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ON FUTURE DRAWDOWNS OF LÉVY PROCESSES

E. J. BAURDOUX, Z. PALMOWSKI, AND M.R. PISTORIUS

Abstract. For a given Lévy process $X = (X_t)_{t\in\mathbb{R}_+}$ and for fixed $s \in \mathbb{R}_+ \cup \{\infty\}$ and $t \in \mathbb{R}_+$ we analyse the *future drawdown extremes* that are defined as follows:

$$\overline{D}^*_{t,s} = \sup_{0\le u\le t} \inf_{u\le w<t+s} (X_w - X_u), \qquad \underline{D}^*_{t,s} = \inf_{0\le u\le t} \inf_{u\le w<t+s} (X_w - X_u).$$

The path-functionals $\overline{D}^*_{t,s}$ and $\underline{D}^*_{t,s}$ are of interest in various areas of application, including financial mathematics and queueing theory. In the case that X has a strictly positive mean, we find the exact asymptotic decay as $x \to \infty$ of the tail probabilities $\mathbb{P}(\overline{D}^*_t < x)$ and $\mathbb{P}(\underline{D}^*_t < x)$ of $\overline{D}^*_t = \lim_{s\to\infty} \overline{D}^*_{t,s}$ and $\underline{D}^*_t = \lim_{s\to\infty} \underline{D}^*_{t,s}$ both when the jumps satisfy the Cramér assumption and in a heavy-tailed case. Furthermore, in the case that the jumps of the Lévy process X are of single sign and X is not subordinator, we identify the one-dimensional distributions in terms of the scale function of X. By way of example, we derive explicit results for the Black-Scholes-Samuelson model.

1. Introduction

In recent times various pricing models with jumps have been put forward to address the shortcomings of diffusion models in representing the risk related to large market movements (see e.g. [8]). Such models allow for a more realistic representation of price dynamics and a greater flexibility in modeling and calibration of the model to market prices and in reproducing a wide variety of implied volatility skews and smiles. An important indicator for the riskiness and effectiveness of an investment strategy is the drawdown, which is the distance of the current value away from the maximum value it has attained to date. Various commonly used trading rules are based on the drawdown (see e.g. [27]), while drawdowns have also been deployed as risk-measure (see [5, 33]) and in the context of portfolio optimisation (see [7, 14]). Drawdown processes (also called reflected processes) are also encountered in various other areas, such as applied probability, mathematical genetics and queueing theory (see [9, 10]). See [21, 23, 32] and references therein for further applications and results concerning drawdown processes.

In this paper we analyse a number of path-functionals of the increments of a given general Lévy process $X = (X_t)_{t\in\mathbb{R}_+}$ that are closely related to the drawdowns and drawups. In particular, we consider the *future drawdown* and *future drawup extremes* that are defined by for given $s, t \in \mathbb{R}_+$ by

$$\overline{D}^*_{t,s} = \sup_{0\le u\le t} \inf_{u\le w<t+s} (X_w - X_u), \qquad \underline{D}^*_{t,s} = \inf_{0\le u\le t} \inf_{u\le w<t+s} (X_w - X_u), \tag{1.1}$$

$$\overline{U}^*_{t,s} = \sup_{0\le u\le t} \sup_{u\le w<t+s} (X_w - X_u), \qquad \underline{U}^*_{t,s} = \inf_{0\le u\le t} \sup_{u\le w<t+s} (X_w - X_u), \tag{1.2}$$

and we denote the infinite-horizon versions by

$$\overline{D}^*_t = \lim_{s\to\infty} \overline{D}^*_{t,s}, \qquad \underline{D}^*_t = \lim_{s\to\infty} \underline{D}^*_{t,s}, \qquad \overline{U}^*_t = \lim_{s\to\infty} \overline{U}^*_{t,s}, \qquad \underline{U}^*_t = \lim_{s\to\infty} \underline{U}^*_{t,s}.$$

The functionals $\overline{D}^*_{t,s}$, $\underline{D}^*_{t,s}$, $\overline{U}^*_{t,s}$ and $\underline{U}^*_{t,s}$ are concerned with the variation in $u \in [0, t]$ of the smallest and largest of the increments $\{X_w - X_u, w \in [u, t + s]\}$. These functionals may be explicitly represented in terms of the (maximal) drawdown and drawup (see Proposition 2.1).

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Since, as is straightforward to check, we have $\underline{D}^*_{t,s} = -\overline{\widehat{U}}^*_{t,s}$ and $\overline{D}^*_{t,s} = -\underline{\widehat{U}}^*_{t,s}$, where $\widehat{\cdot}$ denotes the quantity calculated for the dual process $\widehat{X} = -X$, we may (and often do) restrict ourselves in subsequent analysis to future drawdown extremes, without loss of generality.

The future drawdown and drawup processes arise in various applications, including in financial risk analysis and queueing models. We note that, under an exponential Lévy model $P_t = P_0 \exp(X_t)$ for the stock price, the random variables $\overline{D}^*_{t,s}$ and $\underline{D}^*_{t,s}$ are path-dependent risk indicators: $\overline{D}^*_{t,s}$ and $\underline{D}^*_{t,s}$ are the maximal and minimal values of the *lowest* future log-return $\log(P_w/P_u)$ achieved for w in the time-window $[u, t+s]$, where u is ranging over $[0,t]$. Another application comes from telecommunications and queueing models, where $\overline{U}^*_t = \lim_{s\to\infty} \overline{U}^*_{t,s}$ and $\underline{U}^*_t = \lim_{s\to\infty} \underline{U}^*_{t,s}$ describe the supremum and the infimum of the workload process over a finite time horizon t in a fluid model with netput X, respectively (see [10] for a survey about Lévy-driven queues).

In the mentioned applications it is of interest to obtain the laws of the random variables $\overline{D}^*_{t,s}$, $\underline{D}^*_{t,s}$, $\overline{U}^*_{t,s}$ and $\underline{U}^*_{t,s}$ for finite and infinite horizons s, and in particular the tail-probabilities and their asymptotic behaviour. Restricting ourselves to the case $s = \infty$ we identify the *exact* asymptotic decay as $x \to \infty$ of the tail probabilities $\mathbb{P}(\overline{D}^*_{t,s} < -x)$ and $\mathbb{P}(\underline{D}^*_{t,s} < -x)$ of $\overline{D}^*_{t,s}$ and $\underline{D}^*_{t,s}$. We do so in the distinct cases of a light-tailed and a heavy-tailed Lévy measure. In the former setting we also consider the asymptotics when x and s tend to infinity in a fixed proportion. Furthermore, when the jumps of X are of single sign only and X is not subordinator, we explicitly identify the Laplace transform in time of the one-dimensional distributions in terms of the scale function. As example, we analyze in detail (future) drawdowns and drawups under the Black-Scholes model, identifying in particular the mean of the value $P_t = P_0 \exp(X_t)$ under the measure $\underline{\mathbb{P}}^{(\gamma)}$ defined in (3.21) (for γ given in Assumption 1) and the laws of $\overline{D}^*_t$ and $\underline{D}^*_t$.

Contents. The remainder of the paper is organized as follows. In Section 2 we present the main representation in terms of drawup and drawdown processes. In Section 3 we identify the Cramér asymptotics and describe the associated drawup and drawdown measures in Section 3.1. We analyse the heavy-tailed case in 4. Finally, in Section 5 we derive exact distributions of future drawup and drawdowns in case X has jumps of single sign and we present an application to the Black-Scholes model in Section 5.1.

2. Main representation

Let $(X_t)_{t\in\mathbb{R}_+}$ be a general Lévy process (*i.e.*, a process with stationary and independent increments with cádlág paths such that $X_0 = 0$) defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ with $\mathcal{F}_t = \sigma(\{X_s, s \le t\})$ denoting the completed filtration generated by X. The law of X is determined by its characteristic exponent Ψ which is the map $\Psi : \mathbb{R} \to \mathbb{C}$ that satisfies $\mathbb{E}[\mathrm{e}^{\mathbf{i}\theta X_1}] = \exp(\Psi(\theta))$.

The drawdown and drawup processes of X, $(D_t)_{t\in\mathbb{R}_+}$ and $(U_t)_{t\in\mathbb{R}_+}$, are path-functionals of the increments of X given by

$$D_t = \overline{X}_t - X_t, \qquad U_t = X_t - \underline{X}_t,$$

with $\overline{X}_t = \sup_{0\le s\le t} X_s$ and $\underline{X}_t = \inf_{0\le s\le t} X_s$. We note that the drawdown D_t and drawup U_t at time t are equal to the largest of all increments $X_u - X_t$, $u \in [0,t]$, and the negative of the smallest increment of such increments.

Before turning to the analysis of the future drawdown and drawup extremes, we recall a number of facts concerning drawup and drawdown processes which follow from the fluctuation theory of Lévy processes. First of all, we note that the marginal distributions of the drawup U_t and drawdown D_t, $t \in \mathbb{R}_+$, can be expressed in terms of the marginal distributions of X by deploying the Wiener-Hopf factorisation of X, according to which the characteristic exponent Ψ is related to the marginal distributions of the running supremum and running infimum of X at an exponential random time $\mathbf{e}_q$ of parameter q that is independent of $\mathcal{F}_\infty$ as follows:

$$\frac{q}{q - \Psi(\theta)} = \mathbb{E}[\mathrm{e}^{\mathbf{i}\theta \overline{X}_{\mathbf{e}_q}}]\mathbb{E}[\mathrm{e}^{\mathbf{i}\theta \underline{X}_{\mathbf{e}_q}}], \qquad \theta \in \mathbb{R}, q \in \mathbb{R}_+\backslash\{0\}.$$

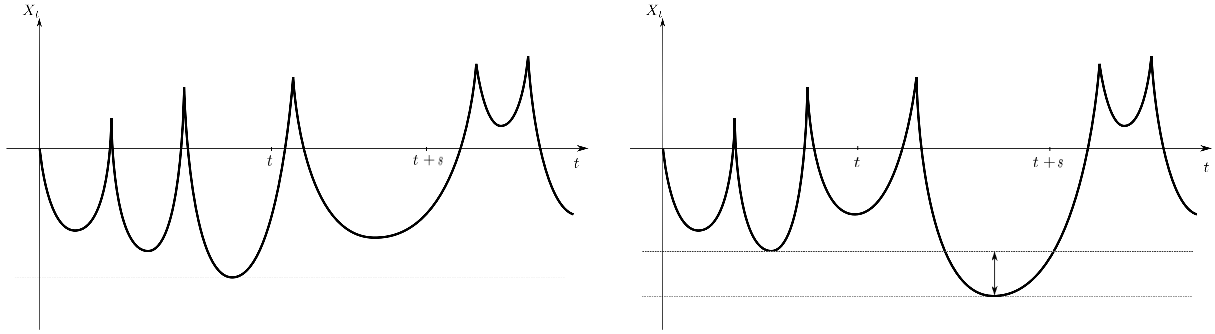

FIGURE 1. Two schematic pictures of a part of the path of X in the cases that (i) the smallest value of X up to time $t+s$ has already been attained before time t so that the path-functional $\underline{D}^*_{t,s}$ is zero (left-hand picture) or (ii) X attains a new minimum between t and $t+s$ and the path-functional $\underline{D}^*_{t,s}$ is strictly negative (right-hand picture).

Using the duality lemma (see e.g. [2, Proposition VI.3]) that U_t has the same law $\overline{X}_t$. Thus the Wiener-Hopf factorisation may be phrased as follows in terms of the drawdown and drawup processes:

$$\frac{q}{q-\Psi(\theta)} = \mathbb{E}[\mathrm{e}^{\mathbf{i}\theta U_{\mathbf{e}_q}}]\mathbb{E}[\mathrm{e}^{-\mathbf{i}\theta D_{\mathbf{e}_q}}], \qquad \theta\in\mathbb{R}, q\in\mathbb{R}_+\backslash\{0\}. \tag{2.1}$$

Moreover, since U_t has the same law $\overline{X}_t$, it follows that, if $\mathbb{E}[X_1]$ is strictly negative, U_t converges in distribution as $t\to\infty$ to a proper random variable U_∞ with the law of all-time supremum $\overline{X}_\infty$. Similarly, if $\mathbb{E}[X_1]$ is strictly positive, D_t having the same law as $\underline{X}_t$ converges to a random variable D_∞ as $t\to\infty$. The Laplace transforms of U_∞ and D_∞ are given explicitly in terms of the Laplace exponents κ and $\widehat{\kappa}$ of the ascending and descending ladder-height processes (L^{-1},H) and $(\widehat{L}^{-1},\widehat{H})$. The ladder time process $L^{-1} = \{L^{-1}_t\}_{t\in\mathbb{R}_+}$ is equal to the right-continuous inverse of a local time L of $(D_t)_{t\in\mathbb{R}_+}$ at zero. The corresponding ladder-height process $H=(H_t)_{t\geq 0}$ is given by $H_t = X(L^{-1}_t)$ for all $t\geq 0$ for which L^{-1}_t is finite, and defined to be $H_t=+\infty$ otherwise. We denote $\kappa(\beta,\theta) = -\log\mathbb{E}[\exp\{-\beta L^{-1}_1-\theta H_1\}\mathbf{1}_{\{H(1)<\infty\}}]$, where, for any set $A\in\mathcal{F}$, $\mathbf{1}_A$ denotes the indicator of the set A. Similarly, the Laplace exponent of the downward ladder-height process $(\widehat{L}^{-1},\widehat{H})$ corresponding to the dual process $\widehat{X}$ of X, $\widehat{X}=-X$, we denote by $\widehat{\kappa}(\beta,\theta) = -\log\mathbb{E}[\exp\{-\beta\widehat{L}^{-1}_1-\theta\widehat{H}_1\}\mathbf{1}_{\{\widehat{H}(1)<\infty\}}]$. Specifically, if $\mathbb{E}[X_1]$ is strictly positive, the Laplace transform of D_∞ is given as follows:

$$\mathbb{E}[\mathrm{e}^{-\theta D_\infty}] = \frac{\widehat{\kappa}(0,0)}{\widehat{\kappa}(0,\theta)}; \tag{2.2}$$

see [19] for details.

A first step in the study of the random variables $\overline{D}^*_{t,s}$, $\underline{D}^*_{t,s}$, $\overline{U}^*_{t,s}$ and $\underline{U}^*_{t,s}$ are the following distributional identities.

Proposition 2.1. *Let $t,s\in\mathbb{R}_+$ and let $\widetilde{U}_s \overset{(d)}{=} U_s$ and $\widetilde{D}_s \overset{(d)}{=} D_s$ be random variables independent of $\mathcal{F}_t$, where $\overset{(d)}{=}$ denotes equality in distribution. Denoting $\overline{U}_t = \sup_{0\leq u\leq t} U_u$, $\overline{D}_t = \sup_{0\leq u\leq t} D_u$, we have the following representations:*

$$\underline{D}^*_{t,s} \overset{(d)}{=} -\max\left\{\widetilde{D}_s + D_t, \overline{D}_t\right\}, \qquad \overline{D}^*_{t,s} \overset{(d)}{=} \min\left\{U_t - \widetilde{D}_s, 0\right\} \tag{2.3}$$

and

$$\overline{U}^*_{t,s} \overset{(d)}{=} \max\left\{\widetilde{U}_s + U_t, \overline{U}_t\right\}, \qquad \underline{U}^*_{t,s} \overset{(d)}{=} \max\{\widetilde{U}_s - D_t, 0\}. \tag{2.4}$$

In particular, when $\mathbb{E}[X_1]\in\mathbb{R}_+\backslash\{0\}$ ($\mathbb{E}[X_1]\in\mathbb{R}\backslash\mathbb{R}_+$), then $\overline{D}^*_t$ and $\underline{D}^*_t$ ($\overline{U}^*_t$ and $\underline{U}^*_t$) are finite $\mathbb{P}$-a.s.

Remark 2.2. **(i)** Extending X from $\mathbb{R}_+$ to a two-sided version on $\mathbb{R}$ and using a time-reversal argument we find that

$$\overline{U}^*_t \overset{(d)}{=} \sup_{0\leq u\leq t}\sup_{-\infty<w\leq u}(X_u - X_w), \qquad \underline{U}^*_t \overset{(d)}{=} \inf_{0\leq u\leq t}\sup_{-\infty<w\leq u}(X_u-X_w). \tag{2.5}$$

Indeed, using the change of variables $u' = t - u$ and $w' = t - w$ we see that

$$\begin{aligned}\sup_{0\le u\le t}\sup_{-\infty<w\le u}(X_u - X_w) &= \sup_{0\le u'\le t}\sup_{w'\ge u'}(X_{t-u'} - X_{t-w'}) \\ &\stackrel{(d)}{=} \sup_{0\le u'\le t}\sup_{w'\ge u'}(X_{w'} - X_{u'}).\end{aligned}$$

The result for $\underline{U}^*_t$ follows similarly.

The random variables $\overline{U}^*_t$ and $\underline{U}^*_t$ arise in a queueing application. Indeed, the workload process Q_u of a queue with net input process X (*i.e.*, input less output) evolves according to the process X reflected at its infimum, *i.e.*, $Q_u = X_u - \inf_{s\le u} X_s$. If we assume that the workload process is stationary (*i.e.*, Q_0 follows the stationary distribution, which is equal to the distribution of $-\inf_{-\infty<s\le 0} X_s$; see [28]), then the workload Q_u is given by:

$$Q_u = \sup_{-\infty<w\le u}(X_u - X_w)$$

and $\overline{U}^*_t$ and $\underline{U}^*_t$ describe the supremum and infimum of the workload process Q over a finite time horizon t, respectively. For details on queues driven by a Lévy process we refer to the survey book [10].

(ii) We note $\mathbb{P}(\underline{U}^*_t = 0) = \mathbb{P}(\int_0^\infty \mathbf{1}_{(X_s\ge 0)}\mathrm{d}s < t)$ (see for example [2, Lemma 15, p. 170] and [2, Theorem 13, p. 169]).

Proof of Proposition 2.1. As noted in the Introduction, it suffices to establish the statements concerning $\overline{D}^*$ and $\underline{D}^*$. Writing $[u, t+s] = [u, t] \cup [t, t+s]$ for given $u, t, s \in \mathbb{R}_+$ we have

$$\underline{D}^*_{t,s} = \inf_{0\le u\le t}\min\left\{\inf_{w\in[t,t+s]}(X_w - X_t) + X_t - X_u, \inf_{u\le w\le t}(X_w - X_u)\right\}.$$

Since $\widetilde{D}_s := -\inf_{t\le w\le t+s}(X_w - X_t)$ is independent of $\mathcal{F}_t$ and is equal in distribution to D_s, we find that $\underline{D}^*_{t,s}$ is equal in distribution to

$$\begin{aligned}\inf_{0\le u\le t}\min\left\{X_t - X_u - \widetilde{D}_s, \inf_{u\le w\le t}(X_w - X_u)\right\} &= \min\left\{-D_t - \widetilde{D}_s, \inf_{0\le u\le t}\inf_{u\le w\le t}(X_w - X_u)\right\} \\ &= -\max\left\{D_t + \widetilde{D}_s, \sup_{0\le w\le t}\sup_{0\le u\le w}(X_u - X_w)\right\},\end{aligned}$$

which yields the first identity in (2.3).

For the second identity in (2.3) we note that the function $u \mapsto \inf_{u\le w\le t+s}(X_w - X_u)$ attains its supremum over $[0, t]$ at G_{t-} or G_t where $G_t = \sup\{u \le t : X_u = \underline{X}_t\}$. In the case that $G_{t+s} \le t$ (*i.e.*, when $\underline{X}_{t+s} = \underline{X}_t$) we have $G_t = G_{t+s}$ (see Figure 1, left-hand picture) and $\overline{D}^*_{t,s} = 0$, while in the case that $G_{t+s} > t$ (see Figure 1, right-hand picture) we find

$$\overline{D}^*_{t,s} = \underline{X}_{t+s} - \underline{X}_t < 0.$$

Hence, writing $\underline{X}_{t+s} = \min\{\inf_{t\le u\le t+s}(X_u - X_t) + X_t, \underline{X}_t\}$ we deduce that

$$\overline{D}^*_{t,s} \stackrel{(d)}{=} \min\left\{X_t - \underline{X}_t + \inf_{0\le w\le s}\widetilde{X}_w, 0\right\},$$

where $\widetilde{X}$ denotes an independent copy of X, from which the expression for $\overline{D}^*_{t,s}$ follows.

Taking $s \to \infty$ in (2.3) and noting that $-\inf_{s\ge 0} X_s$ is finite $\mathbb{P}$-a.s. if $\mathbb{E}[X_1] \in \mathbb{R}_+\backslash\{0\}$ we conclude that also $\overline{D}^*_t$ and $\underline{D}^*_t$ are $\mathbb{P}$-a.s. finite. $\square$

3. Asymptotic future drawdown — the light-tailed case

In this section we study the asymptotics of the tail probabilities $\mathbb{P}(-\overline{D}_t^* > x)$ and $\mathbb{P}(-\underline{D}_t^* > x)$ in the case that the Lévy measure is light-tailed. More specifically, in this section we will make the following assumptions.

Assumption 1. *The Cramér condition holds,* i.e.,

$$\text{there exists a } \gamma \in \mathbb{R}_+\backslash\{0\} \text{ satisfying } \mathbb{E}[\mathrm{e}^{-\gamma X_1}] = 1, \tag{3.1}$$

The mean of X_1 is positive and finite, $\mathbb{E}[X_1] \in \mathbb{R}_+\backslash\{0\}$, and $\mathbb{E}[\mathrm{e}^{-\gamma X_1}|X_1|] \in \mathbb{R}_+\backslash\{0\}$.

Assumption 2. *X has non-monotone paths and either 0 is regular for $\mathbb{R}_+\backslash\{0\}$ or the Lévy measure of X is non-lattice.*

Under condition (3.1) the characteristic exponent Ψ can be extended to the strip $\mathcal{S}_\gamma = \{\theta \in \mathbb{C} : \Im(\theta) \in [0,\gamma]\}$ of the complex plane, by analytical continuation and continuous extension. The Laplace exponent $\psi(\theta) = \log \mathbb{E}[\mathrm{e}^{\theta X_1}]$ of X is finite on the maximal domain $\Theta = \{\theta \in \mathbb{R} : \psi(\theta) < \infty\}$, which contains the interval $[-\gamma, 0]$. Restricted to the interior Θ^o, the map $\theta \mapsto \psi(\theta)$ is convex and differentiable, with derivative $\psi'(\theta)$.[1]

Under (3.1) the Wiener–Hopf factorisation (2.1) remains valid for θ in the strip $\mathcal{S}_\gamma$.

Lemma 3.1. *If Assumption 1 is satisfied, we have*

$$\mathbb{E}[\mathrm{e}^{\gamma D_{\mathbf{e}_q}}] < \infty. \tag{3.2}$$

Proof. It follows from the Wiener–Hopf factorisation (2.1) that

$$\mathbb{E}[\mathrm{e}^{-\mathbf{i}\theta D_{\mathbf{e}_q}}] = q(q-\Psi(\theta))^{-1}\mathbb{E}[\mathrm{e}^{\mathbf{i}\theta U_{\mathbf{e}_q}}]^{-1} \tag{3.3}$$

for all θ in the interior of the strip $\mathcal{S}_\gamma$. We note that $\mathbb{E}[\mathrm{e}^{\mathbf{i}\theta U_{\mathbf{e}_q}}]$ is continuous and strictly positive on the set $\mathcal{A} = \{\theta : -\mathbf{i}\theta \in [0,\gamma]\}$. Moreover, $\Psi(\theta)$ can be analytically extended to $\mathcal{A}$. Indeed, note that $\Psi(\theta) = \Psi_1(\theta) + \Psi_2(\theta)$ where $\Psi_1(\theta)$ is entire function by [29, Lem. 25.6, p. 160] and $\Psi_2(\theta) = \int_{|x|>1} e^{-\gamma x}\,\widehat{\mathcal{V}}(\mathrm{d}x)$ is finite by Assumption 1 and [19, Thm. 3.6, p. 76] for a Lévy measure $\widehat{\mathcal{V}}$ of X. This, combined with the fact $\Psi(\mathbf{i}\gamma) = 0$, yields (3.2). □

In [3] it was shown that under Assumptions 1 and 2, Cramér's estimate holds for the Lévy process X, *i.e.*,

$$\mathbb{P}(D_\infty > y) \simeq C_\gamma \mathrm{e}^{-\gamma y}, \qquad C_\gamma = \frac{\widehat{\kappa}(0,0)}{\gamma\left[\frac{\partial}{\partial\theta}\widehat{\kappa}(0,-\theta)\right]_{|\theta=\gamma}} > 0, \qquad \text{as } y \to \infty, \tag{3.4}$$

where we write $f(x) \simeq g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. Cramér's estimate can be extended to the decay of the finite time probability $\mathbb{P}(D_s > x)$ when x, s jointly tend to infinity in some fixed proportion, that is when we have $x = vs + \mathrm{o}(s^{1/2})$. The proportions v are to be positive and lie in the range of ψ'. This leads to the following definition.

Definition 3.2. *A proportion $v \in \mathbb{R}_+\backslash\{0\}$ is* feasible *if there exists a $\xi_v \in \Theta^o$ such that $\psi'(\xi_v) = -v$.*

More specifically, it was shown in [25] that if the proportion v is feasible and satisfies $0 < v < -\psi'(-\gamma)$ the Höglund's estimates hold for X, *i.e.*, if Assumptions 1 and 2 are satisfied, then for x and s tending to infinity such that $x = vs + \mathrm{o}(s^{1/2})$ we have

$$\mathbb{P}(D_s > x) \sim C_\gamma \mathrm{e}^{-\gamma x}, \tag{3.5}$$

where we write $f \sim g$ if $\lim_{x,s\to\infty, x=vs+\mathrm{o}(s^{1/2})} f(x,s)/g(x,s) = 1$.

Using the representations in Proposition 2.1 we identify the exact asymptotic decay of the tail probabilities of $\underline{D}_{t,s}^*$ and $\overline{D}_{t,s}^*$ as follows:

[1] For $\theta \in \Theta\backslash\Theta^o$, $\psi'(\theta)$ is understood to be $\lim_{\eta\to\theta,\eta\in\Theta^o}\psi'(\eta)$.

Theorem 3.3. *Suppose that Assumptions 1 and 2 hold, and let* $t \in \mathbb{R}_+\backslash\{0\}$.

(i) *Then the following limit hold true:*

$$\mathbb{P}(-\underline{D}^*_t > x) \simeq C_\gamma \mathbb{E}[\mathrm{e}^{\gamma D_t}]\,\mathrm{e}^{-\gamma x}, \quad x \to \infty \tag{3.6}$$

and

$$\mathbb{P}(-\overline{D}^*_t > x) \simeq C_\gamma \mathbb{E}[\mathrm{e}^{-\gamma U_t}]\,\mathrm{e}^{-\gamma x} \quad x \to \infty. \tag{3.7}$$

(ii) *Let* $0 < v < -\psi'(-\gamma)$. *If* x *and* s *tend to infinity such that* $x = vs + \mathrm{o}(s^{1/2})$ *for some feasible proportion* v *then we have the following limits:*

$$\mathbb{P}(-\overline{D}^*_{t,s} > x) \quad \sim \quad C_\gamma \mathbb{E}[\mathrm{e}^{-\gamma U_t}]\,\mathrm{e}^{-\gamma x}, \tag{3.8}$$

$$\mathbb{P}(-\underline{D}^*_{t,s} > x) \quad \sim \quad C_\gamma \mathbb{E}[\mathrm{e}^{\gamma D_t}]\,\mathrm{e}^{-\gamma x}. \tag{3.9}$$

Remark 3.4. In specific cases the Wiener–Hopf factors are known in explicit analytical form, so that the constants in (3.6) can be identified.

(i) If X is spectrally positive, then $C_\gamma = 1$ and

$$\mathbb{E}[\mathrm{e}^{\gamma D_{\mathbf{e}_q}}] = \frac{\widehat{\Phi}(q)}{\widehat{\Phi}(q) - \gamma}, \qquad q > 0, \tag{3.10}$$

where $\gamma = \widehat{\Phi}(0)$, with $\widehat{\Phi}(q)$, $q \geq 0$, the largest root of the equation $\widehat{\psi}(\theta) = q$ where $\widehat{\psi}(\theta) = \log \mathbb{E}[\mathrm{e}^{-\theta X_1}]$ is the Laplace exponent of the dual process $\widehat{X} = -X$. These expressions hold since $D_{\mathbf{e}_q}$ has the same law $\widehat{\overline{X}}_{\mathbf{e}_q}$ and hence follows an exponential distribution with parameter $\widehat{\Phi}(q)$. By inverting the Laplace transforms in q we find the following explicit expression in terms of the one-dimensional distributions of X:

$$\mathbb{E}[\mathrm{e}^{\gamma D_t}] = 1 + \gamma \int_0^t \mathbb{E}[\mathrm{e}^{-\gamma X_z} X_z^-] z^{-1} \mathrm{d}z, \tag{3.11}$$

where $X_t^- = \min\{X_t, 0\}$. Indeed, note that $\mathbb{E}[\mathrm{e}^{\gamma D_t}] = \mathbb{E}[\mathrm{e}^{\gamma \widehat{U}_t}]$. Moreover, on account of Kendall's identity ($\mathbb{P}(\tau_x^+ \in \mathrm{d}t) = \frac{x}{t}\mathbb{P}(\widehat{X}_t \in \mathrm{d}x)$ for $x, t \in \mathbb{R}_+\backslash\{0\}$ and the first passage time $\tau_x^+ = \inf\{t \geq 0 : \widehat{X}_t > x\}$), it follows that

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{E}[\mathrm{e}^{-\gamma \widehat{X}_t} \widehat{X}_t^+] t^{-1} \mathrm{d}t = \frac{1}{\widehat{\Phi}(q) + \gamma}, \tag{3.12}$$

where $\widehat{X}_t^+ = \max\{\widehat{X}_t^+, 0\}$. Further, from [19, eq. (8.2)] and fact that $\widehat{\psi}(\gamma) = \psi(-\gamma) = 0$,

$$\mathbb{E}[\mathrm{e}^{-\gamma U_{\mathbf{e}_q}}] = \mathbb{E}[\mathrm{e}^{\gamma \widehat{\underline{X}}_{\mathbf{e}_q}}] = \frac{q}{q - \widehat{\psi}(\gamma)}\left[1 - \frac{\gamma}{\widehat{\Phi}(q)}\right] = 1 - \frac{\gamma}{\widehat{\Phi}(q)}. \tag{3.13}$$

Hence, we have

$$\mathbb{E}[\mathrm{e}^{-\gamma U_t}] = 1 - \gamma \int_0^t \mathbb{E}[X_z^-] z^{-1}\,\mathrm{d}z.$$

(ii) If X is spectrally negative, then we have $C_\gamma = \frac{\psi'(0)}{|\psi'(-\gamma)|}$ and

$$\mathbb{E}[\mathrm{e}^{-\gamma U_{\mathbf{e}_q}}]^{-1} = \mathbb{E}[\mathrm{e}^{\gamma D_{\mathbf{e}_q}}] = \frac{\Phi(q) + \gamma}{\Phi(q)}, \tag{3.14}$$

where γ and $\Phi(q)$, $q \geq 0$, are the largest roots of $\psi(-\theta) = 0$ and $\psi(\theta) = q$ for the Laplace exponent $\psi(\theta) = \log \mathbb{E}[\mathrm{e}^{\theta X_1}]$. Hence

$$\mathbb{E}[\mathrm{e}^{\gamma D_t}] = 1 + \gamma \int_0^t \mathbb{E}[X_z^+] z^{-1} \mathrm{d}z, \qquad \mathbb{E}[\mathrm{e}^{-\gamma U_t}] = 1 - \gamma \int_0^t \mathbb{E}[\mathrm{e}^{-\gamma X_z} X_z^+] z^{-1}\,\mathrm{d}z. \tag{3.15}$$

(iii) The Wiener–Hopf factors may also be identified for the meromorphic Lévy processes [17, Def. 1]:

$$\mathbb{E}[\mathrm{e}^{\gamma U_{\mathbf{e}_q}}] = \prod_{n\geq 1} \frac{1-\frac{\gamma}{\rho_n}}{1-\frac{\gamma}{\zeta_n(q)}}, \qquad \mathbb{E}[\mathrm{e}^{\gamma D_{\mathbf{e}_q}}] = \prod_{n\geq 1} \frac{1-\frac{\gamma}{\hat\rho_n}}{1-\frac{\gamma}{\hat\zeta_n(q)}},$$

where $\{-\mathbf{i}\rho_n, \mathbf{i}\hat\rho_n\}_{n\geq 1}$ are the poles of Ψ (which is meromorphic) and $\{-\mathbf{i}\zeta_n(q), \mathbf{i}\hat\zeta_n(q)\}_{n\geq 1}$ are the roots of $q+\Psi(\theta)=0$. The above Laplace transforms in q can be numerically inverted giving $\mathbb{E}[\mathrm{e}^{\gamma U_t}]$ and $\mathbb{E}[\mathrm{e}^{\gamma D_t}]$ (see for details [17, Sec. 8]).

Proof of Theorem 3.3. **(i)** From Proposition 2.1 it follows that for $s,t\in\mathbb{R}_+$,

$$\mathbb{P}(-\underline{D}^*_{t,s}\leq x) = \int_{[0,x]} \mathbb{P}(D_s \leq x-z)\mathbb{P}(D_t\in \mathrm{d}z, \overline{D}_t\leq x) \Leftrightarrow \tag{3.16}$$

$$\mathbb{P}(-\underline{D}^*_{t,s}> x) = \mathbb{P}(\overline{D}_t > x) + \int_{[0,x]} \mathbb{P}(D_s > x-z)\mathbb{P}(D_t\in \mathrm{d}z, \overline{D}_t\leq x).$$

By letting $s\to\infty$ in (3.16) we arrive at the identity

$$\mathbb{P}(-\underline{D}^*_{t}> x) = \mathbb{P}(\overline{D}_t > x) + \int_{[0,x]} \mathbb{P}(D_\infty > x-z)\mathbb{P}(D_t\in \mathrm{d}z, \overline{D}_t\leq x). \tag{3.17}$$

Denote by $\mathbb{P}^{(\gamma)}$ the Cramér measure which is defined on $(\Omega,\mathcal{F}_t)$ by $\mathbb{P}^{(\gamma)}(A)=\mathbb{E}[\mathrm{e}^{-\gamma X_t}\mathbf{1}_A]$, $A\in\mathcal{F}_t$. The Cramér asymptotic decay (3.4) implies that

$$\mathrm{e}^{\gamma x}\mathbb{P}(D_\infty > x) = \mathbb{E}^{(\gamma)}[\mathrm{e}^{-\gamma(\widehat{X}_{\tau_x^+}-x)}] \simeq C_\gamma, \quad \text{as } x\to\infty. \tag{3.18}$$

In view of the facts that $t\mapsto \overline{X}_t$ is non-decreasing and $D_{T_x^D}-x\geq 0$ for $T_x^D=\inf\{t\geq 0: D_t>x\}$ and any $x\in\mathbb{R}_+\backslash\{0\}$, we find[2]

$$\begin{aligned}
\mathbb{P}(\overline{D}_t > x) &= \mathbb{P}(T_x^D<t) = \mathrm{e}^{-\gamma x}\mathbb{E}^{(\gamma)}[\mathrm{e}^{\gamma(X_{T_x^D}+x)}\mathbf{1}_{\{T_x^D<t\}}]\\
&= \mathrm{e}^{-\gamma x}\mathbb{E}^{(\gamma)}[\mathrm{e}^{-\gamma(D_{T_x^D}-x-\overline{X}_{T_x^D})}\mathbf{1}_{\{T_x^D<t\}}]\\
&\leq \mathrm{e}^{-\gamma x}\mathbb{E}^{(\gamma)}[\mathrm{e}^{\gamma\overline{X}_t}\mathbf{1}_{\{T_x^D<t\}}] = \mathrm{o}(\mathrm{e}^{-\gamma x}), \quad \text{as } x\to\infty, \qquad (3.19)
\end{aligned}$$

where the expectation in (3.19) converges to zero by virtue of the dominated convergence theorem and the facts that $\mathbb{E}^{(\gamma)}[\mathrm{e}^{\gamma\overline{X}_t}]<\infty$ (by Lemma 3.1) and $T_x^D\to\infty$ $\mathbb{P}^{(\gamma)}$-a.s. as $x\to\infty$ (as $X_t\to-\infty$ as $t\to\infty$, $\mathbb{P}^{(\gamma)}$-a.s.). Combining (3.17) with (3.19), the Cramér asymptotics (3.18) and the dominated convergence theorem yield

$$\lim_{x\to\infty}\mathrm{e}^{\gamma x}\mathbb{P}(-\underline{D}^*_t>x) = C_\gamma\int_{\mathbb{R}_+}\mathrm{e}^{\gamma z}\mathbb{P}(D_t\in\mathrm{d}z) = C_\gamma\mathbb{E}[\mathrm{e}^{\gamma D_t}], \qquad t\in\mathbb{R}_+.$$

As far as $\overline{D}^*_t$ is concerned, we deduce from Proposition 2.1, the Cramér asymptotics (3.4), Lemma 3.1 and the dominated convergence theorem that

$$\begin{aligned}
\mathbb{P}(\overline{D}^*_t > x) &= \int_{\mathbb{R}_+}\mathbb{P}(D_\infty > x+z)\mathbb{P}(U_t\in\mathrm{d}z) \qquad (3.20)\\
&\simeq C_\gamma\mathrm{e}^{-\gamma x}\int_{\mathbb{R}_+}\mathrm{e}^{-\gamma z}\mathbb{P}(U_t\in\mathrm{d}z) = C_\gamma\mathrm{e}^{-\gamma x}\mathbb{E}[\mathrm{e}^{-\gamma U_t}].
\end{aligned}$$

(ii) Let v be a feasible proportion. The proof follows by a line of reasoning that is analogous to the one given in part (i), deploying Höglund's estimate (3.5) instead of Cramér's estimate. In particular, combining (3.5), (3.16), (3.19) and the dominated convergence theorem shows that when $0<v<-\psi'(-\gamma)$

$$\mathrm{e}^{\gamma x}\mathbb{P}(-\underline{D}^*_{t,s}>x)\sim C_\gamma\int_{[0,\infty)}\mathrm{e}^{\gamma z}\mathbb{P}(D_t\in\mathrm{d}z) = C_\gamma\mathbb{E}[\mathrm{e}^{\gamma D_t}].$$

□

[2] $f(x)=o(g(x))$ for $x\to\infty$ if $|f(x)/g(x)|\to 0$ as $x\to\infty$.

3.1. **Asymptotic drawdown and drawup measures.** Conditional on $-\overline{D}^*_{t,s}$ being large, for fixed $s,t \in \mathbb{R}_+$, or on $-\underline{D}^*_{t,s}$ being large, X_t admits a limit in distribution, as we show next. These limits are given by the "drawup-measures" $\overline{\mathbb{P}}^{(s)}$ and the "drawdown measures" $\underline{\mathbb{P}}^{(s)}$, $s \in \Theta$, that are defined as follows on the measurable space $(\Omega, \mathcal{F}_t)$:

$$\overline{\mathbb{P}}^{(s)}(A) = \mathbb{E}\left[\frac{\mathrm{e}^{-sU_t}}{\mathbb{E}[\mathrm{e}^{-sU_t}]}\mathbf{1}_A\right], \qquad \underline{\mathbb{P}}^{(s)}(A) = \mathbb{E}\left[\frac{\mathrm{e}^{sD_t}}{\mathbb{E}[\mathrm{e}^{sD_t}]}\mathbf{1}_A\right], \qquad A \in \mathcal{F}_t. \tag{3.21}$$

Corollary 3.5. *Suppose Assumptions 1 and 2 hold, and let $t \in \mathbb{R}_+\backslash\{0\}$.*

(i) *Then, conditional on $\{\overline{D}^*_t < -x\}$ and on $\{\underline{D}^*_t < -x\}$, X_t converges in distribution as $x \to \infty$:*

$$\mathbb{P}[X_t \le x| -\underline{D}^*_t > x] \quad \simeq \quad \underline{\mathbb{P}}^{(\gamma)}[X_t \le x], \tag{3.22}$$

$$\mathbb{P}[X_t \le x| -\overline{D}^*_t > x] \quad \simeq \quad \overline{\mathbb{P}}^{(\gamma)}[X_t \le x]. \tag{3.23}$$

(ii) *Let $0 < v < -\psi'(-\gamma)$. If x and s tend to infinity such that $x = vs + \mathrm{o}(s^{1/2})$ where v is feasible then the following limits hold true:*

$$\mathbb{P}[X_t \le x| -\underline{D}^*_{t,s} > x] \quad \sim \quad \underline{\mathbb{P}}^{(\gamma)}[X_t \le x], \tag{3.24}$$

$$\mathbb{P}[X_t \le x| -\overline{D}^*_{t,s} > x] \quad \sim \quad \overline{\mathbb{P}}^{(\gamma)}[X_t \le x]. \tag{3.25}$$

Proof of Corollary 3.5. **(i)** By following a similar line of reasoning as the proof of Theorem 3.3 it is straightforward to show that for $\theta \in [0,\gamma]$, as $x \to \infty$,

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{\theta X_t}\mathbf{1}_{\{-\underline{D}^*_t > x\}}] \quad &\simeq \quad C_\gamma \mathrm{e}^{-\gamma x}\mathbb{E}[\mathrm{e}^{\theta X_t + \gamma D_t}],\\
\mathbb{E}[\mathrm{e}^{\theta X_t}\mathbf{1}_{\{-\overline{D}^*_t > x\}}] \quad &\simeq \quad C_\gamma \mathrm{e}^{-\gamma x}\mathbb{E}[\mathrm{e}^{\theta X_t - \gamma U_t}].
\end{aligned}$$

Bayes' lemma then yields the stated identities. The proof of (ii) is similar and is omitted. ☐

4. Asymptotic future drawdown — the heavy-tailed case

We continue the study of the asymptotic behaviour of the tail probabilities of $\overline{D}^*_t$ and $\underline{D}^*_t$ in the case that the Lévy measure $\mathcal{V}$ of $\widehat{X} = -X$ belongs to the class $\mathcal{S}^{(\alpha)}$ of convolution-equivalent measures which, we recall, is a subset of the class $\mathcal{L}^{(\alpha)}$ defined as follows.

Definition 4.1. *(Class $\mathcal{L}^{(\alpha)}$) For a parameter $\alpha \in \mathbb{R}_+$ we say that measure G with tail $\overline{G}(u) := G((u,\infty))$ belongs to class $\mathcal{L}^{(\alpha)}$ if*

- **(i)** *$\overline{G}(u) > 0$ for each $u \in \mathbb{R}_+$,*
- **(ii)** $\lim_{u\to\infty} \frac{\overline{G}(u-y)}{\overline{G}(u)} = \mathrm{e}^{\alpha y}$ *for each $y \in \mathbb{R}$, and G is nonlattice,*
- **(iii)** $\lim_{n\to\infty} \frac{\overline{G}(n-1)}{\overline{G}(n)} = \mathrm{e}^{\alpha}$ *if G is lattice (then assumed of span 1).*

Definition 4.2. *(Class $\mathcal{S}^{(\alpha)}$) We say that G belongs to class $\mathcal{S}^{(\alpha)}$ if*

- **(i)** *$G \in \mathcal{L}^{(\alpha)}$;*
- **(ii)** *for some $M_0 \in \mathbb{R}_+$, we have*

$$\lim_{u\to\infty} \frac{\overline{G^{*2}}(u)}{\overline{G}(u)} = 2M_0, \tag{4.1}$$

where $\overline{G^{*2}}(u) = G^{*2}(u,\infty)$ and $*$ denotes convolution.

The asymptotics are derived under conditions on the Lévy measure Π of the downward ladder height process $\widehat{H}$, which according to the Vigon [31] identity is related to the Lévy measures $\mathcal{V}$ of $\widehat{X}$ by

$$\overline{\Pi}(z) = \Pi((z,\infty)) = -\int_{\mathbb{R}\backslash\mathbb{R}_+} \overline{\mathcal{V}}(u-y)V(\mathrm{d}y), \qquad z \in \mathbb{R}_+,$$

for the renewal measure $V(\mathrm{d}y) = \int_0^\infty \mathbb{P}(H_t \in \mathrm{d}y)\mathrm{d}t$ and $\overline{\mathcal{V}}(y) = \mathcal{V}(y,\infty)$. Throughout this section we assume that for some fixed $\alpha \in \mathbb{R}_+\backslash\{0\}$ the following three conditions hold true:

$$\overline{\Pi} \in \mathcal{S}^{(\alpha)}; \tag{4.2}$$

$$\widehat{\psi}(\alpha) = \psi(-\alpha) \in \mathbb{R}\backslash\mathbb{R}_+; \tag{4.3}$$

$$\widehat{\kappa}(0,0) + \widehat{\kappa}(0,-\alpha) \in \mathbb{R}_+\backslash\{0\}. \tag{4.4}$$

Theorem 4.3. *Assume that* $\mathbb{E}[X_1] \in \mathbb{R}_+\backslash\{0\}$ *and let* $t \in \mathbb{R}_+\backslash\{0\}$. *Under conditions* (4.2)–(4.4) *we have:*

$$\mathbb{P}(-\underline{D}_t^* > x) \simeq \mathrm{const}_t^+\overline{\Pi}(x), \qquad \mathbb{P}(-\overline{D}_t^* > x) \simeq \mathrm{const}_t^-\overline{\Pi}(x),$$

where functions const_t^+ *and* $\mathrm{const}_t^- \in \mathbb{R}_+$ *are given by*

$$\mathrm{const}_t^+ = \mathbb{E}[\mathrm{e}^{\alpha\overline{\widehat{X}}_t}] + \int_{[0,t]} \mathbb{E}\left[\mathrm{e}^{\alpha\underline{\widehat{X}}_{t-z}}\right]^{-1}\mu(\mathrm{d}z) = \mathbb{E}[\mathrm{e}^{-\alpha\underline{X}_t}] + \int_{[0,t]} \mathbb{E}\left[\mathrm{e}^{-\alpha\overline{X}_{t-z}}\right]^{-1}\mu(\mathrm{d}z), \tag{4.5}$$

and

$$\mathrm{const}_t^- = \mathbb{E}[\mathrm{e}^{-\alpha\underline{\widehat{X}}_t}] = \mathbb{E}[\mathrm{e}^{\alpha\overline{X}_t}],$$

with the Borel measure μ *on* $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ *given by*

$$\mu(\mathrm{d}z) = \int_0^\infty \mathbb{P}(\widehat{L}_m^{-1} \in \mathrm{d}z)\mathrm{e}^{-\widehat{\kappa}(0,-\alpha)m}\left[1 - m\widehat{\kappa}(0,-\alpha)\right]\mathrm{d}m. \tag{4.6}$$

Remark 4.4. **(i)** By straightforward calculations it can be verified that

$$(\mathcal{L}\mu)(q) = \frac{1}{q}\cdot\frac{\widehat{\kappa}(q,0)}{(\widehat{\kappa}(q,0)+\widehat{\kappa}(0,-\alpha))^2}, \tag{4.7}$$

where $\mathcal{L}\mu$ denotes the Laplace-Stieltjes transform of the measure μ.

(ii) If $\mathcal{V} \in \mathcal{S}^{(\alpha)}$ for $\alpha > 0$ then (4.2) holds and

$$\overline{\Pi}(x) \simeq \frac{1}{\kappa(0,-\alpha)}\overline{\mathcal{V}}(x);$$

see [15, Proposition 5.3].

(iii) If X is spectrally positive, then from (3.11) and (3.13):

$$\mathbb{E}[\mathrm{e}^{-\alpha\underline{X}_t}] = 1 + \alpha\int_0^t \mathbb{E}[\mathrm{e}^{-\alpha X_z}X_z^-]z^{-1}\mathrm{d}z, \qquad \mathbb{E}[\mathrm{e}^{\pm\alpha\overline{X}_t}] = \mathrm{e}^{t\psi(\mp\alpha)} \pm \alpha\int_0^t \mathrm{e}^{(t-z)\psi(\mp\alpha)}\mathbb{E}X_z^- z^{-1}\,\mathrm{d}z. \tag{4.8}$$

Moreover, since $\widehat{\kappa}(q,0) = q/\widehat{\Phi}(q)$ and $\widehat{\kappa}(0,-\alpha) = -\widehat{\psi}(\alpha)/(\widehat{\Phi}(0)+\alpha)$, we have

$$q\,(\mathcal{L}\mu)(q) = \frac{\widehat{\kappa}(q,0)}{(\widehat{\kappa}(q,0)+\widehat{\kappa}(0,-\alpha))^2} = \frac{q\widehat{\Phi}(q)}{(q-\widehat{\Phi}(q)^2\frac{\widehat{\psi}(\alpha)}{\widehat{\Phi}(0)+\alpha})^2}.$$

Proof of Theorem 4.3. We first prove the statement concerning $\underline{D}_t^*$. The starting point of the proof is to take the identity noted earlier in (3.17) and replace the fixed time t by an independent exponential random variable $\mathbf{e}_q$ with parameter q, which yields

$$\mathbb{P}(-\underline{D}_{\mathbf{e}_q}^* > x) = \mathbb{P}(\overline{D}_{\mathbf{e}_q} > x) + \int_{[0,x]} \mathbb{P}(D_\infty > x-z)\mathbb{P}(D_{\mathbf{e}_q} \in \mathrm{d}z, \overline{D}_{\mathbf{e}_q} \le x). \tag{4.9}$$

We show that both terms on the right-hand side of (4.9) are asymptotically equivalent to the tail-measure $\overline{\Pi}(x)$ of the ladder process $\widehat{H}$ as $x \to \infty$ and identify the constant. As before we denote the first upward and downward passage times of $\widehat{X}$ across the level x by $\tau_x^+ = \inf\{t \ge 0 : \widehat{X}_t > x\}$ and $\tau_x^- = \inf\{t \ge 0 : \widehat{X}_t < x\}$.

To establish this result it suffices to show asymptotic equivalence of the two terms on the right-hand side of (4.9) to the probability $\mathbb{P}(\tau_x^+ < \mathbf{e}_q)$, since it is known from [15, Theorem 4.1] and [24, Lemma 5.4, eq. (5.6)] that under the conditions stated in the theorem

$$\mathbb{P}(\tau_x^+ < \mathbf{e}_q) \simeq \frac{\widehat{\kappa}(q,0)}{(\widehat{\kappa}(q,0)+\widehat{\kappa}(0,-\alpha))^2}\cdot\overline{\Pi}(x), \qquad q \ge 0, \tag{4.10}$$

with the interpretation $\mathbb{P}(\tau_x^+ < \infty) = \mathbb{P}(\tau_x^+ < \mathbf{e}_0)$ for $q = 0$. Note that the constant in (4.10) is strictly positive for all $q \geq 0$ by the condition (4.4) and $\widehat{\kappa}(0,0) > 0$ (as $\mathbb{E}[\widehat{X}_1]$ is strictly negative by the assumption that $\mathbb{E}[X_1] > 0$).

We treat both terms separately, starting with the first term. We first derive upper and lower bounds for the ratio $\mathbb{P}(\overline{D}_{\mathbf{e}_q} > x)/\mathbb{P}(\tau_x^+ < \mathbf{e}_q)$. By an application of the strong Markov property and the definition of $\overline{U}$ we have

$$\begin{aligned}
\mathbb{P}(\overline{D}_{\mathbf{e}_q} > x) &\geq \mathbb{P}(\tau_x^+ < \tau_{-\epsilon}^- \wedge \mathbf{e}_q, \overline{D}_{\mathbf{e}_q} > x) + \mathbb{P}(\tau_{-\epsilon}^- < \tau_x^+ \wedge \mathbf{e}_q, \overline{D}_{\mathbf{e}_q} > x) \\
&= \mathbb{P}(\tau_x^+ < \tau_{-\epsilon}^- \wedge \mathbf{e}_q) + \mathbb{P}(\tau_{-\epsilon}^- < \tau_x^+ \wedge \mathbf{e}_q)\mathbb{P}(\overline{D}_{\mathbf{e}_q} > x) \qquad \text{and} \qquad (4.11)\\
\mathbb{P}(\overline{D}_{\mathbf{e}_q} > x) &\leq \mathbb{P}(\tau_x^+ < \tau_{-\epsilon}^- \wedge \mathbf{e}_q) + \mathbb{P}(\tau_{-\epsilon}^- < \tau_x^+ \wedge \mathbf{e}_q)\mathbb{P}(\overline{D}_{\mathbf{e}_q} > x) + A_q \qquad \text{with}\\
A_q &= \mathbb{P}(\underline{\widehat{X}}_{\mathbf{e}_q} > -\epsilon, x+\epsilon \geq \widehat{X}_{\mathbf{e}_q} - \underline{\widehat{X}}_{\mathbf{e}_q} \geq x) = \mathbb{P}(\underline{\widehat{X}}_{\mathbf{e}_q} > -\epsilon)\mathbb{P}(x+\epsilon \geq \overline{\widehat{X}}_{\mathbf{e}_q} \geq x), \qquad (4.12)
\end{aligned}$$

where in the last line we used that $\underline{\widehat{X}}_{\mathbf{e}_q}$ and $\widehat{X}_{\mathbf{e}_q} - \underline{\widehat{X}}_{\mathbf{e}_q}$ are independent (by the Wiener–Hopf factorisation) and $\widehat{X}_{\mathbf{e}_q} - \underline{\widehat{X}}_{\mathbf{e}_q}$ and $\overline{\widehat{X}}_{\mathbf{e}_q}$ have the same distribution. Hence we find from (4.11) and (4.12) that

$$\frac{\mathbb{P}(\overline{D}_{\mathbf{e}_q} > x)}{\mathbb{P}(\tau_x^+ < \mathbf{e}_q)} \geq \frac{\mathbb{P}(\tau_x^+ < \tau_{-\epsilon}^- \wedge \mathbf{e}_q)}{\mathbb{P}(\tau_{-\epsilon}^- \geq \tau_x^+ \wedge \mathbf{e}_q)\mathbb{P}(\tau_x^+ < \mathbf{e}_q)} \qquad \text{and} \tag{4.13}$$

$$\frac{\mathbb{P}(\overline{D}_{\mathbf{e}_q} > x)}{\mathbb{P}(\tau_x^+ < \mathbf{e}_q)} \leq \frac{\mathbb{P}(\tau_x^+ < \tau_{-\epsilon}^- \wedge \mathbf{e}_q)}{\mathbb{P}(\tau_{-\epsilon}^- \geq \tau_x^+ \wedge \mathbf{e}_q)\mathbb{P}(\tau_x^+ < \mathbf{e}_q)} + \frac{\mathbb{P}(\tau_x^+ < \mathbf{e}_q) - \mathbb{P}(\tau_{x+\epsilon}^+ < \mathbf{e}_q)}{\mathbb{P}(\tau_x^+ < \mathbf{e}_q)}. \tag{4.14}$$

The first terms on the right-hand sides of (4.13) and (4.14) may be simplified by using that, by the Markov property, we have

$$\begin{aligned}
\mathbb{P}(\tau_x^+ < \tau_{-\epsilon}^- \wedge \mathbf{e}_q) &= \mathbb{P}(\tau_x^+ < \mathbf{e}_q) - \mathbb{P}(\tau_{-\epsilon}^- < \tau_x^+ < \mathbf{e}_q) \qquad (4.15)\\
&= \mathbb{P}(\tau_x^+ < \mathbf{e}_q) - \mathbb{E}\left[\mathbf{1}_{\{\tau_{-\epsilon}^- < \tau_x^+ \wedge \mathbf{e}_q\}} \mathbb{P}_{\widehat{X}_{\tau_{-\epsilon}^-}}(\tau_x^+ < \mathbf{e}_q)\right].
\end{aligned}$$

Furthermore, since $\overline{\Pi} \in \mathcal{S}^{(\alpha)}$ we note that

$$\lim_{x\to\infty} \frac{\mathbb{P}(\tau_{x+\epsilon}^+ < \mathbf{e}_q)}{\mathbb{P}(\tau_x^+ < \mathbf{e}_q)} = \mathrm{e}^{-\alpha\epsilon}, \qquad \epsilon > 0. \tag{4.16}$$

From the dominated convergence theorem and Definition 4.1(ii)–(iii) it then follows that

$$\lim_{x\to\infty} \frac{\mathbb{E}\left[\mathbf{1}_{\{\tau_{-\epsilon}^- < \tau_x^+ \wedge \mathbf{e}_q\}} \mathbb{P}_{\widehat{X}_{\tau_{-\epsilon}^-}}(\tau_x^+ < \mathbf{e}_q)\right]}{\mathbb{P}(\tau_x^+ < \mathbf{e}_q)} = \mathbb{E}\left[\mathrm{e}^{\alpha \widehat{X}_{\tau_{-\epsilon}^-}} \mathbf{1}_{\{\tau_{-\epsilon}^- < \mathbf{e}_q\}}\right], \tag{4.17}$$

and an application of the Markov property yields

$$\mathbb{E}\left[\mathrm{e}^{\alpha \widehat{X}_{\tau_{-\epsilon}^-}} \mathbf{1}_{\{\tau_{-\epsilon}^- < \mathbf{e}_q\}}\right] = \frac{\mathbb{E}\left[\mathrm{e}^{\alpha \underline{\widehat{X}}_{\mathbf{e}_q}} \mathbf{1}_{\{\tau_{-\epsilon}^- < \mathbf{e}_q\}}\right]}{\mathbb{E}\left[\mathrm{e}^{\alpha \underline{\widehat{X}}_{\mathbf{e}_q}}\right]}. \tag{4.18}$$

Taking first $x \to \infty$ in (4.13) and (4.14) and using (4.15), (4.16), (4.17) and (4.18) and that $\mathbb{P}[\tau_{-\epsilon}^- = \mathbf{e}_q] = 0$ we find

$$\frac{\mathbb{E}\left[\mathrm{e}^{\alpha \underline{\widehat{X}}_{\mathbf{e}_q}} \middle| \tau_{-\epsilon}^- > \mathbf{e}_q\right]}{\mathbb{E}\left[\mathrm{e}^{\alpha \underline{\widehat{X}}_{\mathbf{e}_q}}\right]} \leq \liminf_{x\to\infty} \frac{\mathbb{P}(\overline{D}_{\mathbf{e}_q} > x)}{\mathbb{P}(\tau_x^+ < \mathbf{e}_q)} \leq \limsup_{x\to\infty} \frac{\mathbb{P}(\overline{D}_{\mathbf{e}_q} > x)}{\mathbb{P}(\tau_x^+ < \mathbf{e}_q)} \leq \frac{\mathbb{E}\left[\mathrm{e}^{\alpha \underline{\widehat{X}}_{\mathbf{e}_q}} \middle| \tau_{-\epsilon}^- > \mathbf{e}_q\right]}{\mathbb{E}\left[\mathrm{e}^{\alpha \underline{\widehat{X}}_{\mathbf{e}_q}}\right]} + 1 - \mathrm{e}^{-\alpha\epsilon}.$$

Letting subsequently $\epsilon \downarrow 0$ and using

$$\lim_{\epsilon\downarrow 0} \mathbb{E}\left[\mathrm{e}^{\alpha \underline{\widehat{X}}_{\mathbf{e}_q}} \middle| \tau_{-\epsilon}^- > \mathbf{e}_q\right] = 1,$$

which in turn holds as the conditional expectation is bounded above by 1 and bounded below by $\mathrm{e}^{-\alpha\epsilon}$, we get the following asymptotics:

$$\mathbb{P}(\overline{D}_{\mathbf{e}_q} > x) \simeq B_q\overline{\Pi}(x), \qquad \text{with} \tag{4.19}$$
$$B_q = \frac{\widehat{\kappa}(q,0)}{(\widehat{\kappa}(q,0)+\widehat{\kappa}(0,-\alpha))^2}\frac{1}{\mathbb{E}\left[\mathrm{e}^{\alpha\underline{\widehat{X}}_{\mathbf{e}_q}}\right]}.$$

Next, we turn to the proof of the asymptotic decay of the second term on the right-hand side of (4.9). Note that it equals

$$\int_{[0,x]}\mathbb{P}(\overline{\widehat{X}}_\infty > x-z)\mathbb{P}(D_{\mathbf{e}_q}\in \mathrm{d}z, \overline{D}_{\mathbf{e}_q}\le x)$$
$$= \left(\int_{[0,y']}+\int_{(y',x-y']}+\int_{(x-y',x]}\right)\mathbb{P}(\overline{\widehat{X}}_\infty > x-z)\mathbb{P}(D_{\mathbf{e}_q}\in \mathrm{d}z, \overline{D}_{\mathbf{e}_q}\le x). \tag{4.20}$$

We next show that the second and third integral of the right-hand side of (4.20) tend to zero as we let first x and then y tend to infinity. Indeed, concerning the second integral we use (4.2), Definition 4.1(ii)–(iii) and (4.10) to show that

$$\lim_{x\to\infty}\frac{\int_{(y',x-y']}\mathbb{P}(\overline{\widehat{X}}_\infty > x-z)\mathbb{P}(D_{\mathbf{e}_q}\in \mathrm{d}z, \overline{D}_{\mathbf{e}_q}\le x)}{\mathbb{P}(\tau_x^+<\infty)} = \int_{(y',\infty)}\mathrm{e}^{\alpha z}\mathbb{P}(\overline{\widehat{X}}_{\mathbf{e}_q}\in \mathrm{d}z),$$

which tends to 0 as $y'\to\infty$.

For the third integral, we obtain the bound

$$\begin{aligned}\int_{(x-y',x]}\mathbb{P}(\overline{\widehat{X}}_\infty > x-z)\mathbb{P}(D_{\mathbf{e}_q}\in \mathrm{d}z, \overline{D}_{\mathbf{e}_q}\le x) &\le \mathbb{P}(\overline{\widehat{X}}_\infty > y')\mathbb{P}(\overline{\widehat{X}}_{\mathbf{e}_q} > x-y')\\ &\le \mathbb{P}(\tau_{y'}^+<\infty)\mathbb{P}(\tau_{x-y'}^+<\infty).\end{aligned}$$

After dividing the integral in the display by $\mathbb{P}(\tau_x^+<\infty)$ and letting first $x\to\infty$ and then $y'\to\infty$, it tends to zero.

Finally, the first integral on the right-hand side of (4.20) is asymptotically of the same order as the left-hand side. Indeed, using (4.2) and Definition 4.1(ii)–(iii), (4.10) and the dominated convergence theorem we find

$$\lim_{x\to\infty}\frac{\int_{[0,y']}\mathbb{P}(\overline{\widehat{X}}_\infty > x-z)\mathbb{P}(D_{\mathbf{e}_q}\in \mathrm{d}z, \overline{D}_{\mathbf{e}_q}\le x)}{\mathbb{P}(\tau_x^+<\infty)} = \int_{[0,y']}\mathrm{e}^{\alpha z}\mathbb{P}(D_{\mathbf{e}_q}\in \mathrm{d}z), \tag{4.21}$$

which converges to $\int_0^\infty \mathrm{e}^{\alpha z}\mathbb{P}(D_{\mathbf{e}_q}\in \mathrm{d}z) = \mathbb{E}[\mathrm{e}^{\alpha\overline{X}_{\mathbf{e}_q}}] := \widetilde{B}_q$ as $y'\to\infty$.

By combining the previous estimates we have the following asymptotics of the tail probability $\mathbb{P}(-\underline{D}^*_{\mathbf{e}_q} > x)$:

$$\lim_{x\to\infty}\frac{\mathbb{P}(-\underline{D}^*_{\mathbf{e}_q} > x)}{q\overline{\Pi}(x)} = q^{-1}(B_q+\widetilde{B}_q). \tag{4.22}$$

Noting that the right-hand side of (4.22) is a pointwise limit of Laplace transforms of measures and is itself such a Laplace transform, it follows from (an extension of) the continuity theorem (see [13, Theorem 15.5.2]) that the corresponding measures also converge to the limiting measure with Laplace transform given by $q^{-1}(B_q+\widetilde{B}_q)$. Hence the first assertion of the theorem follows by inverting the Laplace transform $q^{-1}(B_q+\widetilde{B}_q)$ (see Remark 4.4).

Concerning $\overline{D}^*_t$, note that by (3.20) we have

$$\mathbb{P}(-\overline{D}^*_t > x) = \int_{(-\infty,0]}\mathbb{P}(\tau^+_{x+z}<\infty)\mathbb{P}(\underline{\widehat{X}}_t\in \mathrm{d}z).$$

Asymptotics (4.10), the dominated convergence theorem and part (ii) and (iii) of Definition 4.1 establish that the asymptotic decay of $\mathbb{P}(-\overline{D}^*_t > x)$ is as stated. □

5. Exact distributions

From Proposition 2.1 it follows that the distributions of $\overline{D}^*_{t,s}$, $\underline{D}^*_{t,s}$, $\overline{U}^*_{t,s}$ and $\underline{U}^*_{t,s}$ can be identified if one is able to identify the law of the finite time supremum and the resolvent of the Lévy process reflected at its infimum. In the case of a spectrally one-sided Lévy process X such explicit expressions are provided by existing fluctuation theory.

In this section we suppose that X is spectrally negative (as noted in the Introduction, the case of spectrally positive Lévy process follows from by considering the dual of X). Many fluctuation results for X can be conveniently formulated in terms of its scale function $W^{(q)}$ that is defined as the unique continuous increasing function on $\mathbb{R}_+$ with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\lambda)-q} \quad \text{for any } \lambda > \Phi(q).$$

Note that by convexity of the Laplace exponent ψ its right inverse $\Phi(q)$ is well-defined for all $q \geq 0$. Moreover, let $Z^{(q)}$ denote the function on $\mathbb{R}_+$ given by

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\mathrm{d}y, \quad x \in \mathbb{R}_+,$$

let $\mathbf{e}_\beta$ be an exponentially distributed random variable with parameter $\beta > 0$ (independent of $\mathbf{e}_q$ and X).

Proposition 5.1. *Let $x \in \mathbb{R}_+$.* **(i)** *If $\mathbb{E}[X_1] \in \mathbb{R} \cup \{-\infty\} \backslash \mathbb{R}_+$ then*

$$\begin{aligned}
\mathbb{P}(\overline{U}^*_{\mathbf{e}_q,\mathbf{e}_\beta} > x) &= \frac{1}{Z^{(q)}(x)}\left[1 + q\int_0^x \mathrm{e}^{-\Phi(\beta)z} W^{(\beta)}(z)\mathrm{d}z\right] \qquad \textit{and}\\
\mathbb{P}(\underline{U}^*_{\mathbf{e}_q,\mathbf{e}_\beta} > x) &= \frac{q}{q-\beta}\mathrm{e}^{-\Phi(\beta)x}\frac{\Phi(\beta)-\Phi(q)}{\Phi(q)}.
\end{aligned}$$

(ii) *If $\mathbb{E}[X_1] \in \mathbb{R}_+\backslash\{0\}$ then*

$$\begin{aligned}
\mathbb{P}(-\overline{D}^*_{\mathbf{e}_q,\mathbf{e}_\beta} > x) &= \Phi(q)\int_0^\infty \mathrm{e}^{-\Phi(q)z} Z^{(\beta)}(x+z)\mathrm{d}z - \frac{\beta}{\Phi(\beta)}\Phi(q)\int_0^\infty \mathrm{e}^{-\Phi(q)z} W^{(\beta)}(x+z)\mathrm{d}z \qquad \textit{and}\\
\mathbb{P}(-\underline{D}^*_{\mathbf{e}_q,\mathbf{e}_\beta} > x) &= q\frac{\beta}{\Phi(\beta)}\int_{[0,x]} (W^{(\beta)}(x-z) - \beta Z^{(\beta)}(x-z))W^{(q)}(z)\mathrm{d}z\\
&\quad - \frac{\beta}{\Phi(\beta)}\frac{W^{(q)}(x)}{W^{(q)\prime}_+(x)}\int_{[0,x]} (W^{(\beta)}(x-z) - \beta Z^{(\beta)}(x-z))W^{(q)}(\mathrm{d}z)\\
&\quad + Z^{(q)}(x) - q\frac{W^{(q)}(x)^2}{W^{(q)\prime}_+(x)},
\end{aligned}$$

where $W^{(q)\prime}_+(x)$ denotes the right-derivative of $W^{(q)}$ at x.

The proof of Proposition 5.1 is based on the representations derived in Proposition 2.1 and the form of the q-resolvent measures R^U_x and R^D_x of U and D killed upon crossing the level $x > 0$, which are defined by

$$R^U_x(\mathrm{d}y) = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}(U_t \in \mathrm{d}y, T^U_x > t)\mathrm{d}t \qquad \text{and} \qquad R^D_x(\mathrm{d}y) = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}(D_t \in \mathrm{d}y, T^D_x > t)\mathrm{d}t,$$

where T^U_x and T^D_x are the first-passage times of U and D over x, $T^U_x = \inf\{t \geq 0 : U_t > x\}$, $T^D_x = \inf\{t \geq 0 : D_t > x\}$. In [26, Theorem 1] it was shown that these resolvent measures have a density a version of which is given by

$$R^U_x(\mathrm{d}y) = \frac{W^{(q)}(x-y)}{Z^{(q)}(x)}\mathrm{d}y, \qquad y \in [0,x], \tag{5.1}$$

$$R^D_x(\mathrm{d}y) = W^{(q)}(x)\frac{W^{(q)}(\mathrm{d}y)}{W^{(q)\prime}_+(x)} - W^{(q)}(y)\mathrm{d}y, \qquad y \in [0,x]. \tag{5.2}$$

Proof. Recall that by Proposition 2.1 we have

$$\mathbb{P}(\overline{U}^*_{\mathbf{e}_q,\mathbf{e}_\beta} > x) = \mathbb{E}\left[\mathrm{e}^{-qT_x^U}\right] + q\int_{[0,x]} \mathbb{P}(\overline{X}_{\mathbf{e}_\beta} > x-z) R_x^U(\mathrm{d}z),$$

where by [26, Proposition 2],

$$\mathbb{E}\left[\mathrm{e}^{-qT_x^U}\right] = \frac{1}{Z^{(q)}(x)}$$

and

$$\mathbb{P}(U_{\mathbf{e}_\beta} > x-z) = \mathbb{P}(\overline{X}_{\mathbf{e}_\beta} > x-z) = \mathrm{e}^{-\Phi(\beta)(x-z)}.$$

Similarly,

$$\mathbb{P}(\underline{U}^*_{\mathbf{e}_q,\mathbf{e}_\beta} > x) = \int_0^\infty \mathbb{P}(U_{\mathbf{e}_\beta} > x+z)\mathbb{P}(D_{\mathbf{e}_q} \in \mathrm{d}z),$$

where by [20]:

$$\mathbb{P}(D_{\mathbf{e}_q} \in \mathrm{d}z) = \mathbb{P}(-\underline{X}_{\mathbf{e}_q} \in \mathrm{d}z) = \frac{q}{\Phi(q)}\, W^{(q)}(\mathrm{d}z) - qW^{(q)}(z)\,\mathrm{d}z, \quad z \in \mathbb{R}_+.$$

Straightforward calculations complete the proof of (i).

The proof of (ii) follows by a similar reasoning using the identity

$$\mathbb{E}\left[\mathrm{e}^{-qT_x^D}\right] = Z^{(q)}(x) - q\frac{W^{(q)}(x)^2}{W_+^{(q)\prime}(x)};$$

see [26, Proposition 2]. □

Corollary 5.2. *Let* $x \in \mathbb{R}_+$. **(i)** *If* $\mathbb{E}[X_1] \in \mathbb{R}\cup\{-\infty\}\backslash\mathbb{R}_+$

$$\begin{aligned}\mathbb{P}(\overline{U}^*_{\mathbf{e}_q} > x) &= \frac{1}{Z^{(q)}(x)}\left[1 + q\int_0^x \mathrm{e}^{-\Phi(0)z}W^{(q)}(z)\mathrm{d}z\right] \qquad \textit{and}\\ \mathbb{P}(\underline{U}^*_{\mathbf{e}_q} > x) &= \mathrm{e}^{-\Phi(0)x}\frac{\Phi(q)-\Phi(0)}{\Phi(q)}.\end{aligned}$$

(ii) *If* $\mathbb{E}[X_1] \in \mathbb{R}_+\backslash\{0\}$ *then*

$$\begin{aligned}\mathbb{P}(-\overline{D}^*_{\mathbf{e}_q} > x) &= 1 - \psi'(0)\Phi(q)\int_0^\infty \mathrm{e}^{-\Phi(q)z}W(x+z)\mathrm{d}z \qquad \textit{and}\\ \mathbb{P}(-\underline{D}^*_{\mathbf{e}_q} > x) &= 1 + q\psi'(0)\int_{[0,x]} W(x-z)W^{(q)}(z)\mathrm{d}z\\ &\quad -\psi'(0)\frac{W^{(q)}(x)}{W_+^{(q)\prime}(x)}\int_{[0,x]} W(x-z)W^{(q)}(\mathrm{d}z).\end{aligned}$$

Proof. Note that by negative drift condition $\mathbb{E}[X_1] \in \mathbb{R}\cup\{-\infty\}\backslash\mathbb{R}_+$ we have that $\psi'(0) = \mathbb{E}[X_1] < 0$ and by convexity of ψ we can conclude that $\Phi(0) > 0$. Moreover, since U_∞ has the same law as $\overline{X}_\infty$, which follows an exponential distribution with parameter $\Phi(0)$, we have for any $x \in \mathbb{R}_+$

$$\begin{aligned}\mathbb{P}(\overline{U}^*_{\mathbf{e}_q} \le x) &= \int_{[0,x]}\int_0^y \Phi(0)\mathrm{e}^{-\Phi(0)z}\mathrm{d}z\mathbb{P}(z + U_{\mathbf{e}_q} \in \mathrm{d}y, \mathbf{e}_q < T_x^U)\\ &= \frac{q\Phi(0)}{Z^{(q)}(x)}\int_0^x\int_0^y \mathrm{e}^{-\Phi(0)z}W^{(q)}(x-y+z)\mathrm{d}z\,\mathrm{d}y\\ &= \frac{1}{Z^{(q)}(x)}\left[q\int_0^x (1-\mathrm{e}^{-\Phi(0)y})W^{(q)}(y)\,\mathrm{d}y\right].\end{aligned}$$

Furthermore, from (3.10),

$$\begin{aligned}\mathbb{P}(\underline{U}^*_{\mathbf{e}_q} \le x) &= \mathbb{P}(\widetilde{U}^*_0 - D_{\mathbf{e}_q} \le x)\\ &= \Phi(0)\int_{\mathbb{R}_+}\int_{-y}^x \mathrm{e}^{-\Phi(0)(z+y)}\mathrm{d}z\mathbb{P}(D_{\mathbf{e}_q} \in \mathrm{d}y) = 1 - \mathrm{e}^{-\Phi(0)x}\frac{\Phi(q)-\Phi(0)}{\Phi(q)}.\end{aligned}$$

The proof of (ii) follows by a similar reasoning, using the form of the resolvent and the fact that D_∞ has the same law as $-\underline{X}_\infty$, which is given by $\mathbb{P}[-\underline{X}_\infty < x] = \psi'(0)^{-1}W(x)$ for $x \in \mathbb{R}_+$ (see *e.g.* [20]), where we use fact that $\psi'(0) = \mathbb{E}[X_1] > 0$. □

Remark 5.3. **(i)** By inverting the Laplace transform we find that

$$\mathbb{P}(\underline{U}_t^* > x) = \mathrm{e}^{-\Phi(0)x}\left(1 - \Phi(0)\mathbb{E}[U_t]\right).$$

(ii) Straightforward calculations show that the double Laplace transforms $\mathcal{L}_U(r,s)$ and $\mathcal{L}_D(r,s)$ of $\mathbb{P}(\underline{U}_T^* \leq u)$ and $\mathbb{P}(-\underline{D}_T^* \leq u)$ in T and u are given by:

$$\mathcal{L}_U(r,s) = \frac{\Phi(0)(\Phi(s)+r)}{(\Phi(0)+r)s\Phi(s)}, \qquad \mathcal{L}_D(r,s) = r\psi'(0)\Phi(s)\frac{\psi(r)-s}{s^2\psi(r)(r-\Phi(s))}.$$

This agrees with the forms of $\mathcal{L}_D(r,s)$ and $\mathcal{L}_U(r,s)$ obtained in [9].

(iii) In the literature numerical methods have been developed for the evaluation of scale functions, based on Markov chain approximation (see [22]) or Laplace inversion (see [18, 30]), which may be used for numerical evaluation of the expressions given in Proposition 5.1.

(iv) From the proofs of the propositions above it is clear that we can identify the bivariate Laplace transform of $\overline{U}_{t,s}^*$, $\underline{U}_{t,s}^*$, $\overline{D}_{t,s}^*$ and $\underline{D}_{t,s}^*$ with respect of t and s as long as the laws of $\overline{X}_{\mathbf{e}_q}$, $\underline{X}_{\mathbf{e}_q}$ and resolvents of reflected process R_a^U, R_a^D are known. This could be done not only for spectrally one-sided Lévy processes. For example, one can consider the Kou model, where the log-price $X = (X_t)_{t\in\mathbb{R}_+}$ is modelled by a jump-diffusion with constant drift μ and volatility $\sigma > 0$, with the upward and downward jumps arriving at rate λ_+ and λ_- with sizes following exponential distributions with mean $1/\alpha_+$ and $1/\alpha_-$,

$$X_t = \mu t + \sigma W_t + \sum_{j=1}^{N_t^+} U_j^+ - \sum_{j=1}^{N_t^-} U_j^-,$$

where $N^\pm$ are independent standard Poisson processes with rates $\lambda^\pm$, independent of a Brownian motion W, and $U_i^\pm \sim \mathrm{Exp}(\alpha^\pm)$ are independent. Then the important ingredients are identified in [1, Lemma 1 and Proposition 3] (also applied for the dual process).

5.1. **(Future) drawdowns and drawups under Black–Scholes model.** Consider a risky asset whose price process $P = (P_t)_{t\in\mathbb{R}_+}$ is given as follows:

$$P_t = P_0 \exp(X_t), \qquad t \in \mathbb{R}_+, \tag{5.3}$$

where $X = (X_t)_{t\in\mathbb{R}_+}$ is a Lévy process. In the case of the Black–Scholes model, P is a geometric Brownian motion, with rate of appreciation $\mu \in \mathbb{R}$ and the volatility σ, and $X = (X_t)_{t\in\mathbb{R}_+}$ is given by the linear Brownian motion

$$X_t = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t.$$

Let $\mu > \sigma^2/2$. This model is widely used in practice as a benchmark for other models.

For this model we have $\psi(\theta) = \sigma^2\theta^2/2 + (\mu - \sigma^2/2)\theta$, $\Phi(q) = -\omega + \delta(q)$ with

$$\delta(q) = \sigma^{-2}\sqrt{(\mu - \sigma^2/2)^2 + 2\sigma^2 q}$$

and $\omega = \frac{\mu}{\sigma^2} - \frac{1}{2}$ and

$$W^{(q)}(x) = \frac{1}{\delta(q)\sigma^2}\left[\mathrm{e}^{(-\omega+\delta(q))x} - \mathrm{e}^{-(\omega+\delta(q))x}\right],$$

$$Z^{(q)}(x) = \frac{q}{\delta(q)\sigma^2}\left[\frac{1}{-\omega+\delta(q)}\mathrm{e}^{(-\omega+\delta(q))x} + \frac{1}{\omega+\delta(q)}\mathrm{e}^{-(\omega+\delta(q))x}\right].$$

Hence from Corollary 5.2 we have

$$\begin{aligned}
\mathbb{P}(-\underline{D}^*_{\mathbf{e}_q} > x) &= 1 + \frac{1}{\sigma^4\delta(q)\omega}(\mu - \sigma^2/2)(Z^{(q)}(x) - 1) \\
&+ \frac{q}{\sigma^4\delta(q)\omega}(\mu - \sigma^2/2)\left(\frac{1}{\delta(q)-\omega}\mathrm{e}^{-(\delta(q)+\omega)x} - \frac{1}{\delta(q)+\omega}\mathrm{e}^{(\delta(q)-\omega)x} - \frac{2\omega}{\delta^2(q)-\omega^2}\mathrm{e}^{-2\omega x}\right) \\
&+ (\mu - \sigma^2)\frac{W^{(q)}(x)}{W^{(q)\prime}(x)}\left(\frac{\delta(q)+\omega}{\delta(q)-\omega}\mathrm{e}^{-(\delta(q)+\omega)x} + \frac{\delta(q)-\omega}{\delta(q)+\omega}\mathrm{e}^{(\delta(q)-\omega)x} - \frac{2\delta(q)}{\delta^2(q)-\omega^2}\mathrm{e}^{-2\omega x}\right)
\end{aligned}$$

and

$$\mathbb{P}(-\overline{D}^*_{\mathbf{e}_q} > x) = \frac{-\omega + \delta(q)}{\omega + \delta(q)}\mathrm{e}^{-2\omega x}, \qquad q > 0.$$

Hence we find for $t \in \mathbb{R}_+$

$$\mathbb{P}(-\overline{D}^*_t > x) = \mathbb{E}[\mathrm{e}^{-2\omega U_t}]\mathrm{e}^{-2\omega x}.$$

Moreover,

$$\begin{aligned}
\overline{\mathbb{E}}^{(\gamma)}[P_t] &= P_0\frac{\mathbb{E}\left[\mathrm{e}^{-\gamma U_t + X_t}\right]}{\mathbb{E}\left[\mathrm{e}^{-\gamma U_t}\right]} = P_0\mathrm{e}^{\psi(1)t}\frac{\mathbb{E}^{(1)}[\mathrm{e}^{-\gamma U_t}]}{\mathbb{E}\left[\mathrm{e}^{-\gamma U_t}\right]}, \\
\underline{\mathbb{E}}^{(\gamma)}[P_t] &= P_0\mathrm{e}^{\psi(1)t}\frac{\mathbb{E}^{(1)}[\mathrm{e}^{\gamma D_t}]}{\mathbb{E}\left[\mathrm{e}^{\gamma D_t}\right]},
\end{aligned}$$

where $\overline{\mathbb{E}}^{(\gamma)}$ and $\underline{\mathbb{E}}^{(\gamma)}$ are the expectations with respect of measures $\overline{\mathbb{P}}^{(\gamma)}$ and $\underline{\mathbb{P}}^{(\gamma)}$ given in (3.21) (for γ given in Assumption 1), respectively, and the measure $\mathbb{P}^{(1)}$ is defined via $\mathbb{P}^{(1)}(A) = \mathbb{E}[\mathrm{e}^{X_t - \psi(1)t}\mathbf{1}_A]$ for $A \in \mathcal{F}_t$ and $\gamma = 2\omega$. Under $\mathbb{P}^{(1)}$ we have

$$X_t = \left(\mu - \frac{3}{2}\sigma^2\right) + \sigma W_t.$$

We note that $\mathbb{E}[e^{-2\omega U_t}] = \mathbb{E}[e^{-\gamma U_t}]$ and $\mathbb{E}^{(1)}[e^{-\gamma U_t}]$ may be identified using [4, (1.1.3), p. 250] and $\mathbb{E}[e^{\gamma D_t}]$ and $\mathbb{E}^{(1)}[e^{\gamma D_t}]$ using [4, (1.1.3), (1.2.3) p. 250-251].

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