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Pseudo Maximum Likelihood Estimation of Spatial Autoregressive Models with Increasing Dimension

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Abstract

Pseudo maximum likelihood estimates are developed for higher-order spatial autoregressive models with increasingly many parameters, including models with spatial lags in the dependent variables both with and without a linear or nonlinear regression component, and regression models with spatial autoregressive disturbances. Consistency and asymptotic normality of the estimates are established. Monte Carlo experiments examine finite-sample behaviour.

JEL classifications: C21, C31, C36

Keywords: Spatial autoregression; increasingly many parameters; consistency; asymptotic normality; pseudo Gaussian maximum likelihood; finite sample performance

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1 Introduction

Spatial autoregressive (SAR) models, introduced by Cliff and Ord (1973), can describe spatial dependence parsimoniously even when data are irregularly-spaced or when economic (not necessarily geographic) distances between units are known, and information on locations is unavailable. They have been widely used in modelling economic and geographic data. The first-order SAR model, which involves a single weight matrix, consisting of (inverse) distances, and a single correlation parameter, has been the focus of much research. Greater flexibility, at the cost of less parsimony, is afforded by higher-order SAR models, which incorporate two or more weight matrices and corresponding parameters. These have been studied in both theoretical and applied research. Brandsma and Ketellapper (1979) introduced a second-order model, and discussed its estimation. Blommestein (1983, 1985), Blommestein and Koper (1992, 1997), Anselin and Smirnov (1996), LeSage and Pace (2011), Elhorst, Lacombe and Piras (2012) and others explored various issues in the specification and estimation of higher order SAR models, the latter two references listing a number of others. A recent purely empirical study is in Kolympiris, Kalaitzandonakes, and Miller (2011). A book length exposition can be found in Anselin (1988).

In the present paper we investigate large sample statistical inference on higher order SAR models, in which the number of parameters is allowed to increase slowly with sample size, denoted n, a type of setting previously studied by Gupta and Robinson (2015). From this perspective we find it convenient to consider four specifications that have somewhat different theoretical as well as practical implications. For an $n \times 1$ vector y_n of observations and an integer $p_n \geq 1$, possibly regarded as increasing as n increases, let W_{in}, $i = 1, \ldots, p_n$, be $n \times n$ known weight matrices whose elements are inverse economic distances, let $\lambda_{0n} = (\lambda_{01}, \ldots, \lambda_{0p_n})'$, the prime denoting transposition, be a vector of unknown parameters, and let u be an $n \times 1$ vector of independent, zero-mean, homoscedastic unobservable random variables. The basic p_nth-order SAR model, denoted SAR(p_n), is

$$y_n = \sum_{i=1}^{p_n} \lambda_{0i} W_{in} y_n + u. \tag{1.1}$$

Let l_n be a $n \times 1$ vector of ones and let τ_0 be an unknown scalar. The SAR(p_n) with intercept is

$$y_n = \sum_{i=1}^{p_n} \lambda_{0i} W_{in} y_n + \tau_0 l_n + u. \tag{1.2}$$

For given integers $k_n \geq 1$ (possibly regarded as increasing with n) and fixed $q \geq 1$ let β_{0n} be an unknown $k_n \times 1$ vector, let δ_0 be a known or unknown $q \times 1$ vector and let $X_n(\delta_0)$ be an $n \times k_n$ matrix of functions of δ_0 and of explanatory variables, with reference to the latter suppressed. The SAR(p_n) with regressors is

$$y_n = \sum_{i=1}^{p_n} \lambda_{0i} W_{in} y_n + X_n(\delta_0) \beta_{0n} + u. \tag{1.3}$$

Finally, for an $n \times 1$ vector v_n of unobservable random variables, the regression with SAR(p_n) errors is

$$y_n = X_n(\delta_0)\beta_{0n} + v_n,\ v_n = \sum_{i=1}^{p_n} \lambda_{0i} W_{in} v_n + u. \tag{1.4}$$

These models correspond to versions of p_nth-order autoregressive time series models, where competing approaches to introducing both autocorrelation and explanatory variables are mirrored by (1.3) and (1.4).

When τ_0 is known (1.2) nests (1.1) (which is sometimes referred to as 'pure SAR'), while (1.2) is nested in both (1.3) and (1.4) when $X_n(\delta_0)$ contains a subvector l_n, although estimation methods differ. Indeed (1.1) and (1.2) are not consistently estimable by least squares or instrumental variables, unlike (1.3) and (1.4). In most spatial autoregression literature, SAR(1) versions of these models have been studied, and previous higher-order SAR literature has almost exclusively assumed that p_n and k_n are fixed. In the the bulk of the literature on (1.3) and (1.4) the regression component is linear, formally covered by regarding δ_0 as known. However, (1.3) and (1.4) allow for nonlinear regression, which features widely in statistics (cf. eg Jennrich (1969)) and econometrics but apparently not in the SAR literature, even though Xu and Lee (2015) have studied a SAR model with a nonlinear transformation of the dependent variable. For example, the elements of $X_n(\delta_0)$ may be parametric Box-Cox, arcsinh or other nonlinear transformations of basic explanatory variables. The separation of β_{0n} from δ_0 follows much of the nonlinear regression literature in expressing the likely presence of an unknown scaling vector. The n-subscripting in $X_n(\delta_0)$ allows it to depend on spatial lags of explanatory variables, which entail weight matrices. The model (1.4) may be included in (1.3) by replacing $X_n(\delta_0)$ by a function of both δ_0 and λ_{0n}, but (1.4) is of sufficient practical importance to warrant separate consideration.

Interest centres on statistical inference on λ_{0n}, β_{0n} and, when it is unknown, δ_0. Consider what is known or anticipated from the literature that regards p_n and k_n as fixed. In (1.1) and (1.2), despite the linearity in parameters, least squares estimates are well known to be inconsistent, for typical W_{in}, which differ from the lower triangular ones which deliver consistency in the autoregressive time series models formally covered; however, for (1.1) Kelejian and Prucha (1999) established consistency of a generalized method of moments estimate. For the same reason consistency of least squares estimates of all parameters in (1.3) is problematic, though from Lee (2002) (who assumed $p_n = 1$ and linear regression) we may expect consistency to be achieved under certain asymptotic conditions on the W_{in}. Under milder such conditions, again when the regression is linear, use of instrumental variables, when available, can produce closed form consistent estimates in (1.3), see eg Kelejian and Prucha (1998); for nonlinear regression one expects to be able to extend, eg, Amemiya (1974). As under many other relaxations of Gauss-Markov conditions, least squares estimates of β_{0n} in the first equation of (1.4) (or nonlinear least squares estimates of β_{0n} and δ_0) are expected to be consistent, though those of λ_{0n} based on residuals inconsistent; see eg Kelejian and Prucha (1997). When estimates are consistent, one expects them to satisfy a central limit theorem under additional conditions. The models (1.1)-(1.4) are somewhat idealised, some of the literature considering ones that are more general. In 'SARAR' versions of (1.1), (1.2) or (1.3), u is replaced by v_n, defined as in (1.4) but with p_n possibly replaced by some other order r_n, say. However after transformation they are still essentially covered by (1.1)-(1.3), albeit offering more parsimony, having SAR order $p_n r_n$ with coefficients depending on only p_n+r_n unknowns. In a SARAR version of (1.3), Lee and Liu (2010) established asymptotic theory for generalized method of moments estimates, as did Badinger and Egger (2011, 2013), allowing respectively for error heteroscedasticity and panel structure. Spatial ARMA models are not covered in (1.1)-(1.4); in this setting Huang (1984) and Anselin (2001) respectively discussed maximum likelihood estimation and developed Lagrange multiplier tests to determine model order.

A single type of estimate which can be expected to deliver consistency, and asymptotic normality, in (1.1)-(1.4), and without recourse to instrumental variables, is the Gaussian pseudo-maximum likelihood estimate (PMLE). This maximizes what would be the likelihood were u Gaussian, and as well as enjoying the classical asymptotic properties of maximum likelihood, is consistent and asymptotically normal under more general conditions on u, though in some settings the limiting covariance matrix can be affected. Brandsma and Ketellapper (1979) discussed Gaussian maximum likelihood estimation in the SAR(2) version of (1.1), describing, without rigorous proofs, asymptotic statistical properties, see also Huang (1984). These properties were established for the PMLE by Lee (2004) in case of SAR(1) versions of (1.1)-(1.3) with linear regression in the latter model. The PMLE is asymptotically efficient when u is Gaussian, though otherwise more efficient estimates have been justified in fixed parameter dimension SAR models, see Lee and Liu (2010) and Robinson (2010). Note that our allowance for nonlinear regression does not greatly impact on methods and theory for the PMLE, which is in any case only implicitly defined. One well-known aspect of the PMLE is the need to invert an $n \times n$ matrix in the estimation. On the other hand, a general defence of the PMLE is its asymptotic efficiency properties in the Gaussian case, the fact that consistency and the same limit distribution holds under more general conditions than Gaussianity, and the relatively simple and easy-to-compute form of the limiting covariance matrix estimate following the point estimation.

In practice the specification of p_n, and of k_n, may be influenced by the amount of data n available, as is the case with other multiparameter statistical models. A larger data set affords the possibility of achieving reasonably precise inference on a richer model, which may reflect a degree of model uncertainty. Correspondingly, in a number of other multiparameter models, asymptotic statistical theory has been developed with the number of parameters increasing slowly with sample size, cf. Huber (1973), Berk (1974), Sargan (1975), Robinson (1979), Portnoy (1984, 1985), Robinson (2003). Gupta and Robinson (2015) have argued that regarding p_n as increasing with n is natural in SAR models with some kinds of weight matrix, and have established asymptotic theory for least squares and instrumental variables estimates of (1.3) in the linear regression case. A popular alternative approach to models with a large number of parameters is to apply the LASSO, or a similar estimate based on a penalized objective function. This method is especially useful in cases where $p_n + k_n \geq n$.

The present paper establishes consistency and asymptotic normality for the PMLE in the models (1.1)-(1.4) with p_n and k_n allowed to increase slowly with n. Asymptotic theory for implicitly-defined extremum estimates, requiring an initial consistency proof, is unusual in the literature on increasing parameter dimension with sample size, especially so when combined with nonlinear regression. Our proof of consistency of the PMLE is rather delicate, in particular where both numerator and denominator terms increase with k_n (see (A.8) in the appendix), while we also need the volume of the admissible autoregressive parameter space to remain bounded as p_n diverges. Our results lead to rules of statistical inference which are also valid when p_n and k_n are regarded as fixed, and to some extent provide a novel contribution in this setting also. In particular we know of no asymptotic theory for the PMLE in the models (1.1)-(1.4) with fixed $p_n > 1$ and k_n. We keep the dimension q of δ_0 fixed as otherwise the regression would effectively be nonparametric.

The following section covers models (1.1) and (1.2), with (1.3) and (1.4) covered in Sections 3 and 4, respectively. Section 5 contains a Monte Carlo study of finite sample performance. Proofs are included in two Appendices and an additional online supplementary appendix.

2 SAR with and without intercept

We can rewrite (1.1) as

$$S_n y_n = u \tag{2.1}$$

where $S_n = I_n - \sum_{i=1}^{p_n} \lambda_{0i} W_{in}$. The notation S_n follows a convention we adopt for evaluation of objects at true parameters: $A(\alpha_0) \equiv A$ for any matrix, vector or scalar A and any true parameter α_0. In the sequel we suppress reference to n for individual parameters to simplify notation. We now introduce some basic assumptions.

Assumption 1. *$u = (u_1, \ldots, u_n)'$ has independently distributed elements with zero mean, finite variance σ_0^2 and finite third and fourth moments μ_3 and μ_4 respectively.*

Assumption 2. *For $i = 1, \ldots, p_n$, the diagonal elements of each W_{in} are zero and the off-diagonal elements of W_{in} are uniformly $\mathscr{O}\left(h_n^{-1}\right)$, where h_n is a positive sequence which is bounded away from zero and which may be bounded or divergent, with $n/h_n \to \infty$ as $n \to \infty$ in the latter case.*

It is possible to employ different h_{in} for each of the W_{in}, some bounded and some divergent. However we maintain Assumption 2 for notational simplicity. For any rectangular matrix A, we define $\|A\| = \left\{\bar{\zeta}\left(A'A\right)\right\}^{\frac{1}{2}}$, where $\bar{\zeta}(B)$ (respectively $\underline{\zeta}(B)$) is the largest (smallest) eigenvalue of a square, symmetric matrix B.

Definition For $i = 1, \ldots, p_n$, W_{in} are said to have *'single nonzero diagonal block'* structure if, for some set of $m_i \times m_i$ matrices V_{in} such that $\sum_{i=1}^{p_n} m_i = n$, W_{in} has V_{in} as the ith diagonal block and zeros elsewhere.

Let c, C denote throughout generic positive constants, arbitrarily small and large, respectively.

Assumption 3. *S_n is non-singular and*

$$\left\|S_n^{-1}\right\| + \max_{i=1,\ldots,p_n} \|W_{in}\| \leq C, \tag{2.2}$$

for all sufficiently large n.

The first part of this assumption ensures that (2.1) can be solved for y_n, asymptotically. The restriction on $\left\|S_n^{-1}\right\|$ limits spatial correlation because the covariance matrix of y_n is $\sigma_0^2 S_n^{-1} S_n^{-1\prime}$, while the restrictions on the $\|W_{in}\|$ are satisfied if, for each i, the elements of W_{in} decline fast enough with n. A sufficient condition for the non-singularity of S_n is

$$\left\|\sum_{i=1}^{p_n} \lambda_{0i} W_{in}\right\| < 1. \tag{2.3}$$

Depending on the structure of W_{in} more primitive sufficient conditions can be given for (2.3). Denote by $\lambda = (\lambda_1, \ldots, \lambda_{p_n})'$ and σ^2 any admissible values of λ_{0n} and σ_0^2 and let $\|a\|_1 = \sum_{i=1}^{s} |a_i|$ for any s-dimensional vector a. In the 'single nonzero diagonal block' case we have $\|\sum_{i=1}^{p_n} \lambda_{0i} W_{in}\| \leq \max_{i=1,\ldots,p_n} (|\lambda_{0i}| \|V_{in}\|)$, in which case one could take the parameter space Λ_n for λ to be such that

$$\max_{i=1,\ldots,p_n} |\lambda_i| < 1, \tag{2.4}$$

and take normalized V_{in} such that $\|V_{in}\| = 1$. For more general W_{in} we have $\|\sum_{i=1}^{p_n} \lambda_{0i} W_{in}\| \leq \max_{i=1,\ldots,p_n} \|W_{in}\| \sum_{i=1}^{p_n} |\lambda_{0i}|$, and then we may choose Λ_n such that

$$\|\lambda\|_1 < 1, \tag{2.5}$$

and normalize the W_{in} such that $\|W_{in}\| \equiv 1$. In any case, for the identification of the λ_i some normalization of the W_{in} is necessary, so this operation is essentially costless. A similar discussion applies after Assumption 12 below, with row-sum norm used instead. Define the negative Gaussian log-likelihood function as

$$\log\left(2\pi\sigma^2\right) - 2n^{-1} \log|S_n(\lambda)| + \sigma^2 n^{-1} y_n' S_n(\lambda) S_n(\lambda) y_n, \tag{2.6}$$

for nonsingular $S_n(\lambda) = I_n - \sum_{i=1}^{p_n} \lambda_i W_{in}$. For given λ, (2.6) is minimised with respect to σ^2 by

$$\bar{\sigma}_n^2(\lambda) = n^{-1} y_n' S_n(\lambda) S_n(\lambda) y_n. \tag{2.7}$$

Define the PMLEs of λ_{0n}, σ_0^2 as $\hat{\lambda}_n = \arg\min_{\lambda \in \Lambda_n} \mathcal{Q}_n(\lambda), \hat{\sigma}_n^2 \equiv \bar{\sigma}_n^2\left(\hat{\lambda}_n\right)$ respectively, where

$$\mathcal{Q}_n(\lambda) = \log \bar{\sigma}_n^2(\lambda) + n^{-1} \log \left| S_n^{-1}(\lambda) S_n^{-1\prime}(\lambda) \right|, \tag{2.8}$$

with Λ_n satisfying

Assumption 4. *Λ_n is a subset of $\mathbb{R}^{p_n}$ such that, for some fixed $\varepsilon \in (0,1)$, $-\varepsilon \leq \lambda_i \leq 1-\varepsilon$, for $i = 1, \ldots, p_n$ when the W_{in} have 'single nonzero diagonal block' structure and $\|\lambda\|_1 \leq 1-\varepsilon$ if not.*

Assumption 4 reflects the necessity in our proof that the volume of Λ_n remain bounded as $n \to \infty$, and the likelihood that the λ_{0i} are non-negative, but could be replaced by others. The construction of a compact parameter space requires some care when dimension can increase. The usual Cartesian product of closed and bounded intervals that forms a compact parameter space in the fixed dimension setting will not, in general, yield a region with bounded volume when dimension increases. The Associate Editor handling our paper has pointed out that by analogy with results shown in other settings (see eg Pötscher and Prucha (1997) pp. 29-31 and references therein, and Kuersteiner and Prucha (2015)), the compactness requirement of Assumption 4 might be relaxed and the arbitrary choice of ε avoided by, more naturally, choosing Λ_n as (2.4) in the 'single nonzero diagonal block' case, and as (2.5) otherwise. The only drawback to optimizing over an open set would appear to be that $\hat{\lambda}_n$ might sometimes not exist. On the other hand with compact Λ_n, if $\hat{\lambda}_n$ falls on its boundary it is likely that shrinking ε would change $\hat{\lambda}_n$. This may suggest that n is too small for asymptotics to be relevant, and/or the parameter space has been chosen too small or the model is misspecified. Typically there will be no option to collect further data, while employing an alternative method of estimation in the hope that the outcome will lie within the boundary seems an over-reaction, especially as one can choose ε so small that shrinking it would not affect $\hat{\lambda}_n$ to any desired number of decimal places, or indeed make any statistically significant difference. Our use of $\|\lambda\|_1 \leq 1-\varepsilon$, or indeed (2.5), in non-'single nonzero diagonal block' cases is nevertheless still unsatisfactory because, with the restriction on the W_{in}, it is a crude sufficient condition for (2.3), compared to the precise conditions for stationarity of autoregressive time series in terms of the locations of zeros of the autoregressive polynomial. Further work to relax Assumption 4 in our increasing parameter dimension setting would be desirable.

Note that though we treat the W_{in} as known, in reality the scaling of distances is arbitrary and different scalings are used in the literature. Some scaling, such as $\|W_{in}\| = 1$, is necessary in order to identify the λ_{0i} and correspondingly specify a suitable Λ_n. We could replace each W_{in} by cW_{in} for $c \in (0, \infty)$ and Λ_n by $c^{-1}\Lambda_n$, but for identification we must choose one scaling and one parameter space.

Assumption 5. *$\lambda_{0n} \in \Lambda_n$, for all sufficiently large n.*

Denote

$$\sigma_n^2(\lambda) = n^{-1}\sigma_0^2 tr\left(S_n^{-1\prime} S_n'(\lambda) S_n(\lambda) S_n^{-1}\right). \tag{2.9}$$

Assumption 6. *For $\lambda \in \Lambda_n$ and all sufficiently large n, $c \leq \sigma_n^2(\lambda) \leq C$.*

$\sigma_n^2(\lambda)$ is nonnegative by inspection and finite by Assumptions 3 and 4. For a generic matrix A define $\|A\|_F = \{tr(A'A)\}^{\frac{1}{2}}$ and introduce

Assumption 7. *For any $\eta > 0$,*

$$\underline{\lim}_{n\to\infty} \inf_{\lambda \in \overline{\mathcal{N}}_n^{\lambda}(\eta)} n^{-1} \|T_n(\lambda)\|_F^2 / |T_n(\lambda)|^{2/n} > 1, \tag{2.10}$$

where $T_n(\lambda) = S_n(\lambda)S_n^{-1}$, $\overline{\mathcal{N}}_n^{\lambda}(\eta) = \Lambda_n \setminus \mathcal{N}_n^{\lambda}(\eta)$, $\mathcal{N}_n^{\lambda}(\eta) = \{\lambda : \|\lambda - \lambda_{0n}\| < \eta\} \cap \Lambda_n$.

The ratio in (2.10) is guaranteed ≥ 1 due to the inequality between arithmetic and geometric means. Assumption 7 is an identification condition related to the uniqueness of the covariance matrix of y_n, introduced in Delgado and Robinson (2015) who discussed it and compared it to the identification condition employed by Lee (2004) in his asymptotic theory.

Theorem 2.1. *Let (1.1) and Assumptions 1-7 hold, and p_n be allowed to diverge as $n \to \infty$. Then*

$$\left\|\hat{\lambda}_n - \lambda_{0n}\right\| \xrightarrow{p} 0, \text{ as } n \to \infty.$$

Theorem 2.2. *Let (1.1) and Assumptions 1-7 hold, and p_n be allowed to diverge as $n \to \infty$ such that $p_n^2/nh_n \to 0$. Then $\hat{\sigma}_n^2 - \sigma_0^2 = o_p(1)$, as $n \to \infty$.*

Multimodality can be a potential problem with implicitly defined extremum estimates, see eg Warnes and Ripley (1987) in a rather different spatial context. It is plausible that the likelihood of it could increase with increasing p_n or decreasing n, or perhaps with 'increasing nonlinearity'. However on the one hand one could get multimodality when $p = 1$, and on the other, normal multiple linear regression is always unimodal if $k_n < n$. Certainly the smaller the gap between n and p_n the flatter we might expect the objective function to be, but this a local rather than global issue. The problem does not necessarily go away with large n, as even if the objective function is asymptotically uniquely optimised asymptotic sub-optimal modes are not ruled out. For $p = 1$ Hillier and Martellosio (2013) are able to establish unimodality if W_{1n} has real eigenvalues (amongst other conditions), although their approach relies on an explicit analysis of the second derivative of the likelihood function and seems difficult to extend when $p > 1$. One way to mitigate the problem is by searching over a sufficiently fine grid before any iteration, though the larger p_n is the more expensive this is.

To establish asymptotic normality, we denote by $H_n(\lambda, \sigma^2)$ the second derivative matrix of (2.6) and define it in (A.18) in Appendix A. Writing $P_{1n}(\lambda)$, $P_{2n}(\lambda)$ for the $p_n \times p_n$ matrices with (i,j)-th element given by $tr(G_{jn}(\lambda)G_{in}(\lambda))$, $tr\left(G_{jn}'(\lambda)G_{in}(\lambda)\right)$, respectively, with $G_{in}(\lambda) =$

$W_{in}S_n^{-1}(\lambda)$ for $i = 1, \ldots, p_n$, we deduce (details in Appendix A) that

$$\Xi_n = \mathbb{E}\left(H_n\right) = 2n^{-1}\left(P_{1n} + P_{2n}\right). \tag{2.11}$$

Write F_n for the $n \times p_n$ matrix with (i,j)-th element $c_{ii,jn}$, where $c_{pq,in}$ is the (p,q)-th element of $G_{in} + G'_{in}$, and define $\Omega_n = \left(\mu_4 - 3\sigma_0^4\right)\sigma_0^{-4}n^{-1}F'_nF_n$. The covariance matrix of the first derivative of (2.6) is $n^{-1}\left(2\Xi_n + \Omega_n\right)$. The following assumption is standard:

Assumption 8. *λ_{0n} is in the interior of Λ_n, for all sufficiently large n.*

If h_n diverges with n, we need to account for the correct normalisation that will yield a central limit theorem as follows:

Assumption 9. *$h_n \to \infty$ as $n \to \infty$. $\overline{\lim\limits_{n\to\infty}}\, \overline{\zeta}\left(h_n\Xi_n\right) < \infty$ and $\underline{\lim\limits_{n\to\infty}}\, \underline{\zeta}\left(h_n\Xi_n\right) > 0$.*

Assumption 10. *h_n is bounded as $n \to \infty$. $\overline{\lim\limits_{n\to\infty}}\, \overline{\zeta}\left(\Xi_n^{-1}\Omega_n\Xi_n^{-1}\right) < \infty$, $\underline{\lim\limits_{n\to\infty}}\, \underline{\zeta}\left(2\Xi_n^{-1} + \Xi_n^{-1}\Omega_n\Xi_n^{-1}\right) > 0$ and $\underline{\lim\limits_{n\to\infty}}\, \underline{\zeta}\left(\Xi_n\right) > 0$.*

The rank conditions here strongly restrict the W_{in} in higher-order SAR models, even with fixed p_n. Such problems are transparently avoided with weight matrices having 'single nonzero diagonal block' structure. Blommestein (1985) discusses the possibility of 'circularity' when W_{in} represent orders of contiguity, causing rank condition failure. By way of an illustration, W_{1n} could assign 1 to an element if the relevant units share a common boundary, W_{2n} could assign 1 to an element if the relevant units do not share a boundary with each other but have a common neighbour, and so on. In this case, there is a risk of high-order W_{in} 'circling' back to W_{1n}.

Assumption 11. *For some $\chi > 0$, $\mathbb{E}\left|u_i\right|^{4+\chi} \le C$, $i = 1, \ldots, n$.*

For any $s \times q$ matrix $A = [a_{ij}]$ define $\left\|A\right\|_R = \max_{i=1,\ldots,s}\sum_{j=1}^{q}\left|a_{ij}\right|$, the maximum absolute row-sum norm.

Assumption 12. *S_n is non-singular and*

$$\left\|S_n^{-1}\right\|_R + \left\|S_n'^{-1}\right\|_R + \max_{i=1,\ldots,p_n}\left(\left\|W_{in}\right\|_R + \left\|W'_{in}\right\|_R\right) \le C, \tag{2.12}$$

for all sufficiently large n.

This strengthens Assumption 3 due to the inequality $\left\|A\right\|^2 \le \left\|A\right\|_R\left\|A'\right\|_R$.

Denote throughout by Ψ_n a matrix of constants with full and fixed row rank, and columns equal in number to the parameters for which a central limit theorem is being established. Our next theorem covers the unbounded h_n case, establishing asymptotic normality of a fixed number of linear combinations of $\hat{\lambda}_n - \lambda_{0n}$.

Theorem 2.3. *Let (1.1) and Assumptions 1, 2, 4, 6-9, 11 and 12 hold, $h_n \to \infty$ as $n \to \infty$, p_n be allowed to diverge as $n \to \infty$ such that*

$$\frac{p_n^5}{nh_n} + \frac{p_n}{h_n} + \frac{p_n^4 h_n^2}{n} + \frac{p_n^{2+\frac{8}{\chi}} h_n^{1+\frac{4}{\chi}}}{n} \to 0 \text{ as } n \to \infty. \tag{2.13}$$

Then

$$\frac{n^{\frac{1}{2}}}{h_n^{\frac{1}{2}} p_n^{\frac{1}{2}}} \Psi_n \left(\hat{\lambda}_n - \lambda_{0n} \right) \xrightarrow{d} N\left(0, \Delta_1\right), \text{ as } n \to \infty,$$

where $\Delta_1 = 2 \lim_{n\to\infty} p_n^{-1} \Psi_n \left(h_n \Xi_n\right)^{-1} \Psi_n'$.

First, note that $\chi > 4$ implies that the last term on the LHS of (2.13) is dominated by the third one. If $G_{jn}G_{in} = 0$ and $G_{jn}'G_{in} = 0$ for $i \neq j$, as with 'single nonzero diagonal block' weight matrices, then any finite-dimensional subset of estimates will be asymptotically distributed as independent normal random variables with mean zero and variances $\left\{\lim_{n\to\infty} \left(h_n/n\right) tr\left(G_{in}^2 + G_{in}'G_{in}\right)\right\}^{-1}$. If p_n is fixed then the restrictions on p_n in (2.13) are redundant. In this case the same proof, considering a single linear combination, implies $\left(n^{\frac{1}{2}}/h_n^{\frac{1}{2}}\right)\left(\hat{\lambda}_n - \lambda_0\right) \xrightarrow{d} N\left(0, 2\lim_{n\to\infty}\left(h_n\Xi_n\right)^{-1}\right)$, by the Cramer-Wold device. We may derive similar results for fixed parameter spaces from the subsequent central limit theorems in this section. The following theorem takes h_n to be bounded, complementing Theorem 2.3.

Theorem 2.4. *Let (1.1) and Assumptions 1, 2, 4, 6-8, 10-12 hold, and p_n be allowed to diverge as $n \to \infty$ such that*

$$\frac{p_n^5}{n} + \frac{p_n^{2+\frac{8}{\chi}}}{n} \to 0, \text{ as } n \to \infty. \tag{2.14}$$

Then

$$\frac{n^{\frac{1}{2}}}{p_n^{\frac{1}{2}}} \Psi_n \left(\hat{\lambda}_n - \lambda_{0n} \right) \xrightarrow{d} N\left(0, \Delta_2\right), \text{ as } n \to \infty,$$

where $\Delta_2 = \lim_{n\to\infty} p_n^{-1} \Psi_n \left(2\Xi_n^{-1} + \Xi_n^{-1}\Omega_n\Xi_n^{-1}\right) \Psi_n'$.

The parameter growth restrictions may be simplified if moment conditions are strengthened. For instance when $\chi \geq 8/3$ in Assumption 11, (2.14) only requires $p_n^5/n \to 0$. Covariance matrix estimation for Theorems 2.3 and 2.4 can be based on $H_n\left(\lambda, \sigma_n^2\right)$ and $\Omega_n\left(\lambda, \sigma_n^2\right)$ evaluated at $\hat{\lambda}_n$, $\hat{\sigma}_n^2$ and empirical moments.

We now turn from model (1.1) to the slightly more general (1.2). For any admissible λ, τ and σ^2 and nonsingular $S_n(\lambda)$ the negative Gaussian pseudo log-likelihood function is $\log\left(2\pi\sigma^2\right) - 2n^{-1}\log\left|S_n\left(\lambda\right)\right| + \left(n\sigma^2\right)^{-1}\left\|S_n\left(\lambda\right)y_n - l_n\tau\right\|^2$, which for given λ is minimised with respect to τ and σ^2 by $\bar{\tau}_n\left(\lambda\right) = n^{-1}l_n'S_n\left(\lambda\right)y_n$ and $\bar{\sigma}_n^2\left(\lambda\right) = n^{-1}y_n'S_n'\left(\lambda\right)M_{l_n}S_n\left(\lambda\right)y_n$, where we write $M_A = I_n - A(A'A)^{-1}A'$ for any $n\times s$ matrix A of rank s, with I_n denoting the $n\times n$ identity matrix. The PMLE of λ_0 is $\hat{\lambda}_n = \arg\min_{\lambda\in\Lambda_n} \mathcal{Q}_n\left(\lambda\right)$, where $\mathcal{Q}_n\left(\lambda\right) = \log\bar{\sigma}_n^2\left(\lambda\right) + n^{-1}\log\left|S_n^{-1}\left(\lambda\right)S_n^{-1\prime}\left(\lambda\right)\right|$, and the PMLEs of τ_0 and σ_0^2 are $\hat{\tau}_n = \bar{\tau}_n\left(\hat{\lambda}_n\right)$ and $\hat{\sigma}_n^2 = \bar{\sigma}_n^2\left(\hat{\lambda}_n\right)$ respectively. The first and second derivatives evaluated at $\left(\lambda_{0n}', \tau_0, \sigma_0^2\right)$ are written ξ_n^I and H_n^I respectively. Both now include derivatives with respect to τ, and explicit expressions can be obtained by taking $X_n = l_n$ in (1.3). The covariance matrix of the first derivative of the likelihood function is $n^{-1}\left(2\Xi_n^I + \Omega_n^I\right)$, with $\Xi_n^I = \mathbb{E}\left(H_n^I\right)$.

A feature of this model noted by Lee (2004) is potential multicollinearity. For example, if the W_{in} are row-normalised (with non-negative elements) then $W_{in}l_n = l_n$, so that $G_{in}l_n\tau_0 = \tau_0 l_n \left(1 - \sum_{i=1}^{p_n} \lambda_{0i}\right)^{-1}$ for each i. It follows that $M_{l_n}G_{in}l_n\tau_0 = 0$ for every i and multicollinearity ensues. Indeed when h_n diverges and $p_n = o\left(h_n\right)$, $\left\|\Xi_n^I\right\| = o(1)$, as $n \to \infty$, implying that $\underline{\zeta}\left(\Xi_n^I\right) = o(1)$ also (see Lee (2004) for justification when $p_n \equiv 1$, extension to divergent p_n being obvious). While consistency as established in the following section is preserved as long as Assumption 7 continues to hold (τ_0 is identified if λ_{0n} is identified), the central limit theorem entails a different norming.

Theorem 2.5. *Let (1.2) and Assumptions 1-7 hold, and p_n be allowed to diverge as $n \to \infty$. Then*

$$\left\|\left(\hat{\lambda}_n', \hat{\tau}_n\right) - \left(\lambda_{0n}', \tau_0\right)\right\| \xrightarrow{p} 0, \text{ as } n \to \infty.$$

Theorem 2.6. *Let (1.2) hold with $h_n \to \infty$ as $n \to \infty$. Let Assumptions 1, 2, 4, 6-8, 11 and 12 hold, $\underline{\zeta}\left(\Xi_n^I\right) \to 0$ as $n \to \infty$, $\overline{\lim}_{n\to\infty} \overline{\zeta}\left(\left(h_n\Xi_n^I\right)^{-1} h_n\Omega_n^I \left(h_n\Xi_n^I\right)^{-1}\right) < \infty$, $\underline{\lim}_{n\to\infty} \underline{\zeta}\left(h_n\Xi_n^I\right) > 0$, $\underline{\lim}_{n\to\infty} \underline{\zeta}\left(2\left(h_n\Xi_n^I\right)^{-1} + \left(h_n\Xi_n^I\right)^{-1} h_n\Omega_n^I \left(h_n\Xi_n^I\right)^{-1}\right) > 0$, and p_n be allowed to diverge as $n \to \infty$ such that*

$$\frac{p_n^5}{nh_n} + \frac{p_n^4 h_n^2}{n} + \frac{p_n^{2+\frac{8}{\chi}} h_n^{1+\frac{4}{\chi}}}{n} \to 0, \text{ as } n \to \infty. \tag{2.15}$$

Then

$$\frac{n^{\frac{1}{2}}}{h_n^{\frac{1}{2}} p_n^{\frac{1}{2}}} \Psi_n \left(\left(\hat{\lambda}_n', \hat{\tau}_n\right)' - \left(\lambda_{0n}', \tau_0\right)'\right) \xrightarrow{d} N\left(0, \Delta_3\right), \text{ as } n \to \infty,$$

where $\Delta_3 = \lim_{n\to\infty} p_n^{-1}\Psi_n \left(2\left(h_n\Xi_n^I\right)^{-1} + \left(h_n\Xi_n^I\right)^{-1} h_n\Omega_n^I \left(h_n\Xi_n^I\right)^{-1}\right)\Psi_n'$.

If either multicollinearity does not arise or if h_n is bounded the asymptotic distribution of the PMLE for the parameters of (1.2) is covered under the theorems of the following section.

3 SAR with regressors

We now consider (1.3). Let $X_n(\delta)$ have i-th row $x_{in}'(\delta) = \left(x_{i1n}(\delta), \ldots, x_{ik_nn}(\delta)\right)$, for known functions $x_{ijn}(\delta)$, $j = 1, \ldots, k_n$, and unknown vector $\delta = \left(\delta_1, \ldots, \delta_q\right)'$. When δ_0 is known the regression is linear. A nonlinear example with $k_n = q = 1$ is the Box-Cox choice $x_{in}(\delta) = \left(z_{in}^\delta - 1\right)/\delta$ for a positive explanatory variable z_{in}. Generally, the vector β_{0n} is distinguished from δ_0, playing a similar scaling role as in a linear model (and unlike δ_0, β_{0n} need not be assumed an element of a prescribed compact set, cf Robinson (1972)). Recall also that q is assumed fixed as n increases.

With $X_n \equiv X_n(\delta_0)$ we have $S_n y_n = X_n\beta_{0n} + u$ and, denoting by $\theta = \left(\lambda', \beta', \delta'\right)'$ any admissible values of $\theta_{0n} = \left(\lambda_{0n}', \beta_{0n}', \delta_0'\right)'$, redefine the negative Gaussian pseudo log-likelihood function as

$$\log\left(2\pi\sigma^2\right) - 2n^{-1}\log\left|S_n(\lambda)\right| + \sigma^{-2}n^{-1}\left\|S_n(\lambda)\,y_n - X_n(\delta)\,\beta\right\|^2, \tag{3.1}$$

for nonsingular $S_n(\lambda)$. For given $\gamma = (\lambda', \delta')'$, (3.1) is minimised with respect to β and σ^2 by

$$\bar{\beta}_n(\gamma) = \left(X_n'(\delta)\,X_n(\delta)\right)^{-1} X_n'(\delta)\,S_n(\lambda)\,y_n, \tag{3.2}$$

$$\bar{\sigma}_n^2(\gamma) = n^{-1}y_n'S_n'(\lambda)\,M_n(\delta)S_n(\lambda)\,y_n, \tag{3.3}$$

with $M_n(\delta) = I_n - X_n(\delta)\left(X_n'(\delta)X_n(\delta)\right)^{-1} X_n'(\delta)$. The PMLE of γ_0 is $\hat{\gamma}_n = \arg\min_{\gamma\in\Gamma_n}\mathcal{Q}_n(\gamma)$, where we have redefined

$$\mathcal{Q}_n(\gamma) = \log\bar{\sigma}_n^2(\gamma) + n^{-1}\log\left|S_n^{-1}(\lambda)\,S_n^{-1\prime}(\lambda)\right|, \tag{3.4}$$

$\Gamma_n = \Lambda_n \times \mathcal{D}$, with $\mathcal{D}$ a compact subset of $\mathbb{R}^q$ and $\hat{\delta}_n \equiv \hat{\delta}$. The PMLEs of β_{0n} and σ_0^2 are defined as $\bar{\beta}_n(\hat{\gamma}_n) \equiv \hat{\beta}_n$ and $\bar{\sigma}_n^2(\hat{\gamma}_n) \equiv \hat{\sigma}_n^2$ respectively.

Assumption 13. $\delta_0 \in \mathcal{D}$.

Assumption 14. *$x_{ijn}(\delta)$ are uniformly bounded constants, $i = 1,\ldots,n$, $j = 1,\ldots,k_n$, $\delta \in \mathcal{D}$, and*

$$\varliminf_{n\to\infty} n^{-1}\sup_{\delta\in\mathcal{D}}\underline{\zeta}\left(X_n'(\delta)X_n(\delta)\right) > 0, \text{ as } n \to \infty. \tag{3.5}$$

(3.5) is an asymptotic non-multicollinearity condition.

Assumption 15. *The $x_{ijn}(\delta)$ are uniformly continuous on $\mathcal{D}$: that is, for any $\varepsilon > 0$ and any $\delta_* \in \mathcal{D}$, there exists $\rho > 0$ such that* $\varlimsup_{n\to\infty}\max_{1\le i\le n, 1\le j\le k_n}\sup_{\|\delta-\delta_*\|<\rho;\ \delta\in\mathcal{D}}\left|x_{ijn}(\delta) - x_{ijn}(\delta_*)\right| < \varepsilon$.

Assumption 16. *When δ_0 is unknown,*

$$\|\beta_{0n}\| \sim k_n^{1/2} \text{ as } n \to \infty, \tag{3.6}$$

and for any $\eta > 0$,

$$\varliminf_{n\to\infty}\inf_{(\lambda',\,\delta')'\in\,\Lambda_n\times\overline{\mathcal{N}}_n^{\delta}(\eta)} n^{-1}\beta_{0n}'X_n'T_n'(\lambda)M_n(\delta)\,T_n(\lambda)X_n\beta_{0n}/\left\|\beta_{0n}\right\|^2 > 0. \tag{3.7}$$

We deal in this paper with the relatively challenging case when $\|\beta_{0n}\|$ is unbounded, and control over this is provided by (3.6). The proof with finitely many, but at least one, nonzero β_{0n} elements would be simpler. We could rewrite (3.7) as

$$\varliminf_{n\to\infty}\inf_{(\lambda',\,\beta',\,\delta')'\in\,\Lambda_n\times\mathbb{R}^{k_n}\times\overline{\mathcal{N}}_n^{\delta}(\eta)} n^{-1}\left\|X_n(\delta)\,\beta - T_n(\lambda)X_n\beta_{0n}\right\|^2/\left\|\beta_{0n}\right\|^2 > 0, \tag{3.8}$$

which is analogous to the identification condition for the nonlinear regression model $y_n = X_n(\delta)\,\beta_{0n} + u$ (take $\lambda = \lambda_{0n}$) with a parametric linear factor in Robinson (1972), and (3.8) may be easier to comprehend than (3.7). A sufficient condition is: for any $\eta > 0$

$$\underline{\lim}_{n\to\infty} \inf_{(\lambda',\,\delta')'\in\,\Lambda_n\times\overline{\mathcal{N}}_n^{\,\delta}(\eta)} n^{-1}\underline{\zeta}\left(X_n' T_n'(\lambda) M_n\left(\delta\right) T_n(\lambda) X_n\right) > 0. \tag{3.9}$$

Theorem 3.1. *Let (1.3) and Assumptions 1-7, 13-16 hold, and p_n, k_n be allowed to diverge as $n \to \infty$ such that*

$$\frac{k_n}{n} \longrightarrow 0, \text{ as } n \to \infty. \tag{3.10}$$

Then

$$\left\|\hat{\theta}_n - \theta_{0n}\right\| \xrightarrow{p} 0, \text{ as } n \to \infty.$$

As discussed after Theorem 2.1 the same proof holds when p_n and k_n remain fixed, and the restriction on k_n in (3.10) becomes redundant. The conditions of the theorem can be compared to those in Gupta and Robinson (2015). The requirement of finite fourth moments for u_i is not imposed by them for consistency of the IV and OLS estimates, where second moments suffice. On the other hand, the only restriction imposed on h_n here is that it be bounded away from zero uniformly in n. For $\epsilon > 0$, define $\mathcal{N}^\delta(\epsilon) = \{\delta : \|\delta - \delta_0\| < \epsilon\}$.

Assumption 17. *For some $\epsilon > 0$, $\partial x_{ijn}\left(\delta\right)/\partial \delta_l$ exist and are uniformly bounded in absolute value for all $\delta \in \mathcal{N}^\delta(\epsilon) \cap \mathcal{D}$, $i = 1\ldots, n$, $j = 1, \ldots, k_n$, $l = 1, \ldots, q$. As $n \to \infty$, $\overline{\lim}_{n\to\infty} n^{-1}\bar{\zeta}\left(X_n' X_n\right) < \infty$.*

This assumption implies $\sup_{\delta\in\mathcal{N}^\delta(\epsilon)\cap\mathcal{D}} \left\|\partial x_{ijn}\left(\delta\right)/\partial\delta\right\| < C$.

Theorem 3.2. *Let (1.3) and Assumptions 1-7, 13-17 hold, and p_n, k_n be allowed to diverge as $n \to \infty$ such that $p_n k_n^4 \left(p_n + k_n\right)/n \to 0$ as $n \to \infty$. Then $\hat{\sigma}_n^2 - \sigma_0^2 = o_p\left(1\right),$ as $n \to \infty$. If δ_0 is known (i.e. the regression is linear), the sufficient rate can be improved to $p_n^2 k_n^3/n \to 0$ as $n \to \infty$.*

Assumption 18. *For some $\epsilon > 0$, $\partial^2 x_{ijn}\left(\delta\right)/\partial\delta_{l_1}\partial\delta_{l_2}$ and $\partial^3 x_{ijn}\left(\delta\right)/\partial\delta_{l_1}\partial\delta_{l_2}\partial\delta_{l_3}$ exist and are uniformly bounded in absolute value for all $\delta \in \mathcal{N}^\delta(\epsilon) \cap \mathcal{D}$, $i = 1, \ldots, n$, $j = 1, \ldots, k_n$, $l_1, l_2, l_3 = 1, \ldots, q$. As $n \to \infty$,*

$$\overline{\lim}_{n\to\infty} n^{-1} \max_{l=1,,\ldots,q} \bar{\zeta}\left\{\left(\partial X_n'/\partial\delta_l\right)\left(\partial X_n/\partial\delta_l\right)\right\} \quad < \quad \infty, \tag{3.11}$$

$$\overline{\lim}_{n\to\infty} n^{-1} \max_{l_1,l_2=1,,\ldots,q} \bar{\zeta}\left\{\left(\partial^2 X_n'/\partial\delta_{l_1}\partial\delta_{l_2}\right)\left(\partial^2 X_n/\partial\delta_{l_1}\partial\delta_{l_2}\right)\right\} \quad < \quad \infty. \tag{3.12}$$

Together (3.11) and (3.12) imply $n^{-\frac{1}{2}}\left(\left\|\partial X_n/\partial\delta_{l_1}\right\|, \left\|\partial^2 X_n/\partial\delta_{l_1}\partial\delta_{l_2}\right\|\right) = \mathscr{O}(1)$, uniformly in $l_1, l_2 = 1\ldots, q$.

Let $\Pi_n\left(\theta\right)$ be the $n \times q$ matrix with i-th column $\left(\partial X_n(\delta)/\partial\delta_i\right)\beta$, where the matrix is differentiated element-by-element. Redefine H_n to be the second derivative matrix of (3.1), so

$$\Xi_n = \mathbb{E}(H_n) = 2\sigma_0^{-2} n^{-1} \begin{bmatrix} \sigma_0^2 (P_{1n} + P_{2n}) + A_n' A_n & A_n' X_n & A_n' \Pi_n \\ * & X_n' X_n & X_n' \Pi_n \\ * & * & \Pi_n' \Pi_n \end{bmatrix}, \tag{3.13}$$

where $A_n = [a_{1n}, \ldots, a_{p_n n}]$ with $a_{jn} = G_{jn} X_n \beta_{0n}$. Assumption 14 implies $a_{ijn} = \mathscr{O}(k_n)$, uniformly in $i = 1, \ldots, n$, $j = 1, \ldots, p_n$, where a_{ijn} is the (i,j)-th element of A_n. More details on derivatives are in Appendix A, where their components are used in the proofs of the central limit theorems stated below. Define $L_n = n^{-1}\left([A_n, X_n, \Pi_n]' [A_n, X_n, \Pi_n]\right)$, which equals

$$\frac{\sigma_0^2}{2} \Xi_n - \sigma_0^2 \begin{bmatrix} P_{1n} + P_{2n} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Assumption 19. $\varliminf_{n\to\infty} \underline{\zeta}(L_n) > 0$ *and* $\varlimsup_{n\to\infty} \overline{\zeta}(L_n) < \infty$.

Theorem 3.3. *Let $h_n \to \infty$ as $n \to \infty$, (1.3) and Assumptions 1, 2, 4, 6-8, 12, 14-19 hold, δ_0 be in the interior of $\mathcal{D}$, and p_n, k_n be allowed to diverge as $n \to \infty$ such that*

$$\frac{p_n^2 k_n^6}{n} (p_n + k_n) + \frac{p_n^3 k_n^2}{h_n^2} \longrightarrow 0, \text{ as } n \to \infty. \tag{3.14}$$

Then

$$\frac{n^{\frac{1}{2}}}{(p_n + k_n)^{\frac{1}{2}}} \Psi_n \left(\hat{\theta}_n - \theta_{0n}\right) \xrightarrow{d} N(0, \Delta_4), \text{ as } n \to \infty,$$

where $\Delta_4 = \sigma_0^2 \lim_{n\to\infty} (p_n + k_n)^{-1} \Psi_n L_n^{-1} \Psi_n'$.

The matrix

$$n^{-1}\left[W_{1n} y_n, \ldots, W_{p_n n} y_n, X_n\left(\hat{\delta}\right), \Pi_n\left(\hat{\theta}_n\right)\right]' \left[W_{1n} y_n, \ldots, W_{p_n n} y_n, X_n\left(\hat{\delta}\right), \Pi_n\left(\hat{\theta}_n\right)\right]$$

and $\hat{\sigma}_n^2$ can replace L_n and σ_0^2 respectively to obtain a consistent estimate of Δ_4. When p_n and k_n are fixed we obtain $n^{\frac{1}{2}}\left(\hat{\theta}_n - \theta_0\right) \xrightarrow{d} N\left(0, \sigma_0^2 \lim_{n\to\infty} L_n^{-1}\right)$ via the Cramér-Wold device, as discussed after Theorem 2.3. Similar comments apply after the other central limit theorems presented subsequently both in this section and the next one. If h_n is bounded as $n \to \infty$ a more complicated analysis is required because the information equality does not hold asymptotically. Define

$$\Omega_n = \sigma_0^{-4} n^{-1} \begin{bmatrix} 2\mu_3 (F_n' A_n + A_n' F_n) + \left(\mu_4 - 3\sigma_0^4\right) F_n' F_n & 2\mu_3 F_n' X_n & 2\mu_3 F_n' \Pi_n \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}. \tag{3.15}$$

Again $n^{-1}(2\Xi_n + \Omega_n)$ is the covariance matrix of the first derivative of (3.1). The asymptotic distribution relies on the following non-multicollinearity and boundedness condition:

Assumption 20. $\overline{\lim}_{n\to\infty} \bar{\zeta}\left(\Xi_n^{-1}\Omega_n\Xi_n^{-1}\right) < \infty$, $\underline{\lim}_{n\to\infty} \underline{\zeta}\left(2\Xi_n^{-1} + \Xi_n^{-1}\Omega_n\Xi_n^{-1}\right) > 0$ *and* $\underline{\lim}_{n\to\infty} \underline{\zeta}\left(\Xi_n\right) > 0$.

Theorem 3.4. *Let h_n be bounded as $n \to \infty$, (1.3) and Assumptions 1, 2, 4, 6-8, 11, 12, 14-18, 20 hold, δ_0 be in the interior of $\mathcal{D}$, and p_n, k_n be allowed to diverge as $n \to \infty$ such that*

$$\frac{p_n^2 k_n^4}{n}\left(p_n^3 + k_n^3 + p_n k_n^2\right) + \frac{\left(p_n k_n\right)^{2+\frac{8}{\chi}}}{n} \longrightarrow 0, \text{ as } n \to \infty. \tag{3.16}$$

Then

$$\frac{n^{\frac{1}{2}}}{\left(p_n + k_n\right)^{\frac{1}{2}}}\Psi_n\left(\hat{\theta}_n - \theta_{0n}\right) \xrightarrow{d} N\left(0, \Delta_5\right), \text{ as } n \to \infty,$$

where $\Delta_5 = \lim_{n\to\infty}\left(p_n + k_n\right)^{-1}\Psi_n\left(2\Xi_n^{-1} + \Xi_n^{-1}\Omega_n\Xi_n^{-1}\right)\Psi_n'$.

It may be shown that if $\chi \geq 8/3$ then $p_n^5 k_n^7/n = o(1)$ suffices for (3.16) to hold while if $\chi \geq 8/5$ and p_n is fixed then $k_n^7/n \to 0$ is sufficient.

For linear regression, i.e. when δ_0 is known, Gupta and Robinson (2015) show that the asymptotic covariance matrix of a fixed number of linear combinations of IV estimates is given by $\Delta_{IV} = \sigma_0^2 \lim_{n\to\infty}\left(p_n + k_n\right)^{-1} n^{-1}\Psi_n\left(\left[A_n, X_n\right]' \mathscr{P}\left(\left[Z_n, X_n\right]\right)\left[A_n, X_n\right]\right)^{-1}\Psi_n'$, where $\mathscr{P}(A) = A(A'A)^{-1}A'$ for a matrix A with full column rank. On the other hand, when u in (1.3) is normally distributed, $\Omega_n = 0$ and $\Delta_5 = 2\lim_{n\to\infty}\left(p_n + k_n\right)^{-1}\Psi_n\Xi_n^{-1}\Psi_n'$, where Ξ_n in (3.13) now no longer contains the blocks with Π_n. Straightforward calculations show that $\lim_{n\to\infty}\left(p_n + k_n\right)^{-1}\Psi_n\left(2^{-1}\Xi_n - \left[A_n, X_n\right]' \mathscr{P}\left(\left[Z_n, X_n\right]\right)\left[A_n, X_n\right]\right)\Psi_n'$ equals

$$\sigma_0^2 \lim_{n\to\infty}\left(p_n + k_n\right)^{-1} n^{-1}\Psi_n\left(\begin{bmatrix} \sigma_0^2\left(P_{1n} + P_{2n}\right) & 0 \\ 0 & 0 \end{bmatrix} + \left[A_n, X_n\right]' M_{\left[Z_n, X_n\right]}\left[A_n, X_n\right]\right)\Psi_n',$$

which is the sum of two nonnegative definite matrices, implying that $\Delta_{IV} \geq \Delta_5$.

4 Regression with SAR errors

We can write (1.4) as

$$S_n\left(\lambda_0\right) y_n = X_n\left(\gamma_0\right)\beta_0 + u, \tag{4.1}$$

where with some abuse of notation $X_n(\gamma) = S_n(\lambda) X_n(\delta)$. Thus consider $\mathcal{Q}_n(\gamma)$ defined as before but with

$$\begin{aligned} \overline{\sigma}_n^2(\gamma) &= n^{-1} y_n' S_n'(\lambda) M_n(\gamma) S_n(\lambda) y_n, \\ M_n(\gamma) &= I_n - X_n(\gamma)\left(X_n'(\gamma) X_n(\gamma)\right)^{-1} X_n'(\gamma). \end{aligned}$$

Write $X_n = X_n(\gamma_0)$ and introduce

Assumption 21. *When δ_0 is unknown, (3.6) holds and for any $\eta > 0$*

$$\varliminf_{n\to\infty} \inf_{(\lambda',\delta')\in\Lambda_n\times\overline{\mathcal{N}}_n^{\delta}(\eta)} n^{-1}\beta_{0n}'X_n'T_n'(\lambda)M_n(\gamma)T_n(\lambda)X_n\beta_{0n}/\|\beta_{0n}\|^2 > 0.$$

Theorem 4.1. *Let (1.4) and Assumptions 1-7, 13-15 and 21 hold, and p_n, k_n be allowed to diverge as $n\to\infty$ such that*

$$\frac{k_n}{n} \longrightarrow 0, \text{ as } n\to\infty. \tag{4.2}$$

Then

$$\left\|\hat{\theta}_n - \theta_{0n}\right\| \xrightarrow{p} 0, \text{ as } n\to\infty.$$

Under similar regularity conditions as in the previous section we may obtain the asymptotic distribution of $\hat{\theta}_n = \left(\hat{\lambda}_n', \hat{\beta}_n'\right)'$. We provide the derivatives of the redefined (3.1) in Appendix A, from which

$$\Xi_n = 2\sigma_0^{-2}n^{-1}\begin{bmatrix} \sigma_0^2(P_{1n}+P_{2n}) & 0 & 0 \\ * & X_n'S_n'S_nX_n & X_n'S_n'\Pi_n \\ * & * & \Pi_n'\Pi_n \end{bmatrix}, \tag{4.3}$$

which is block diagonal between λ and $(\beta',\delta')'$ and, notably, the top left block can have spectral norm going to zero when $h_n\to\infty$ because it is identical to (2.11), which entailed a different norming in Theorems 2.3 and 2.6.

Assumption 22. *For some $\epsilon > 0$, $\partial x_{ijn}(\gamma)/\partial\gamma_l$ exist and are uniformly bounded in absolute value for all $\gamma\in\mathcal{N}^{\gamma}(\epsilon)\cap\Gamma$, $i = 1\ldots,n$, $j = 1,\ldots,k_n$, $l = 1,\ldots,p_n+q$. As $n\to\infty$, $\varlimsup_{n\to\infty} n^{-1}\bar{\zeta}(X_n'X_n) < \infty$.*

Assumption 23. *For some $\epsilon > 0$, $\partial^2 x_{ijn}(\gamma)/\partial\gamma_{l_1}\partial\gamma_{l_2}$ and $\partial^3 x_{ijn}(\gamma)/\partial\gamma_{l_1}\partial\gamma_{l_2}\partial\gamma_{l_3}$ exist and are uniformly bounded in absolute value for all $\gamma\in\mathcal{N}^{\gamma}(\epsilon)\cap\mathcal{D}$, $i = 1,\ldots,n$, $j = 1,\ldots,k_n$, $l_1,l_2,l_3 = 1,\ldots,p_n+q$. As $n\to\infty$,*

$$\varlimsup_{n\to\infty} n^{-1}\max_{l=1,,\ldots,p_n+q}\bar{\zeta}\{(\partial X_n'/\partial\gamma_l)(\partial X_n/\partial\gamma_l)\} < \infty, \tag{4.4}$$

$$\varlimsup_{n\to\infty} n^{-1}\max_{l_1,l_2=1,,\ldots,p_n+q}\bar{\zeta}\left\{\left(\partial^2 X_n'/\partial\gamma_{l_1}\partial\gamma_{l_2}\right)\left(\partial^2 X_n/\partial\gamma_{l_1}\partial\gamma_{l_2}\right)\right\} < \infty. \tag{4.5}$$

In the two central limit theorems stated below, identification conditions are taken to hold for the changed definitions of Ξ_n and Ω_n in this section. These definitions are described in Appendix A, but a feature of the next theorem is the differential norming that implies a slower rate of convergence for $\hat{\lambda}_n$ as compared to $\left(\hat{\beta}_n', \hat{\delta}_n'\right)'$. Define $\Phi_n = diag[h_nI_{p_n}, I_{k_n}, I_q]$ and write $B_n^{\Phi} = \Phi_n^{\frac{1}{2}}B_n\Phi_n^{\frac{1}{2}}$ for a generic matrix B_n.

Theorem 4.2. *Let $h_n\to\infty$ as $n\to\infty$, (1.4) and Assumptions 1, 2, 4, 6-8, 11, 12, 14, 15 and 21-23 hold, δ_0 be in the interior of $\mathcal{D}$, $\varlimsup_{n\to\infty}\bar{\zeta}\left(\Xi_n^{\Phi-1}\Omega_n^{\Phi}\Xi_n^{\Phi-1}\right) < \infty$, $\varliminf_{n\to\infty}\underline{\zeta}\left(\Xi_n^{\Phi}\right) > 0$,*

$\underline{\lim}_{n\to\infty} \underline{\zeta} \left(2\Xi_n^{\Phi-1} + \Xi_n^{\Phi-1}\Omega_n^{\Phi}\Xi_n^{\Phi-1}\right) > 0$, *and (2.13), (3.14) hold if* p_n, k_n *are allowed to diverge as* $n \to \infty$. *Then*

$$\frac{n^{\frac{1}{2}}}{(p_n + k_n)^{\frac{1}{2}}} \Psi_n \Phi_n^{-\frac{1}{2}} \left(\hat{\theta}_n - \theta_{0n}\right) \xrightarrow{d} N\left(0, \Delta_6\right), \text{ as } n \to \infty,$$

where $\Delta_6 = \lim_{n\to\infty} (p_n + k_n)^{-1} \Psi_n \left(2\Xi_n^{\Phi-1} + \Xi_n^{\Phi-1}\Omega_n^{\Phi}\Xi_n^{\Phi-1}\right) \Psi_n'$.

Theorem 4.3. *Let* h_n *be bounded as* $n \to \infty$, *(1.4) and Assumptions 1, 2, 4, 6-8, 11, 12, 14, 15 and 20-23 hold,* δ_0 *be in interior of* $\mathcal{D}$, *and (3.16) hold if* p_n, k_n *are allowed to diverge as* $n \to \infty$. *Then*

$$\frac{n^{\frac{1}{2}}}{(p_n + k_n)^{\frac{1}{2}}} \Psi_n \left(\hat{\theta}_n - \theta_{0n}\right) \xrightarrow{d} N\left(0, \Delta_7\right), \text{ as } n \to \infty,$$

where $\Delta_7 = \lim_{n\to\infty} (p_n + k_n)^{-1} \Psi_n \left(2\Xi_n^{-1} + \Xi_n^{-1}\Omega_n\Xi_n^{-1}\right) \Psi_n'$.

5 Finite-sample performance

In this section we study the finite-sample properties of the estimates considered above in a Monte Carlo study, in two distinct settings considered earlier eg in Gupta and Robinson (2015). In the first setting, we consider Case (1991, 1992), where weight matrices take the 'single nonzero diagonal block' specification

$$W_{kn}^f = diag\left[0, \ldots, \underbrace{V_m}_{k-th \text{ diagonal block}}, \ldots, 0\right], \; k = 1, \ldots, p, \tag{5.1}$$

with $V_m = (m-1)^{-1}(l_m l_m' - I_m)$. In the second setting the weight matrices are

$$W_{kn}^c = \left(\|W_{kn}^*\|\right)^{-1} W_{kn}^*, \tag{5.2}$$

with W_{kn}^* the symmetric circulant matrix with first row elements given by

$$w_{1j,kn}^* = \begin{cases} 0 & \text{if } j = 1 \text{ or } j = k+2, \ldots, n-k; \\ 1 & \text{if } j = 2, \ldots, k+1 \text{ or } j = n-k+1, \ldots, n. \end{cases} \tag{5.3}$$

Thus W_{kn}^c is also a symmetric circulant matrix with first row elements given by $w_{1j,kn}^*/2k$. In both experiments we took $p = 2, 4, 6$. We first analyse the pure and intercept SAR cases. y_n was generated using (1.1) or (1.2) in each of the 1000 replications. We chose $\lambda_{01} = 0.7, \lambda_{02} = 0.8, \lambda_{03} = 0.5, \lambda_{04} = 0.8, \lambda_{05} = 0.4$ and $\lambda_{06} = 0.3$, when using W_{kn}^f while the values chosen when using W_{kn}^c were $\lambda_{01} = 0.1, \lambda_{02} = 0.2, \lambda_{03} = 0.2, \lambda_{04} = 0.1, \lambda_{05} = 0.1$ and $\lambda_{06} = 0.2$ (because a sufficient condition for S_n^{-1} to exist in this case is $\|\lambda\|_1 < 1$). One set of u_i was generated as independent draws from $N(0,1)$ (here PMLE is MLE), and another set as independent draws

$u \sim N(0, I_n)$

n		108		216		432	
	p	Bias	MSE	Bias	MSE	Bias	MSE
W^c_{kn}	2	0.0169	0.0267	0.0036	0.0138	0.0017	0.0069
	4	0.0464	0.1300	0.0592	0.0861	0.0181	0.0404
	6	0.0449	0.2325	0.1068	0.2298	0.0284	0.1058
s		12		24		36	
W^f_{kn}	2	0.0396	0.0132	0.0177	0.0040	0.0114	0.0023
	4	0.1047	0.0710	0.0453	0.0198	0.0288	0.0105
	6	0.2017	0.1982	0.0962	0.0703	0.0593	0.0352

$u \sim t_6$

n		108		216		432	
	p	Bias	MSE	Bias	MSE	Bias	MSE
W^c_{kn}	2	0.0141	0.0274	0.0026	0.0135	0.0012	0.0069
	4	0.0501	0.1277	0.0499	0.0880	0.0121	0.0364
	6	0.0350	0.2296	0.0917	0.2189	0.0356	0.1099
s		12		24		36	
W^f_{kn}	2	0.0343	0.0114	0.0178	0.0040	0.0108	0.0023
	4	0.0991	0.0685	0.0441	0.0180	0.0262	0.0093
	6	0.2001	0.2014	0.0923	0.0661	0.0574	0.0336

Table 5.1: Monte Carlo (average) absolute bias and (average) MSE for PMLE, model (1.1)

from t_6 ($\sigma_0^2 = 3/2$), having thicker tails.

Tables 5.1 and 5.2 display Monte Carlo (absolute) bias and MSE for (1.1) and (1.2) respectively, with $\tau_0 = 1$. Table 5.2 considers only W^c_{kn}, the inclusion of an intercept not being possible with W^f_{kn} (cf Kelejian, Prucha, and Yuzefovich (2006)). Averages (averaging over bias and MSE for λ_{0i}, $i = 1, \ldots, p$) are reported for the spatial parameter estimates to conserve space. We report results for $m = 16$ ($m = 8, 24$ were also simulated) only when using W^f_{kn}, and also take the number of districts s (implying $n = 16s$, and more generally $n = ms$) to grow faster than p. Indeed Theorems 2.3 and 2.4 indicate that when m is either bounded or divergent the PMLE is $s^{\frac{1}{2}}/p^{\frac{1}{2}}$-consistent for the farmer-district setting, and in any case $p^{2+\frac{8}{\chi}} m^{\frac{4}{\chi}}/s + p^4 m/s \to 0$ as $p, m, s \to \infty$ is necessary for (2.13) to hold asymptotically. We take $s = 12, 24, 36$, and this implies the need to combine spatial weight matrices by imposing the same spatial parameter for some blocks. When $p = 2$ we combine into two groups with six blocks, twelve blocks and eighteen blocks each when $s = 12, 24, 36$, respectively. When $p = 4$ (respectively $p = 6$) we combine into four groups (respectively six groups) with three, six and nine blocks each (respectively two, four and six blocks each). When using W^c_{kn} we take $n = 108, 216, 432$.

In Table 5.1, bias and MSE decline with sample size for both MLE and PMLE of (1.1), using

$u \sim N(0, I_n)$	n	108		216		432	
p		Bias	MSE	Bias	MSE	Bias	MSE
2	λ	0.0275	0.0277	0.0088	0.0141	0.0043	0.0069
	τ	0.0386	0.0420	0.0181	0.0192	0.0079	0.0092
4	λ	0.0445	0.1327	0.0607	0.0844	0.0177	0.0404
	τ	0.4455	1.7600	0.4286	1.5636	0.1772	0.6106
6	λ	0.0356	0.2187	0.0856	0.2115	0.0272	0.1030
	τ	1.3373	7.0599	1.4352	5.7450	0.5206	2.0209
$u \sim t_6$							
p		Bias	MSE	Bias	MSE	Bias	MSE
2	λ	0.0253	0.0286	0.0080	0.0137	0.0027	0.0069
	τ	0.0384	0.0468	0.0207	0.0225	0.0093	0.0107
4	λ	0.0433	0.1270	0.0511	0.0865	0.0099	0.0350
	τ	0.3932	1.5885	0.3694	1.3702	0.0952	0.3675
6	λ	0.0370	0.2233	0.0715	0.1998	0.0271	0.1044
	τ	1.3176	7.3067	1.5214	6.1180	0.5600	2.1321

Table 5.2: Monte Carlo absolute bias and MSE for PMLE, model (1.2), with W^c_{kn} only.

either W^c_{kn} or W^f_{kn}, although with the former the decline in bias is not necessarily monotonic. Generally biases for W^f_{kn} exceed those for W^c_{kn} but MSEs and thus variances tend to be smaller. Table 5.2 indicates a similar, non-monotonic, pattern of reduction for (1.2). However the bias and MSE for $\hat{\tau}_n$ can be very high for large p, eg for $p = 6$ they are unacceptable even when $n = 432$.

Tables 5.3 and 5.4 similarly display Monte Carlo size and power for (1.1) and (1.2) respectively, with nominal size 5%. Power was computed using the false null hypothesis $\lambda_i, \tau = 0.5$, for each i. With W^c_{kn} in (1.1), size approaches the nominal value non-monotonically with n, but with (1.2) the behaviour is rather more erratic. For $p = 2$ the oversizing is moderate, but dramatically worsens for $p = 4, 6$. However in each case it gets closer to the nominal size as n increases, although not necessarily monotonically. On the contrary, with W^f_{kn} there is considerable undersizing. Larger values of s give little indication of an approach to the nominal 5%. The sizes are better for larger p, the best results arising when $p = 6$. The behaviour is similar across $N(0, 1)$ or t_6 disturbances. On the other hand power increases monotonically in each of the various settings, and would be much higher for the $p = 4, 6$ cases if not for $\lambda_{03} = 0.5$ (true under the false null), which effectively caps power at around 83%.

When generating y_n using (1.3), we set $k_n = 2$ and $\beta_{01} = 1$, $\beta_{02} = 0.7$. In X_n we took $x_{i1n}(\delta) = (z^{\delta}_{i1} - 1)/\delta$ and $x_{i2n}(\delta) = z_{i2}$, with $(z_{i1}, z_{i2})' \sim U(0, 5)$ (generated independently of each other), $i = 1, \ldots, n$, with 1000 replications, and $\delta_0 = 0.7$. With W^f_{kn} equal blocks of size m were used, while three different m were chosen for each p: 48, 96 and 144. We also simulated a model with $x_{i1n}(\delta) = \mathrm{e}^{\delta z_{i1}}$ and $x_{i2n}(\delta) = z_{i2}$ and found similar results.

$u \sim N(0, I_n)$							
n		108		216		432	
	p	Size	Power	Size	Power	Size	Power
W^c_{kn}	2	0.0475	0.5800	0.0460	0.7935	0.0400	0.9470
	4	0.0660	0.2715	0.0998	0.4047	0.0760	0.5830
	6	0.0335	0.1860	0.1197	0.3043	0.1027	0.4040
s		12		24		36	
W^f_{kn}	2	0.0085	0.5835	0.0060	0.7805	0.0060	0.8855
	4	0.0103	0.3520	0.0083	0.4925	0.0073	0.5867
	6	0.0300	0.2035	0.0242	0.2807	0.0215	0.3422
$u \sim t_6$							
n		108		216		432	
	p	Size	Power	Size	Power	Size	Power
W^c_{kn}	2	0.0560	0.5695	0.0425	0.7970	0.0490	0.9480
	4	0.0583	0.2745	0.0985	0.4233	0.0660	0.5775
	6	0.0313	0.1818	0.1148	0.3062	0.1078	0.4085
s		12		24		36	
W^f_{kn}	2	0.0090	0.5910	0.0060	0.7755	0.0070	0.8770
	4	0.0123	0.3602	0.0067	0.4080	0.0070	0.5900
	6	0.0303	0.2087	0.0230	0.2855	0.0237	0.3480

Table 5.3: Monte Carlo average size and average power for PMLE, model (1.1)

We now discuss the results for $\hat{\theta}_n$ in Tables 5.5 and 5.6, which report Monte Carlo bias and MSE for $u \sim N(0, I_n)$ and $u \sim t_6$ respectively. It is interesting to note that for W^f_{kn} increasing m mostly improves the estimates of the spatial parameters, for fixed p. However, Lee (2004) showed that the PMLE is inconsistent if $p = 1$ when m alone increases, while simulations conducted by Hillier and Martellosio (2013) also suggest convergence to a nondegenerate distribution. Similar results will undoubtedly apply if $p > 1$, but fixed, and m alone increases. On the other hand, the block-diagonality of W^f_{kn} implies that the number of observations available to estimate the λ_{0i} increase one-to-one with m. Generally bias and MSE improve with n, as expected. For W^c_{kn} the results are as expected. Bias and MSE reduce with larger n and smaller p, and also with larger n for fixed p, and seem acceptable.

Tables 5.7 and 5.8 report Monte Carlo size and power for $u \sim N(0, I_n)$ and $u \sim t_6$ respectively. Now power is calculated using the incorrect null hypothesis $\theta_i = 0.6$, for each i. Under normality, sizes when using W^f_{kn} are always between 4.3% and 8.2% but those for W^c_{kn} range from 2.35% to 7.1%. When the disturbances are non-normal matters are similar, although there are instances ($p = 2, 6$) of severe undersizing for λ_{0i} with W^c_{kn} that persists for all values of n. On the other hand, the power tends to increase (but not always monotonically) with large n and small p for

$u \sim N(0, I_n)$	n	108		216		432	
p		Size	Power	Size	Power	Size	Power
2	λ	0.0275	0.5950	0.0088	0.8055	0.0043	0.9505
	τ	0.0750	0.8460	0.0560	0.9920	0.0570	1.0000
4	λ	0.0445	0.2742	0.0607	0.4072	0.0177	0.5837
	τ	0.2630	0.7070	0.2000	0.9320	0.1260	1.0000
6	λ	0.0356	0.1817	0.0856	0.2982	0.0272	0.4038
	τ	0.4050	0.4960	0.5330	0.7830	0.2960	0.9110
$u \sim t_6$							
p		Size	Power	Size	Power	Size	Power
2	λ	0.0253	0.5885	0.0080	0.8090	0.0027	0.9540
	τ	0.0540	0.7950	0.0630	0.9720	0.0610	1.0000
4	λ	0.0433	0.2785	0.0511	0.4265	0.0099	0.5825
	τ	0.2360	0.6210	0.1840	0.9100	0.0890	0.9920
6	λ	0.0370	0.1778	0.0715	0.2968	0.0271	0.4060
	τ	0.4090	0.4820	0.5540	0.7680	0.2970	0.9920

Table 5.4: Monte Carlo size and power for PMLE, model (1.2), with W_{kn}^c only.

W_{kn}^c but large m, p, for W_{kn}^f, due to the increase in n afforded by increasing p. Power for δ_0 tends to be low across the board, in part because its true value is 0.7 and the postulated value is 0.6. This factor doubtless also plays a role in the lower power for β_{02} generally as compared to that for β_{01}.

Finally, Table 5.9 compares $\hat{\theta}_n$ with the IV estimate of Gupta and Robinson (2015) (denoted $\check{\theta}_n$) when W_{kn}^c are employed, $x_{i1n}(\delta) = z_{i1}$ also (i.e. linear regressive SAR) and $(z_{i1}, z_{i2}) \sim U(0,1)$ to match their design. We consider u generated from both $N(0, I_n)$ and t_6 distributions. We report relative average MSE (RAMSE) separately for the autoregression and regression components, defining these as average MSE$(\hat{\lambda}_n)$/average MSE$(\check{\lambda}_n)$ and average MSE$(\hat{\beta}_n)$/average MSE$(\check{\beta}_n)$, using the instruments $\{W_{jn}^c z_{i1}, W_{jn}^c z_{i2}\}$, $j = 1, \ldots, p$. The PMLE does very well in general. The IV estimates outperform the PMLE for the regression coefficients β_{01} and β_{02} in 4 out of 6 cases when $p = 6$, but fare much worse for the spatial parameters in all cases. Experiments in which the u were generated from a $\chi_6^2 - 6$ (this having $\sigma_0^2 = 12$, and also being non-symmetric) distribution followed the same pattern.

It is particularly interesting to note that the PMLE outperforms the IV estimate by such a large margin even when the disturbances are non-normal. A possible explanation for this is that the IV estimate relies on instruments derived from a power series expansion of S_n^{-1}, and the estimates are sensitive to the strength of these instruments.

W^c_{kn}	n	108		216		432	
p		Bias	MSE	Bias	MSE	Bias	MSE
2	λ	0.0036	0.0110	0.0009	0.0053	0.0003	0.0028
	δ	0.0184	0.0462	0.0212	0.0220	0.0016	0.0089
	β_1	0.0135	0.0238	0.0150	0.0116	0.0025	0.0049
	β_2	0.0018	0.0038	0.0002	0.0017	0.0002	0.0010
4	λ	0.0085	0.0546	0.0029	0.0268	0.0028	0.0132
	δ	0.0176	0.0470	0.0215	0.0222	0.0018	0.0090
	β_1	0.0172	0.0242	0.0171	0.0116	0.0037	0.0049
	β_2	0.0048	0.0043	0.0010	0.0020	0.0006	0.0011
6	λ	0.0073	0.1186	0.0070	0.0648	0.0070	0.0312
	δ	0.0194	0.0492	0.0221	0.0227	0.0023	0.0092
	β_1	0.0220	0.0251	0.0196	0.0118	0.0050	0.0050
	β_2	0.0068	0.0046	0.0024	0.0021	0.0010	0.0012
W^f_{kn}	m	48		96		144	
p		Bias	MSE	Bias	MSE	Bias	MSE
2	λ	0.0009	0.0002	0.0018	0.0003	0.0010	0.0002
	δ	0.0038	0.0214	0.0090	0.0232	0.0090	0.0156
	β_1	0.0083	0.0264	0.0026	0.0129	0.0055	0.0078
	β_2	0.0042	0.0052	0.0066	0.0025	0.0032	0.0019
4	λ	0.0044	0.0006	0.0020	0.0003	0.0012	0.0002
	δ	0.0129	0.0230	0.0070	0.0117	0.0023	0.0074
	β_1	0.0022	0.0129	0.0036	0.0061	0.0001	0.0040
	β_2	0.0109	0.0026	0.0059	0.0013	0.0022	0.0008
6	λ	0.0059	0.0010	0.0027	0.0005	0.0020	0.0003
	δ	0.0142	0.0158	0.0037	0.0074	0.0046	0.0050
	β_1	0.0046	0.0079	0.0000	0.0040	0.0014	0.0028
	β_2	0.0087	0.0020	0.0039	0.0008	0.0028	0.0006

Table 5.5: Monte Carlo absolute bias and MSE for MLE ($u \sim N(0, I_n)$), model (1.3) with $x_{i1n}(\delta) = (z_{i1}^{\delta} - 1)/\delta$.

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W_{kn}^c	n	108		216		432	
p		Bias	MSE	Bias	MSE	Bias	MSE
2	λ	0.0024	0.0134	0.0010	0.0068	0.0033	0.0031
	δ	0.0263	0.0708	0.0148	0.0304	0.0076	0.0143
	β_1	0.0183	0.0358	0.0112	0.0170	0.0053	0.0074
	β_2	0.0028	0.0057	0.0010	0.0026	0.0018	0.0012
4	λ	0.0110	0.0622	0.0112	0.0319	0.0069	0.0165
	δ	0.0250	0.0728	0.0165	0.0308	0.0080	0.0145
	β_1	0.0229	0.0366	0.0143	0.0171	0.0067	0.0074
	β_2	0.0059	0.0066	0.0018	0.0030	0.0013	0.0014
6	λ	0.0140	0.1371	0.0112	0.0743	0.0046	0.0390
	δ	0.0261	0.0748	0.0159	0.0308	0.0083	0.0147
	β_1	0.0283	0.0371	0.0169	0.0173	0.0081	0.0074
	β_2	0.0092	0.0071	0.0035	0.0033	0.0007	0.0015
W_{kn}^f	m	48		96		144	
p		Bias	MSE	Bias	MSE	Bias	MSE
2	λ	0.0036	0.0009	0.0030	0.0004	0.0009	0.0003
	δ	0.0384	0.0881	0.0239	0.0378	0.0072	0.0238
	β_1	0.0132	0.0411	0.0133	0.0195	0.0041	0.0125
	β_2	0.0079	0.0084	0.0098	0.0041	0.0048	0.0026
4	λ	0.0070	0.0009	0.0036	0.0005	0.0018	0.0003
	δ	0.0291	0.0380	0.0165	0.0180	0.0086	0.0102
	β_1	0.0117	0.0196	0.0077	0.0096	0.0026	0.0055
	β_2	0.0159	0.0042	0.0070	0.0019	0.0023	0.0013
6	λ	0.0083	0.0015	0.0041	0.0007	0.0039	0.0005
	δ	0.0142	0.0237	0.0097	0.0102	0.0119	0.0082
	β_1	0.0024	0.0127	0.0018	0.0055	0.0037	0.0040
	β_2	0.0130	0.0027	0.0045	0.0013	0.0061	0.0009

Table 5.6: Monte Carlo absolute bias and MSE for PMLE ($u \sim t_6$), model (1.3) with $x_{i1n}(\delta) = (z_{i1}^{\delta} - 1)/\delta$.

W^c_{kn}	n	108		216		432	
p		Size	Power	Size	Power	Size	Power
2	λ	0.0335	0.9610	0.0240	0.9985	0.0285	1.0000
	δ	0.0590	0.0600	0.0440	0.0940	0.0460	0.1240
	β_1	0.0640	0.7250	0.0520	0.9140	0.0380	0.9980
	β_2	0.0490	0.3530	0.0350	0.6070	0.0460	0.8930
4	λ	0.0335	0.5093	0.0318	0.7045	0.0275	0.8790
	δ	0.0660	0.0600	0.0520	0.0920	0.0460	0.1230
	β_1	0.0620	0.7210	0.0520	0.9120	0.0380	0.9980
	β_2	0.0500	0.3130	0.0430	0.5580	0.0520	0.8440
6	λ	0.0235	0.3148	0.0273	0.4598	0.0263	0.6620
	δ	0.0710	0.0570	0.0480	0.1020	0.0440	0.1230
	β_1	0.0640	0.7170	0.0510	0.9130	0.0390	0.9980
	β_2	0.0510	0.2830	0.0450	0.5290	0.0550	0.8200
W^f_{kn}	m	48		96		144	
p		Size	Power	Size	Power	Size	Power
2	λ	0.0660	1.0000	0.0565	0.9995	0.0650	1.0000
	δ	0.0760	1.0000	0.0530	0.0700	0.0570	0.1100
	β_1	0.0810	0.8000	0.0680	0.9170	0.0470	0.9830
	β_2	0.0550	0.3470	0.0570	0.5790	0.0720	0.7120
4	λ	0.0548	0.9590	0.0475	0.9960	0.0550	1.0000
	δ	0.0510	0.0740	0.0650	0.1270	0.0560	0.1760
	β_1	0.0690	0.9150	0.0610	0.9960	0.0520	1.0000
	β_2	0.0560	0.6090	0.0480	0.8510	0.0450	0.9470
6	λ	0.0623	0.9862	0.0542	0.9993	0.0540	1.0000
	δ	0.0550	0.1180	0.0570	0.1810	0.0530	0.2830
	β_1	0.0480	0.9810	0.0560	1.0000	0.0560	1.0000
	β_2	0.0700	0.7490	0.0430	0.9530	0.0590	0.9870

Table 5.7: Monte Carlo size and power for MLE ($u \sim N(0, I_n)$), model (1.3) with $x_{i1n}(\delta) = (z_{i1}^{\delta} - 1)/\delta$.

W_{kn}^c	n	108		216		432	
p		Size	Power	Size	Power	Size	Power
2	λ	0.0265	0.9070	0.0260	0.9940	0.0185	1.0000
	δ	0.0650	0.0660	0.0570	0.0550	0.0510	0.0950
	β_1	0.0650	0.5700	0.0600	0.8150	0.0400	0.9810
	β_2	0.0490	0.2540	0.0420	0.4620	0.0360	0.7820
4	λ	0.0238	0.4343	0.0240	0.6228	0.0245	0.8275
	δ	0.0630	0.0670	0.0570	0.0550	0.0460	0.1030
	β_1	0.0640	0.5690	0.0530	0.8120	0.0390	0.9790
	β_2	0.0480	0.2360	0.0450	0.4260	0.0380	0.7330
6	λ	0.0135	0.2687	0.0215	0.3852	0.0227	0.5775
	δ	0.0670	0.0680	0.0540	0.0600	0.0480	0.0970
	β_1	0.0620	0.5650	0.0560	0.8130	0.0400	0.9800
	β_2	0.0510	0.2220	0.0470	0.4010	0.0410	0.7130
W_{kn}^f	m	48		96		144	
p		Size	Power	Size	Power	Size	Power
2	λ	0.0735	0.9115	0.0770	0.9870	0.0620	0.9985
	δ	0.0660	0.0650	0.0540	0.0710	0.0610	0.0750
	β_1	0.0630	0.5610	0.0500	0.7750	0.0620	0.9190
	β_2	0.0640	0.2580	0.0640	0.4300	0.0580	0.5610
4	λ	0.0665	0.9133	0.0643	0.9838	0.0560	0.9968
	δ	0.0550	0.0700	0.0540	0.1020	0.0420	0.1230
	β_1	0.0590	0.7730	0.0540	0.9620	0.0390	0.9990
	β_2	0.0810	0.4700	0.0550	0.6900	0.0540	0.8090
6	λ	0.0602	0.9585	0.0555	0.9965	0.0513	0.9995
	δ	0.0590	0.0780	0.0450	0.1220	0.0600	0.2270
	β_1	0.0620	0.9180	0.0380	0.9990	0.0530	1.0000
	β_2	0.0650	0.6240	0.0550	0.8270	0.0480	0.9550

Table 5.8: Monte Carlo size and power for PMLE ($u \sim t_6$), model (1.3) with $x_{i1n}(\delta) = (z_{i1}^{\delta} - 1)/\delta$.

	n	108	216	432	108	216	432
p			$u \sim N(0, I_n)$			$u \sim t_6$	
2	λ	0.0472	0.0488	0.0507	0.0362	0.0287	0.0284
	β	0.5212	0.5554	0.6202	0.4931	0.5028	0.5649
4	λ	0.0339	0.0413	0.0399	0.0239	0.0231	0.0233
	β	0.4152	0.4706	0.5404	0.4630	0.4022	0.4357
6	λ	0.0353	0.0683	0.0601	0.0300	0.0536	0.0382
	β	0.8069	3.5825	1.5249	0.9315	3.4552	1.3950

Table 5.9: Monte Carlo RAMSE between PMLE and IV with W_{kn}^c, $(z_{i1}, z_{i2}) \sim U(0, 1)$.

Appendices

A Proofs of theorems

Proof of Theorem 2.1. This is omitted as it can be deduced from the proof of Theorem 3.1 below, ignoring components of formulae and steps that are not relevant. □

Proof of Theorem 2.2. In supplementary material. □

We drop n subscripts in the appendices. The following inequalities will be useful: $\|A\| \leq \|A\|_F$, $\|A\|^2 \leq \|A\|_R \|A'\|_R$, $\|AB\|_F \leq \|A\|_F \|B\|$. In the sequel write $\nu = n^{\frac{1}{2}}/a^{\frac{1}{2}}$, where a is the number of columns in Ψ. Thus in Section 2, $a = p$ or $p+1$, in Section 3, $a = p+k+q$ and in Section 4, $a = p+k$. Further, for any matrix, vector $E\left(\theta, \sigma^2\right)$, $\tilde{E}$ denotes evaluation at a generic estimate $\left(\tilde{\theta}', \tilde{\sigma}^2\right)'$ and $\tilde{\Delta}^E = \tilde{E} - E$. We can express (1.3) as $y = R\lambda_0 + X\beta_0 + u$ with $R = [W_1 y, \ldots, W_p y]$. Because Assumption 3 implies

$$y = S^{-1} X\beta_0 + S^{-1} u, \tag{A.1}$$

we have $R = A + B$, with $B = [G_{1n}u, \ldots, G_{p_n n}u]$, and for (1.1) the reduced form (A.1) holds with $X = 0$. The proofs of Theorems 3.3 and 3.4 should be read before the next two proofs, which are in any case in the supplementary appendix. We introduce them at this point to follow the order of the paper.

Proof of Theorem 2.3. In supplementary material. □

Proof of Theorem 2.4. In supplementary material. □

Proof of Theorem 2.5. This is omitted for the same reason as Theorem 2.1's proof. □

Proof of Theorem 2.6. This is similar to the proof of Theorem 2.3 and therefore omitted. □

Proof of Theorem 3.1. The property $\left\|\hat{\beta} - \beta_0\right\| \xrightarrow{p} 0$ follows using arguments below, the closed form expression (see (3.2)) for $\hat{\beta}$ as a function of $\hat{\gamma}$, and the property $\|\hat{\gamma} - \gamma_0\| \xrightarrow{p} 0$, so we focus on proving the latter. From (3.4), (A.1)

$$\begin{aligned} \mathcal{Q}(\gamma) - \mathcal{Q} &= \log \overline{\sigma}^2(\gamma)/\overline{\sigma}^2 - n^{-1}\log\left|T'(\lambda)T(\lambda)\right| \\ &= \log \overline{\sigma}^2(\gamma)/\sigma^2(\lambda) - \log \overline{\sigma}^2/\sigma_0^2 + \log r(\lambda), \end{aligned} \tag{A.2}$$

where

$$\sigma^2(\lambda) = n^{-1}\sigma_0^2 \|T(\lambda)\|_F^2, \ \overline{\sigma}^2 = \overline{\sigma}^2(\gamma_0) = n^{-1}u'Mu,$$

using (3.3) and writing $r(\lambda) = n^{-1} \|T(\lambda)\|_F^2 / |T(\lambda)|^{2/n}$. From (A.1)

$$\begin{aligned}\overline{\sigma}^2(\gamma) &= n^{-1}\left\{S^{-1'}(X\beta_0+u)\right\}' S'(\lambda)M(\delta)S(\lambda)S^{-1}(X\beta_0+u)\\ &= c(\gamma)+d(\gamma)+e(\gamma),\end{aligned}$$

where

$$\begin{aligned}c(\gamma) &= n^{-1}\beta_0' X' T'(\lambda)M(\delta)T(\lambda)X\beta_0,\\ d(\gamma) &= n^{-1}\sigma_0^2 tr\left(T'(\lambda)M(\delta)T(\lambda)\right),\\ e(\gamma) &= n^{-1}tr\left(T'(\lambda)M(\delta)T(\lambda)\left(uu'-\sigma_0^2 I\right)\right)+2n^{-1}\beta_0'X'T'(\lambda)M(\delta)T(\lambda)u.\end{aligned}$$

Then

$$\begin{aligned}\log\frac{\overline{\sigma}^2(\gamma)}{\sigma^2(\lambda)} &= \log\frac{\overline{\sigma}^2(\gamma)}{(c(\gamma)+d(\gamma))}+\log\frac{c(\gamma)+d(\gamma)}{\sigma^2(\lambda)}\\ &= \log\left(1+\frac{e(\gamma)}{c(\gamma)+d(\gamma)}\right)+\log\left(1+\frac{c(\gamma)-f(\gamma)}{\sigma^2(\lambda)}\right),\end{aligned}$$

where

$$f(\gamma) = n^{-1}\sigma_0^2 tr\left(T'(\lambda)\left(I-M(\delta)\right)T(\lambda)\right).$$

Then from (A.2) and a standard kind of argument for proving consistency of implicitly defined extremum estimates

$$\begin{aligned}P\left(\|\hat{\gamma}-\gamma_0\| \in \overline{\mathcal{N}}^{\gamma}(\eta)\right) &= P\left(\inf_{\gamma\in\overline{\mathcal{N}}^{\gamma}(\eta)}\mathcal{Q}(\gamma)-\mathcal{Q}\leq 0\right)\\ &\leq P\left(\log\left(1+\sup_{\gamma\in\overline{\mathcal{N}}^{\gamma}(\eta)}\left|\frac{e(\gamma)}{c(\gamma)+d(\gamma)}\right|\right)+\left|\log\left(\overline{\sigma}^2/\sigma_0^2\right)\right|\right.\\ &\qquad\left.\geq\inf_{\gamma\in\overline{\mathcal{N}}^{\gamma}(\eta)}\left(\log\left(1+\frac{c(\gamma)-f(\gamma)}{\sigma^2(\lambda)}\right)+\log r(\lambda)\right)\right),\end{aligned}$$

where $\overline{\mathcal{N}}^{\gamma}(\eta) = \Gamma\backslash\mathcal{N}^{\gamma}(\eta)$, $\mathcal{N}^{\gamma}(\eta) = \{\gamma : \|\gamma-\gamma_0\| < \eta; \gamma\in\Gamma\}$. From Assumptions 1 and 16 it follows that $\overline{\sigma}^2/\sigma_0^2 \xrightarrow{p} 1$, so using $\log(1+x) = x + o(x)$ as $x \to 0$ it suffices to show that as $n \to \infty$

$$\sup_{\gamma\in\overline{\mathcal{N}}_n^{\gamma}(\eta)}\left|\frac{e(\gamma)}{c(\gamma)+d(\gamma)}\right| \xrightarrow{p} 0, \tag{A.3}$$

$$\sup_{\gamma\in\overline{\mathcal{N}}_n^{\gamma}(\eta)}\left|\frac{f(\gamma)}{\sigma^2(\lambda)}\right| \longrightarrow 0, \tag{A.4}$$

$$\varliminf_{n\to\infty}\inf_{\gamma\in\overline{\mathcal{N}}_n^{\gamma}(\eta)}\left\{\frac{c(\gamma)}{\sigma^2(\lambda)}+\log r(\lambda)\right\} > 0. \tag{A.5}$$

Now $\overline{\mathscr{N}}^{\gamma}(\eta) \subseteq \left\{\Lambda \times \overline{\mathscr{N}}^{\delta}(\eta/2)\right\} \cup \left\{\overline{\mathscr{N}}^{\lambda}(\eta/2) \times \mathcal{D}\right\}$, so

$$\begin{aligned}
\inf_{\gamma \in \overline{\mathscr{N}}^{\gamma}(\eta)} \left\{\frac{c(\gamma)}{\sigma^2(\lambda)} + \log r(\lambda)\right\} &\geq \min\left\{\inf_{\Lambda \times \overline{\mathscr{N}}^{\delta}(\eta/2)} \frac{c(\gamma)}{\sigma^2(\lambda)}, \inf_{\overline{\mathscr{N}}^{\lambda}(\eta/2)} \log r(\lambda)\right\} \\
&\geq \min\left\{\inf_{\Lambda \times \overline{\mathscr{N}}^{\delta}(\eta/2)} \frac{c(\gamma)}{C}, \inf_{\overline{\mathscr{N}}^{\lambda}(\eta/2)} \log r(\lambda)\right\},
\end{aligned}$$

from Assumption 6, whence Assumptions 7 and 16 imply (A.5). Again using Assumption 6, uniformly in γ, $\left|f(\gamma)/\sigma^2(\lambda)\right| \leq |f(\gamma)|/c$ and

$$\begin{aligned}
|f(\gamma)| &\leq Ctr\left(T'(\lambda)X(\delta)\left(X'(\delta)X(\delta)\right)^{-1}X'(\delta)T(\lambda)\right)/n \\
&= \mathscr{O}\left(tr\left(X'(\delta)X(\delta)\right)/n^2\right) = \mathscr{O}(k/n)
\end{aligned}$$

uniformly, by Assumption 14, to check (A.4).

Finally consider (A.3). We first prove pointwise convergence. For any fixed $\gamma \in \overline{\mathscr{N}}^{\gamma}(\eta)$ and large enough n, $c(\gamma) \geq c\|\beta_0\|^2$ from Assumption 16, $d(\gamma) \geq c$ because $n^{-1}\sigma_0^2 tr\left(T'(\lambda)T(\lambda)\right) \geq c$ and $tr\left(T'(\lambda)\left(I - M(\delta)\right)T(\lambda)\right) = \mathscr{O}(k/n)$. Thus $e(\gamma)/\left(c(\gamma) + d(\gamma)\right) = \mathscr{O}_p\left(|e(\gamma)|\right)$, where $e(\gamma)$ has mean 0 and variance

$$\mathscr{O}\left(\|T'(\lambda)M(\delta)T(\lambda)/n\|_F^2 + \sum_{i=1}^{n}\left(t_i'(\lambda)M(\delta)t_i(\lambda)/n\right)^2 + \|\beta_0'X'T'(\lambda)M(\delta)T(\lambda)/n\|^2\right),$$

where $t_i(\lambda)$ is the ith column of $T(\lambda)$. Since $\|M(\delta)\| = 1$ and Assumptions 4 and 12 imply (we give a bound for the general case, that the same bound holds for the 'single nonzero diagonal block' case is simple to check)

$$\|T(\lambda)\| \leq C\|S(\lambda)\| \leq C \max_{i=1,\ldots,p}\|W_i\|\|\lambda\|_1 = \mathscr{O}(1), \tag{A.6}$$

the first component is $\mathscr{O}\left(\|T(\lambda)/n\|_F^2\right) = \mathscr{O}\left(n^{-1}\right)$. The second one is $\mathscr{O}\left(\sum_{i=1}^{n}\|t_i(\lambda)\|^2/n^2\right) = \mathscr{O}\left(\|T(\lambda)/n\|_F^2\right) = \mathscr{O}\left(n^{-1}\right)$ likewise. The final component is $\mathscr{O}\left(\|X\beta_0/n\|^2\right) = \mathscr{O}\left(\|\beta_0\|^2/n\right) = \mathscr{O}(k/n)$, from (3.6). Thus pointwise convergence is established.

To complete the proof of (A.3) we employ an equicontinuity argument. For arbitrary $\varepsilon > 0$ and finitely many $\gamma_* = (\lambda_*', \delta_*')'$, the neighbourhoods $\|\lambda - \lambda_*\|_1 < \varepsilon \times \|\delta - \delta_*\| < \varepsilon$ form a sub-cover of the compact $\Gamma = \Lambda \times \mathcal{D}$ in the product topology formed from the sum of $\|\cdot\|_1$ and $\|\cdot\|$ distances. It remains to prove that

$$\sup_{\|\lambda-\lambda_*\|_1 < \varepsilon \times \|\delta-\delta_*\| < \varepsilon}\left|\frac{e(\gamma)}{c(\gamma)+d(\gamma)} - \frac{e(\gamma_*)}{c(\gamma_*)+d(\gamma_*)}\right| \xrightarrow{p} 0.$$

Write

$$\frac{e(\gamma)}{c(\gamma)+d(\gamma)}-\frac{e(\gamma_*)}{c(\gamma_*)+d(\gamma_*)}=\frac{e(\gamma)-e(\gamma_*)}{c(\gamma)+d(\gamma)}+e(\gamma_*)\left(\frac{c(\gamma_*)-c(\gamma)+d(\gamma_*)-d(\gamma)}{(c(\gamma)+d(\gamma))(c(\gamma_*)+d(\gamma_*))}\right)$$

whence, denoting the two components of $e(\gamma)$ by $e_1(\gamma)$, $e_1(\gamma)$, the left side is bounded in absolute value by

$$\frac{|e_1(\gamma)-e_1(\gamma_*)|}{d(\gamma)}+\frac{|e_2(\gamma)-e_2(\gamma_*)|}{c(\gamma)}+\frac{|e(\gamma_*)|}{c(\gamma)\,c(\gamma_*)}|c(\gamma_*)-c(\gamma)|+\frac{|e(\gamma_*)|}{d(\gamma)\,d(\gamma_*)}|d(\gamma_*)-d(\gamma)|. \tag{A.7}$$

We prove that

$$\sup_{\|\lambda-\lambda_*\|_1<\varepsilon\times\|\delta-\delta_*\|<\varepsilon}\frac{|e_2(\gamma)-e_2(\gamma_*)|}{c(\gamma)}\xrightarrow{p}0. \tag{A.8}$$

This part of the proof is relatively delicate due to both numerator and denominator increasing with k. The proof for the first term in (A.7) does not involve this feature and uses other arguments in the proof of (A.8). For the third term in (A.7),

$$\frac{|e(\gamma_*)|}{c(\gamma)\,c(\gamma_*)}|c(\gamma_*)-c(\gamma)|\le\frac{|e(\gamma_*)|}{c(\gamma_*)}\left(1+\frac{c(\gamma_*)}{c(\gamma)}\right)\xrightarrow{p}0$$

uniformly on $\|\lambda-\lambda_*\|_1<\varepsilon\times\|\delta-\delta_*\|<\varepsilon$, from the pointwise convergence of $e(\gamma)/(c(\gamma)+d(\gamma))$ and the fact that numerator and denominator of $c(\gamma_*)/c(\gamma)$ are uniformly of the same order of magnitude, namely k, the result for the numerator being straightforward and that for the denominator a consequence of Assumption 16. The fourth term in (A.7) is uniformly $o_p(1)$ by similar arguments.

To prove (A.8), note that

$$e_2(\gamma)-e_2(\gamma_*)=2n^{-1}\beta_0'\left(X'(\delta)\,T'(\lambda)M(\delta)\,T(\lambda)\right)-X'(\delta_*)\,T'(\lambda_*)M(\delta_*)\,T(\lambda_*))\,u,$$

which can be written

$$\begin{aligned}&2n^{-1}\beta_0'\left\{(X(\delta)-X(\delta_*))'\,T'(\lambda)M(\delta)\,T(\lambda)\right.\\&\left.+X'(\delta_*)\left(T'(\lambda)M(\delta)\,T(\lambda)-T'(\lambda_*)M(\delta_*)\,T(\lambda_*)\right)\right\}u.\end{aligned} \tag{A.9}$$

The first of the two terms in braces has spectral norm bounded by $\|X(\delta)-X(\delta_*)\|\,\|T(\lambda)\|^2$, and by Assumption 15,

$$\|X(\delta)-X(\delta_*)\|^2\le\sum_{i=1}^{n}\sum_{j=1}^{k}(x_{ij}(\delta)-x_{ij}(\delta_*))^2=O\left(kn\varepsilon^2\right), \tag{A.10}$$

for sufficiently small $\|\delta-\delta_*\|$.

Thus due to $\|u\|=\mathcal{O}_p\left(n^{1/2}\right)$, it follows that $2n^{-1}\beta_0'(X(\delta)-X(\delta_*))'\,T'(\lambda)M(\delta)\,T(\lambda)\,u$ is uniformly $\mathscr{O}_p\left(\|\beta_0\|\, k^{1/2}\varepsilon\right)$. Looking at the second term in braces in (A.9), write $T'(\lambda)M(\delta)\,T(\lambda) - T'(\lambda_*)M(\delta_*)\,T(\lambda_*)$ as

$$(T(\lambda)-T(\lambda_*))'\,M(\delta)\,T(\lambda)+T'(\lambda_*)\,(M(\delta_*)-M(\delta))\,T(\lambda)+T'(\lambda_*)M(\delta_*)\,(T(\lambda)-T(\lambda_*)),$$

whose spectral norm is bounded by

$$\begin{aligned} & \|T(\lambda)-T(\lambda_*)\|\,(\|T(\lambda)\|+\|T(\lambda_*)\|)+\|T(\lambda_*)\|\,\|M(\delta_*)-M(\delta)\|\,\|T(\lambda)\| \\ = \;& \mathscr{O}\left(\|T(\lambda)-T(\lambda_*)\|+\|M(\delta_*)-M(\delta)\|\right). \end{aligned} \tag{A.11}$$

Now

$$\begin{aligned} \|T(\lambda)-T(\lambda_*)\| &\le \sum_{i=1}^{p}|\lambda_i-\lambda_{*i}|\,\|W_i\|\,\left\|S^{-1}\right\| \\ &\le C\max_{i=1,\dots,p}\|W_i\|\,\|\lambda-\lambda_*\|_1\le C\varepsilon \end{aligned} \tag{A.12}$$

uniformly on $\|\lambda-\lambda_*\|_1<\varepsilon\times\|\delta-\delta_*\|<\varepsilon$. Representing $M(\delta_*)-M(\delta)$ as a sum of terms each with factor $X(\delta)-X(\delta_*)$, or its transpose, with bounds for these typified by

$$n^{-1}\,\|X(\delta)-X(\delta_*)\|\,\left\|\left(X'(\delta)\,X(\delta)/n\right)^{-1}\right\|\,\|X(\delta)\|,$$

where $\|X(\delta)\|\le Cn^{1/2}$, we deduce $\|M(\delta_*)-M(\delta)\|=\mathscr{O}\left(n^{-1/2}\,\|X(\delta)-X(\delta_*)\|\right)=\mathscr{O}\left(k^{1/2}\varepsilon\right)$, from (A.10). Thus from (A.12), (A.11) has the same bound, so arguing much as before the contribution from the second term in braces in (A.9) is $\mathscr{O}\left(\|\beta_0\|\,k^{1/2}\varepsilon\right)$. Thus (A.9)$=\mathscr{O}_p\left(\|\beta_0\|\,k^{1/2}\varepsilon\right)$, and since Assumption 16 implies that as $n\to\infty$, $c(\gamma)\ge c\,\|\beta_0\|^2$ uniformly and $\|\beta_0\|^{-1}=\mathscr{O}\left(k^{-1/2}\right)$, the left side of (A.8) is $\mathscr{O}_p\left(\|\beta_0\|^{-1}\,k^{1/2}\varepsilon\right)=\mathscr{O}_p(\varepsilon)$, whence (A.8) follows from arbitrariness of ε, and the proof is completed. □

Proof of Theorem 3.2. In supplementary material. □

Proof of Theorem 3.3. Let $\xi\left(\lambda,\sigma^2\right)$ denote the first derivative vector of (3.1), evaluated at $\left(\lambda,\sigma^2\right)$. Defining $\mathcal{R}^y(\theta)=R\lambda+X(\delta)\beta-y$, the derivative of (3.1) at any admissible $\left(\theta,\sigma^2\right)$ is

$$\xi\left(\theta,\sigma^2\right)=\left(\varphi'(\theta,\sigma^2),\;2\sigma^{-2}n^{-1}\mathcal{R}^{y\prime}(\theta)\,X(\delta),\;2\sigma^{-2}n^{-1}\mathcal{R}^{y\prime}(\theta)\,\Pi(\theta)\right)', \tag{A.13}$$

where

$$\varphi\left(\theta,\sigma^2\right)=2\sigma^{-2}n^{-1}\left(\sigma^2trG_1(\lambda)+y'W_1'\mathcal{R}^y(\theta),\dots,\sigma^2trG_p(\lambda)+y'W_p'\mathcal{R}^y(\theta)\right)'. \tag{A.14}$$

Noting that $\mathcal{R}^y=-u$, denoting $C_i=G_i+G_i'$ and

$$\phi=\sigma_0^{-2}n^{-1}\left(\sigma_0^2trC_1-u'C_1u,\dots,\sigma_0^2trC_p-u'C_pu\right)', \tag{A.15}$$

so

$$\xi = (\phi', 0, 0)' - 2\sigma_0^{-2} t - 2\sigma_0^{-2} \ell, \tag{A.16}$$

with

$$t = n^{-1} [A, X, 0]' u, \qquad \ell = n^{-1} [0, 0, \Pi]' u. \tag{A.17}$$

Note that φ and ϕ are not identical, hence the different notations. Denote by $K_1(\theta)$ and $K_2(\theta)$ the $k \times q$ and $q \times q$ matrices with i-th column $(\partial X'(\delta)/\partial \delta_i) \mathcal{R}^y(\theta)$ and (i,j)-th element $\mathcal{R}^{y\prime}(\theta) \left(\partial^2 X(\delta)/\partial \delta_i \partial \delta_j\right) \beta$, respectively. The matrix of second derivatives of (3.1) at any admissible point in the parameter space, denoted $H\left(\theta, \sigma^2\right)$, is

$$2\sigma^{-2} n^{-1} \begin{bmatrix} \sigma^2 P_1(\lambda) + R'R & R'X(\delta) & R'\Pi(\theta) \\ * & X'(\delta) X(\delta) & X'(\delta)\Pi(\theta) + K_1(\theta) \\ * & * & \Pi'(\theta)\Pi(\theta) + K_2(\theta) \end{bmatrix}, \tag{A.18}$$

whence (2.11) and (3.13) follow.

For any non-null fixed-dimensional vector of constants α, we can use $\hat{\xi} = 0$ and the mean value theorem to write

$$\nu \alpha' \Psi \left(\hat{\theta} - \theta_0\right) = -\nu \alpha' \Psi \bar{H}^{-1} \xi,$$

for some $\bar{\theta}$ such that $\left\|\bar{\theta} - \theta_0\right\| \le \left\|\hat{\theta} - \theta_0\right\|$, where $\bar{\theta}$ may be different across rows of $\bar{H}^{-1}$. The RHS equals $\sum_{i=1}^4 \Upsilon_i - \nu \alpha' \Psi L^{-1}(t + \ell)$ with

$$\Upsilon_1 = 2\sigma_0^{-2} \nu \alpha' \Psi \bar{H}^{-1} \bar{\Delta}^H H^{-1}(t + \ell), \ \Upsilon_2 = 2\sigma_0^{-2} \nu \alpha' \Psi \Xi^{-1}(H - \Xi) H^{-1}(t + \ell),$$
$$\Upsilon_3 = \nu \alpha' \Psi L^{-1}\left(\sigma_0^2 \Xi/2 - L\right)\left(\sigma_0^2 \Xi/2\right)^{-1}(t + \ell), \ \Upsilon_4 = -\nu \alpha' \Psi \bar{H}^{-1} \phi.$$

We will demonstrate that $\Upsilon_i = o_p(1)$, $i = 1, 2, 3, 4$. First, $\mathbb{E}\|\ell\|^2 = \sigma_0^2 n^{-2} \sum_{r=1}^n \|\pi_r\|^2$, where π_r is the r-th column of Π'. Now

$$\|\pi_r\|^2 = \sum_{i=1}^q \left\{\beta_0' (\partial x_r(\delta_0)/\partial \delta_i)\right\}^2 \le \|\beta_0\|^2 \sum_{i=1}^q \sum_{l=1}^k (\partial x_{rl}(\delta_0)/\partial \delta_i)^2 \le Ck^2,$$

by Assumption 17. Thus

$$\|\ell\| = \mathscr{O}_p\left(n^{-\frac{1}{2}} k\right), \tag{A.19}$$

by Markov's inequality. By Lemma B.1 we have

$$|\Upsilon_1| \le 2\sigma_0^{-2} \nu \|\alpha\| \|\Psi\| \left\|\bar{H}^{-1}\right\| \left\|\bar{\Delta}^H\right\| \left\|H^{-1}\right\| (\|t\| + \|\ell\|),$$

where the second factor in norms is $\mathscr{O}\left((p+k)^{\frac{1}{2}}\right)$, the third and fifth are bounded for sufficiently large n by Lemma B.3 (i), the fourth is $\mathscr{O}_p\left(\left\|\bar{\Delta}^H\right\|\right) = \mathscr{O}_p\left(\max\left\{p^2 k/n^{\frac{1}{2}} h, p^{\frac{1}{2}} k^{\frac{5}{2}}/n^{\frac{1}{2}}, pk^2/n^{\frac{1}{2}}\right\}\right)$ by Lemma B.1 (i) and the last is $\mathscr{O}_p\left(p^{\frac{1}{2}} k/n^{\frac{1}{2}}\right)$ (because $\|t\| = \mathscr{O}\left(p^{\frac{1}{2}} k/n^{\frac{1}{2}}\right)$ by (A.13) of Gupta and Robinson (2015)), so $\Upsilon_1 = \mathscr{O}_p\left(\max\left\{p^{\frac{5}{2}}k^2/n^{\frac{1}{2}}h, pk^{\frac{7}{2}}/n^{\frac{1}{2}}, p^{\frac{3}{2}}k^3/n^{\frac{1}{2}}\right\}\right)$, which is negligible by (3.14), noting that $p^{\frac{5}{2}}k^2/n^{\frac{1}{2}}h = \left(p^{\frac{3}{2}}k/h\right)\left(pk/n^{\frac{1}{2}}\right)$. Similarly $\Upsilon_2 = \mathscr{O}_p\left(p^{\frac{3}{2}}k^2/n^{\frac{1}{2}}\right)$ which is negligible by (3.14) and Lemma B.2 (i), and $\Upsilon_3 = \mathscr{O}_p\left(p^{\frac{3}{2}}k/h\right)$ by Lemma B.2 (ii), which is negligible by (3.14). Finally, $\mathbb{E}\left\|\phi\right\|^2 = \sum_{i=1}^p \operatorname{var}\left(n^{-1}u'C_iu\right) = \mathscr{O}\left(p/nh\right)$, (shown like (B.23) in the supplementary appendix) so that

$$\|\phi\| = \mathscr{O}_p\left(n^{-\frac{1}{2}}h^{-\frac{1}{2}}p^{\frac{1}{2}}\right), \tag{A.20}$$

by Chebyshev's inequality. So Υ_4 has modulus bounded by $\nu\left\|\Psi\right\|\left\|\bar{H}^{-1}\right\|\left\|\phi\right\|$ times a constant, where the second factor is $\mathscr{O}\left((p+k)^{\frac{1}{2}}\right)$, the third is bounded for sufficiently large n by Lemma B.3 (i) and the last is $\mathscr{O}_p\left(p^{\frac{1}{2}}/n^{\frac{1}{2}}h^{\frac{1}{2}}\right)$. Thus $\Upsilon_4 = \mathscr{O}_p\left(p^{\frac{1}{2}}/h^{\frac{1}{2}}\right)$ which is negligible by (3.14). Then we only need to find the asymptotic distribution of $\nu\alpha'\Psi L^{-1}(t+\ell)$. The theorem now follows by a standard Lindeberg central limit theorem argument. The asymptotic covariance matrix exists, and is positive definite, by Assumption 19. The proof of the consistency of its estimate is omitted. $\square$

Proof of Theorem 3.4. Here we redefine $\mathcal{R}^y(\lambda) = R\lambda - y$ and obtain $\xi = \phi$. Also $H\left(\lambda, \sigma^2\right) = 2n^{-1}P_1(\lambda) + 2\sigma^{-2}n^{-1}R'R$, whence the formulae for H and Ξ follow. Then proceeding as in the proof of Theorem 3.3, we can write

$$\nu\alpha'\Psi\left(\hat{\theta} - \theta_0\right) = \nu\alpha'\Psi\left(\bar{H}^{-1} - \Xi^{-1}\right)\xi - \nu\alpha'\Psi\Xi^{-1}\xi. \tag{A.21}$$

Lemma B.3 (i) indicates that the first term on the RHS of (A.21) is bounded in modulus by a constant times

$$\begin{aligned}&\nu\left\|\Psi\right\|\left(\|t\| + \|\ell\| + \|\phi\|\right)\left(\left\|\bar{\Delta}^H\right\| + \|H - \Xi\|\right) = \\ &\mathscr{O}_p\left(n^{\frac{1}{2}}\max\left\{p^{\frac{1}{2}}k/n^{\frac{1}{2}}, p^{\frac{1}{2}}/n^{\frac{1}{2}}h^{\frac{1}{2}}\right\}\max\left\{p^2k/n^{\frac{1}{2}}h, p^{\frac{1}{2}}k^{\frac{5}{2}}/n^{\frac{1}{2}}, pk^2/n^{\frac{1}{2}}, pk/n^{\frac{1}{2}}\right\}\right),\end{aligned}$$

by (A.19), (A.20) and Lemma B.1 (i). This is negligible by (3.16). Thus we establish the asymptotic distribution of the second term on the RHS of (A.21), which has zero mean and variance $a^{-1}\Psi\left(2\Xi^{-1} + \Xi^{-1}\Omega\Xi^{-1}\right)\Psi'$. Hence we consider the asymptotic normality of

$$\frac{-n^{\frac{1}{2}}\alpha'\Psi\Xi^{-1}\xi}{\left\{\alpha'\Psi\left(2\Xi^{-1} + \Xi^{-1}\Omega\Xi^{-1}\right)\Psi'\alpha\right\}^{\frac{1}{2}}}, \tag{A.22}$$

where α is any fixed-dimensional vector of constants. Write $\varsigma = \left\{\alpha'\Psi\left(2\Xi^{-1} + \Xi^{-1}\Omega\Xi^{-1}\right)\Psi'\alpha\right\}^{\frac{1}{2}}$ for the denominator of (A.22). Then

$$\varsigma \geq \left\|\Psi'\alpha\right\|\left\{\underline{\zeta}\left(2\Xi^{-1} + \Xi^{-1}\Omega\Xi^{-1}\right)\right\}^{\frac{1}{2}} \geq c\left\|\Psi'\alpha\right\| \tag{A.23}$$

by Assumption 20. The numerator of (A.22) can be written as

$$-2\sigma_0^{-2}n^{-\frac{1}{2}}m'u - \sigma_0^{-2}n^{-\frac{1}{2}}u'Du + n^{-\frac{1}{2}}trD \tag{A.24}$$

where $D = \sum_{j=1}^{p}\left(\alpha'\Psi\zeta^j\right)C_j,\ m = \sum_{j=1}^{p}\left(\alpha'\Psi\zeta^j\right)a_j + \sum_{j=p+1}^{p+k}\left(\alpha'\Psi\zeta^j\right)\chi_{j-p}$, with ζ^j and χ_j denoting the j-th columns of Ξ^{-1} and X respectively. We also denote by d_{ij} and m_i the (i,j)-th and i-th elements of D and m respectively. Using (A.24), we can write (A.22) as $-\sum_{i=1}^{n} w_i$, with

$$w_i = \sigma_0^{-2}n^{-\frac{1}{2}}\varsigma^{-1}\left(u_i^2 - \sigma_0^2\right)d_{ii} + 2\sigma_0^{-2}n^{-\frac{1}{2}}\varsigma^{-1}u_i\sum_{j<i}u_jd_{ij} + 2\sigma_0^{-2}n^{-\frac{1}{2}}\varsigma^{-1}m_iu_i. \tag{A.25}$$

$\{w_i, i = 1,\ldots,n, n\geq 1\}$ forms a martingale difference sequence by Assumption 14, so Theorem 2 of Scott (1973) implies $\sum_{i=1}^{n} w_i \xrightarrow{d} N(0,1)$ if

$$\sum_{i=1}^{n}\mathbb{E}\left\{w_i^2 1\left(w_i \geq \epsilon\right)\right\} \xrightarrow{p} 0,\quad \text{for all } \epsilon > 0, \tag{A.26}$$

$$\sum_{i=1}^{n}\mathbb{E}\left(w_i^2 \mid u_j, j<i\right) \xrightarrow{p} 1. \tag{A.27}$$

To show (A.26) we can check the sufficient Lyapunov condition

$$\sum_{i=1}^{n}\mathbb{E}\left|w_i\right|^{2+\frac{\chi}{2}} \xrightarrow{p} 0. \tag{A.28}$$

The c_r inequality, (11) and (A.23) indicate that the left side is bounded by a constant times

$$\frac{\sum_{i=1}^{n}\left|d_{ii}\right|^{2+\frac{\chi}{2}}}{n^{1+\frac{\chi}{4}}\left\|\Psi'\alpha\right\|^{2+\frac{\chi}{2}}} + \frac{\sum_{i=1}^{n}\mathbb{E}\left|\sum_{j<i}u_jd_{ij}\right|^{2+\frac{\chi}{2}}}{n^{1+\frac{\chi}{4}}\left\|\Psi'\alpha\right\|^{2+\frac{\chi}{2}}} + \frac{\sum_{i=1}^{n}\left|m_i\right|^{2+\frac{\chi}{2}}}{n^{1+\frac{\chi}{4}}\left\|\Psi'\alpha\right\|^{2+\frac{\chi}{2}}}. \tag{A.29}$$

The first term in (A.29) is bounded by

$$\max_i\left|d_{ii}\right|^{2+\frac{\chi}{2}}/n^{\frac{\chi}{4}}\left\|\Psi'\alpha\right\|^{2+\frac{\chi}{2}}, \tag{A.30}$$

while the third term is bounded by

$$\max_i\left|m_i\right|^{2+\frac{\chi}{2}}/n^{\frac{\chi}{4}}\left\|\Psi'\alpha\right\|^{2+\frac{\chi}{2}}. \tag{A.31}$$

By the Burkholder, von Bahr/Esseen and elementary ℓ_p-norm inequalities, the second term in

(A.29) is bounded by a constant times

$$\max_i \left| \sum_{j<i} d_{ij}^2 \right|^{1+\frac{\chi}{4}} / n^{\frac{\chi}{4}} \left\| \Psi' \alpha \right\|^{2+\frac{\chi}{2}}. \tag{A.32}$$

Now, writing e_i for the n-dimensional vector with unity in the i-th position and zeros elsewhere, we can write $\sum_{j=1}^n d_{ij}^2 = e_i' D^2 e_i \leq \|D\|^2$ which is bounded by

$$\begin{aligned}
\left\| \sum_{j=1}^p \left(\alpha' \Psi \zeta^j\right) C_j \right\|^2 &\leq Cp^2 \left(\max_j \|C_j\|\right)^2 \left(\max_j \left\|\zeta^j\right\|\right)^2 \|\Psi'\alpha\|^2 \leq C \left\|\Xi^{-1}\right\|^2 p^2 \|\Psi'\alpha\|^2 \\
&= Cp^2 \|\Psi'\alpha\|^2 \left\{\underline{\zeta}(\Xi)\right\}^{-2} \leq Cp^2 \|\Psi'\alpha\|^2, \qquad \text{(A.33)}
\end{aligned}$$

using Assumption 20. Also, we can use (A.33) to bound

$$|d_{ii}| \leq \left(\sum_{j=1}^n d_{ij}^2\right)^{\frac{1}{2}} \leq Cp \|\Psi'\alpha\|. \tag{A.34}$$

(A.33) and (A.34) imply that (A.30) and (A.32) are both $\mathscr{O}\left(p^{2+\frac{\chi}{2}}/n^{\frac{\chi}{4}}\right)$. This is negligible by (3.16). Next

$$|m_i| \leq \sum_{j=1}^p \left|\alpha'\Psi\zeta^j\right| |a_{ij}| + \sum_{j=p+1}^{p+k} \left|\alpha'\Psi\zeta^j\right| |x_{ij}| = \mathscr{O}\left(k(p+1)\|\Psi'\alpha\|\right), \tag{A.35}$$

using Assumptions 14, 20. Then (A.31) is $\mathscr{O}_p\left(p^{2+\frac{\chi}{2}} k^{2+\frac{\chi}{2}}/n^{\frac{\chi}{4}}\right)$, which is negligible by (3.16). Hence (A.28) is proved.

We now show (A.27). Write $\sum_{i=1}^n \mathbb{E}\left(w_i^2 \mid u_j, j<i\right) - 1 = 4(f_1+f_2+f_3)$ with $f_1 = \sigma_0^{-2} n^{-1} \varsigma^{-2} \sum_i \sum_j \sum_{k\ (j,k<i, j\neq k)} d_{ij} d_{ik} u_j u_k$, $f_2 = \sigma_0^{-2} n^{-1} \varsigma^{-2} \sum_i \sum_{j<i} d_{ij}^2 \left(u_j^2 - \sigma_0^2\right)$ and $f_3 = \sigma_0^{-4} n^{-1} \varsigma^{-2} \sum_i \left(\sigma_0^2 m_i + \mu_3 d_{ii}\right) \sum_{j<i} d_{ij} u_j$. All sums and maxima are taken over 1 to n unless otherwise stated. f_1 has zero mean and variance bounded by $n^{-2}\varsigma^{-4}$ times

$$\begin{aligned}
C \sum_{h,i,j,k\ (j,k<i,h)} |d_{ij} d_{ik} d_{hj} d_{hk}| &\leq C \sum_{h,i,j,k} |d_{ij} d_{ik}| \left(d_{hj}^2 + d_{hk}^2\right) \\
&\leq C \left(\max_i \sum_k |d_{ik}|\right) \left(\max_j \sum_i |d_{ij}|\right) \sum_{i,j} d_{ij}^2 \\
&= C \|D\|_R^2 \|D\|_F^2 \leq C \|\Psi'\alpha\|^4 np^4, \qquad \text{(A.36)}
\end{aligned}$$

by (A.33) and because, for each $i = 1, \ldots, n$, $\left(\sum_{j=1}^n d_{ij}^2\right)^{\frac{1}{2}} \leq \sum_{j=1}^n |d_{ij}| \leq \|D\|_R \leq Cp \|\Psi'\alpha\|$

by Assumption 3. (A.23) and (A.36), together with Markov's inequality, imply that $f_1 = \mathscr{O}_p\left(p^2/n^{\frac{1}{2}}\right)$, which is negligible by (3.16). Next, f_2 has zero mean and variance bounded by $n^{-2}\varsigma^{-4}$ times

$$C\sum_{i,h}\sum_{j<i,h} d_{ij}^2 d_{hj}^2 \leq C\sum_{i,h,j} d_{ij}^2 d_{hj}^2 \leq C\left(\max_j \sum_h d_{hj}^2\right)\|D\|_F^2 \leq C\left\|\Psi'\alpha\right\|^4 np^4, \tag{A.37}$$

by (A.33). (A.23) and (A.37), together with Markov's inequality, imply that $f_2 = \mathscr{O}_p\left(p^2/n^{\frac{1}{2}}\right)$ which is negligible by (3.16). Finally f_3 has zero mean and variance bounded by $n^{-2}\varsigma^{-4}$ times

$$\begin{aligned} C &\sum_i \left(\sigma_0^2 m_i + \mu_3 d_{ii}\right)^2 \sum_{j<i} d_{ij}^2 \leq C\left(\max_i m_i^2 + \max_i d_{ii}^2\right)\|D\|_F^2 \\ \leq\ & C\left(\max_i m_i^2 + \max_i \sum_j d_{ij}^2\right)\|D\|_F^2 = \mathscr{O}\left(\left\|\Psi'\alpha\right\|^4\left(k^2+1\right)np^4\right), \end{aligned} \tag{A.38}$$

by (A.33) and (A.35). (A.23) and (A.38), together with Markov's inequality, imply that $f_3 = \mathscr{O}_p\left(p^2 k/n^{\frac{1}{2}}\right)$, which is negligible by (3.16). The asymptotic covariance matrix exists, and is positive definite, by Assumption 20. □

Proofs of Theorems 4.1, 4.2 and 4.3 . In supplementary material. □

B Technical Lemmas

All proofs are contained in the supplementary appendix.

Lemma B.1. *(i) Under the conditions of Theorem 3.3 or 3.4,*

$$\left\|\hat{\Delta}^H\right\| = \mathscr{O}_p\left(n^{-\frac{1}{2}}p^{\frac{1}{2}}k\left(h^{-1}p^{\frac{3}{2}} + k^{\frac{3}{2}} + p^{\frac{1}{2}}k\right)\right).$$

(ii) Under the conditions of Theorem 2.3, 2.4 or 2.6,

$$h\left\|\hat{\Delta}^H\right\| = \mathscr{O}_p\left(n^{-\frac{1}{2}}h^{-\frac{1}{2}}p^2 + n^{-\frac{1}{2}}h^{\frac{1}{2}}p\right),$$

or, equivalently, $\left\|\hat{\Delta}^H\right\| = \mathscr{O}_p\left(n^{-\frac{1}{2}}h^{-\frac{3}{2}}p^2 + n^{-\frac{1}{2}}h^{-\frac{1}{2}}p\right)$.

The same bounds hold if we replace $\left\|\hat{\Delta}^H\right\|$ *by* $\left\|\bar{\Delta}^H\right\|$, *where* $\left\|\bar{\theta}-\theta_0\right\| \leq \left\|\hat{\theta}-\theta_0\right\|$.

Lemma B.2. *Suppose that Assumptions 1-14 hold. Then*

(i) $\|H-\Xi\| = \mathscr{O}_p\left(p/n^{\frac{1}{2}}\right)$ *for SAR without regressors and bounded* h, $\|H-\Xi\| = \mathscr{O}_p\left(p/n^{\frac{1}{2}}h^{\frac{1}{2}}\right)$ *for SAR without regressors and divergent* h *and* $\|H-\Xi\| = \mathscr{O}_p\left(pk/n^{\frac{1}{2}}\right)$ *for SAR with regressors.*

(ii) $\left\| L - \sigma_0^2 \Xi/2 \right\| = \mathscr{O}\left(p/h\right).$

Lemma B.3. *Let Assumptions 1-19 hold.*

(i) If (3.14) holds, then

$$\begin{aligned}\left\| \hat{H}^{-1} \right\| &= \mathscr{O}_p\left(\left\| H^{-1} \right\|\right) = \mathscr{O}_p\left(\left\| \Xi^{-1} \right\|\right) = \mathscr{O}_p\left(\left\{\underline{\zeta}(L)\right\}^{-1}\right) = \mathscr{O}_p(1),\\ \left\| \hat{H} \right\| &= \mathscr{O}_p\left(\|H\|\right) = \mathscr{O}_p\left(\|\Xi\|\right) = \mathscr{O}_p\left(\overline{\zeta}(L)\right) = \mathscr{O}_p(1).\end{aligned}$$

If h is bounded and Assumption 20 holds together with (3.16), then

$$\begin{aligned}\left\| \hat{H}^{-1} \right\| &= \mathscr{O}_p\left(\left\| H^{-1} \right\|\right) = \mathscr{O}_p\left(\left\{\underline{\zeta}(\Xi)\right\}^{-1}\right) = \mathscr{O}_p(1),\\ \left\| \hat{H} \right\| &= \mathscr{O}_p\left(\|H\|\right) = \mathscr{O}_p\left(\overline{\zeta}(\Xi)\right) = \mathscr{O}_p(1)\end{aligned}$$

(ii) If $\underline{\lim}_{n\to\infty} \underline{\zeta}(h\Xi) > 0$ and (2.13) holds, then

$$\left\| \left(h\hat{H}\right)^{-1} \right\| = \mathscr{O}_p\left(\left\| (hH)^{-1} \right\|\right) = \mathscr{O}_p\left(\left\{\underline{\zeta}\left(h\Xi\right)\right\}^{-1}\right) = \mathscr{O}_p(1).$$

(iii) If h is bounded, $\underline{\lim}_{n\to\infty} \underline{\zeta}(\Xi) > 0$ and (2.14) holds , then

$$\left\| \hat{H}^{-1} \right\| = \mathscr{O}_p\left(\left\| H^{-1} \right\|\right) = \mathscr{O}_p\left(\left\{\underline{\zeta}\left(\Xi\right)\right\}^{-1}\right) = \mathscr{O}_p(1).$$

The same bounds hold if we replace $\left\| \hat{H} \right\|$ by $\left\| \bar{H} \right\|$, where $\left\| \bar{\theta} - \theta_0 \right\| \leq \left\| \hat{\theta} - \theta_0 \right\|$.

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