xi(dG_t - \beta^{Y^*}dI_t) + \tfrac{r}{2}\zeta d\langle G - \beta^{Y^*}I\rangle_t \\
&+ \tfrac{1}{2}(\Gamma^{\bar{G}} + r\bar{\rho})d\langle \bar{G}\rangle_t + (\Gamma^{G\bar{G}} + r\bar{\rho}Z)d\langle G, \bar{G}\rangle_t + (\Gamma^{\bar{G}I} + r\bar{\rho}U)d\langle \bar{G}, I\rangle_t,
\end{aligned} \qquad (4.10)
$$

where C from (3.18), *Z, U, ξ, ζ are the same as in* (3.18), *and*

$$
\Gamma^{\bar{G}} > \tfrac{1}{Z^2}\Gamma^G, \quad \Gamma^{G\bar{G}} = \tfrac{1-Z}{Z}\Gamma^G, \quad \Gamma^{\bar{G}I} = \tfrac{1}{Z}\Gamma^{GI}, \quad \Gamma^G = -rZ\mathscr{C}_b, \quad \Gamma^{GI} = rZ\mathscr{D}_b\beta^{Y^*}.
$$

5 Conclusions

We find equilibrium asset prices in a model with OU dynamics for the dividend processes, in a market in which CARA investors hire CARA portfolio managers. The optimal contract involves rewarding the manager for return in excess of a benchmark portfolio value, and for quadratic deviation thereof. The latter provides incentives to Agent to take on specific risk of individual stocks. We find that the stocks in large supply have high risk premia, and the stocks in low supply have low risk premia, and this effect is stronger as agency friction increases. However, this effect is of a lower order of magnitude than when only the contracts without the quadratic variation terms are allowed, as in BVW (2014). Therefore, introducing the quadratic variation term in the contract mitigates the price/return distortion of asset prices in equilibrium. It would be of interest to study, in the future, the problem with dividends modeled as square-root processes, in which case the contract terms would change with the state of the economy, and the volatility would also depend on the agency frictions in equilibrium. This would require numerically solving the corresponding HJB equations.

6 Proofs

6.1 Properties of $\bar{W}_t$

The following lemma provides two properties of the continuation value in our setting.

Lemma 6.1 *For any $t \geq 0$ and admissible Ξ,*

(i) $\partial_{\bar{W}_t} \bar{\mathscr{V}}_t(\Xi) = -r\bar{\rho}\bar{\mathscr{V}}_t(\Xi)$*;*

(ii) $\lim_{t\to\infty} \mathbb{E}\big[e^{-\bar{\delta}t}\bar{\mathscr{V}}_t(\Xi)\big] = 0$.

Proof: We denote $\bar{\mathscr{V}}_t(\Xi)$ by $\bar{\mathscr{V}}_t(\bar{W}_t)$ to emphasize its dependence on $\bar{W}_t$. Let $(\bar{c}'_s)_{s\geq t}$ be Agent's optimal consumption stream from t onwards. Note that $\bar{c}'$ is financed by a wealth process starting from $\bar{W}_t$ at time t. Therefore $\bar{c}' - r\bar{W}_t$ can be financed by a wealth process starting from 0 at time t. Agent's exponential utility function implies that

$$\bar{\mathscr{V}}_t(0) \geq e^{r\bar{\rho}\bar{W}_t}\bar{\mathscr{V}}_t(\bar{W}_t).$$

Above inequality is in fact an equality, i.e., $\bar{c}' - r\bar{W}_t$ is optimal for $\bar{\mathscr{V}}_t(0)$. Assuming otherwise, there exists another consumption stream $\bar{c}''$ whose associated value is strictly larger than $e^{r\bar{\rho}\bar{W}_t}\bar{\mathscr{V}}_t(\bar{W}_t)$. Since $\bar{c}''$ is financed by a wealth process starting from 0 at time t, $\bar{c}'' + r\bar{W}_t$ can be financed by a wealth process starting from $\bar{W}_t$ at time t. Moreover, the expected utility associated to $\bar{c}'' + r\bar{W}_t$ is strictly larger than $\bar{\mathscr{V}}_t(\bar{W}_t)$, contradicting the optimality of $\bar{c}'$ for $\bar{\mathscr{V}}_t(\bar{W}_t)$. Therefore, $\bar{\mathscr{V}}_t(\bar{W}_t) = e^{-r\bar{\rho}\bar{W}_t}\bar{\mathscr{V}}_t(0)$, confirming item (i).

For item (ii), definition of $\bar{\mathscr{V}}_t(\Xi)$ yields

$$\mathbb{E}\big[e^{-\bar{\delta}t}\bar{\mathscr{V}}_t(\Xi)\big] = \mathbb{E}\Big[\int_t^\infty e^{-\bar{\delta}s}u_A(\bar{c}^t_s)ds\Big],$$

where $\bar{c}^t$ is Agent's optimal consumption stream from t onwards. Then, item (ii) follows from applying the monotone convergence theorem on the right-hand side.

6.2 Proof of Lemma 3.2

First order condition for $\bar{c}$ and Y in (3.8) gives

$$u_A'(\bar{c}) = X \quad \text{and} \quad \Gamma^G \Sigma_R Y = -Z(\mu - r) - \Gamma^{GI} \Sigma_R \eta.$$

Since the optimization problem on the right-hand side of (3.8) is concave in $\bar{c}$ and Y, and $Z \geq b$, we have that $\Xi^* = (\bar{c}^*, m^*, Y^*)$ in (3.11) is the optimizer for H.

For an arbitrary Agent's admissible strategy $\Xi = (\bar{c}, m, Y)$, consider the process

$$\tilde{V}_t(\Xi) = \int_0^t e^{-\bar{\delta}s} u_A(\bar{c}_s) ds + e^{-\bar{\delta}t} \bar{V}_t(\Xi), \quad t \geq 0,$$

where $\bar{V}(\Xi)$ is defined via (3.9). The definition of H in (3.8) implies that $\tilde{V}(\Xi)$ is a local supermartingale. Taking a localizing sequence $\{\tau_n\}_n$ for this local supermartingale and arbitrary $T \in \mathbb{R}$, we obtain

$$\mathbb{E}\Big[\int_0^{T\wedge\tau_n} e^{-\bar{\delta}s} u_A(\bar{c}_s) ds\Big] + \mathbb{E}\Big[e^{-\bar{\delta}(T\wedge\tau_n)} \bar{V}_{T\wedge\tau_n}(\Xi)\Big] = \mathbb{E}[\tilde{V}_{T\wedge\tau_n}(\Xi)] \leq \tilde{V}_0(\Xi) = \bar{V}_0. \tag{6.1}$$

Sending n, and then T to infinity, applying the monotone convergence theorem to the first term on the left-hand side, and (3.10) to the second term, we obtain

$$\mathbb{E}\Big[\int_0^{\infty} e^{-\bar{\delta}s} u_A(\bar{c}_s) ds\Big] \leq \bar{V}_0.$$

For strategy $\Xi^* = (\bar{c}^*, m^*, Y^*)$, $\tilde{V}$ is a local martingale. Then, the inequality in (6.1) is an equality. Sending n, and then T to infinity and using the transversality condition for $\bar{V}(\Xi^*)$, optimality of Ξ^* is confirmed. Thus, $\bar{V}_0$ is Agent's optimal value at time 0. A similar argument works for $\bar{V}_t$.

6.3 Proof of Lemma 3.3

Introduce $\hat{V}_t = \bar{V}_0 e^{-r\bar{\rho}\bar{W}_t}$, where $\bar{W}$ follows (2.5) with F in (3.14). We claim that $\hat{V} = \bar{V}(\Xi)$. Therefore, when Agent is offered the contract F in (3.14), and with everything else remaining the same, his continuation value satisfies the viability condition (3.9). To prove the claim notice first that Hamiltonian H in (3.8) can be written as

$$H = X\Big[\tfrac{1}{\bar{\rho}}\log(X) - \tfrac{1}{\bar{\rho}} + (ZY^* + U\eta)'(\mu - r) + \tfrac{1}{2}\Gamma^G (Y^*)'\Sigma_R Y^* + \tfrac{1}{2}\Gamma^I \eta'\Sigma_R \eta + \Gamma^{GI}(Y^*)'\Sigma_R \eta\Big]. \tag{6.2}$$

Next, we also notice that SDE (3.9) for $\bar{V}(\Xi)$ has locally Lipschitz coefficients on $(-\infty, 0)$, hence it admits a unique strong solution before the solution hitting either $-\infty$ or 0. On the other hand, applying Itô's formula to $\hat{V}$, we have

$$
\begin{aligned}
d\hat{V}_t =& X_t\Big[(r\bar{W}_t + bm_t - \bar{c}_t)dt + dF_t\Big] - \tfrac{1}{2}r\bar{\rho}X_t d\langle Z\cdot G + U\cdot I\rangle_t \\
=& X_t\Big[(r\bar{W}_t + bm_t - \bar{c}_t)dt + Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma^I d\langle I\rangle_t + \Gamma_t^{GI} d\langle G, I\rangle_t - \bar{H}_t\Big] \\
=& \bar{\delta}\hat{V}_t + X_t\Big[dP_t + Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma_t^I d\langle I\rangle_t + \Gamma_t^{GI} d\langle G, I\rangle_t - (-r\bar{W}_t + \bar{H}_t - \tfrac{\bar{\delta}}{r\bar{\rho}})\Big] \\
=& \bar{\delta}\hat{V}_t + X_t\Big[dP_t + Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma_t^I \langle I\rangle_t + \Gamma_t^{GI} d\langle G, I\rangle_t\Big] - H_t dt,
\end{aligned}
$$

where $X_t = -r\bar{\rho}\hat{V}_t$ and the fourth identity follows from $-r\bar{W}_t + \frac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) = \frac{1}{\bar{\rho}}\log(-r\bar{\rho}\hat{V}_t) = \frac{1}{\bar{\rho}}\log(X)$ and (6.2). Thus, $\hat{V}$ satisfies (3.9) and it does not hit $-\infty$ or 0 in finite time, since $\bar{W}$ does not hit $-\infty$ nor ∞ in finite time. Therefore, $\hat{V}$ is the unique solution of (3.9).

6.4 Proof of Theorem 3.5

6.4.1 Step 1: Preparation

Given $Z, U, \Gamma^G, \Gamma^{GI}$ satisfying $Z \geq b$ and $\Gamma^G < 0$, we denote

$$
\bar{U} = \tfrac{U}{Z}, \quad \bar{\Gamma}^G = -\tfrac{Z}{\Gamma^G}, \quad \text{and} \quad \bar{\Gamma}^{GI} = -\tfrac{\Gamma^{GI}}{\Gamma^G}. \tag{6.3}
$$

Then, Agent's optimal strategy Ξ^* from (3.11) is

$$
\bar{c}^* = (u_A')^{-1}(-r\bar{\rho}\bar{V}(\Xi^*)), \quad m^* = 0, \quad Y^* = \bar{\Gamma}^G\alpha + \bar{\Gamma}^{GI}\eta, \tag{6.4}
$$

where $\alpha = \{\alpha_t\}_{t\geq 0}$ with $\alpha_t = \Sigma_R^{-1}(\mu_t - r)$.

When Agent employs the optimal strategy Ξ^*, using (3.14) and (3.15), we see that the contract takes the form

$$
dF_t^* = -\Big(\tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{\bar{\delta}}{r\bar{\rho}} - \tfrac{1}{\bar{\rho}} - \tfrac{1}{2}r\bar{\rho}Z_t^2(Y_t^* + \bar{U}_t\eta)'\Sigma_R(Y_t^* + \bar{U}_t\eta)\Big)dt + Z_t(Y_t^* + \bar{U}_t\eta)'(\gamma dB_t^p + \sigma dB_t^e).
$$

Then, Principal's wealth process (3.12) follows

$$
\begin{aligned}
dW_t =& \Big(rW_t - c_t + \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{\bar{\delta}}{r\bar{\rho}} - \tfrac{1}{\bar{\rho}} - \tfrac{1}{2}r\bar{\rho}Z_t^2(Y_t^* + \bar{U}_t\eta)'\Sigma_R(Y_t^* + \bar{U}_t\eta)\Big)dt \\
&+ (Y_t^* + y_t\eta)'dR_t - Z_t(Y_t^* + \bar{U}_t\eta)'(\gamma dB_t^p + \sigma dB_t^e).
\end{aligned} \tag{6.5}
$$

6.4.2 Step 2: Principal's HJB equation

For constant μ, we conjecture that Principal's value function is given as

$$
V(w) = Ke^{-r\rho w}, \tag{6.6}
$$

for some constant $K < 0$. This value function is expected to satisfy the following HJB equation

$$\begin{aligned}\delta V = \sup_{Z\geq b,\bar{U},\bar{\Gamma}^G,\bar{\Gamma}^{GI},c,y} &\Big\{u_p(c) + V_w\Big[rw - c + \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{\bar{\delta}}{r\bar{\rho}} - \tfrac{1}{\bar{\rho}}\Big] \\ &+ V_w\Big[(Y^* + y\eta)'(\mu - r) - \tfrac{1}{2}r\bar{\rho}Z^2(Y^* + \bar{U}\eta)'\Sigma_R(Y^* + \bar{U}\eta)\Big] \\ &+ \tfrac{1}{2}V_{ww}\big[(Y^* + y\eta) - Z(Y^* + \bar{U}\eta)\big]'\Sigma_R\big[(Y^* + y\eta) - Z(Y^* + \bar{U}\eta)\big]\Big\}.\end{aligned} \tag{6.7}$$

Note that $Y^* + y\eta = \bar{\Gamma}^G\alpha + (\bar{\Gamma}^{GI} + y)\eta$ and $Y^* + \bar{U}\eta = \bar{\Gamma}^G\alpha + (\bar{\Gamma}^{GI} + \bar{U})\eta$. Therefore, instead of optimizing over $\bar{U}, \bar{\Gamma}^{GI}$, and y individually, we can optimize over $\bar{\Gamma}^{GI} + y, \bar{U} - y$, and still obtain the same maximum value. This means that Principal is indifferent with respect to which amount y to invest in the index. For notational simplicity, we assume Principal chooses

$$y = 0. \tag{6.8}$$

The maximizer of c in (6.7) is

$$c = (u_P')^{-1}(V_w). \tag{6.9}$$

Plugging (6.6), (6.8), and (6.9) back into (6.7), and taking into account that $K < 0$, we reduce (6.7) to

$$\begin{aligned}r - \delta = \sup_{Z\geq b,\bar{U},\bar{\Gamma}^{GI},\bar{\Gamma}^G} &\Big\{\tfrac{\rho}{\bar{\rho}}(\bar{\delta} - r) + r\rho\Big[(Y^*)'(\mu - r) + \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{1}{\rho}\log(-r\rho K)\Big] \\ &- \tfrac{1}{2}r^2\rho^2\big[(Y^*)'\Sigma_R Y^*\big] + r^2\rho^2 Z\big[(Y^* + \bar{U}\eta)'\Sigma_R Y^*\big] \\ &- \tfrac{1}{2}r^2\rho(\bar{\rho} + \rho)Z^2\big[(Y^* + \bar{U}\eta)'\Sigma_R(Y^* + \bar{U}\eta)\big]\Big\}.\end{aligned} \tag{6.10}$$

The first order condition of optimality for $\bar{U}$ in (6.10) yields

$$(\rho + \bar{\rho})Z[\eta'\Sigma_R\eta]\bar{U} = [\rho - (\rho + \bar{\rho})Z][\eta'\Sigma_R Y^*].$$

Since we assume that $\eta'\Sigma_R\eta > 0$, the concavity in $\bar{U}$ of the maximization problem in (6.10) implies that the maximizer in $\bar{U}$ is

$$\bar{U} = \frac{\rho - (\rho + \bar{\rho})Z}{(\rho + \bar{\rho})Z}\frac{Covar^{Y^*,\eta}}{Var^{\eta}}. \tag{6.11}$$

Using (6.4), the first order condition for $\bar{\Gamma}^{GI}$ in (6.10) is

$$0 = \eta'(\mu - r) - r\mathscr{C}_b\big[Covar^{\alpha,\eta}\bar{\Gamma}^G + Var^{\eta}\bar{\Gamma}^{GI}\big] + rZ(\rho - (\rho + \bar{\rho})Z)Var^{\eta}\bar{U}. \tag{6.12}$$

Plugging in (6.11) for $\bar{U}$, the previous equation is transformed into

$$0 = \eta'(\mu - r) + r(\mathscr{D}_b - \mathscr{C}_b)\big[Covar^{\alpha,\eta}\bar{\Gamma}^G + Var^{\eta}\bar{\Gamma}^{GI}\big], \tag{6.13}$$

where

$$\mathscr{C}_b = \rho(1-Z)^2 + \bar{\rho}Z^2 \quad \text{and} \quad \mathscr{D}_b = \frac{(\rho - (\rho+\bar{\rho})Z)^2}{\rho+\bar{\rho}}. \tag{6.14}$$

Similarly, the first order condition for $\bar{\Gamma}^G$ in (6.10) is

$$0 = \alpha'(\mu - r) - r\mathscr{C}_b\big[Var^{\alpha}\bar{\Gamma}^G + Covar^{\eta,\alpha}\bar{\Gamma}^{GI}\big] + rZ(\rho - (\rho+\bar{\rho})Z)Covar^{\eta,\alpha}\bar{U}.$$

Plugging in the expression (6.11) for $\bar{U}$, the previous equation is transformed into

$$0 = \alpha'(\mu - r) + r(\mathscr{D}_b - \mathscr{C}_b)Covar^{\eta,\alpha}\bar{\Gamma}^{GI} + r\left[\mathscr{D}_b\frac{(Covar^{\eta,\alpha})^2}{Var^{\eta}} - \mathscr{C}_bVar^{\alpha}\right]\bar{\Gamma}^G. \tag{6.15}$$

Solving (6.13) and (6.15) for $\bar{\Gamma}^G$ and $\bar{\Gamma}^{GI}$, and using

$$Covar^{\eta,\alpha} = \eta'\Sigma_R\alpha = \eta'\Sigma_R\Sigma_R^{-1}(\mu - r) = \eta'(\mu - r),$$

$$Var^{\alpha} = \alpha'\Sigma_R\alpha = \alpha'(\mu - r),$$

we obtain

$$\bar{\Gamma}^G = \frac{1}{r\mathscr{C}_b}, \tag{6.16}$$

$$\bar{\Gamma}^{GI} = \frac{\mathscr{D}_b}{r\mathscr{C}_b(\mathscr{C}_b - \mathscr{D}_b)}\frac{\eta'(\mu - r)}{Var^{\eta}}. \tag{6.17}$$

On the right-hand side of (6.10), the function to be maximized tends to negative infinity when either $|\bar{\Gamma}^G| \to \infty$ or $|\bar{\Gamma}^{GI}| \to \infty$. Therefore, $\bar{\Gamma}^G$ and $\bar{\Gamma}^{GI}$ obtained in (6.16) and (6.17) are the maximizers for the maximization problem in (6.10). Moreover, since $\mu - r$ is a constant vector, $Y^* = \bar{\Gamma}^G\alpha + \bar{\Gamma}^{GI}\eta$ is a constant vector as well.

Another form of $\bar{\Gamma}^{GI}$ that will be useful later can be obtained by plugging (6.16) back into (6.12) and using (6.11). This gives

$$\bar{\Gamma}^{GI} = \frac{\mathscr{D}_b}{\mathscr{C}_b}\frac{Covar^{Y^*,\eta}}{Var^{\eta}}. \tag{6.18}$$

Finally, the unconstrained first order condition for Z in (6.10) gives

$$0 = \rho\big[(Y^* + \bar{U}\eta)'\Sigma_RY^*\big] - (\rho+\bar{\rho})Z\big[(Y^* + \bar{U}\eta)'\Sigma_R(Y^* + \bar{U}\eta)\big].$$

Plugging the expression of $\bar{U}$ from (6.11) into the previous equation, we can solve it and get $Z = \frac{\rho}{\rho+\bar{\rho}}$. Since the maximization problem in (6.7) is concave in Z, under the constraint $Z \geq b$ optimal Z is

$$Z = \max\left\{\tfrac{\rho}{\rho+\bar{\rho}}, b\right\}. \tag{6.19}$$

6.4.3 Step 3: Optimal contract

Plugging (6.11), (6.16), and (6.18) back to (6.3) yields

$$U = -(Z - \tfrac{\rho}{\rho+\bar\rho})\beta^{Y^*}, \quad \Gamma^G = -rZ\mathscr{C}_b, \quad \text{and} \quad \Gamma^{GI} = rZ\mathscr{D}_b\beta^{Y^*}, \quad \text{where} \quad \beta^{Y^*} = \frac{Covar^{Y^*,\eta}}{Var^{\eta}}.$$

Combining the previous expressions with (3.14), we obtain

$$ZdG_t + U_t dI_t = \tfrac{\rho}{\rho+\bar\rho}dG_t + \xi(dG_t - \beta^{Y^*}dI_t),$$
$$\tfrac{1}{2}\Gamma^G d\langle G\rangle_t + \Gamma^{GI}d\langle G,I\rangle_t + \tfrac{1}{2}r\bar\rho Z^2 d\langle G+\bar U I\rangle_t = \tfrac{r}{2}\zeta\left[d\langle G\rangle_t - 2d\langle G,\beta^{Y^*}I\rangle_t\right] + \tfrac{1}{2}r\bar\rho U^2 d\langle I\rangle_t,$$

where

$$\xi = (b - \tfrac{\rho}{\rho+\bar\rho})_+ \quad \text{and} \quad \zeta = (\rho+\bar\rho)Z(1-Z)(b - \tfrac{\rho}{\rho+\bar\rho})_+.$$

In order to have $\langle G - \beta^{Y^*}I\rangle_t$ instead of $\langle G\rangle_t - 2\langle G,\beta^{Y^*}I\rangle_t$ in the above expression, we introduce

$$\tfrac{1}{2}\Gamma^I = \tfrac{r}{2}\left[\xi - \bar\rho(Z - \tfrac{\rho}{\rho+\bar\rho})^2\right](\beta^{Y^*})^2.$$

Then,

$$\tfrac{1}{2}\Gamma^G d\langle G\rangle_t + \Gamma^{GI}d\langle G,I\rangle_t + \tfrac{1}{2}\Gamma^I d\langle I\rangle_t + \tfrac{1}{2}r\bar\rho Z^2 d\langle G+\bar U I\rangle_t = \tfrac{r}{2}\zeta d\langle G - \beta^{Y^*}I\rangle_t.$$

On the other hand,

$$\begin{aligned}\tfrac{\bar\delta}{r\bar\rho} + \bar H =& \tfrac{1}{\bar\rho}\log(-r\bar\rho\bar V_0) + \tfrac{\bar\delta}{r\bar\rho} - \tfrac{1}{\bar\rho} + (ZY^* + U\eta)'(\mu - r)\\ &+ \tfrac{r}{2}\zeta(Y^* - \beta^{Y^*}I)'\Sigma_R(Y^* - \beta^{Y^*}I) - \tfrac{r}{2}\bar\rho(ZY^* + U\eta)'\Sigma_R(ZY^* + U\eta).\end{aligned}$$

Collecting above results and combining them with (3.14), we obtain

$$dF_t = Cdt + \tfrac{\rho}{\rho+\bar\rho}dG_t + \xi(dG_t - \beta^{Y^*}dI_t) + \tfrac{r}{2}\zeta d\langle G - \beta^{Y^*}I\rangle_t, \tag{6.20}$$

where

$$\begin{aligned}C =& -\tfrac{1}{\bar\rho}\log(-r\bar\rho\bar V_0) - \tfrac{\bar\delta}{r\bar\rho} + \tfrac{1}{\bar\rho} - (ZY^* + U\eta)'(\mu - r)\\ &- \tfrac{r}{2}\zeta(Y^* - \beta^{Y^*}I)'\Sigma_R(Y^* - \beta^{Y^*}I) + \tfrac{r}{2}\bar\rho(ZY^* + U\eta)'\Sigma_R(ZY^* + U\eta).\end{aligned} \tag{6.21}$$

6.4.4 Step 4: Verifications

Let us verify that $\Theta = (c, F, y)$, defined by (6.9), (6.20), and (6.8), is viable. First, when Agent employs the strategy Ξ^* with constant Y^*, we have from (3.9), (3.8) that

$$d\bar V_t(\Xi^*) = -r\bar\rho\bar V_t(\Xi^*)[ZY^* + U\eta]'(\gamma dB_t^p + \sigma dB_t^e) + (\bar\delta - r)\bar V_t(\Xi^*)dt.$$

Therefore, $\bar V(\Xi^*)$ is given by

$$\bar V_t(\Xi^*) = \bar V_0 e^{(\bar\delta - r)t}\mathscr{E}\Big(-r\bar\rho\int_0^t (ZY^* + U\eta)'(\gamma dB_s^p + \sigma dB_s^\sigma)\Big).$$

We need to show that $\bar{V}(\Xi^*)$ satisfies the transversality condition (3.10). To this end, take any $T \in \mathbb{R}$ and any sequence of stopping times $\{\tau_n\}_n$ converging to infinity. Since Y^* is a constant vector, then Z and U are constants, therefore the above stochastic exponential is a martingale, hence the family

$$\left\{\mathscr{E}\Big(-r\bar{\rho}\int_0^{T\wedge\tau_n}(Z\theta+U\eta)'(\gamma dB_s^p+\sigma dB_s^\sigma)\Big)\right\}_n \qquad \text{is uniformly integrable in } n.$$

As a result,

$$\lim_{n\to\infty}\mathbb{E}\Big[e^{-\bar{\delta}T\wedge\tau_n}\bar{V}_{T\wedge\tau_n}(\Xi^*)\Big]=\bar{V}_0e^{-rT}\mathbb{E}\Big[\mathscr{E}\Big(-r\bar{\rho}\int_0^T(Z\theta+U\eta)'(\gamma dB_s^p+\sigma dB_s^\sigma)\Big)\Big]=\bar{V}_0e^{-rT},$$

which vanishes when $T\to\infty$. Therefore Lemma 3.2 shows that $\Xi^*=(\bar{c}^*,m^*,Y^*)$ in (3.11) is Agent's optimal strategy in $\Xi(\Theta)$.

Next we need to show that Θ is adapted to $\mathbf{F}^{G^*,I}$. Combining (6.6) and (6.9) yields

$$c=-\tfrac{1}{\rho}\log(-r\rho K)+rW.$$

Plugging this expression for c into (3.12) we obtain

$$dW_t=\tfrac{1}{\rho}\log(-r\rho K)dt+dG_t^*-dF^*.$$

We have seen in Lemma 3.3 that F^* is adapted to $\mathbf{F}^{G^*,I}$, thus W and c are adapted to the same filtration.

It remains to check the transversality condition (3.13) is satisfied. To this end, applying Itô's formula to $V_t=V(W_t)$, and using (6.7) and (6.9), we obtain

$$dV_t=(\delta-r)V_tdt-r\rho V_t[Y^*-(ZY^*+U\eta)]'[\gamma dB_t^p+\sigma dB_t^e].$$

Therefore

$$V_t=V_0e^{(\delta-r)t}\mathscr{E}\Big(-r\rho\int_0^t[Y^*-(ZY^*+U\eta))'[\gamma dB_s^p+\sigma dB_s^e]\Big).$$

The same argument leading to verify (3.10) above yields that (3.13) is satisfied. This concludes the proof of viability for Principal's strategy Θ.

Let us now verify the optimality of Principal's strategy Θ. For arbitrary Principal's viable strategy $\tilde{\Theta}=(\tilde{c},\tilde{F},\tilde{y})$ and its associated Agent's optimal strategy $\tilde{\Xi}^*$, consider the process

$$\tilde{V}_t=\int_0^t e^{-\delta s}u_P(\tilde{c}_s)ds+e^{-\delta t}V(W_t),$$

where $V(w)$ is defined in (6.6). From the HJB equation (6.7), we obtain that $\tilde{V}$ is a local super-martingale. Using the same localization argument as in the proof of Lemma 3.2 together with the transversality condition (3.13), we obtain

$$\mathbb{E}\Big[\int_0^\infty e^{-\delta s}u_P(\tilde{c}_s)ds\Big]\le\tilde{V}_0=V(W_0),$$

where the inequality is equality when Principal chooses Θ. This verifies the optimality of Θ.

6.5 Proof of Theorem 3.8

6.5.1 Step 1: Equilibrium asset prices

In equilibrium with $y=0$, we necessarily have $Y^*=\theta$. Then, (6.4) combined with (6.16) and (6.18) yields[9]

$$\theta = \frac{1}{r\mathscr{C}_b}\alpha_t + \frac{\mathscr{D}_b}{\mathscr{C}_b}\beta^{\theta}\eta. \tag{6.22}$$

Next, recall that $\alpha_t = \Sigma_R^{-1}(\mu_t - r)$. Left-multiplying (6.22) by $r\mathscr{C}_b\eta'\Sigma_R$ leads to

$$r\mathscr{C}_b\eta'\Sigma_R\theta = \eta'(\mu_t - r) + r\mathscr{D}_b Covar^{\theta,\eta}.$$

Note that all the terms above equation are constants except for the first term on the right-hand side, which is $\eta'(p_tA_1 + e_tA_2 + A_3)$. Since this equation has to hold for all values of p_t and e_t in equilibrium, it is necessary to have

$$A_1 = 0 \quad \text{and} \quad A_2 = 0.$$

Hence μ is a constant. Recalling the definition of A_1 and A_2 in (2.14), we then obtain

$$a_{pi} = \frac{a_i}{r+\kappa^p}, \quad a_{ei} = \frac{1}{r+\kappa_i^e}, \quad i = 1,\ldots,N. \tag{6.23}$$

In order to determine a_0, we left-multiply both sides of (6.22) by $r\mathscr{C}_b\Sigma_R$, and using $A_1 = A_2 = 0$ it follows that

$$A_3 = \mu - r = r\Sigma_R\big(\mathscr{C}_b\theta - \mathscr{D}_b\beta^{\theta}\eta\big). \tag{6.24}$$

Note that $\mathscr{C}_b - \mathscr{D}_b = \frac{\rho\bar{\rho}}{\rho+\bar{\rho}}$. Thus, the previous equation can be rewritten as

$$A_3 = \mu - r = r\Sigma_R\Big(\tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}}\theta + \mathscr{D}_b(\theta - \beta^{\theta}\eta)\Big). \tag{6.25}$$

Recalling the definition of A_3 from (2.14), (6.25) yields

$$a_0 = \frac{1}{r}\kappa^p\bar{p}a_p + \frac{1}{r}(\kappa^e)'\bar{e}a_e - \tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}}\Sigma_R\theta - \mathscr{D}_b\Sigma_R(\theta - \beta^{\theta}\eta). \tag{6.26}$$

Plugging (6.16), (6.17), and $\mathscr{C}_b - \mathscr{D}_b = \frac{\rho\bar{\rho}}{\rho+\bar{\rho}}$ back to (6.4), we confirm (3.19). Left-multiplying both sides of (6.25) by θ', we obtain the excess return of the portfolio:

$$\theta'(\mu - r) = r\tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}}Var^{\theta} + r\mathscr{D}_b\Big(Var^{\theta} - \tfrac{(Covar^{\theta,\eta})^2}{Var^{\eta}}\Big). \tag{6.27}$$

Left-multiplying η' both sides of (6.25) by η' , we obtain the excess return of the index:

$$\eta'(\mu - r) = r\tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}}Covar^{\theta,\eta}. \tag{6.28}$$

Finally, since a_{pi} and a_{ei} obtained in (6.23) are positive, all entries of Σ_R are positive. Moreover all entries of η are positive. Therefore $\eta'\Sigma_R\eta > 0$ is also confirmed.

[9]Equation (6.22) remains the same when Principal chooses nonzero index investment, i.e., $y \neq 0$. For any y, Principal can control variables $\bar{\Gamma}^{GI} + y$ and $\bar{U} - y$ for the maxmization problem in (6.7). Then, equations (6.11), (6.12), and (6.18) remain valid when $\bar{\Gamma}^{GI}$, $\bar{U}$, and Y^* are replaced by $\bar{\Gamma}^{GI} + y$, $\bar{U} - y$, and $Y^* + y\eta$, respectively. Moreover, equation (6.22) follows from the market clearing condition $Y^* + y\eta = \theta$.

6.5.2 Step 2: Participation constraint and Principal's value

We now determine Principal's optimal choice of Agent's value at time 0, i.e., $\bar{V}_0$, so that Agent is willing to take this contract. and that Principal is willing to issue the contract F.

If Agent does not take the contract, his value function $\bar{V}^u$ is expected to satisfy the following HJB equation

$$\bar{\delta}\bar{V}^u = \sup_{\bar{c}^u, Y}\left\{u_A(\bar{c}^u) + \bar{V}^u_{\bar{w}}(r\bar{w} + Y'(\mu - r) - \bar{c}^u) + \tfrac{1}{2}\bar{V}^u_{\bar{w}\bar{w}}Y'\Sigma_R Y\right\}, \tag{6.29}$$

We conjecture that $\bar{V}^u$ takes the form

$$\bar{V}^u(\bar{w}) = \bar{K}^u e^{-r\bar{\rho}\bar{w}},$$

for some constant $\bar{K}^u < 0$. The first order conditions for the maximization of $\bar{c}^u$ and Y give

$$\begin{aligned}\bar{c}^u &= -\tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{K}^u) + r\bar{w},\\ Y &= \tfrac{1}{r\bar{\rho}}\Sigma_R^{-1}(\mu - r),\end{aligned}$$

which are optimizers for the right-hand side of (6.29) due to concavity. Plugging above $\bar{c}^u$ and Y back to (6.29) yields

$$\log(-r\bar{\rho}\bar{K}^u) = \tfrac{r-\bar{\delta}}{r} - \tfrac{1}{2r}(\mu - r)'\Sigma_R^{-1}(\mu - r). \tag{6.30}$$

An argument similar to the proof of Lemma 3.2 verifies the optimality of $(\bar{c}^u, Y)$. Since $\bar{W}_0 = 0$, Principal can set $\bar{V}_0 = \bar{K}^u$. In this case, Agent is indifferent with respect to taking the contract or not, in which case we assume he chooses to work for Principal. Plugging (6.30) into (6.21) yields

$$\begin{aligned}C =& \tfrac{1}{2r}\tfrac{1}{\bar{\rho}}(\mu - r)'\Sigma_R^{-1}(\mu - r) - (ZY^* + U\eta)'(\mu - r)\\ &- \tfrac{r}{2}\zeta(Y^* - \beta^{Y^*}I)'\Sigma_R(Y^* - \beta^{Y^*}I) + \tfrac{r}{2}\bar{\rho}(ZY^* + U\eta)'\Sigma_R(ZY^* + U\eta).\end{aligned} \tag{6.31}$$

Let us determine K in (6.6). First, plugging $\mu - r$ from (6.24) into the right-hand side of (6.30), we obtain

$$\log(-r\bar{\rho}\bar{V}_0) = \tfrac{r-\bar{\delta}}{r} - \tfrac{r}{2}\left[\mathscr{C}_b^2 Var^\theta + (\mathscr{D}_b^2 - 2\mathscr{C}_b\mathscr{D}_b)\tfrac{(Covar^{\theta,\eta})^2}{Var^\eta}\right]. \tag{6.32}$$

Second, plugging (6.11) back into (6.10) and using $Y^* = \theta$, we obtain

$$\tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{1}{\rho}\log(-r\rho K) + \theta'(\mu - r) = \tfrac{1}{\rho} + \tfrac{1}{\bar{\rho}} - \tfrac{\delta}{r\rho} - \tfrac{\bar{\delta}}{r\bar{\rho}} + \tfrac{r}{2}\mathscr{C}_b Var^\theta - \tfrac{r}{2}\mathscr{D}_b\tfrac{(Covar^{\theta,\eta})^2}{Var^\eta}. \tag{6.33}$$

Plugging (6.27) and (6.32) back to (6.33), we obtain

$$\tfrac{1}{\rho}\log(-r\rho K) = \tfrac{1}{\rho} - \tfrac{\delta}{r\rho} + \tfrac{r}{2}\tfrac{1}{\bar{\rho}}(\mathscr{C}_b^2 - \bar{\rho}\mathscr{C}_b)Var^\theta + \tfrac{r}{2}\tfrac{1}{\bar{\rho}}(\mathscr{D}_b^2 - 2\mathscr{C}_b\mathscr{D}_b + \bar{\rho}\mathscr{D}_b)\tfrac{(Covar^{\theta,\eta})^2}{Var^\eta}. \tag{6.34}$$

If Principal does not hire Agent, her value function V^u is expected to satisfy the following HJB equation

$$\delta V^u = \sup_{c^u, y}\left\{u_p(c^u) + V^u_w(rw + y\eta'(\mu - r) - c^u) + \tfrac{1}{2}V^u_{ww}y^2\eta'\Sigma_R\eta\right\}. \tag{6.35}$$

We conjecture that V^u takes the form

$$V^u(w) = K^u e^{-r\rho w},$$

for some constant $K^u < 0$. The first order conditions for the maximization of c^u and y give

$$c^u = -\tfrac{1}{\rho}\log(-r\rho K^u) + rw,$$
$$y = \tfrac{1}{r\rho}\tfrac{\eta'(\mu-r)}{\eta'\Sigma_R\eta},$$

which are optimizers for the right-hand side of (6.35) due to concavity. Plugging above c^u and y back to (6.35), we obtain

$$\tfrac{1}{\rho}\log(-r\rho K^u) = \tfrac{1}{\rho} - \tfrac{\delta}{r\rho} - \tfrac{1}{2r}\tfrac{1}{\rho}\tfrac{(\eta'(\mu-r))^2}{\eta'\Sigma_R\eta}. \tag{6.36}$$

Using $\eta'(\mu - r)$ from (6.28), we obtain

$$\tfrac{1}{\rho}\log(-r\rho K^u) = \tfrac{1}{\rho} - \tfrac{\delta}{r\rho} - \tfrac{r}{2}\tfrac{1}{\rho}\big(\tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}}\big)^2\tfrac{(Covar^{\theta,\eta})^2}{Var^{\eta}}. \tag{6.37}$$

An argument similar to the proof of Lemma 3.2 verifies the optimality of (c^u, y).

Comparing (6.34) and (6.37), we see that $V(W_0) \geq V^u(W_0)$ if and only if

$$C_3 Var^{\theta} + C_4\tfrac{(Covar^{\theta,\eta})^2}{Var^{\eta}} \geq \tfrac{r}{2}\big(\tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}}\big)^2\tfrac{(Covar^{\theta,\eta})^2}{Var^{\eta}}, \tag{6.38}$$

where constants C_3 and C_4 are

$$C_3 = \tfrac{r}{2}\tfrac{\rho}{\bar{\rho}}\mathscr{C}_b(\bar{\rho} - \mathscr{C}_b) \quad \text{and} \quad C_4 = \tfrac{r}{2}\tfrac{\rho}{\bar{\rho}}\mathscr{D}_b(2\mathscr{C}_b - \mathscr{D}_b - \bar{\rho}).$$

Note that $C_3 = \frac{r}{2}\big(\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\big)^2 - C_4$ and $C_3 \geq 0$ when $0 \leq b \leq 1$. Then, (6.38) is equivalent to $Var^{\theta} \geq \frac{(Covar^{\theta,\eta})^2}{Var^{\eta}}$, which holds by Cauchy-Schwarz inequality. Therefore, $V(W_0) \geq V^u(W_0)$ and Principal is willing to hire Agent.

6.6 Calculation for Figure 4

Given the excess return in BVW (2014)

$$\mu - r = r\bar{\rho}\Sigma_R(Z\theta + U\eta), \tag{6.39}$$

where Z and U are as in Theorem 3.5 item (a), and Agent's optimal holding $Y^* = \theta$, Principal's value function is

$$V_{BVW}(W_0) = -e^{-r\rho W_0 - C_{BVW}},$$

where

$$\begin{aligned} C_{BVW} =& -1 + \tfrac{\delta}{r} + \log(r\rho) - \tfrac{r}{2}\big(\tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}}\big)^2\theta'\Sigma_R\theta \\ & - r\rho\big(b - \tfrac{\rho}{\rho+\bar{\rho}}\big)_+\big[\tfrac{\bar{\rho}^2}{\rho+\bar{\rho}} - (\bar{\rho} + \tfrac{1}{2}\rho)\big(b - \tfrac{\rho}{\rho+\bar{\rho}}\big)_+\big]\big(Var^{\theta} - \tfrac{(Covar^{\eta,\theta})^2}{Var^{\eta}}\big). \end{aligned} \tag{6.40}$$

The term $\log(r\rho)$ in above eqaution corresponds to the term $\log(r)$ in BVW (2014), Equation (A.104). The difference is due to Principal's utility being equal to $-e^{-\rho c}$ in BVW (2014), while being equal to $-\frac{1}{\rho}e^{-\rho c}$ in the present paper.

Taking the excess return (6.39), we want to calculate Principal's value if she uses our contract. First, (6.39) yields

$$\eta'(\mu-r) = r\tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}}Covar^{\eta,\theta}, \quad \alpha = r\bar{\rho}(Z\theta+U\eta).$$

Plugging above two identities back into (6.4), and using $\bar{\Gamma}^G$ and $\bar{\Gamma}^{GI}$ from (6.16) and (6.17), we obtain

$$Y^* = \frac{\bar{\rho}Z}{\mathscr{C}_b}\theta + \frac{\mathscr{D}_b - \bar{\rho}(b-\frac{\rho}{\rho+\bar{\rho}})_+}{\mathscr{C}_b}\beta^\theta\eta. \tag{6.41}$$

Using the above expression and the fact that $\mathscr{C}_b = \frac{\rho\bar{\rho}}{\rho+\bar{\rho}} + \mathscr{D}_b$, we have

$$Covar^{Y^*,\eta} = Covar^{\theta,\eta}.$$

Hence (6.11) yields

$$\bar{U} = \tfrac{\rho-(\rho+\bar{\rho})Z}{(\rho+\bar{\rho})Z}\beta^\theta.$$

Plugging the previous expression of $\bar{U}$ back into (6.10), a calculation shows that

$$\tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{1}{\rho}\log(-r\rho K) + (Y^*)'(\mu-r) = \tfrac{1}{\rho} + \tfrac{1}{\bar{\rho}} - \tfrac{\delta}{r\rho} - \tfrac{\bar{\delta}}{r\bar{\rho}} + \tfrac{r}{2}\mathscr{C}_b Var^{Y^*} - \tfrac{r}{2}\mathscr{D}_b\tfrac{(Covar^{\eta,Y^*})^2}{Var^\eta}.$$

Combining (6.39), (6.41), and the fact that $\mathscr{C}_b = \frac{\rho\bar{\rho}}{\rho+\bar{\rho}} + \mathscr{D}_b$, we show that

$$\begin{aligned}
&\frac{r}{2}\mathscr{C}_b Var^{Y^*} - \frac{r}{2}\mathscr{D}_b\frac{(Covar^{\eta,Y^*})^2}{Var^\eta} = \frac{1}{2}(Y^*)'(\mu-r)\\
&= \frac{r}{2}\frac{\bar{\rho}^2Z^2}{\mathscr{C}_b}Var^\theta + \frac{r}{2}\frac{-\bar{\rho}(b-\frac{\rho}{\rho+\bar{\rho}})_+\mathscr{C}_b + \bar{\rho}Z\mathscr{D}_b - \bar{\rho}^2(b-\frac{\rho}{\rho+\bar{\rho}})_+Z}{\mathscr{C}_b}\frac{(Covar^{\theta,\eta})^2}{Var^\eta}.
\end{aligned}$$

On the other hand, using (6.30), (6.39), and $\bar{K}^u = \bar{V}_0$, we obtain

$$\log(-r\bar{\rho}\bar{V}_0) = 1 - \tfrac{\bar{\delta}}{r} - \frac{r}{2}\bar{\rho}^2Z^2Var^\theta - \frac{r}{2}\Big[-2Z(b-\tfrac{\rho}{\rho+\bar{\rho}})_+\bar{\rho}^2 + (b-\tfrac{\rho}{\rho+\bar{\rho}})_+^2\bar{\rho}^2\Big]\frac{(Covar^{\theta,\eta})^2}{Var^\eta}.$$

Combining the previous three equations, Principal's value, when she employs the contract in (3.17), is

$$V_{CX}(W_0) = Ke^{-r\rho W_0} = -e^{-r\rho W_0 - C_{CX}},$$

where

$$\begin{aligned}
C_{CX} =& -1 + \tfrac{\delta}{r} + \log(r\rho)\\
&+ \frac{r}{2}\frac{\rho\bar{\rho}Z^2}{\mathscr{C}_b}\Big[\tfrac{\bar{\rho}^2}{\rho+\bar{\rho}} - (\rho+\bar{\rho})\big(b-\tfrac{\rho}{\rho+\bar{\rho}}\big)_+^2\Big]Var^\theta\\
&+ \frac{r}{2}\Big[\tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}}(\rho-\bar{\rho})\big(b-\tfrac{\rho}{\rho+\bar{\rho}}\big)_+ + \rho\bar{\rho}\big(b-\tfrac{\rho}{\rho+\bar{\rho}}\big)_+^2 + \frac{\rho\bar{\rho}Z}{\mathscr{C}_b}\big(\mathscr{D}_b - \bar{\rho}\big(b-\tfrac{\rho}{\rho+\bar{\rho}}\big)_+\big)\Big]\frac{(Covar^{\theta,\eta})^2}{Var^\eta}.
\end{aligned} \tag{6.42}$$

Figure 4 compares the certainty equivalences C_{BVW}/ρ and C_{CX}/ρ for Principal under the two contracts.

6.7 Proofs for Section 4

6.7.1 Proof of Lemma 4.1

Let us first motivate the contract form (4.2) from the martingale principle. Given $\mathbf{F}^{G,I}$-adapted processes $Z, U, \Gamma^G, \Gamma^I, \Gamma^{GI}$ such that $Z \geq b$, $\Gamma^G \neq 0$, and $\Gamma^G - \frac{1}{2}r\bar{\rho}Z^2 < 0$, consider a contract

$$dF_t = Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma_t d\langle I\rangle_t + \Gamma_t^{GI} d\langle G, I\rangle_t - \bar{H}_t dt,$$

where $\bar{H}$ will be determined as follows. Suppose that Agent's continuation value is given by

$$\bar{V}_t = \bar{V}_0 e^{-r\bar{\rho}\bar{W}_t},$$

where $\bar{W}$ is as in (4.1). It is expected from the martingale principle that the process $\tilde{V}_t = e^{-\bar{\delta}t}\bar{V}_t + \int_0^t e^{-\bar{\delta}s}u_A(\bar{c}_s)ds$ is a supermartingale for arbitrary strategy $\Xi = (\bar{c}, m, Y, \bar{Y})$, and it is a martingale for the optimal strategy. Itô's formula implies that the drift of $\tilde{V}$ (divided throughout by $e^{-\bar{\delta}t}X_t$) is

$$\begin{aligned}
&\tfrac{\bar{\delta}}{r\bar{\rho}} + \tfrac{u_A(\bar{c})}{X} + r\bar{W} + bm - \bar{c} + \bar{Y}'(\mu - r) + ZY'(\mu - r) - Zm + U\eta'(\mu - r) \\
&+ \tfrac{1}{2}\Gamma^G Y'\Sigma_R Y + \tfrac{1}{2}\Gamma^I \eta'\Sigma_R\eta + \Gamma^{GI}Y'\Sigma_R\eta - \tfrac{1}{2}r\bar{\rho}(ZY + U\eta + \bar{Y})'\Sigma_R(ZY + U\eta + \bar{Y}) - \bar{H},
\end{aligned}$$

where $X = -r\bar{\rho}\bar{V}$. This drift being negative for all strategies and zero for some strategy implies that

$$\begin{aligned}
\bar{H} = &\sup_{m,Y,\bar{Y},\bar{c}} \Big\{ \tfrac{\bar{\delta}}{r\bar{\rho}} + \tfrac{u_A(\bar{c})}{X} + r\bar{W} + bm - \bar{c} + \bar{Y}'(\mu - r) + ZY'(\mu - r) - Zm + U\eta'(\mu - r) \\
&\quad + \tfrac{1}{2}\Gamma^G Y'\Sigma_R Y + \tfrac{1}{2}\Gamma^I \eta'\Sigma_R\eta + \Gamma^{GI}Y'\Sigma_R\eta - \tfrac{1}{2}r\bar{\rho}(ZY + U\eta + \bar{Y})'\Sigma_R(ZY + U\eta + \bar{Y}) \Big\} \\
= &\tfrac{\bar{\delta}}{r\bar{\rho}} + \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) - \tfrac{1}{\bar{\rho}} \\
&+ \sup_{Y,\bar{Y}} \Big\{ (ZY + U\eta + \bar{Y})'(\mu - r) + \tfrac{1}{2}\Gamma^G Y'\Sigma_R Y + \tfrac{1}{2}\Gamma^I\eta'\Sigma_R\eta + \Gamma^{GI}Y'\Sigma_R\eta \\
&\quad - \tfrac{1}{2}r\bar{\rho}(ZY + U\eta + \bar{Y})'\Sigma_R(ZY + U\eta + \bar{Y}) \Big\}.
\end{aligned}$$

Given that $\Gamma^G - \frac{1}{2}r\bar{\rho}Z^2 < 0$, the optimization problem above is concave. Then the optimizers Y^* and $\bar{Y}^*$ satisfy the first order conditions

$$Y^* = -\frac{Z}{\Gamma^G - r\bar{\rho}Z^2}\Sigma_R^{-1}(\mu - r) - \frac{\Gamma^{GI} - r\bar{\rho}ZU}{\Gamma^G - r\bar{\rho}Z^2}\eta + \frac{r\bar{\rho}Z}{\Gamma^G - r\bar{\rho}Z^2}\bar{Y}^*, \tag{6.43}$$

$$ZY^* + U\eta + \bar{Y}^* = \tfrac{1}{r\bar{\rho}}\Sigma^{-1}(\mu - r), \tag{6.44}$$

whose solutions are given in (4.4). Suppose the transversality condition (2.7) is satisfied when Agent employs the strategy Ξ^*. The optimality of Ξ^* follows from a proof similar to Lemma 3.2.

6.7.2 Proof of Theorem 4.2

When Agent employs the optimal strategy Ξ^*, combining (4.2), (6.43), and (6.44), we get

$$\begin{aligned} dF_t =& \Big[-\tfrac{\bar{\delta}}{r\bar{\rho}} - \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{1}{\bar{\rho}} - (\bar{Y}_t^*)'(\mu - r) + \tfrac{1}{2}\tfrac{1}{r\bar{\rho}}(\mu - r)'\Sigma_R^{-1}(\mu - r)\Big]\, dt \\ &+ (Z_t Y_t^* + U_t\eta_t)'(\gamma dB_t^p + \sigma dB_t^e). \end{aligned}$$

The Principal's wealth process then follows

$$\begin{aligned} dW_t =& (rW_t - c_t)dt + (Y_t^*)'dR_t + y_t dI_t - dF_t \\ =& \big[rW_t - c_t + \tfrac{\bar{\delta}}{r\bar{\rho}} + \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) - \tfrac{1}{\bar{\rho}}\big]dt + (Y_t^* + y_t\eta + \bar{Y}_t^*)'(\mu - r)dt \\ &- \tfrac{1}{2}\tfrac{1}{r\bar{\rho}}(\mu - r)'\Sigma_R^{-1}(\mu - r)\, dt + \big[(Y^* + y\eta) - (ZY^* + U\eta)\big]'(\gamma dB_t^p + \sigma dB_t^e). \end{aligned}$$

We conjecture that Principal's value function takes the form $V(W) = Ke^{-r\rho W}$ for some constant $K < 0$. Then V satisfies the following HJB equation

$$\begin{aligned} r - \delta = \sup_{Z \geq b, U, \Gamma' s} \Big\{ & \tfrac{\rho}{\bar{\rho}}(\bar{\delta} - r) + r\rho\big[(Y^* + y\eta + \bar{Y}^*)'(\mu - r) + \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{1}{\rho}\log(-r\rho K)\big] \\ &+ r\rho\big[-\tfrac{1}{2}\tfrac{1}{r\bar{\rho}}(\mu - r)'\Sigma_R^{-1}(\mu - r)\big] \\ &- \tfrac{1}{2}(r\rho)^2\big[(Y^* + y\eta + \bar{Y}^*) - (ZY^* + U\eta + \bar{Y}^*)\big]'\Sigma_R\big[(Y^* + y\eta + \bar{Y}^*) - (ZY^* + U\eta + \bar{Y}^*)\big]\Big\}, \end{aligned}$$

where $ZY^* + U\eta + \bar{Y}^*$ is given in (6.44) and

$$Y^* + y\eta + \bar{Y}^* = \tfrac{1}{r\bar{\rho}}\Sigma^{-1}(\mu - r) + \Big[(Z - 1)\tfrac{\Gamma^{GI}}{\Gamma^G} - (U - y)\Big]\eta. \tag{6.45}$$

Using (6.44), we observe that the right-hand side of the HJB equation is a maximization problem in terms of $Y^* + y\eta + \bar{Y}^*$. Denoting $A = (Z - 1)\frac{\Gamma^{GI}}{\Gamma^G} - (U - y)$ and using (6.44), this maximization problem becomes equivalent to

$$\sup_{Z \geq b, U, \Gamma' s} \Big\{ \tfrac{\rho + \bar{\rho}}{\bar{\rho}}\big[\tfrac{1}{r\bar{\rho}}\Sigma_R^{-1}(\mu - r) + A\eta\big]'(\mu - r) - \tfrac{1}{2}r\rho\big[\tfrac{1}{r\bar{\rho}}\Sigma_R^{-1}(\mu - r) + A\eta\big]'\Sigma_R\big[\tfrac{1}{r\bar{\rho}}\Sigma_R^{-1}(\mu - r) + A\eta\big]\Big\}.$$

Instead of optimizing over $Z \geq b, U$, and $\Gamma' s$, we optimize over A to obtain the optimizer

$$A = \tfrac{1}{r\rho}\tfrac{\eta'(\mu - r)}{\eta'\Sigma_R\eta}.$$

Therefore any optimizers $Z, U, \Gamma' s$ satisfy (4.5). The verification for both Agent and Principal's optimization problems is similar to Section 6.4.4.

6.7.3 Proof of Theorem 4.3

When Agent employs the optimal strategy in (4.8) and (4.9), using (4.6), (4.7), and noticing that (4.6) does not contain $\bar{y} \cdot I$, we obtain

$$\begin{aligned} dF^* =& -\Big[\tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{\bar{\delta}}{r\bar{\rho}} - \tfrac{1}{\bar{\rho}} + \bar{y}^*\eta'(\mu - r) - \tfrac{1}{2}r\bar{\rho}(Z_t Y_t^* + U_t\eta + \bar{y}^*\eta)'\Sigma_R(Z_t Y^* + U_t\eta + \bar{y}^*\eta)\Big]dt \\ &+ (Z_t Y_t^* + U_t\eta)'(\gamma dB_t^p + \sigma dB_t^e). \end{aligned}$$

Therefore, Principal's wealth process follows

$$\begin{aligned} dW_t = & \left[rW_t - c_t + \tfrac{1}{\bar\rho}\log(-r\bar\rho\bar V_0) + \tfrac{\bar\delta}{r\bar\rho} - \tfrac{1}{\bar\rho}\right]dt \\ & + \left[(Y_t^* + y_t\eta + \bar y_t^*\eta)'(\mu - r) - \tfrac{1}{2}r\bar\rho(Z_tY_t^* + U_t\eta + \bar y_t^*\eta)'\Sigma_R(Z_tY^* + U_t\eta + \bar y_t^*\eta)\right]dt \\ & + \left[(Y_t^* + y_t\eta) - (Z_tY_t^* + U_t\eta)\right]'(\gamma dB_t^p + \sigma dB_t^e). \end{aligned}$$

We conjecture that Principal's value function is given by

$$V(w) = Ke^{-r\rho w}, \tag{6.46}$$

for some constant $K < 0$. The value function is expected to satisfy the following HJB equation

$$\begin{aligned} \delta V = \sup_{Z \geq b, U, \Gamma's, c, y} & \Big\{u_p(c) + V_w\left[rw - c + \tfrac{1}{\bar\rho}\log(-r\bar\rho\bar V_0) + \tfrac{\bar\delta}{r\bar\rho} - \tfrac{1}{\bar\rho}\right] \\ & + V_w\left[(Y^* + y\eta + \bar y^*\eta)'(\mu - r) - \tfrac{1}{2}r\bar\rho(ZY^* + U\eta + \bar y^*\eta)'\Sigma_R(ZY^* + U\eta + \bar y^*\eta)\right] \\ & + \tfrac{1}{2}V_{ww}\left[(Y^* + y\eta) - (ZY^* + U\eta)\right]'\Sigma_R\left[(Y^* + y\eta) - (ZY^* + U\eta)\right]\Big\}. \end{aligned} \tag{6.47}$$

Recalling (4.8) and (4.9), we have

$$\begin{aligned} Y^* + y\eta + \bar y^*\eta &= -\tfrac{Z}{\Gamma^G}\alpha - \left(\tfrac{\Gamma^{GI}}{\Gamma^G} - y\right)\eta - \left(\tfrac{\Gamma^{G\bar G}}{\Gamma^G} - 1\right)\bar y^*\eta, \\ ZY^* + U\eta + \bar y^*\eta &= Z(Y^* + y\eta + \bar y^*\eta) + (U - Zy - (Z-1)\bar y^*)\eta, \\ (Y^* + y\eta) - (ZY^* + U\eta) &= (Y^* + y\eta + \bar y^*\eta) - (ZY^* + U\eta + \bar y^*\eta), \end{aligned}$$

where $\alpha = \Sigma_R^{-1}(\mu - r)$. Therefore, instead of optimizing over $Z, \bar U, \Gamma'$s, and y individually, we can optimize over Z, $\Gamma^{GI} - y\Gamma^G$, $\Gamma^{G\bar G} - \Gamma^G$ and $U - Zy - (Z-1)\bar y^*$. We rename the last three new variables as Γ^{GI}, $\Gamma^{G\bar G}$, and U. As a result, choosing $y = \bar y^* = 0$, with appropriate choice of Z, U, and $\Gamma's$, the right-hand side of (6.47) remains the same. Writing (4.8) and (4.9) in terms of these new variables and using $y = \bar y^* = 0$, we obtain

$$\begin{aligned} Y^* &= -\frac{Z}{\Gamma^G}\alpha - \frac{\Gamma^{GI}}{\Gamma^G}\eta, \\ \bar y^* &= \frac{\Gamma^{G\bar G}Z - (1-Z)\Gamma^G}{\Gamma^G\Gamma^{\bar G} - (\Gamma^{G\bar G} + \Gamma^G)^2}\frac{\eta'(\mu - r)}{\eta'\Sigma_R\eta} + \frac{\Gamma^{GI}(\Gamma^{G\bar G} + \Gamma^G) - \Gamma^G\Gamma^{\bar G I}}{\Gamma^G\Gamma^{\bar G} - (\Gamma^{G\bar G} + \Gamma^G)^2}. \end{aligned}$$

For given Z, Γ^G, Γ^{GI} with $Z > b, \Gamma^G < 0$, we can take

$$\Gamma^{G\bar G} = \tfrac{1-Z}{Z}\Gamma^G, \quad \Gamma^{\bar G} < \tfrac{1}{Z^2}\Gamma^G, \quad \Gamma^{\bar G I} = \tfrac{1}{Z}\Gamma^{GI}.$$

Then, $\Gamma^G\Gamma^{\bar G} - (\Gamma^{G\bar G})^2 > 0$ and $\bar y^* = 0$ are satisfied. This reduces to the case in which Agent is not allowed to invest in the index privately. Hence, the remainder of the proof is the same as before.

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