Reducing Model Risk With Goodness-of-fit
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06 June 2017

Agenda

I. An overview of Copula Theory
II. Copulas and Model Risk
III. Goodness-of-fit methods for copulas
IV. Presentation of the new method

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Measuring Dependence

Copula’s Definition

• A Mathematical Approach…

“\(d\)-dimensional copula is a multivariate distribution function on \([0,1]^d\) with uniform marginals.”

• A Conceptual Approach…

“a mixing of distributional functions which allows for flexibility in the dependence structure.”
Copulas and Tail Dependence

• Copulas allow for flexibility in their dependence structure; incorporating tail dependence in the model fitting procedure is of upmost importance for risk management professionals

• Internal models: Gaussian and Student-t Copulas

• Other interesting copulas: Empirical, Vine and Archimedean Copulas.

<table>
<thead>
<tr>
<th>Copula</th>
<th>Lower Tail Dependence, $\lambda_L$</th>
<th>Upper Tail Dependence, $\lambda_U$</th>
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<tbody>
<tr>
<td>Gumbel</td>
<td>0</td>
<td>$\geq 0$</td>
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<tr>
<td>Frank</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Clayton</td>
<td>$\geq 0$</td>
<td>0</td>
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<tr>
<td>Generalised Clayton</td>
<td>$\geq 0$</td>
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Copulas Gone Wrong

• Recent failures due to erroneous copula usage:

$$C(u, v) = \phi_2(\phi^{-1}(u), \phi^{-1}(v), \rho) \text{ for } -1 \leq \rho \leq 1$$

Photo: AP photo/Richard Drew
https://www.wired.com/2009/02/wp-quant/
The Model Risk Problem

“...model risk ... is the potential for adverse consequences from decisions based on incorrect or misused model outputs and reports.”

Federal Reserve (2011)

Sources of Model Risk:
- Incorrect Model Use
- Expert Judgements
- Model Changes

• The Model Risk Problem with Copulas is:
  Selecting the wrong copula because of using the wrong selection criteria.

Limitations of Copula

<table>
<thead>
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<th>General Limitations</th>
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<tr>
<td>Data Limitations</td>
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<td>Parameter Fitting</td>
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<td>Computational Cost</td>
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<td>Possibility for Overconfidence</td>
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<th>Copula Specific Limitations</th>
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<td>Practicality</td>
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<td>Use Test</td>
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<td>Communication</td>
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Model Risk ≠ Model Error

Reflects the lack of knowledge in our ability to fully capture all forms of uncertainty in the model.

Assumes the existence of a true model that we can measure our deviances from.

For us, there is no such thing as a model error problem.

Goodness-of-fit and Model Risk

- Our Objective: to reduce model risk by developing a system that can select a copula and thus reduce uncertainty in the dependency structure between the risks.

- A definition for Goodness-of-fit

  “the degree to which observed data matches the values expected by theory”
Hypothesis Test

• The hypothesis test under discussion is

\[ H_0 : C \in C_0 \]
\[ H_1 : C \notin C_0 \]

where the copula family is represented by \( C_0 = \{ C_\theta : \theta \in \Theta \} \) and \( \Theta \) is the parameter space [Berg, 2009].

Current Goodness-of-fit Approaches

Cramér–von Mises, [Berg, 2009]

• Examines the squared deviances between the suggested copula \( C(u) \) and the empirical copula \( C^*(u) \).

• Test Statistic (one sample case)

\[ \int_{-\infty}^{\infty} (C^*(u) - C(u))^2 \, dC(u) \]

Limitations

Computational Expense \| Limitations in the Tail of the Distribution
Current Goodness-of-fit Approaches

Anderson–Darling test, [Berg, 2009]

• An extension of the Cramér–von Mises test, and places more weights on the tails of the distribution:

$$n \int_{-\infty}^{\infty} (C^*(u) - C(u))^2 w_{AD} \ dC(u)$$

where $w_{AD} = \left[ C(u) \ (1 - C(u)) \right]^{-1}$

Limitations
Computational Expense \ Requires knowledge of Critical Values

Current Goodness-of-fit Approaches

Kolmogorov–Smirnov test, [Berg, 2009]

• Quantifies the distance between the suggested copula $C(u)$ and the empirical copula $C^*(u)$

• Test statistic

$$\sup |C(u) - C^*(u)|$$

Limitations
Computational Expense \ Requires large dataset \ Distribution must be fully specified
Current Goodness-of-fit Approaches

Other tests

### Ranks
- For any sample $x_j$, $\frac{R_{j1}}{n+1}, \ldots, \frac{R_{jd}}{n+1}$ where $R_{ji}$ is the rank of $x_{ji}$ in $x_j$.
- Can be thought of as pseudo-samples from the copula.

### Rosenblatt's Transform
- Transforms a set of dependent variables into independent uniform variables.
- $V_i = R(Z_i)$ where $R(Z_i) = \mathbb{P}(Z_d \leq x_d | Z_1 = z_1, \ldots, Z_{d-1} = z_{d-1})$.

### AIC
- More of a measure of model quality.
- Trade-off between goodness-of-fit of a model and its complexity.
- $2k - 2 \ln L$ where $k$ is the number of parameters and $L$ is the likelihood.

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The New Approach
Overview: New Approach

- The approach discussed in my paper is a complete reformulation of the goodness-of-fit problem.
- By finding a suitable approximation (see paper) to a given copula we can determine the relevant copula family.
- In order to achieve this we need some classical results from the field of uncertainty quantification.

Overview: New Approach

- **Convex Relaxation**
  - A trade-off between data usage and numerical computation, we **aim to find a weaker algorithm**.
Benefits of the New Model

Idowu's Approach

- A non-parametric technique
- Avoids the curse of multi-dimensionality
- Reduces computational expense and time

Ongoing work

- Great scope for implementation in the financial sector
- Development of a computational package
- For further details of the corresponding mathematics and implementation of the approach see [Idowu, 2017] – Working Paper.
Further Reading

• Victory Idowu is an academic working on Uncertainty Quantification and Model Risk research with an emphasis in Actuarial science

• Other areas of research include:
  – Structured Expert Judgement
  – Model Validation (see The Model Validator’s Manifesto).

Questions

Comments

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References

