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CONVEX DUALITY FOR EPSTEIN-ZIN STOCHASTIC DIFFERENTIAL UTILITY

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ABSTRACT. This paper introduces a dual problem to study a continuous-time consumption and investment problem with incomplete markets and Epstein-Zin stochastic differential utilities. Duality between the primal and dual problems is established. Consequently, the optimal strategy of this consumption and investment problem is identified without assuming several technical conditions on market models, utility specifications, and agent's admissible strategies. Meanwhile, the minimizer of the dual problem is identified as the utility gradient of the primal value and is economically interpreted as the "least favorable" completion of the market.

1. INTRODUCTION

Classical asset pricing theory in the representative agent framework assumes that the representative agent's preference is modeled by a time-additive Von Neumann-Morgenstein utility. This specification restricts the relationship between risk aversion and intertemporal substitutability, leading to a rich literature on asset pricing anomalies, such as the low risk premium and high risk-free rate. To disentangle risk aversion and intertemporal substitutability, the notion of recursive utility was introduced by Kreps and Porteus (1978), Epstein and Zin (1989), Weil (1990), among others. Its continuous-time analogue, stochastic differential utility, was defined by Epstein (1987) for deterministic settings and by Duffie and Epstein (1992a) for stochastic environments. The connection between recursive utility and stochastic differential utility has also been rigorously established by Kraft and Seifried (2014) recently. Recursive utility and its continuous-time analogue generalize time-additive utility and provide a flexible framework to tackle the aforementioned asset pricing anomalies, cf. Bansal and Yaron (2004), Bhamra et al. (2010), and Benzoni et al. (2011), among others.

The asset pricing theory for recursive utility and stochastic differential utility builds on the optimal consumption and investment problems. For Epstein-Zin utility, a specification widely used in the aforementioned asset pricing applications, its continuous-time optimal consumption and investment problems have been studied by Schroder and Skiadas (1999, 2003), Chacko and Viceira (2005), Kraft et al. (2013), Kraft et al. (2017), and Xing (2017). These studies mainly utilize stochastic control techniques, either the Hamilton-Jacobi-Bellman equation (HJB) in Markovian settings or the backward stochastic differential equation (BSDE) in non-Markovian settings, to tackle these optimization problems directly. We call this class of methods the *primal approach*. However, the HJB equations that arise from these problems are typically nonlinear, and

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BSDEs are usually nonstandard. Therefore, currently available results obtained via the primal approach still come with unsatisfactory restrictions on either market models, utility specifications, or agent's admissible actions.

In contrast, for portfolio optimization problems for time-additive utility, a martingale (or duality) approach was introduced by Pliska (1986), Cox and Huang (1989), Karatzas et al. (1987), Karatzas et al. (1991), and He and Pearson (1991). Instead of tackling the primal optimization problem directly, a dual problem was introduced whose solution leads to the solution of the primal problem via the first-order condition. For time-additive utility, this dual approach allows unnecessary assumptions to be stripped away and portfolio optimization problems to be solved with minimal assumptions on market models and utilities, cf. Kramkov and Schachermayer (1999, 2003) for terminal consumption and Karatzas and Žitković (2003) for intertemporal consumption.

This paper proposes a dual problem for an optimal consumption and investment problem in incomplete markets with Epstein-Zin utility. It is a minimization problem of a convex functional of state price densities (deflators). Similar to the primal problem, the dual value process aggregates the state price density and future evolution of the dual value process. Hence, the dual problem also takes a recursive form; we call it the *stochastic differential dual*. Similar to time-additive utility, the solution of this dual problem can be economically interpreted as the *least favorable completion* of the market; i.e., the agent's optimal portfolio does not contain the fictitious assets that are introduced to complete the market, cf. He and Pearson (1991) and Karatzas et al. (1991).

In contrast to time-additive utility, the convex functional appearing in the dual problem does not follow directly from applying the Fenchel-Legendre transformation to the utility function. Instead, we utilize a variational representation of recursive utility, introduced by Geoffard (1996), El Karoui et al. (1997) and Dumas et al. (2000), to transform the primal problem to a min-max problem, which leads to a variational representation of the dual problem. This dual variational representation can be transformed back to a recursive form thanks to the homothetic property of Epstein-Zin utility in the consumption variable.

We assume that the risk aversion γ and elasticity of intertemporal substitution (EIS) ψ of Epstein-Zin utility satisfy either $\gamma\psi \geq 1, \psi > 1$ or $\gamma\psi \leq 1, \psi < 1$. This class includes a large portion of cases where Epstein-Zin utility is known to exist in the literature; see Proposition 2.1. Moreover, this class allows the dual variable ν for the min-max problem to be chosen from an admissible class of processes that are uniformly bounded from below. For Epstein-Zin utility, this notion of admissibility is easier to check than the integrability conditions used in El Karoui et al. (1997) and Dumas et al. (2000).

The dual problem gives rise to an inequality between the primal value function and the concave conjugate of the dual value function. When this inequality is an identity, there is *duality* between the primal and dual problems, or there is *no duality gap*. Consider market models whose investment opportunities are driven by some state variables. We obtain duality in two situations: 1) non-Markovian models with bounded market price of risk, together with $\gamma\psi \geq 1, \psi > 1$ or $\gamma\psi \leq 1, \psi < 1$; 2) Markovian models with unbounded market price of risk, including the Heston model, the $3/2$ model, and the Kim-Omberg model, when $\gamma, \psi > 1$. The latter market and utility specifications are widely used in the aforementioned asset pricing applications.

The duality between primal and dual problems allow us to simultaneously verify the primal and dual optimizers. On the primal side, technical conditions on utilities and market models in Kraft et al. (2013) are removed. Moreover, Xing (2017) introduced a permissible class of strategies which is needed to verify optimality of the candidate strategy. Thanks to duality, this permissible class is removed, and the primal optimality is established in the standard admissible class, which consists of all nonnegative self-financing wealth processes. On the dual side, the super-differential of the primal value is identified as the minimizer of the dual problem, extending this well-known result from time-additive utility to stochastic differential utility. In the primal approach, the super-differential of the primal value is identified via the *utility gradient* approach by Duffie and Skiadas (1994). In this approach, one needs to show that the sum of the deflated wealth process and the integral of the deflated consumption stream form a martingale for the candidate optimal strategy. This martingale property now becomes a direct consequence of duality.

The remainder of the paper is organized as follows. A dual problem is introduced for Epstein-Zin utility in Section 2. The main results are presented in Section 3, where duality is established for two market and utility settings. In the second setting, we first introduce two abstract conditions that lead to duality. These abstract conditions are then specified as explicit parameter conditions in three examples. All proofs are presented in Section 4 and basic facts about the Epstein-Zin aggregator are recorded in the appendix.

2. Consumption investment optimization and its dual

2.1. **Epstein-Zin preferences.** Let $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathcal{F}, \mathbb{P})$ be a filtered probability space whose filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual assumptions of completeness and right-continuity.

Let $\mathcal{C}$ be the class of nonnegative progressively measurable processes defined on $[0,T]\times\Omega$. For $c\in\mathcal{C}$ and $t<T$, c_t represents the consumption rate at the time t, and c_T stands for the lump sum consumption at the time T. We consider an agent whose preference over $\mathcal{C}$-valued consumption streams is described by a continuous time stochastic differential utility of *Kreps-Porteus* or *Epstein-Zin* type. To describe this preference, let $\delta>0$ represent the discounting rate, $0<\gamma\neq 1$ be the relative risk aversion, and $0<\psi\neq 1$ be the elasticity of intertemporal substitution (EIS). Define the Epstein-Zin aggregator f via

$$f(c,u) := \delta\frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)u)^{1-\frac{1}{\theta}} - \delta\theta u, \qquad \text{for } c>0 \text{ and } (1-\gamma)u>0, \tag{2.1}$$

where $\theta := \frac{1-\gamma}{1-1/\psi}$. Given a bequest utility $U_T(c) = \epsilon\frac{c^{1-\gamma}}{1-\gamma}$, where $\epsilon>0$ represents the weight of the bequest utility comparing to the intertemporal consumption, the *Epstein-Zin utility* for the consumption stream $c\in\mathcal{C}$ over a time horizon T is a semimartingale $(U^c_t)_{0\leq t\leq T}$ which satisfies

$$U^c_t = \mathbb{E}_t\Big[U_T(c_T) + \int_t^T f(c_s, U^c_s)ds\Big], \qquad \text{for all } t\in[0,T]. \tag{2.2}$$

Here $\mathbb{E}_t[\cdot]$ stands for the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$. This utility specification excludes the cases with unit EIS ($\psi=1$) and zero bequest, which we will comment on in Remarks 2.10 and 3.3 later.

We call a consumption stream c **admissible** if the associated Epstein-Zin utility U^c exists, moreover it is of class (D) and satisfies $(1-\gamma)U^c>0$. The class of admissible consumption streams is denoted by $\mathcal{C}_a$. All existing sufficient conditions for existence of Epstein-Zin utility in the literature are summarized in the following result, which, in particular, implies $\mathcal{C}_a\neq\emptyset$.

PROPOSITION 2.1 *Let the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ be the augmented filtration generated by some Brownian motion.*

(i) (Schroder and Skiadas, 1999, Theorem 1) When either $\gamma > 1, 0 < \psi < 1$, or $0 < \gamma < 1, \psi > 1$, for any $c \in \mathcal{C}$ such that $\mathbb{E}[\int_0^T c_t^\ell dt + c_T^\ell] < \infty$ for all $\ell \in \mathbb{R}$, there exists a unique U^c such that $\mathbb{E}[\operatorname{ess\,sup}_t |U_t^c|^\ell] < \infty$ for every $\ell > 0$.

(ii) (Xing, 2017, Propositions 2.2 and 2.4) When $\gamma, \psi > 1$, for any $c \in \mathcal{C}$ such that $\mathbb{E}[\int_0^T c_t^{1-1/\psi} dt + c_T^{1-\gamma}] < \infty$, there exists a unique U^c of class (D).

For general filtration $(\mathcal{F}_t)_{0\leq t\leq T}$,

(iii) (Seiferling and Seifried, 2015, Theorems 3.1 and 3,3) When $\gamma\psi \geq 1, \psi > 1$, or $\gamma\psi \leq 1, \psi < 1$, for any $c \in \mathcal{C}$ such that $\mathbb{E}[\int_0^T c_t^\ell dt + c_T^\ell] < \infty$ for all $\ell \in \mathbb{R}$, there exists a unique U^c such that $\mathbb{E}[\operatorname{ess\,sup}_t |U_t^c|^\ell] < \infty$ for every $\ell \in \mathbb{R}$.

In all above cases, $(1-\gamma)U > 0$ and U_0^c is concave in c.

Remark 2.2. For our main results in Section 3, we will work with the admissible set $\mathcal{C}_a$, whose associated Epstein-Zin utility admits mild integrability properties. As a result, these main results establish optimality of the optimal consumption stream c^* in a large admissible class. Moreover, to verify the membership of c^* in $\mathcal{C}_a$, one only needs to check that U^{c^*} exists and is of class (D), rather than the integrability assumptions of c presented in Proposition 2.1.

2.2. **Consumption investment optimization.** Consider a model of financial market with assets $S = (S^0, S^1, \dots, S^n)$, where S^0 is the price of a riskless asset, $(S^1, \cdots, S^n)$ are prices for risky assets, and S is assumed to be a semimartingale whose components are all positive.

An agent, starting with an initial capital $w > 0$, invests in this market by choosing a portfolio represented by a predictable, S-integrable process $\pi = (\pi^0, \pi^1, \dots, \pi^n)$. With π_t^i representing the proportion of current wealth invested in asset i at time t, $\pi_t^0 = 1-\sum_{i=1}^n \pi_t^i$ is the proportion invested in the riskless asset. Given an investment strategy π and a consumption stream c, agent's wealth process $\mathcal{W}^{(\pi,c)}$ follows

$$d\mathcal{W}_t^{(\pi,c)} = \mathcal{W}_{t-}^{(\pi,c)} \pi_t^\top \frac{dS_t}{S_{t-}} - c_t dt, \quad \mathcal{W}_0^{(\pi,c)} = w. \tag{2.3}$$

A pair of investment strategy and consumption stream (π, c) is **admissible** if $c \in \mathcal{C}_a$ and $\mathcal{W}^{(\pi,c)}$ is nonnegative. The class of admissible pairs is denoted by $\mathcal{A}$. This admissibility outlaws doubling strategies and ensures existence of the associated Epstein-Zin utility.

The agent aims to maximize her utility at time 0 over all admissible strategies, i.e.,

$$U_0 = \sup_{(\pi,c)\in\mathcal{A}} U_0^c. \tag{2.4}$$

We call (2.4) the **primal problem**.

2.3. **Dual problem.** Rather than tackle the primal problem directly, we introduce a dual problem in this section. To formulate this dual problem, we focus on Epstein-Zin utility satisfying the following parameter restriction:

$$\gamma\psi \geq 1 \quad \text{and} \quad \psi > 1. \tag{2.5}$$

The restriction $\gamma\psi \geq 1$ is equivalent to the convexity of $f(c, u)$ in u, see Lemma A.1 part (i). The case where $f(c, u)$ is concave in u can be treated similarity, see Remark 2.9 below. The convexity (resp. concavity) of $f(c, u)$ in u implies preference for early (resp. late) resolution of uncertainty (cf. Kreps and Porteus (1978) and Skiadas (1998). On the other hand, because $\{\gamma\psi \geq 1, \psi > 1\} = \{\gamma, \psi > 1\} \cup \{0 < \gamma <$

$1.\gamma\psi \geq 1\}$, (2.5) ensures the existence of Epstein-Zin utility due to Proposition 2.1. When $\gamma\psi = 1$, Epstein-Zin utility reduces to the time-additive utility with constant relative risk aversion γ, whose dual problem is well understood. Therefore we focus on $\gamma\psi, \psi > 1$ when we introduce the dual problem and only compare with the time-additive utility case at the end of this section.

The convexity of $f(c,u)$ in u leads to an alternative representation of stochastic differential utility. This **variational representation** was first proposed by Geoffard (1996) in a deterministic continuous-time setting, and extended by El Karoui et al. (1997) and Dumas et al. (2000) to uncertainty. Let us recall the **felicity function** F, defined as the Fenchel-Legendre transformation of f with respect to its second argument:

$$F(c,\nu) := \inf_{(1-\gamma)u>0} \big(f(c,u) + \nu u\big), \quad \text{for } c > 0, \nu > \delta\theta. \tag{2.6}$$

As $\gamma\psi, \psi > 1$ implies $\theta < 1$, we have from Lemma A.1 part (ii) that

$$F(c,\nu) = \delta^\theta \frac{c^{1-\gamma}}{1-\gamma}\left(\frac{\delta\theta - \nu}{\theta - 1}\right)^{1-\theta}, \quad \text{for } c > 0, \nu > \delta\theta,$$

moreover $F(c,\nu)$ is concave in c and concave in ν when $c > 0$ and $\nu > \delta\theta$, f and F satisfy the duality relation

$$f(c,u) = \sup_{\nu>\delta\theta} \big(F(c,\nu) - \nu u\big), \quad \text{for } c > 0, (1-\gamma)u > 0. \tag{2.7}$$

To introduce the variational representation, define

$$\mathcal{V} := \{\nu \,|\, \text{progressively measurable and } \nu > \delta\theta\}. \tag{2.8}$$

For each $\nu \in \mathcal{V}, c \in \mathcal{C}_a, s, t \in [0,T]$, define

$$U_t^{c,\nu} := \mathbb{E}_t\big[\kappa^\nu_{t,T} U_T(c_T) + \int_t^T \kappa^\nu_{t,s} F(c_s,\nu_s)ds\big], \quad \text{where } \kappa^\nu_{t,s} := \exp\Big(-\int_t^s \nu_u du\Big).$$

The following result is a minor extension of (El Karoui et al., 1997, Section 3.2) and (Dumas et al., 2000, Theorem 2.1).

LEMMA 2.3 *For any $c \in \mathcal{C}_a$,*

$$U_0^c = \sup_{\nu\in\mathcal{V}} U_0^{c,\nu}.$$

Using Lemma 2.3, the primal problem in (2.4) is transformed into

$$\begin{aligned} U_0 &= \sup_{(\pi,c)\in\mathcal{A}} \sup_{\nu\in\mathcal{V}} \mathbb{E}\Big[\kappa^\nu_{0,T} U_T(c_T) + \int_0^T \kappa^\nu_{0,s} F(c_s,\nu_s)ds\Big] \\ &= \sup_{\nu\in\mathcal{V}} \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\Big[\kappa^\nu_{0,T} U_T(c_T) + \int_0^T \kappa^\nu_{0,s} F(c_s,\nu_s)ds\Big]. \end{aligned} \tag{2.9}$$

For a given $\nu \in \mathcal{V}$, the inner problem in the second line above can be considered as an optimization problem for a bequest utility U_T and a time-additive intertemporal utility $F(c,\nu)$, parameterized by ν, which can be viewed as a fictitious discounting rate. To present the dual problem of this inner problem, we define the Fenchel-Legendre transform of U_T and F (with respect to its first argument):

(2.10)

$$V_T(d) := \sup_{c>0}(U_T(c) - d\,c), \quad G(d,\nu) := \sup_{c>0}(F(c,\nu) - d\,c), \quad \text{for } d > 0, \nu > \delta\theta.$$

Lemma A.1 part (iii) shows that

$$V_T(d) = \epsilon^{\frac{1}{\gamma}} \frac{\gamma}{1-\gamma} d^{\frac{\gamma-1}{\gamma}}, \quad G(d,\nu) = \delta^{\frac{\theta}{\gamma}} \frac{\gamma}{1-\gamma} d^{\frac{\gamma-1}{\gamma}} \left(\frac{\delta\theta-\nu}{\theta-1}\right)^{\frac{1-\theta}{\gamma}},$$

for $d > 0, \nu > \delta\theta$. Moreover $G(d,\nu)$ is convex in d and concave in ν.

Recall the class of **state price densities** (**supermartingale deflators**):

$$\mathcal{D} := \{D \,|\, D_0 = 1, D > 0, D\mathcal{W}^{(\pi,c)} + \textstyle\int_0^\cdot D_s c_s ds \text{ is a supermartingale for all } (\pi,c) \in \mathcal{A}\}.$$

We assume that

$$\mathcal{D} \neq \emptyset.$$

Introducing the dual problem to the inner problem in the second line of (2.9), we obtain

(2.11)

$$\begin{aligned} U_0 &\leq \sup_{\nu\in\mathcal{V}} \inf_{y>0, D\in\mathcal{D}} \Big\{ \mathbb{E}\Big[\kappa^\nu_{0,T} V_T((\kappa^\nu_{0,T})^{-1} y D_T) + \int_0^T \kappa^\nu_{0,s} G((\kappa^\nu_{0,s})^{-1} y D_s, \nu_s) ds\Big] \\ &\qquad + wy\Big\} \\ &\leq \inf_{y>0, D\in\mathcal{D}} \Big\{ \sup_{\nu\in\mathcal{V}} \mathbb{E}\Big[\kappa^\nu_{0,T} V_T((\kappa^\nu_{0,T})^{-1} y D_T) + \int_0^T \kappa^\nu_{0,s} G((\kappa^\nu_{0,s})^{-1} y D_s, \nu_s) ds\Big] \\ &\qquad + wy\Big\}. \end{aligned}$$

Now the inner problem in the second line of (2.11) can be viewed as a variational problem. In order to transform it back to a recursive form, we use the power function form of V_T and G in d to get

$$\begin{aligned} \kappa^\nu_{0,T} V_T((\kappa^\nu_{0,T})^{-1} y D_T) &= (\kappa^\nu_{0,T})^{\frac{1}{\gamma}} V_T(y D_T) = \kappa^{\frac{\nu}{\gamma}}_{0,T} V_T(y D_T), \\ \kappa^\nu_{0,s} G((\kappa^\nu_{0,s})^{-1} y D_s, \nu_s) &= (\kappa^\nu_{0,s})^{\frac{1}{\gamma}} G(y D_s, \nu_s) = \kappa^{\frac{\nu}{\gamma}}_{0,s} G(y D_s, \nu_s). \end{aligned}$$

Plugging the previous two identities to the second line of (2.11) and introducing

$$V^{yD,\frac{\nu}{\gamma}}_t := \mathbb{E}_t\Big[\kappa^{\frac{\nu}{\gamma}}_{t,T} V_T(y D_T) + \int_t^T \kappa^{\frac{\nu}{\gamma}}_{t,s} G(y D_s, \nu_s) ds\Big],$$

we transform (2.11) to

$$U_0 \leq \inf_{y>0, D\in\mathcal{D}} \big(\sup_{\nu\in\mathcal{V}} V^{yD,\frac{\nu}{\gamma}}_0 + wy\big). \tag{2.12}$$

The inner problem is of variational form. In order to transform it to a recursive form, we introduce the convex conjugate of G with respect to its second variable

$$g(d,v) := \sup_{\nu>\delta\theta} (G(d,\nu) - \nu\, v), \quad \text{for } d > 0, (1-\gamma)v > 0. \tag{2.13}$$

Lemma A.1 part (iv) shows that

$$g(d,v) = \delta^\psi \frac{d^{1-\psi}}{\psi-1} ((1-\gamma)v)^{1-\frac{\gamma\psi}{\theta}} - \delta\theta v, \quad \text{for } d > 0, (1-\gamma)v > 0.$$

Additionally we introduce an analogue of stochastic differential utility for the dual problem.

Definition 2.4. For $y > 0$ and $D \in \mathcal{D}$, an Epstein-Zin **stochastic differential dual** for yD is a semimartingale $(V_t^{yD})_{0 \le t \le T}$ satisfying

$$V_t^{yD} = \mathbb{E}_t\Big[V_T(yD_T) + \int_t^T g(yD_s, \tfrac{1}{\gamma} V_s^{yD}) ds\Big], \quad \text{for all } t \in [0, T]. \tag{2.14}$$

Similar to Epstein-Zin utility, we denote by $\mathcal{D}_a$ the class of state price densities D whose associated stochastic differential duals V^{yD} exist for all $y > 0$, $(1-\gamma)V^{yD} > 0$, and V^{yD} is of class (D). Sufficient conditions for the existence of stochastic differential duals are summarized as follows, which imply $\mathcal{D}_a \neq \emptyset$.

PROPOSITION 2.5 *Let the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ be the augmented filtration generated by some Brownian motion.*

(i) When either $\gamma > 1, 0 < \psi < 1$, or $0 < \gamma < 1, \psi > 1$, for any $y > 0$ and $D \in \mathcal{D}$ such that $\mathbb{E}[\int_0^T D_t^\ell dt + D_T^\ell] < \infty$ for all $\ell \in \mathbb{R}$, there exists a unique V^{yD} satisfying (2.14)*, $(1-\gamma)V^{yD} > 0$, and $\mathbb{E}[\operatorname{ess\,sup}_t |V_t^{yD}|^\ell] < \infty$ for every $\ell > 0$.*

(ii) When $\gamma, \psi > 1$, for any $y > 0$, $D \in \mathcal{D}$ such that $\mathbb{E}[\int_0^T D_t^{1-\psi} dt + D_T^{(\gamma-1)/\gamma}] < \infty$, there exists a unique V^{yD} of class (D) satisfying $(1-\gamma)V^{yD} > 0$ and (2.14).

Coming back to the right-hand side of (2.12), an argument similar to Lemma 2.3 then yields

LEMMA 2.6 *For any $D \in \mathcal{D}_a$ and $y > 0$,*

$$V_0^{yD} = \sup_{\nu \in \mathcal{V}} V_0^{yD,\nu}.$$

Combining the previous result with (2.12), we obtain the following duality inequality, whose right-hand side is called the **dual problem**.

THEOREM 2.7 *Assume that $\gamma\psi \ge 1$ and $\psi > 1$. Then*

$$\sup_{(\pi,c) \in \mathcal{A}} U_0^c \le \inf_{y>0} \big(\inf_{D \in \mathcal{D}_a} V_0^{yD} + w\, y \big). \tag{2.15}$$

A diagram illustrating relationship between various functions introduced above is presented in Figure 1, starting from the primal problem in the upper left corner and ending at the dual problem in the bottom left corner.

Remark 2.8. When $\gamma\psi = 1$, Epstein-Zin utility reduces to time-additive utility with constant relative risk aversion γ. Then (2.2) and (2.14) reduce to the following standard form of primal and dual problems,

$$U_t^c = \mathbb{E}_t\left[e^{-\delta T} U_T(c_T) + \int_t^T \delta e^{-\delta s} \frac{c_s^{1-\gamma}}{1-\gamma} ds\right],$$

$$V_t^{yD} = \mathbb{E}_t\left[e^{-\frac{\delta}{\gamma}T} V_T(yD_T) + \int_t^T \delta^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}s} \tfrac{\gamma}{1-\gamma} (yD_s)^{\frac{\gamma-1}{\gamma}} ds\right].$$

The duality inequality (2.15) follows from duality between the power utility and its convex conjugate, together with the fact that $\mathbb{E}\big[c_T D_T + \int_0^T c_s D_s\big] \le w$.

Remark 2.9. When $f(c, u)$ is concave in u, i.e., $\gamma\psi \le 1$, we can replace the supremum (resp. infimum) in (2.6), (2.7), and (2.13) by infimum (resp. supremum). Then the same statement of Theorem 2.7 holds when $\gamma\psi \le 1$ and $\psi < 1$.

$$
\begin{tikzcd}
 & \text{Recursive} & \text{Variational} \\
\text{Primal} & U_T(c), f(c,u) \arrow[r, "\text{concave conjugate in } u"] & U_T(c), F(c,\nu) \arrow[d, "\text{convex conjugate in } c"] \\
\text{Dual} & V_T(d), g(d,v) & V_T(d), G(d,\nu) \arrow[l, "\text{convex conjugate in } \nu"']
\end{tikzcd}
$$

FIGURE 1. Double Fenchel-Legendre transformation

Remark 2.10. Theorem 2.7 excludes the unit EIS case ($\psi = 1$). This is because the dual domain $\mathcal{V}$ in (2.8) for the variational representation is specifically designed for the non-unit EIS case. For the unit EIS case, the HJB equations in optimal consumption and investment problems are typically linear in the utility variable, hence it is easier to work with the primal approach; see (Schroder and Skiadas, 2003, Section 5.6), Chacko and Viceira (2005), and Kraft et al. (2017).

3. MAIN RESULTS

For a wide class of financial models specified below, this section shows that the inequality (2.15) is actually an identity, i.e., there is *no duality gap*. Moreover optimal primal optimizer (π^*, c^*) and dual optimizer (y^*, D^*) will be identified as follows

$$
\max_{(\pi,c)\in\mathcal{A}} U_0^c = U_0^{c^*} = V_0^{y^* D^*} + wy^* = \min_{y>0}\Big(\min_{D\in\mathcal{D}_a} V_0^{yD} + wy\Big). \tag{3.1}
$$

3.1. **Market setting.** We work with models with Brownian noise. Let $(\mathcal{F}_t)_{0\leq t\leq T}$ be the augmented filtration generated by a $k+n$-dimensional Brownian motion $B = (W, W^\perp)$, where W (resp. $W^\perp$) represents the first k (resp. last n) components. We will also use $(\mathcal{F}_t^W)_{0\leq t\leq T}$ (resp. $(\mathcal{F}_t^{W^\perp})_{0\leq t\leq T}$) as the augmented filtration generated by W (resp. $W^\perp$).

Consider a model of financial market where assets $S = (S^0, S^1, \ldots, S^n)$ have the dynamics

$$
dS_t^0 = S_t^0 r_t dt, \quad dS_t^i = S_t^i\Big[(r_t + \mu_t^i)dt + \sum_{j=1}^n \sigma_t^{ij} dW_t^{\rho,j}\Big], \quad i = 1, \ldots, n, \tag{3.2}
$$

where r, μ, σ and ρ are $\mathcal{F}^W$-adapted processes valued in $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n\times n}, \mathbb{R}^{n\times k}$, respectively, and satisfy $\int_0^T |\alpha_t|^2 dt < \infty$ a.s. for $\alpha = r, \mu, \sigma, \rho$. Moreover $\Sigma := \sigma\sigma'$ is assumed to be invertible. For an $\mathbb{R}^{n\times n}$-valued process $\rho^\perp$ satisfying $\rho\rho' + \rho^\perp(\rho^\perp)' = 1_{n\times n}$ (the n-dimensional identity matrix), the n-dimensional Brownian motion W^ρ is defined as $W^\rho := \int_0^\cdot \rho_s dW_s + \int_0^\cdot \rho_s^\perp dW_s^\perp$. Define

$$
D_t^0 := e^{-\int_0^t r_s ds}\mathcal{E}\Big(-\int \mu_s' \Sigma_s^{-1}\sigma_s dW_s^\rho\Big)_t, \quad t\in[0,T].
$$

It follows that each component of $D^0 S$ is a nonnegative local martingale, hence a supermartingale. The process D^0 is called a *supermartingale deflator*, whose presence excludes arbitrage opportunities, cf. Karatzas and Kardaras (2007).

For $(\pi, c) \in \mathcal{A}$, agent's wealth process $\mathcal{W}^{(\pi,c)}$ follows

$$d\mathcal{W}_t^{(\pi,c)} = \mathcal{W}_t^{(\pi,c)}[(r_t + \pi_t'\mu_t)dt + \pi_t'\sigma_t dW_t^\rho] - c_t dt, \tag{3.3}$$

which implies $D^0 \in \mathcal{D}$.

3.2. **Candidate optimal strategy.** We consider a dynamic version of the primal and dual problem by introducing the **primal** and **dual value processes**

$$\mathbb{U}_t^{\pi,c} := \operatorname*{ess\,sup}_{(\tilde{\pi},\tilde{c})\in\mathcal{A}(\pi,c,t)} U_t^{\tilde{c}} \quad \text{and} \quad \mathbb{V}_t^{yD} := \operatorname*{ess\,inf}_{\tilde{D}\in\mathcal{D}_a(D,t)} V_t^{y\tilde{D}},$$

where

$$\begin{aligned}\mathcal{A}(\pi,c,t) :=& \{(\tilde{\pi},\tilde{c}) \in \mathcal{A} \,:\, (\tilde{\pi},\tilde{c}) = (\pi,c) \text{ on } [0,t]\},\\ \mathcal{D}_a(D,t) :=& \{\tilde{D} \in \mathcal{D}_a \,:\, \tilde{D} = D \text{ on } [0,t]\}.\end{aligned}$$

Due to the homothetic property of Epstein-Zin utilities, we speculate that $\mathbb{U}^{\pi,c}$ and $\mathbb{V}^{yD}$ have the following decomposition:

$$\mathbb{U}_t^{\pi,c} = \tfrac{1}{1-\gamma}(\mathcal{W}_t^{(\pi,c)})^{1-\gamma}e^{Y_t^p} \quad \text{and} \quad \mathbb{V}_t^{yD} = \tfrac{\gamma}{1-\gamma}(yD_t)^{\frac{\gamma-1}{\gamma}}e^{Y_t^d/\gamma}, \tag{3.4}$$

for some processes Y^p and Y^d.

Let us recall the *martingale principle*: $\mathbb{U}^c + \int_0^\cdot f(c_s, \mathbb{U}_s^c)ds$ (resp. $\mathbb{V}^{yD} + \int_0^\cdot g(yD_s, \frac{1}{\gamma}\mathbb{V}_s^{yD})ds$) is a supermartingale (resp. submartingale) for an arbitrary (π, c) (resp. D) and is a martingale for the optimal one. For Markovian models, the martingale principle is a reformulation of the dynamic programming principle. For non-Markovian models, it can be considered as the dynamic programming for BSDEs, cf., eg. Hu et al. (2005). The following result determines the dynamics of Y^p and Y^d using the martingale principle. Moreover, candidate optimal strategies for both primal and dual problems are identified. In particular, the candidate dual optimizer D^* follows

$$dD_t^*/D_t^* = -r_t dt + \xi_t^* dW_t + \eta_t^* dW_t^\perp,$$

for some ξ^* and η^*.

LEMMA 3.1 *The ansatz* (3.4) *and the martingale principle imply that both* (Y^p, Z^p) *and* (Y^d, Z^d)*, for some processes* Z^p *and* Z^d*, satisfy the BSDE*

$$Y_t = \log\epsilon + \int_t^T H(Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0,T], \tag{3.5}$$

where $H : \Omega \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ *is given by*

$$H(y,z) := \tfrac{1}{2}zM_t z' + \tfrac{1-\gamma}{\gamma}\mu_t'\Sigma_t^{-1}\sigma_t\rho_t z' + \theta\tfrac{\delta^\psi}{\psi}e^{-\frac{\psi}{\theta}y} + h_t - \delta\theta. \tag{3.6}$$

Here, suppressing the subscript t,

$$\Sigma := \sigma\sigma', \quad M := 1_{k\times k} + \tfrac{1-\gamma}{\gamma}\rho'\sigma'\Sigma^{-1}\sigma\rho, \quad \textit{and} \quad h := (1-\gamma)r + \tfrac{1-\gamma}{2\gamma}\mu'\Sigma^{-1}\mu. \tag{3.7}$$

The function H, interpreted as the Hamilton of the primal and dual optimization problem, has the following representation

$$
\begin{aligned}
H(y,z) =&(1-\gamma)r_t - \delta\theta + \tfrac{1}{2}|z|^2 + (1-\gamma)\sup_{\bar{c}}\Big[-\bar{c} + \delta e^{-\frac{1}{\theta}y}\tfrac{1}{1-\frac{1}{\psi}}\bar{c}^{1-\frac{1}{\psi}}\Big]\\
&+(1-\gamma)\sup_{\pi}\Big[-\tfrac{\gamma}{2}\pi'\Sigma_t\pi + \pi'(\mu_t+\sigma_t\rho_t z')\Big]\\
=&(1-\gamma)r_t - \delta\theta + \theta\tfrac{\delta^\psi}{\psi}e^{-\frac{\psi}{\theta}y} + \tfrac{1}{2\gamma}|z|^2\\
&+(1-\gamma)\inf_{\mu_t+\sigma_t\rho_t\xi'+\sigma_t\rho_t^\perp\eta'=0}\Big[\tfrac{1}{2\gamma}(|\xi|^2+|\eta|^2) - \tfrac{1}{\gamma}\xi z'\Big],
\end{aligned}
\tag{3.8}
$$

whose optimizers are

$$
\begin{aligned}
\pi(t,z) &= \tfrac{1}{\gamma}\Sigma_t^{-1}(\mu_t+\sigma_t\rho_t\, z'), & \bar{c}(y) &= \delta^\psi e^{-\frac{\psi}{\theta}y},\\
\xi(t,z) &= -(\mu_t' + z\rho_t'\sigma_t')\Sigma_t^{-1}\sigma_t\rho_t + z, & \eta(t,z) &= -(\mu_t'+z\rho_t'\sigma_t')\Sigma_t^{-1}\sigma_t\rho_t^\perp.
\end{aligned}
$$

In the next section, we will start from the BSDE (3.5) and show that it admits a solution (Y,Z). Define

$$
\begin{aligned}
\pi_t^* =&\pi(t,Z_t), \quad \frac{c_t^*}{\mathcal{W}_t^{(\pi^*,c^*)}} = \bar{c}(Y_t),\\
\xi_t^* =&\xi(t,Z_t), \quad \eta_t^* = \eta(t,Z_t), \quad y^* = w^{-\gamma}e^{Y_0}, \quad \text{and}\\
dD_t^*/D_t^* =& -r_tdt - (\mu_t' + Z_t\rho_t'\sigma_t')\Sigma_t^{-1}\sigma_t dW_t^\rho + Z_t dW_t\\
=& -r_tdt + \xi_t^* dW_t + \eta_t^* dW_t^\perp.
\end{aligned}
\tag{3.9}
$$

We call (π^*,c^*) and (y^*,D^*) the candidate optimal strategies for primal and dual problems.

Remark 3.2. The form of (π^*,c^*) has been documented in various settings, see (Schroder and Skiadas, 1999, Theorem 2 and 4) for complete markets, (Kraft et al., 2013, Equation (4.4)), (Kraft et al., 2017, Theorem 6.1), and (Xing, 2017, Equation (2.14)) for Markovian models. The form of D^* can be obtained via the utility gradient approach, cf. (Duffie and Epstein, 1992b, Equation (35)), (Duffie and Skiadas, 1994, Theorem 2), and (Schroder and Skiadas, 1999, Equation (3)). The novelty of this paper is to relate the utility gradient D^* and the minimization problem in (3.8), see Corollary 3.7 below.

Remark 3.3. Lemma 3.1 requires $\epsilon > 0$, in other words, the Epstein-Zin preferences we considered has non-zero bequest utility. The case with zero bequest utility was considered in Schroder and Skiadas (1999). However the utility parameter restriction Equation (8) therein excludes the $\gamma,\psi>1$ case.

3.3. **Models with bounded market price of risk.** This section verifies the identity (3.1), hence confirm the optimality of (π^*,c^*) and D^*. We start with the following restriction on model coefficients.

Assumption 3.4. The processes r and $\mu'\Sigma^{-1}\mu$ are both bounded.

This assumption allows non-Markovian models, but requires the *market price of risk* $\sqrt{\mu'\Sigma^{-1}\mu}$ to be bounded. Markovian models with unbounded market price of risk will be discussed in the next section, where more technical conditions will be imposed. We will also assume the same restriction on utility parameters γ and ψ as in Theorem 2.7 and Remark 2.9.

LEMMA 3.5 *Suppose that* $\gamma\psi \geq 1, \psi > 1$*, or* $\gamma\psi \leq 1, \psi < 1$*, and Assumption 3.4 holds. Then* (3.5) *admit a solution* (Y, Z) *such that* Y *is bounded and* $Z \in H_{BMO}$*.*

Having establish a solution (Y, Z) to (3.5), the following result verifies the optimality of (π^*, c^*) and (y^*, D^*) in (3.9). This result generalizes (Kraft et al., 2017, Theorem 5.1) to non-Markovian models and establishes the optimality of (π^*, c^*) in a large admissible set.

THEOREM 3.6 *Suppose that* $\gamma\psi \geq 1, \psi > 1$*, or* $\gamma\psi \leq 1, \psi < 1$*, and Assumption 3.4 holds. Then,* (π^*, c^*) *and* (y^*, D^*) *satisfy*

$$\max_{(\pi,c)\in\mathcal{A}} U_0^c = U_0^{c^*} = V_0^{y^* D^*} + wy^* = \min_{y>0}\big(\min_{D\in\mathcal{D}_a} V_0^{yD} + wy\big). \tag{3.10}$$

Therefore (π^*, c^*) *is the optimal strategy for the primal problem,* D^* *is the optimal state price density for the dual problem, and* y^* *is the Lagrangian multiplier.*

As a direct consequence of Theorem 3.6, the optimal state price density D^* is identified as the super-differential of the primal value function, coming from the utility gradient approach, cf. Duffie and Epstein (1992b), Duffie and Skiadas (1994).

COROLLARY 3.7 *Let the assumptions of Theorem 3.6 hold. The minimizer* D^* *of the dual problem satisfies*

$$D_t^* = w^\gamma e^{-Y_0} \exp\Big[\int_0^t \partial_u f(c_s^*, U_s^{c^*}) ds\Big] \partial_c f(c_t^*, U_t^{c^*}), \quad t \in [0, T], \tag{3.11}$$

and $\mathcal{W}^{(\pi^*, c^*)} D^* + \int_0^{\cdot} D_s^* c_s^* ds$ *is a martingale.*

3.4. **Models with unbounded market price of risk.** Many widely used market models in the asset pricing literature come with unbounded market price of risk; for example, Heston model in Chacko and Viceira (2005), Kraft (2005), and Liu (2007), Kim-Omberg model in Kim and Omberg (1996) and Wachter (2002). To obtain similar results to Theorem 3.6 and Corollary 3.7, we focus on the utility specification

$$\gamma, \psi > 1,$$

and work with Markovian models, whose investment opportunities are driven by a state variable X satisfying

$$dX_t = b(X_t)dt + a(X_t)dW_t. \tag{3.12}$$

Here X takes value in an open domain $E \subseteq \mathbb{R}^k$, $b : E \to \mathbb{R}^k$ and $a : E \to \mathbb{R}^{k\times k}$. The domain E is assumed to satisfy $E = \cup_n E_n$, where $(E_n)_n$ is a sequence of open domains in E such that $\overline{E}_n$ is compact and $\overline{E}_n \subset E_{n+1}$ for each n. Given functions $r : E \to \mathbb{R}$, $\mu : E \to \mathbb{R}^n$, $\sigma : E \to \mathbb{R}^{n\times n}$, and $\rho : E \to \mathbb{R}^{n\times k}$, the processes r, μ, σ, ρ in (3.2) are the corresponding functions evaluated at X. Instead of Assumption 3.4, these model coefficients satisfy the following assumptions.

Assumption 3.8. r, μ, σ, b, a, and ρ are all locally Lipschitz in E; $A := aa'$ and $\Sigma = \sigma\sigma'$ are positive definite in any compact subdomain of E; dynamics of (3.12) does not reach the boundary of E in finite time; moreover $r + \frac{1}{2\gamma}\mu'\Sigma^{-1}\mu$ is bounded from below on E.

The regularity of coefficients and the non-explosion assumption ensure that the dynamics for X is well-posed, i.e., (3.12) admits a unique E-valued strong solution $(X_t)_{0\leq t\leq T}$. The assumption on the lower bound of $r + \frac{1}{2\gamma}\mu'\Sigma^{-1}\mu$ allows for unbounded market price of risk and is readily satisfied when r is bounded from below.

To present an analogue of Theorem 3.6 and Corollary 3.7, let us first introduce two sets of abstract conditions, which will be verified in two classes of models below.

Assumption 3.9.

(i) $\frac{d\overline{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}\big(\int \frac{1-\gamma}{\gamma}\mu'\Sigma^{-1}\sigma\rho(X_s)dW_s\big)$ defines a probability measure $\overline{\mathbb{P}}$ equivalent to $\mathbb{P}$;

(ii) $\mathbb{E}^{\overline{\mathbb{P}}}\big[\int_0^T h(X_s)ds\big] > -\infty$, where h comes from (3.7).

When all model coefficients are bounded, as in Assumption 3.4, Assumption 3.9 is automatically satisfied. When the market price of risk is unbounded, the last part of Assumption 3.8 and $\gamma > 1$ combined imply that h in (3.7) is bounded from above by $h_{\max} := \max_{x\in E} h(x)$, but not bounded from below. Nevertheless Assumption 3.9 allows us to transform (3.5) under $\overline{\mathbb{P}}$ and present the following result from (Xing, 2017, Proposition 2.9).

LEMMA 3.10 *Let Assumptions 3.8 and 3.9 hold. For $\gamma, \psi > 1$,* (3.5) *admits a solution (Y, Z) such that, for any $t \in [0, T]$,*

$$\mathbb{E}_t^{\overline{\mathbb{P}}}\Big[\int_t^T h(X_s)\,ds\Big] - \delta\theta(T-t) + \theta\frac{\delta^\psi}{\psi}e^{(\delta\psi - \frac{\psi}{\theta}h_{\max})T}(T-t) \leq Y_t \leq \tag{3.13}$$

$$\leq -\delta\theta(T-t) + \log\mathbb{E}_t^{\overline{\mathbb{P}}}\Big[\exp\Big(\int_t^T h(X_s)\,ds\Big)\Big],$$

and $\mathbb{E}^{\overline{\mathbb{P}}}[\int_0^T |Z_s|^2 ds] < \infty$. In particular, because $h \leq h_{max}$, Y is bounded from above.

Having constructed (Y, Z), (π^*, c^*) and D^* in (3.9) are well defined. To verify their optimality, let us introduce an operator $\mathfrak{F}$. For $\phi \in C^2(E)$,

$$\mathfrak{F}[\phi] := \tfrac{1}{2}\sum_{i,j=1}^k A_{ij}\partial^2_{x_ix_j}\phi + \Big(b + \tfrac{1-\gamma}{\gamma}a\rho'\sigma'\Sigma^{-1}\mu\Big)'\nabla\phi + \tfrac{1}{2}\nabla\phi' aMa'\nabla\phi + h, \tag{3.14}$$

where the dependence on x is suppressed on both sides. To understand this operator, note that the solution (Y, Z) to (3.5) is expected to be Markovian, i.e., there exists a function $u : [0, T] \times E \to \mathbb{R}$ such that $Y = u(\cdot, X)$. Then the BSDE (3.5) corresponds the following PDE:

$$\partial_t u + \mathfrak{F}[u] + \theta\tfrac{\delta^\psi}{\psi}e^{-\frac{\psi}{\theta}u} - \delta\theta = 0, \quad u(T, x) = 0.$$

As $\theta < 0$ when $\gamma, \psi > 1$, Y is bounded from above, and hence so is u. Therefore the last two terms in the previous PDE are bounded, and $\mathfrak{F}$ is the unbounded part of the spatial operator.

Assumption 3.11. There exists $\phi \in C^2(E)$ such that

(i) $\lim_{n\to\infty}\inf_{x\in E\setminus E_n}\phi(x) = \infty$;

(ii) $\mathfrak{F}[\phi]$ is bounded from above on E.

The function ϕ in the previous assumption is called a *Lyapunov* function. Its existence facilitates the proof of a verification result, leading to the following result.

THEOREM 3.12 *Suppose that $\gamma, \psi > 1$, and that Assumptions 3.8, 3.9, 3.11 hold. Then the statements of Theorem 3.6 and Corollary 3.7 hold.*

Remark 3.13. The proof of Theorem 3.6 (resp. Theorem 3.12) implies that for any solution (Y, Z) of (3.5) with Y bounded (resp. bounded from above), Y is the same as Y^p and Y^d which are uniquely identified via the primal and dual value functions in (3.4). Therefore the following uniqueness results for (3.5) hold:

- Suppose that $\gamma\psi \geq 1, \psi > 1$, or $\gamma\psi \leq 1, \psi < 1$, and Assumption 3.4 holds. Then (3.5) admits a unique solution (Y, Z) with bounded Y.
- Suppose that $\gamma, \psi > 1$, and that Assumptions 3.8, 3.9, 3.11 hold. Then (3.5) admits a unique solution (Y, Z) with Y bounded from above.

Remark 3.14. The optimality of (π^*, c^*) has been verified in (Xing, 2017, Theorem 2.14) under similar, but more restrictive, conditions. First, Xing (2017) restricts strategies to a *permissible* class which is smaller than the current admissible class $\mathcal{A}$. It is the duality inequality (2.15) that allows us to make this extension. Secondly, (Xing, 2017, Assumption 2.11) is needed to ensure c^* satisfies the integrability condition in Proposition 2.1 (ii). This integrability condition translates to model parameter restrictions, see (Xing, 2017, Proposition 3.2 ii)) for the Heston model and (Xing, 2017, Proposition 3.4 ii)) for the Kim-Omberg model. Rather than forcing c^* to satisfy this integrability condition, which is a sufficient condition for the existence of Epstein-Zin utility, we show that the class (D) Epstein-Zin utility exists for c^*, hence c^* belongs to $\mathcal{C}_a$, which abstractly envelops all Epstein-Zin utilities and, in particular, contains those ones satisfying the integrability condition. As a result the aforementioned model parameter restrictions for the Heston model and the Kim-Omberg model can be removed in the following examples.

Example 3.15 (Heston model). Consider a 1-dimensional process X following

$$dX_t = b(\ell - X_t)dt + a\sqrt{X_t}dW_t,$$

where $b, \ell \geq 0, a > 0$, and $b\ell > \frac{1}{2}a^2$. These parameter restrictions ensure the existence of a strictly positive process X. Given $r_0, r_1 \in \mathbb{R}$, $\lambda_0, \lambda_1, \rho \in \mathbb{R}^n$, and $\sigma : (0, \infty) \to \mathbb{R}^{n\times n}$ which is locally Lipschitz on $(0, \infty)$ and satisfies $\Sigma(x) := \sigma\sigma(x)' > 0$, consider the following asset dynamics

$$\begin{aligned} dS_t^0 &= S_t^0(r_0 + r_1 X_t)dt, \\ dS_t^i &= S_t^i\Big[(r_0 + r_1X_t)dt + \sum_{j=1}^n \sigma^{ij}(X_t)\Big(\Big(\frac{\lambda_0^j}{\sqrt{X_t}} + \lambda_1^j\sqrt{X_t}\Big)dt + dW_t^{\rho,j}\Big)\Big], \\ & i = 1, \dots, n. \end{aligned}$$

This class of models encapsulates several special examples with 1 risky assets $(n = 1)$:

- The Heston model studied in Kraft (2005) and Liu (2007): $\lambda_0 = 0$, $\lambda_1 \in \mathbb{R}$, $\sigma(x) = \sqrt{x}$.
- The inverse Heston model studied in Chacko and Viceira (2005): $\lambda_0 = 0$, $\lambda_1 \in \mathbb{R}, \sigma(x) = \frac{1}{\sqrt{x}}$.
- When $\lambda_0 \neq 0$ and $\sigma(x) = \sqrt{x}$, the previous model is not an affine model, but is in the class of *essentially affine models* proposed by Duffee (2002).

Denote $\Theta(x) := \sigma(x)'\Sigma(x)^{-1}\sigma(x)$. The following result specifies Assumptions 3.8, 3.9, and 3.11 to explicit model parameter restriction.

PROPOSITION 3.16 *Assume $\gamma, \psi > 1$ and the following conditions:*

(i) $r_1 + \inf_{x>0} \frac{1}{2\gamma}\lambda_1'\Theta(x)\lambda_1 \geq 0$*;*
(ii) Either $r_1 > 0$ *or* $\inf_{x>0} \lambda_1'\Theta(x)\lambda_1 > 0$*;*
(iii) $b\ell > \frac{1}{2}a^2$ *and* $b\ell + \inf_{x>0} \frac{1-\gamma}{\gamma}a\lambda_0'\Theta(x)\rho > \frac{1}{2}a^2$.

Then the statements of Theorem 3.6 and Corollary 3.7 hold.

Example 3.17 (3/2 model). Consider a 1-dimensional process X following

$$dX_t = (pX_t + qX_t^2)dt + aX_t^{3/2}dW_t,$$

where $p \in \mathbb{R}, a > 0, q < \frac{1}{2}a^2$. Feller's test ensures the existence of a strictly positive process X satisfying the previous SDE. Using X to model the variance process has received a large amount of empirical support; see the survey in (Carr and Sun, 2007, Page 109).

Given $r_0, r_1 \in \mathbb{R}$, $\lambda, \rho \in \mathbb{R}^n, \sigma \in \mathbb{R}^{n\times n}$, consider the following asset dynamics

$$dS_t^0 = S_t^0(r_0 + r_1 X_t)dt,$$
$$dS_t^i = S_t^i\Big[(r_0 + r_1 X_t)dt + \sum_{j=1}^n \sigma^{ij}\sqrt{X_t}\Big(\lambda_1^j\sqrt{X_t}dt + dW_t^{\rho,j}\Big)\Big], \quad i = 1,\dots,n.$$

Denote $\Theta := \sigma'\Sigma^{-1}\sigma$. The following result provides sufficient conditions for Assumptions 3.8, 3.9, and 3.11.

PROPOSITION 3.18 *Assume $\gamma, \psi > 1$ and the following conditions:*

(i) $r_1 + \frac{1}{2\gamma}\lambda'\Theta\lambda \geq 0$*;*
(ii) Either $r_1 > 0$ *or* $\lambda'\Theta\lambda > 0$*;*
(iii) $q < \frac{1}{2}a^2$ *and* $q + \frac{1-\gamma}{\gamma}a\lambda'\Theta\rho < \frac{1}{2}a^2$.

Then the statements of Theorem 3.6 and Corollary 3.7 hold.

Example 3.19 (Linear diffusion). Consider a 1-dimensional Ornstein-Uhlenbeck process X following

$$dX_t = -bX_t dt + a dW_t,$$

where $a, b > 0$. Given $r_0, r_1 \in \mathbb{R}$, $\lambda_0, \lambda_1, \rho \in \mathbb{R}^n$ and $\sigma \in \mathbb{R}^{n\times n}$ with $\Sigma := \sigma\sigma' > 0$, consider the following asset dynamics

$$dS_t^0 = S_t^0(r_0 + r_1 X_t)dt,$$
$$dS_t^i = S_t^i\Big[(r_0 + r_1 X_t)dt + \sum_{j=1}^n \sigma^{ij}\big(\big(\lambda_0^j + \lambda_1^j X_t\big)dt + dW_t^{\rho,j}\big)\Big], \quad i = 1,\dots,n.$$

This model has been studied by Kim and Omberg (1996) and Wachter (2002) for time separable utilities, and by Campbell and Viceira (1999) for recursive utilities in discrete time. Set $\Theta := \sigma'\Sigma^{-1}\sigma$. The following result from (Xing, 2017, Proposition 3.4) specifies Assumptions 3.8, 3.9, and 3.11 to explicit model parameter restrictions.

PROPOSITION 3.20 *Assume $\gamma, \psi > 1$ and either of the following parameter restrictions hold:*

(i) $r_1 = 0$ *and* $-b + \frac{1-\gamma}{\gamma}a\lambda_1'\Theta\rho < 0$*;*
(ii) $\lambda_1'\Theta\lambda_1 > 0$.

Then the statements of Theorem 3.6 and Corollary 3.7 hold.

4. PROOFS

4.1. **Proof of Lemma 2.3.** Note that $\{\gamma\psi > 1, \psi > 1\} = \{\gamma > 1, \psi > 1\} \cup \{0 < \gamma < 1, \gamma\psi > 1\}$. The proof is split into two cases.

$\underline{\gamma > 1, \psi > 1.}$ We will first prove

$$U_0^c \geq \sup_{\nu\in\mathcal{V}} U_0^{c,\nu}, \quad \text{for any } c \in \mathcal{C}_a. \tag{4.1}$$

To this end, it suffices to focus on $\nu \in \mathcal{V}$ such that $U_0^{c,\nu} > -\infty$. For such ν, because $U_T, F < 0$ when $\gamma > 1$, therefore the process $\mathbb{E}_t\big[\kappa_{0,T}^\nu U_T(c_T) + \int_0^T \kappa_{0,s}^\nu F(c_s,\nu_s)ds\big]$ is of class (D). On the other hand, the class (D) property of U^c for $c \in \mathcal{C}_a$ and the boundedness of κ^ν for $\nu \in \mathcal{V}$ ensure the integrability of $\kappa^\nu_{0,T} U_T(c_T)$, hence the class (D) property of the process $\mathbb{E}_t[\kappa^\nu_{0,T} U_T(c_T)]$. Due to $F < 0$, we have

$$\mathbb{E}_t\Big[\kappa^\nu_{0,T} U_T(c_T) + \int_0^T \kappa^\nu_{0,s} F(c_s, \nu_s) ds\Big] \leq \kappa^\nu_{0,t} U^{c,\nu}_t \leq \mathbb{E}_t[\kappa^\nu_{0,T} U_T(c_T)].$$

The class (D) property of both upper and lower bounds implies the class (D) property of $\kappa^\nu_{0,\cdot} U^{c,\nu}$.

As $U^{c,\nu}_0$ and U^c_0 are finite, both

$$U^c + \int_0^\cdot f(c_s, U^c_s) ds \quad \text{and} \quad \kappa^\nu_{0,\cdot} U^{c,\nu} + \int_0^\cdot \kappa^\nu_{0,s} F(c_s, \nu_s) ds$$

are martingales. It follows that $U^{c,\nu} + \int_0^\cdot F(c_s, \nu_s) - \nu_s U^{c,\nu}_s ds$ is a local martingale. Therefore there exists a local martingale L such that

$$d(U^c_t - U^{c,\nu}_t) - \nu_t (U^c_t - U^{c,\nu}_t) = -dA_t + dL_t,$$

where $A_t = \int_0^t f(c_s, U^c_s) - (F(c_s, \nu_s) - \nu_s U^c_s) ds$ is an increasing process due to (2.7). As a result, $\kappa^\nu_{0,\cdot}(U^c - U^{c,\nu})$ is a local super-martingale. On the other hand, we have seen that $\kappa^\nu_{0,\cdot} U^{c,\nu}$ is of class (D). Moreover, $\kappa^\nu_{0,\cdot} U^c$ is of class (D), thanks to the boundedness of κ^ν and class (D) property of U^c. Hence the local super-martingale $\kappa^\nu_{0,\cdot}(U^c - U^{c,\nu})$ is a supermartingale. Hence

$$U^c_t - U^{c,\nu}_t \geq \mathbb{E}_t\big[\kappa^\nu_{t,T}(U^c_T - U^{c,\nu}_T)\big] = 0. \tag{4.2}$$

Taking supremum in ν, we confirm (4.1).

To show that the inequality in (4.1) is actually an identity, it suffices to identify $\nu \in \mathcal{V}$ such that $U^c \leq U^{c,\nu}_0$. To this end, take $\nu^c := -f_u(c, U^c)$. Lemma A.1 part (i) and $\theta < 1$ ensure $\nu^c \in \mathcal{V}$. It then follows from (2.2) and (2.7) that $\kappa^{\nu^c}_{0,\cdot} U^c + \int_0^\cdot \kappa^{\nu^c}_{0,s} F(c_s, \nu^c_s) ds$ is a local martingale, and hence a submartingale, due to $F < 0$ and the class (D) property of $\kappa^{\nu^c}_{0,\cdot} U^c$. Therefore

$$U^c_0 \leq \mathbb{E}\Big[\kappa^{\nu^c}_{0,T} U_T(c_T) + \int_0^T \kappa^{\nu^c}_{0,s} F(c_s, \nu^c_s) ds\Big] = U^{c,\nu^c}_0, \tag{4.3}$$

which concludes the proof.

$\underline{0 < \gamma < 1, \gamma\psi > 1.}$ In this case, $F, U_T > 0$. We show $U^{c,\nu}_0 < \infty$ for any $c \in \mathcal{C}_a, \nu \in \mathcal{V}$ first. To this end, for $c \in \mathcal{C}_a$ and $\nu \in \mathcal{V}$, define an increasing process $A^{c,\nu} := \int_0^\cdot f(c_s, U^c_s) - (F(c_s, \nu_s) - \nu_s U^c_s) ds$. Equation (2.2) then implies that $U^c + \int_0^\cdot F(c_s, \nu_s) - \nu_s U^c_s ds + A^{c,\nu}$ is a martingale, hence $\kappa^\nu_{0,\cdot} U^c + \int_0^\cdot \kappa^\nu_{0,s} F(c_s, \nu_s) ds$ is a local supermartingale. Taking a localization sequence $(\tau_n)_n$, we have

$$U^c_0 \geq \mathbb{E}\left[\kappa^\nu_{0,\tau_n \wedge T} U^c_{\tau_n \wedge T} + \int_0^{\tau_n \wedge T} \kappa^\nu_{0,s} F(c_s, \nu_s) ds\right].$$

Sending $n \to \infty$ on the right-hand side, the class (D) property of U^c and monotone convergence theorem imply

$$\mathbb{E}\left[\int_0^T \kappa^\nu_{0,s} F(c_s, \nu_s) ds\right] < \infty. \tag{4.4}$$

Combined with $\mathbb{E}[\kappa^\nu_{0,T} U_T(c_T)] < \infty$, we obtain $U^{c,\nu}_0 < \infty$.

Because $F > 0$, we have

$$\mathbb{E}_t[\kappa^\nu_{0,T} U_T(c_T)] \leq \kappa^\nu_{0,t} U^{c,\nu}_t \leq \mathbb{E}_t\Big[\kappa^\nu_{0,T} U_T(c_T) + \int_0^T \kappa^\nu_{0,s} F(c_s, \nu_s) ds\Big].$$

Therefore $\kappa^{\nu}_{0,\cdot} U^{c,\nu}$ is of class (D), and a similar argument as in the previous case confirms (4.1). To show the inequality in (4.1) is an identity, same argument as the previous case shows that $\nu^c \in \mathcal{V}$ and $\kappa^{\nu^c}_{0,\cdot} U^c + \int_0^\cdot \kappa^{\nu^c}_{0,s} F(c_s, \nu^c_s) ds$ is a local martingale. Moreover, due to $F > 0$, (4.4), and the class (D) property of U^c, we obtain that $\kappa^{\nu^c}_{0,\cdot} U^c + \int_0^\cdot \kappa^{\nu^c}_{0,s} F(c_s, \nu^c_s) ds$ is a martingale. Hence (4.3) holds as an identity.

4.2. **Proof of Proposition 2.5.** Let the filtration be generated by some Brownian motion B. Solving (2.14) is equivalent to solve the following BSDE

(4.5)
$$
\begin{aligned}
V^{yD}_t =& \tfrac{\gamma}{1-\gamma} \epsilon^{\frac{1}{\gamma}} (yD_T)^{\frac{\gamma-1}{\gamma}} + \int_t^T \tfrac{\delta^\psi}{\psi-1} (yD_s)^{1-\psi} \left(\tfrac{1-\gamma}{\gamma} V^{yD}_s\right)^{1-\frac{\gamma\psi}{\theta}} - \tfrac{\delta\theta}{\gamma} V^{yD}_s ds \\
&- \int_t^T Z^{yD}_s dB_s.
\end{aligned}
$$

Set $Y_t = \frac{1-\gamma}{\gamma} e^{-\frac{\delta\theta}{\gamma} t} V^{yD}_t$ and $Z_t = \frac{1-\gamma}{\gamma} e^{-\frac{\delta\theta}{\gamma} t} Z^{yD}_t$. The previous BSDE translates to

(4.6)
$$
Y_t = e^{-\frac{\delta\theta}{\gamma} T} \epsilon^{\frac{1}{\gamma}} (yD_T)^{\frac{\gamma-1}{\gamma}} + \int_t^T \left[\delta^\psi \tfrac{\theta}{\gamma\psi} e^{-\delta\psi s} (yD_s)^{1-\psi} Y_s^{1-\frac{\gamma\psi}{\theta}}\right] ds - \int_t^T Z_s dB_s.
$$

(i) $\underline{\gamma > 1, 0 < \psi < 1, \text{ or } 0 < \gamma < 1, \psi > 1}$. Define $\mathcal{Y} = Y^{\frac{\gamma\psi}{\theta}}$ and $\mathcal{Z} = \frac{\gamma\psi}{\theta} Y^{\frac{\gamma\psi}{\theta}-1} Z$. Then $(\mathcal{Y}, \mathcal{Z})$ satisfies

$$
\begin{aligned}
\mathcal{Y}_t =& e^{-\delta\psi T} \epsilon^{\frac{\psi}{\theta}} (yD_T)^{1-\psi} + \int_t^T \left[\delta^\psi e^{-\delta\psi s} (yD_s)^{1-\psi} + \tfrac{1}{2}\left(\tfrac{\theta}{\gamma\psi} - 1\right) \tfrac{\mathcal{Z}_s^2}{\mathcal{Y}_s}\right] ds \\
&- \int_t^T \mathcal{Z}_s dB_s.
\end{aligned}
$$

This is exactly the type of BSDE studied in (Schroder and Skiadas, 1999, Equation (A7)). It then follows from (Schroder and Skiadas, 1999, Theorem A2) that the previous BSDE admits a unique solution $(\mathcal{Y}, \mathcal{Z})$ with $\mathbb{E}[\operatorname{ess\,sup}_t |\mathcal{Y}_t|^\ell] < \infty$ for any $\ell > 0$. (To treat the terminal condition $e^{-\delta\psi T} \epsilon^{\frac{\psi}{\gamma}} (yD_T)^{1-\psi}$, we consider an approximated terminal condition $\zeta + e^{-\delta\psi T} \epsilon^{\frac{\psi}{\theta}} (yD_T)^{1-\psi}$ with $\zeta > 0$ and its associated solution $(\mathcal{Y}^\zeta, \mathcal{Z}^\zeta)$. Proceed as in the proof of (Schroder and Skiadas, 1999, Theorem A2), $\mathcal{Y}$ is constructed as $\lim_{\zeta \downarrow 0} \mathcal{Y}^\zeta$). Coming back to (Y, Z), the statement in (i) is confirmed.

(ii) $\underline{\gamma, \psi > 1}$. Our assumption on D implies the integrability of $e^{-\frac{\delta\theta}{\gamma} T} \epsilon^{\frac{1}{\gamma}} (yD_T)^{\frac{\gamma-1}{\gamma}}$ and $\int_0^T e^{-\delta\psi s} (yD_s)^{1-\psi} ds$. Moreover, because $\gamma, \psi > 1$, we have $\theta < 0$, therefore the generator of (4.6) is decreasing in the Y-component. This is exactly the type of BSDEs studied in (Xing, 2017, Proposition 2.2). Then the statement in (ii) is confirmed following the proof of (Xing, 2017, Proposition 2.2).

4.3. **Proof of Lemma 3.1.** The statement for the primal problem is proved in Xing (2017), see the argument leading to equation (2.14) therein. In particular, because all investment opportunities are driven by W, it suffices to consider the martingale part of Y^p as a stochastic integral with respect to W.

Let us outline the argument for the primal problem. Parameterize c by $c = \bar{c}\mathcal{W}$ and suppose that

$$
dY^p_t = -H^p_t dt + Z^p_t dW_t,
$$

for some processes H^p and Z^p. Calculation shows

(4.7)
$$\begin{aligned} d\mathcal{W}_t^{1-\gamma} &= (1-\gamma)\mathcal{W}_t^{1-\gamma}\big[r_t - \overline{c}_t + \pi_t'\mu_t - \tfrac{\gamma}{2}\pi_t'\Sigma_t\pi_t\big]dt + (1-\gamma)\mathcal{W}_t^{1-\gamma}\pi_t'\sigma_t dW_t^{\rho},\\ de^{Y_t^p} &= e^{Y_t^p}\big[-H_t^p + \tfrac{1}{2}|Z_t^p|^2\big]dt + e^{Y_t^p}Z_t^p dW_t. \end{aligned}$$

Therefore the drift of $\frac{\mathcal{W}^{1-\gamma}}{1-\gamma}e^{Y^p} + \int_0^{\cdot} f(c_s, \frac{\mathcal{W}_s^{1-\gamma}}{1-\gamma}e^{Y_s^p})ds$ is (after suppressing the subscript t)

$$\begin{aligned} \tfrac{\mathcal{W}^{1-\gamma}}{1-\gamma}e^{Y^p}\Big\{&(1-\gamma)r - \delta\theta + \tfrac{1}{2}|Z^p|^2 + (1-\gamma)\big[-\overline{c} + \delta e^{-\frac{1}{\theta}Y^p}\tfrac{1}{1-\frac{1}{\psi}}\overline{c}^{1-\frac{1}{\psi}}\big]\\ &+ (1-\gamma)\big[-\tfrac{\gamma}{2}\pi'\Sigma\pi + \pi'(\mu + \sigma\rho(Z^p)')\big] - H^p\Big\}. \end{aligned}$$

The martingale principle then yields the previous drift to be nonpositive, leading to $H^p = H^p(Y_t^p, Z_t^p)$ with

$$\begin{aligned} H^p(y,z) =&(1-\gamma)r_t - \delta\theta + \tfrac{1}{2}|z|^2 + (1-\gamma)\sup_{\overline{c}}\Big[-\overline{c} + \delta e^{-\frac{1}{\theta}y}\tfrac{1}{1-\frac{1}{\psi}}\overline{c}^{1-\frac{1}{\psi}}\Big]\\ &+ (1-\gamma)\sup_{\pi}\Big[-\tfrac{\gamma}{2}\pi'\Sigma_t\pi + \pi'(\mu_t + \sigma_t\rho_t z')\Big], \end{aligned}$$

whose maximizer is obtained by calculation.

For the dual problem, suppose that

$$dY_t^d = -H_t^d dt + Z_t^d dW_t,$$

for some processes H^d and Z^d. Calculation shows

(4.8)
$$\begin{aligned} dD_t^{\frac{\gamma-1}{\gamma}} &= \tfrac{\gamma-1}{\gamma}D_t^{\frac{\gamma-1}{\gamma}}\big[-r_t - \tfrac{1}{2\gamma}(|\xi_t|^2 + |\eta_t|^2)\big]dt + \tfrac{\gamma-1}{\gamma}D_t^{\frac{\gamma-1}{\gamma}}(\xi_t dW_t + \eta_t dW_t^{\perp}),\\ de^{\frac{Y_t^d}{\gamma}} &= e^{\frac{Y_t^d}{\gamma}}\big[-\tfrac{1}{\gamma}H_t^d + \tfrac{1}{2\gamma^2}|Z_t^d|^2\big]dt + e^{\frac{Y_t^d}{\gamma}}\tfrac{Z_t^d}{\gamma}dW_t. \end{aligned}$$

Therefore the drift of $\frac{\gamma}{1-\gamma}(yD)^{\frac{\gamma-1}{\gamma}}e^{Y^d/\gamma} + \int_0^{\cdot} g\big(yD_s, \frac{1}{1-\gamma}(yD_s)^{\frac{\gamma-1}{\gamma}}e^{Y_s^d/\gamma}\big)ds$ is (after suppressing the subscript t)

$$\begin{aligned} \tfrac{1}{1-\gamma}(yD)^{\frac{\gamma-1}{\gamma}}e^{\frac{Y^d}{\gamma}}\Big\{&(1-\gamma)r - \delta\theta + \theta\tfrac{\delta^{\psi}}{\psi}e^{-\frac{\psi}{\theta}Y^d} + \tfrac{1}{2\gamma}|Z^d|^2\\ &+(1-\gamma)\big[\tfrac{1}{2\gamma}(|\xi|^2 + |\eta|^2) - \tfrac{1}{\gamma}\xi(Z^d)'\big] - H^d\Big\}. \end{aligned}$$

Then the martingale principle implies that the previous drift is nonnegative, leading to $H^d = H^d(Y^d, Z^d)$ with

$$\begin{aligned} H^d(y,z) =&(1-\gamma)r_t - \delta\theta + \theta\tfrac{\delta^{\psi}}{\psi}e^{-\frac{\psi}{\theta}y} + \tfrac{1}{2\gamma}|z|^2\\ &+ (1-\gamma)\inf_{\mu_t + \sigma_t\rho_t\xi' + \sigma_t\rho_t^{\perp}\eta' = 0}\Big[\tfrac{1}{2\gamma}(|\xi|^2 + |\eta|^2) - \tfrac{1}{\gamma}\xi z'\Big]. \end{aligned}$$

It then remains to obtain the minimizer for H^d. To this end, consider the unconstrained problem

$$\tfrac{1}{2\gamma}(|\xi|^2 + |\eta|^2) - \tfrac{1}{\gamma}\xi z' + \lambda\sigma\rho\xi' + \lambda\sigma\rho^{\perp}\eta'.$$

The first order condition yields

$$\xi = z - \gamma\lambda\sigma\rho \quad \text{and} \quad \eta = -\gamma\lambda\sigma\rho^{\perp}.$$

Plugging these optimizers into the constraint $\mu + \sigma\rho\xi' + \sigma\rho^{\perp}\eta' = 0$ yields the Lagrangian multiplier $\lambda = \frac{1}{\gamma}(\mu' + z\rho'\sigma')\Sigma^{-1}$, hence the form of minimizer $\xi(t,z)$ and $\eta(t,z)$.

Plugging $\pi(t,z)$, $\bar{c}(y)$ back into $H^p(y,z)$ and $\xi(t,z), \eta(t,z)$ back into $H^d(y,z)$, we obtain that both $H^p(y,z)$ and $H^d(y,z)$ are the same as $H(y,z)$ in (3.6). Therefore both (Y^p, Z^p) and (Y^d, Z^d) are solution to (3.5).

4.4. **Proof of Lemma 3.5.** Because $\mu'\Sigma^{-1}\mu$ is bounded, $|\mu'\Sigma^{-1}\sigma\rho|^2 \leq \mu'\Sigma^{-1}\mu$ implies that $\frac{1-\gamma}{\gamma}\mu'\Sigma^{-1}\sigma\rho$ is bounded as well. Therefore, $d\overline{\mathbb{P}}/d\mathbb{P} = \mathcal{E}(\int \frac{1-\gamma}{\gamma}\mu_s'\Sigma_s^{-1}\sigma_s\rho_s dW_s)_T$ defines a probability measure $\overline{\mathbb{P}}$ equivalent to $\mathbb{P}$, hence (3.5) can be rewritten as

$$Y_t = \log\epsilon + \int_t^T \mathcal{H}(Y_s, Z_s)ds - \int_t^T Z_s d\overline{W}_s, \tag{4.9}$$

where $\overline{W} = W - \int_0^\cdot \frac{1-\gamma}{\gamma}\rho_s'\sigma_s'\Sigma_s^{-1}\mu_s ds$ is a $\overline{\mathbb{P}}$-Brownian motion by the Girsanov theorem, and

$$\mathcal{H}(y,z) := \tfrac{1}{2}zM_tz' + \theta\tfrac{\delta^\psi}{\psi}e^{-\frac{\psi}{\theta}y} + h_t - \delta\theta.$$

Here because the eigenvalues of $\sigma'\Sigma^{-1}\sigma$ are either 0 or 1, we have $0 \leq z\rho'\sigma'\Sigma^{-1}\sigma\rho z' \leq z\rho'\rho z' \leq |z|^2$. This inequality implies that

$$\begin{aligned} &0 < |z|^2 \leq zM_tz' \leq \tfrac{1}{\gamma}|z|^2, \quad \text{when } 0<\gamma<1,\\ &0 < \tfrac{1}{\gamma}|z|^2 \leq zM_tz' \leq |z|^2, \quad \text{when } \gamma > 1. \end{aligned} \tag{4.10}$$

Therefore the z-term in $\mathcal{H}$ is positive and has quadratic growth. On the other hand, Assumption 3.4 implies that h is bounded. We denote $h_{\min} = \operatorname{ess\,inf}_{t\in[0,T]} h_t$ and $h_{\max} = \operatorname{ess\,sup}_{t\in[0,T]} h_t$. Due to the exponential term in y, we introduce a truncated version of (4.9)

$$Y_t^n = \log\epsilon + \int_t^T \mathcal{H}^n(Y_s^n, Z_s^n)ds - \int_t^T Z_s^n d\overline{W}_s, \quad \text{for } n > 0, \tag{4.11}$$

where the truncated generator

$$\mathcal{H}^n(y,z) := \tfrac{1}{2}zM_tz' + \theta\tfrac{\delta^\psi}{\psi}\left(e^{-\frac{\psi}{\theta}y}\wedge n\right) + h_t - \delta\theta$$

is Lipschitz in y, has quadratic growth in z, and $\mathcal{H}^n(0,0)$ is bounded. This is the quadratic BSDE studied in Kobylanski (2000) and Theorem 2.3 therein implies that (4.11) admits a solution (Y^n, Z^n) with Y^n bounded and $Z^n \in H^2(\overline{\mathbb{P}})$.

Note that $\{\gamma\psi\geq 1, \psi>1\} = \{0<\gamma<1, \gamma\psi\geq 1\}\cup\{\gamma,\psi>1\}$ and $\{\gamma\psi\leq 1, \psi<1\} = \{0<\gamma<1, \psi<1\}\cup\{\gamma>1, \gamma\psi\leq 1\}$. We split the following discussion into two cases.

Case $0<\gamma<1, \gamma\psi\geq 1$, or $\gamma>1, \gamma\psi\leq 1$: In this case $0<\theta\leq 1$, hence the second term in $\mathcal{H}^n$ is positive, therefore $\mathcal{H}^n(y,z)\geq h_{\min}-\delta\theta$ for all n. Comparison theorem for quadratic BSDE (cf. (Kobylanski, 2000, Theorem 2.6)) yields that $Y_t^n \geq \log\epsilon + (h_{\min}-\delta\theta)(T-t) \geq \log\epsilon - (h_{\min}-\delta\theta)_- T$, for all t and n, where $f_- = -\min\{f, 0\}$. As a result, $\exp(-\frac{\psi}{\theta}Y^n) \leq \epsilon^{-\psi/\theta}\exp(\frac{\psi}{\theta}(h_{\min}-\delta\theta)_- T)$ for all n. Take $N := \epsilon^{-\psi/\theta}\exp(\frac{\psi}{\theta}(h_{\min}-\delta)_- T)$. For any $n\geq N$, $\mathcal{H}(Y^n, Z^n) = \mathcal{H}^n(Y^n, Z^n)$, therefore, $(Y,Z) := (Y^n, Z^n)$ is a solution to (4.9).

Case $\gamma,\psi>1$, or $0<\gamma,\psi<1$: In this case $\theta<0$, hence the second term in $\mathcal{H}^n$ is negative. As a result, (4.10) implies $\mathcal{H}^n(y,z)\leq \frac{1}{2}\max\{1,\frac{1}{\gamma}\}|z|^2 + h_{\max} - \delta\theta$. Consider the BSDE

$$\overline{Y}_t^n = \log\epsilon + \int_t^T \Big[\tfrac{1}{2}\max\{1, \tfrac{1}{\gamma}\}|\overline{Z}_s^n|^2 + h_{\max} - \delta\theta\Big] ds - \int_t^T \overline{Z}_s^n d\overline{W}_s,$$

which has the solution $\overline{Y}_t^n = \log\epsilon + (h_{\max} - \delta\theta)(T-t)$ and $\overline{Z}_t^n = 0$. Then the comparison theorem for quadratic BSDE yields that $Y_t^n \leq \overline{Y}_t^n \leq \log\epsilon + (h_{\max} - \delta\theta)_+ T$, for all t and n, where $f_+ = \max\{f, 0\}$. As a result, $\theta < 0$ implies that $\exp(-\frac{\psi}{\theta}Y^n) \leq \epsilon^{-\psi/\theta}\exp(-\frac{\psi}{\theta}(h_{\max} - \delta\theta)_+ T)$ for all n. Take $N := \epsilon^{-\psi/\theta}\exp(-\frac{\psi}{\theta}(h_{\max} - \delta\theta)_+ T)$. For any $n \geq N$, $\mathcal{H}(Y^n, Z^n) = \mathcal{H}^n(Y^n, Z^n)$, therefore, $(Y, Z) := (Y^n, Z^n)$ is a solution to (4.9).

Finally, we will show $Z \in H_{\text{BMO}}$ in both cases. For any stopping time τ, (4.9) and $Z \in H^2(\overline{\mathbb{P}})$ imply

$$\tfrac{1}{2}\mathbb{E}_\tau^{\overline{\mathbb{P}}}\Big[\int_\tau^T Z_s M_s Z_s' ds\Big] = Y_\tau - \log\epsilon - \mathbb{E}_\tau^{\overline{\mathbb{P}}}\Big[\int_\tau^T \theta\tfrac{\delta^\psi}{\psi}e^{-\frac{\psi}{\theta}Y_s} + h_s - \delta\theta\, ds\Big].$$

Because Y and h are bounded, the right-hand side of the previous identity is bounded by some constant C, which does not depend on τ. Therefore $\mathbb{E}_\tau[\int_\tau^T Z_s M_s Z_s' ds] \leq 2C$ for any stopping time τ. Combining the previous inequality with (4.10), we confirm $Z \in H_{\text{BMO}}(\overline{\mathbb{P}})$. As $\mu'\Sigma^{-1}\sigma\rho$ is bounded, hence it belongs to $H_{\text{BMO}}(\mathbb{P})$. It then follows from (Kazamaki, 1994, Theorem 3.6) that $Z \in H_{\text{BMO}}(\mathbb{P})$.

4.5. **Proof of Theorem 3.6.** For any solution (Y, Z) of (3.5) with bounded Y, and π^*, c^*, D^* defined in (3.9), let us define

$$\mathbb{U}_t^* = \tfrac{1}{1-\gamma}(\mathcal{W}_t^*)^{1-\gamma}e^{Y_t} \quad\text{and}\quad \mathbb{V}_t^{y,*} = \tfrac{\gamma}{1-\gamma}(yD_t^*)^{\frac{\gamma-1}{\gamma}}e^{Y_t/\gamma}, \tag{4.12}$$

where $\mathcal{W}^* = \mathcal{W}^{(\pi^*, c^*)}$. It is clear that $(1-\gamma)\mathbb{U}^* > 0$ and $(1-\gamma)\mathbb{V}^{y,*} > 0$. We will prove $\mathbb{U}^*, \mathbb{V}^{y,*}$ are of class (D), and

$$\mathbb{U}_t^* = \mathbb{E}_t\Big[\int_t^T f(c_s^*, \mathbb{U}_s^*)ds + U_T(\mathcal{W}_T^*)\Big], \tag{4.13}$$

$$\mathbb{V}_t^{y,*} = \mathbb{E}_t\Big[\int_t^T g(yD_s^*, \tfrac{1}{\gamma}\mathbb{V}_s^{y,*})ds + V_T(yD_T^*)\Big], \tag{4.14}$$

for any $y > 0$ and $t \in [0, T]$. Therefore $(\pi^*, c^*) \in \mathcal{A}$ and $D^* \in \mathcal{D}_a$. Take $y = y^* = w^{-\gamma}e^{Y_0}$ and denote $\mathbb{V}^* = \mathbb{V}^{y^*,*}$. We have from $\mathcal{W}_0^* = w$ and $D_0^* = 1$ that

$$\mathbb{U}_0^* = \tfrac{1}{1-\gamma}w^{1-\gamma}e^{Y_0} = \tfrac{\gamma}{1-\gamma}(y^*)^{\frac{\gamma-1}{\gamma}}e^{Y_0/\gamma} + wy^* = \mathbb{V}_0^* + wy^* = \inf_{y>0}(\mathbb{V}_0^{y,*} + wy).$$

Combining this identity with (2.15), (3.10) is confirmed.

<u>$\mathbb{U}^*$ is of class (D) and it satisfies (4.13):</u> Using (4.7), where (Y^p, Z^p) is replaced by (Y, Z), H from (3.8), and (π^*, c^*) from (3.9), we obtain

(4.15)

$$\begin{aligned}
&d\,(\mathcal{W}_t^*)^{1-\gamma}e^{Y_t} \\
&= -(\mathcal{W}_t^*)^{1-\gamma}e^{Y_t}\Big(\delta\theta\,(c_s^*)^{1-\frac{1}{\psi}}\,\big((\mathcal{W}_t^*)^{1-\gamma}e^{Y_t}\big)^{-\frac{1}{\theta}} - \delta\theta\Big)dt \\
&\quad + (\mathcal{W}_t^*)^{1-\gamma}e^{Y_t}\,[(1-\gamma)(\pi_t^*)'\sigma_t dW_t^\rho + Z_t dW_t] \\
&= -(\mathcal{W}_t^*)^{1-\gamma}e^{Y_t}\big(\theta\delta^\psi e^{-\frac{\psi}{\theta}Y_t} - \delta\theta\big)dt + (\mathcal{W}_t^*)^{1-\gamma}e^{Y_t}\,[(1-\gamma)(\pi_t^*)'\sigma_t dW_t^\rho + Z_t dW_t]\,.
\end{aligned}$$

This implies

$$(\mathcal{W}_t^*)^{1-\gamma}\, e^{Y_t} = w^{1-\gamma} e^{Y_0} \exp\Big(-\int_0^t \big(\delta^\psi \theta e^{-\frac{\psi}{\theta}Y_s} - \delta\theta\big) ds\Big) Q_t, \tag{4.16}$$

where

(4.17)

$$\begin{aligned} Q_t &= \mathcal{E}\Big(\int (1-\gamma)(\pi_s^*)'\sigma_s dW_s^\rho + \int Z_s dW_s\Big)_t = \mathcal{E}\Big(\int L_s dW_s + \int L_s^\perp dW_s\Big)_t,\\ L &= \tfrac{1-\gamma}{\gamma}\mu'\Sigma^{-1}\sigma\rho + ZM, \quad L^\perp = \tfrac{1-\gamma}{\gamma}(\mu' + Z\rho'\sigma')\Sigma^{-1}\sigma\rho^\perp. \end{aligned}$$

Because Y is bounded, the first three terms on the right-hand side of (4.16) are bounded uniformly for $t \in [0,T]$. For the exponential local martingale Q, note that $\mu'\Sigma^{-1}\sigma\rho\rho'\sigma'\Sigma^{-1}\mu \leq \mu'\Sigma^{-1}\mu$ and $ZMM'Z' \leq 2[1 + (\frac{1-\gamma}{\gamma})^2]|Z|^2$. The boundedness of $\mu'\Sigma^{-1}\mu$ and $Z \in H_{\text{BMO}}$ imply $L \in H_{\text{BMO}}$ as well. A similar argument yields $L^\perp \in H_{\text{BMO}}$. It then follows from (Kazamaki, 1994, Theorem 2.3) that Q is a martingale, hence is of class (D). Coming back to (4.16), we have confirmed that $(\mathcal{W}^*)^{1-\gamma}e^Y$ is of class (D), and so is $\mathbb{U}^*$.

To verify (4.13), (4.15) shows that $\mathbb{U}^* + \int_0^\cdot f(c_s^*, \mathbb{U}_s^*)ds$ is a local martingale. Taking a localizing sequence $(\sigma_n)_{n\geq 1}$, we obtain

$$\mathbb{U}_t^* + \delta\theta\mathbb{E}_t\Big[\int_t^{T\wedge\sigma_n} \mathbb{U}_s^* ds\Big] = \mathbb{E}_t\Big[\mathbb{U}_{T\wedge\sigma_n}^* + \int_t^{T\wedge\sigma_n} \delta\frac{(c_s^*)^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)\mathbb{U}_s^*)^{1-\frac{1}{\theta}} ds\Big],$$

on $\{t < \sigma_n\}$. Sending $n \to \infty$, the monotone convergence theorem and the class (D) property of $\mathbb{U}^*$ yield

$$\mathbb{U}_t^* + \delta\theta\mathbb{E}_t\Big[\int_t^T \mathbb{U}_s^* ds\Big] = \mathbb{E}_t\Big[U_T(\mathcal{W}_T^*) + \int_t^T \delta\frac{(c_s^*)^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)\mathbb{U}_s^*)^{1-\frac{1}{\theta}} ds\Big].$$

The class (D) property of $\mathbb{U}^*$ ensures $\delta\theta\mathbb{E}_t[\int_t^T \mathbb{U}_s^* ds] < \infty$ a.s.. Subtracting it from both sides of the previous equation, (4.13) is confirmed.

<u>$\mathbb{V}^{y,*}$ is of class (D) and satisfies (4.14):</u> Using (4.8) together with D^* from (3.9), where (Y^d, Z^d) is replaced by (Y, Z), we obtain

$$\begin{aligned} d(D_t^*)^{\frac{\gamma-1}{\gamma}} e^{\frac{Y_t}{\gamma}} &= -\tfrac{\theta}{\gamma\psi}\delta^\psi (D_t^*)^{\frac{\gamma-1}{\gamma}} e^{(1-\frac{\gamma\psi}{\theta})\frac{Y_t}{\gamma}} dt + \tfrac{\delta\theta}{\gamma}(D_t^*)^{\frac{\gamma-1}{\gamma}} e^{\frac{Y_t}{\gamma}} dt\\ &\quad + (D_t^*)^{\frac{\gamma-1}{\gamma}} e^{\frac{Y_t}{\gamma}}[(1-\gamma)(\pi_t^*)'\sigma_t dW_t^\rho + Z_t dW_t] \end{aligned}$$

The previous SDE for $(D^*)^{\frac{\gamma-1}{\gamma}} e^{Y_t/\gamma}$ has the following solution

$$(D_t^*)^{\frac{\gamma-1}{\gamma}} e^{\frac{Y_t}{\gamma}} = e^{\frac{Y_0}{\gamma}} \exp\Big(-\tfrac{\theta}{\gamma\psi}\delta^\psi \int_0^t e^{-\frac{\psi}{\theta}Y_s} ds + \tfrac{\delta\theta}{\gamma}t\Big) Q_t, \tag{4.18}$$

where Q_t comes from (4.17). Because Y is bounded, the second term on the right-hand side is bounded uniformly for $t \in [0,T]$. Therefore the class (D) property of Q implies the same property of $(D^*)^{\frac{\gamma-1}{\gamma}} e^{Y/\gamma}$ and $\mathbb{V}^{y,*}$. The discussion after (4.8) and the optimality of ξ^* and η^* imply that $\mathbb{V}^{y*} + \int_0^\cdot g(yD_s^*, \frac{1}{\gamma}\mathbb{V}_s^{y*})ds$ is a local martingale. A similar localization argument as the previous step confirms (4.14).

Remark 4.1. A careful examination reveals that the previous proof only requires $-Y/\theta$ to be bounded from above and Q to be a martingale. Indeed, when $-Y/\theta$ is bounded from above, both the third term on the right-hand side of (4.16) and the second term on the right-hand side of (4.18) are bounded. Combining with the class (D) property of Q, we reach the same conclusion. We record this observation here for future reference.

4.6. **Proof of Corollary 3.7.** We will prove that D^* given in (3.11) satisfies the SDE of D^* in (3.9). As this SDE clearly admits an unique solution, this unique solution must be given by (3.11).

Denote $\mathcal{W}^{\pi^*,c^*}$ by $\mathcal{W}^*$ and U^{c^*} by U^*. Calculation using (2.1), (3.9) and (4.12) shows that

$$\begin{aligned}D^*_t =&w^\gamma e^{-Y_0}\exp\Big[\int_0^t \delta(\theta-1)((1-\gamma)U^*_s)^{-\frac{1}{\theta}}(c^*_s)^{1-\frac{1}{\psi}}ds-\delta\theta t\Big]\\ &\cdot\delta((1-\gamma)U^*_t)^{1-\frac{1}{\theta}}(c^*_t)^{-\frac{1}{\psi}}\\ =&\exp\Big[\int_0^t(\theta-1)\delta^\psi e^{-\frac{\psi}{\theta}Y_s}ds-\delta\theta t\Big]\frac{(\mathcal{W}^*_t)^{-\gamma}e^{Y_t}}{w^{-\gamma}e^{Y_0}}.\end{aligned}$$

On the other hand, set $\overline{c}^* = c^*/\mathcal{W}^*$. Calculation using (3.5) and (3.9) yields

$$\begin{aligned}d(\mathcal{W}^*)^{-\gamma} =&(\mathcal{W}^*)^{-\gamma}\Big[-\gamma(r-\overline{c}^*+(\pi^*)'\mu)+\tfrac{\gamma(\gamma+1)}{2}(\pi^*)'\Sigma\pi^*\Big]dt\\ &-\gamma(\mathcal{W}^*)^{-\gamma}(\pi^*)'\sigma dW^\rho\\ =&(\mathcal{W}^*)^{-\gamma}\Big[-\gamma(r-\tilde{c}^*)+\tfrac{1-\gamma}{2\gamma}\mu'\Sigma^{-1}\mu+\tfrac{1}{\gamma}\mu'\Sigma^{-1}\sigma\rho Z'\\ &+\tfrac{1+\gamma}{2\gamma}Z\rho'\sigma'\Sigma^{-1}\sigma\rho Z'\Big]dt-\gamma(\mathcal{W}^*)^{-\gamma}(\pi^*)'\sigma dW^\rho\\ de^Y =&e^Y\Big[-H(Y,Z)+\tfrac12 ZZ'\Big]dt+e^YZdW.\end{aligned}$$

Combining the previous three identities, we confirm

$$\begin{aligned}dD^* =&D^*\Big[-\gamma(r-\overline{c}^*)+(\theta-1)\delta^\psi e^{-\frac{\psi}{\theta}Y}-\delta\theta\\ &+\tfrac{1-\gamma}{\gamma}\mu'\Sigma^{-1}\mu+\tfrac{1-\gamma}{\gamma}\mu'\Sigma^{-1}\sigma\rho Z'+\tfrac12 ZMZ'-H(t,Y,Z)\Big]dt\\ &+D^*[-\gamma(\pi^*)'\sigma dW^\rho+ZdW]\\ =&D^*\Big[-r+\big(\theta-1-\tfrac{\theta}{\psi}+\gamma\big)\delta^\psi e^{-\frac{\psi}{\theta}Y}\Big]dt+D^*\left[-\gamma(\pi^*)'\sigma dW^\rho+ZdW\right]\\ =&-rD^*dt+D^*\left[-\gamma(\pi^*)'\sigma dW^\rho+ZdW\right],\end{aligned}$$

where the third identity follows from $\theta+\gamma-1-\frac{\theta}{\psi}=0$.

For the second statement, when (3.10) holds, the first inequality in (2.11) must be an identity. Hence $\mathbb{E}\big[\mathcal{W}^*_TD^*_T+\int_0^T D^*_sc^*_sds\big]=w$, which implies the martingale property of $D^*\mathcal{W}^*+\int_0^\cdot D^*_sc^*_sds$, because this process is already a supermartingale.

4.7. **Proof of Theorem 3.12.** For any solution (Y,Z) of (3.5) with Y bounded from above, we have $-Y/\theta$ bounded from above. On the other hand, (Xing, 2017, Lemma B.2) proved that Q from (4.17) is a martingale when Assumptions 3.8, 3.9, 3.11 hold. Therefore the statement readily follows from Remark 4.1.

4.8. **Proof of Proposition 3.16.** This proof is similar to (Xing, 2017, Proposition 3.2), whose Assumption 2.11 is no longer needed here, see Remark 3.14.

Assumption 3.8: One only needs to check $r(x)+\frac{1}{2\gamma}\mu(x)'\Sigma(x)^{-1}\mu(x)$ is bounded from below on $(0,\infty)$. To this end,

$$r(x)+\tfrac{1}{2\gamma}\mu(x)'\Sigma(x)^{-1}\mu(x) = r_0+\tfrac{1}{\gamma}\lambda_0'\Theta(x)\lambda_1+\tfrac{1}{2\gamma}\lambda_0'\Theta(x)\lambda_0\frac{1}{x}+(r_1+\tfrac{1}{2\gamma}\lambda_1'\Theta(x)\lambda_1)x.$$

Note that Θ is bounded on $(0,\infty)$, because its eigenvalues are either 0 or 1. Therefore $r(x)+\frac{1}{2\gamma}\mu(x)'\Sigma(x)^{-1}\mu(x)$ is bounded from below on $(0,\infty)$ thanks to condition (i).

Assumption 3.9: Note $\frac{1-\gamma}{\gamma}\mu(x)'\Sigma(x)^{-1}\sigma(x)\rho = \frac{1-\gamma}{\gamma}\big(\frac{\lambda_0}{\sqrt{x}}+\lambda_1\sqrt{x}\big)'\Theta(x)\rho$. Consider the martingale problem associated to

$$\overline{\mathcal{L}} := \Big[b\ell+\tfrac{1-\gamma}{\gamma}a\lambda_0'\Theta(x)\rho-\Big(b-\tfrac{1-\gamma}{\gamma}a\lambda_1'\Theta(x)\rho\Big)x\Big]\partial_x+\tfrac{1}{2}a^2x\partial_x^2, \quad \text{on } (0,\infty).$$

Because $\Theta(x)$ is bounded and $b\ell+\inf_{x>0}\frac{1-\gamma}{\gamma}a\lambda_0'\Theta(x)\rho > \frac{1}{2}a^2$ in condition (iii), Feller's test of explosion implies that the previous martingale problem is well-posed. Then (Cheridito et al., 2005, Remark 2.6) implies that the stochastic exponential in Assumption 3.9 (i) is a $\mathbb{P}-$martingale, hence $\overline{\mathbb{P}}$ is well defined.

For Assumption 3.9 (ii),

(4.19)
$$h(x) = (1-\gamma)r_0+\tfrac{1-\gamma}{\gamma}\lambda_0'\Theta(x)\lambda_1+\tfrac{1-\gamma}{2\gamma}\lambda_0'\Theta(x)\lambda_0\tfrac{1}{x}+\big[(1-\gamma)r_1+\tfrac{1-\gamma}{2\gamma}\lambda_1'\Theta(x)\lambda_1\big]x.$$

Note that X and $\tilde{X}=X^{-1}$ have the following dynamics under $\overline{\mathbb{P}}$:

$$\begin{aligned}
dX_t &= \Big[b\ell+\tfrac{1-\gamma}{\gamma}a\lambda_0'\Theta(x)\rho-\Big(b-\tfrac{1-\gamma}{\gamma}a\lambda_1'\Theta\rho\Big)X_t\Big]+a\sqrt{X_t}d\overline{W}_t,\\
d\tilde{X}_t &= \Big[\big(b-\tfrac{1-\gamma}{\gamma}a\lambda_1'\Theta\rho\big)\tilde{X}_t-\big(b\ell+\tfrac{1-\gamma}{\gamma}a\lambda_0'\Theta\rho-a^2\big)\tilde{X}_t^2\Big]dt-a\tilde{X}^{3/2}d\bar{W}_t,
\end{aligned}$$

where $\overline{W}$ is a $\overline{\mathbb{P}}-$Brownian motion. As Θ is bounded, one has $\mathbb{E}^{\overline{\mathbb{P}}}[\int_0^T X_s ds]<\infty$. On the other hand, note that $-\big(b\ell+\inf_{x>0}\frac{1-\gamma}{\gamma}a\lambda_0'\Theta(x)\rho-a^2\big)<\frac{1}{2}a^2$ follows from $b\ell+\inf_{x>0}\frac{1-\gamma}{\gamma}a\lambda_0'\Theta(x)\rho>\frac{1}{2}a^2$. It follows from (Carr and Sun, 2007, Theorem 4 and Appendix 4) and the comparison theorem for SDE that $\mathbb{E}^{\overline{\mathbb{P}}}[\int_0^T \tilde{X}_s ds]<\infty$. As a result, $\mathbb{E}^{\overline{\mathbb{P}}}[\int_0^T h(X_s)ds]>-\infty$ follows from (4.19) and $\gamma>1$.

Assumption 3.11: The operator $\mathfrak{F}$ in (3.14) reads

$$\begin{aligned}
\mathfrak{F}[\phi](x) =&\tfrac{1}{2}a^2x\partial_x^2\phi+\Big(b\ell+\tfrac{1-\gamma}{\gamma}a\lambda_0'\Theta(x)\rho-\Big(b-\tfrac{1-\gamma}{\gamma}a\lambda_1'\Theta(x)\rho\Big)x\Big)\partial_x\phi\\
&+\tfrac{1}{2}Ma^2x(\partial_x\phi)^2+(1-\gamma)r_0+\tfrac{1-\gamma}{\gamma}\lambda_0'\Theta(x)\lambda_1+\tfrac{1-\gamma}{2\gamma}\lambda_0'\Theta(x)\lambda_0\tfrac{1}{x}\\
&+(1-\gamma)\Big(r_1+\tfrac{1}{2\gamma}\lambda_1'\Theta(x)\lambda_1\Big)x,
\end{aligned}$$

where $M=1+\frac{1-\gamma}{\gamma}\rho'\Theta(x)\rho>0$ because $\rho'\Theta\rho\leq 1$. Consider

$$\phi(x) = -\underline{c}\log x+\overline{c}x,$$

for two positive constants $\underline{c}$ and $\overline{c}$, which will be determined later. It is clear that $\phi(x)\uparrow\infty$ when $x\downarrow 0$ or $x\uparrow\infty$. On the other hand, calculation shows

$$\begin{aligned}
\mathfrak{F}[\phi](x) =&C(x)+\Big[\tfrac{1}{2}a^2\underline{c}+\tfrac{1}{2}a^2\underline{c}^2M-\underline{c}\big(b\ell+\tfrac{1-\gamma}{\gamma}a\lambda_0'\Theta(x)\rho\big)+\tfrac{1-\gamma}{2\gamma}\lambda_0'\Theta(x)\lambda_0\Big]\tfrac{1}{x}\\
&+\Big[-\Big(b-\tfrac{1-\gamma}{\gamma}a\lambda_1'\Theta(x)\rho\Big)\overline{c}+\tfrac{1}{2}a^2\overline{c}^2M+(1-\gamma)r_1+\tfrac{1-\gamma}{2\gamma}\lambda_1'\Theta(x)\lambda_1\Big]x,
\end{aligned}$$

where C is a bounded function. Because $b\ell+\inf_{x>0}\frac{1-\gamma}{\gamma}a\lambda_0'\Theta(x)\rho>\frac{1}{2}a^2$ and $\frac{1-\gamma}{2\gamma}\lambda_0'\Theta(x)\lambda_0\leq 0$, the coefficient of $\frac{1}{x}$ is negative for sufficiently small $\underline{c}$. When r_1 or $\inf_{x>0}\lambda_1'\Theta(x)\lambda_1>0$, because $\gamma>1$ and $\Theta(x)$ is bounded, the coefficient of

x is negative for sufficiently small $\overline{c}$. Therefore, these choices of $\underline{c}$ and $\overline{c}$ imply that $\mathfrak{F}[\phi](x) \downarrow -\infty$ when $x \downarrow 0$ or $x \uparrow \infty$, hence $\mathfrak{F}[\phi]$ is bounded from above on $(0, \infty)$, verifying Assumption 3.11.

4.9. **Proof of Proposition 3.18.** Note that

$$r(x) + \tfrac{1}{2\gamma}\mu(x)'\Sigma^{-1}\tfrac{1}{x}\mu(x) = r_0 + (r_1 + \tfrac{1}{2\gamma}\lambda'\Theta\lambda)x.$$

Therefore Assumption 3.8 follows from condition (i). For Assumption 3.9, $\frac{1-\gamma}{\gamma}\mu(x)'\Sigma^{-1}\frac{1}{x}\sigma\sqrt{x}\rho = \frac{1-\gamma}{\gamma}\lambda'\Theta\rho\sqrt{x}$. Consider the martingale problem associated to

$$\overline{\mathcal{L}} := \Big[px + \Big(q + \tfrac{1-\gamma}{\gamma}a\lambda'\Theta\rho\Big)x^2\Big]\partial_x + \tfrac{1}{2}a^2x^3\partial_x^2, \quad \text{on } (0, \infty).$$

Due to $q + \frac{1-\gamma}{\gamma}a\lambda'\Theta\rho < \frac{1}{2}a^2$ from condition (iii), Feller's test of explosion ensures that the previous martingale problem is well-posed. Then the same argument as the previous subsection ensures that $\overline{\mathbb{P}}$ is well defined. On the other hand,

$$h(x) = (1-\gamma)r_0 + \big[(1-\gamma)r_1 + \tfrac{1-\gamma}{2\gamma}\lambda'\Theta\lambda\big]x.$$

Then $\mathbb{E}^{\overline{\mathbb{P}}}[\int_0^T h(X_s)ds] > -\infty$ follows from $\gamma > 1$ and $\mathbb{E}^{\overline{\mathbb{P}}}[\int_0^T X_s ds] < \infty$ from (Carr and Sun, 2007, Theorem 4 and Appendix 4).

Comparing to the previous subsection, the main difference here is the choice of Lyapunov function ϕ. To this end,

$$\begin{aligned}\mathfrak{F}[\phi](x) =& \tfrac{1}{2}a^2x^3\partial_x^2\phi + \Big(px + \Big(q + \tfrac{1-\gamma}{\gamma}a\lambda'\Theta\rho\Big)x^2\Big)\partial_x\phi + \tfrac{1}{2}Ma^2x^3(\partial_x\phi)^2\\ &+ (1-\gamma)r_0 + (1-\gamma)[r_1 + \tfrac{1-\gamma}{2\gamma}\lambda'\Theta\lambda]x,\end{aligned}$$

where $M = 1 + \frac{1-\gamma}{\gamma}\rho'\Theta\rho > 0$. Consider $\phi \in C^2(0,\infty)$ such that

$$\phi(x) = -\underline{c}\log x \quad \text{for } 0 < x < 1 \quad \text{and} \quad \phi(x) = \overline{c}\log x \quad \text{for } x > 2,$$

where $\underline{c}$ and $\overline{c}$ are two positive constants to be determined later. Clearly $\phi(x) \uparrow \infty$ when $x \downarrow 0$ or $x \uparrow \infty$. When $0 < x < 1$, calculation shows

$$\mathfrak{F}[\phi](x) = C + \Big[\tfrac{1}{2}a^2\underline{c} - \Big(q + \tfrac{1-\gamma}{\gamma}a\lambda'\Theta\rho\Big)\underline{c} + \tfrac{1}{2}a^2M\underline{c}^2 + (1-\gamma)r_1 + \tfrac{1-\gamma}{2\gamma}\lambda'\Theta\lambda\Big]x,$$

for some constant C. Therefore, for any $\overline{c} > 0$, $\lim_{x\downarrow 0}\mathfrak{F}[\phi](x)$ is bounded from above. On the other hand, when $x > 2$,

$$\mathfrak{F}[\phi](x) = C + \Big[-\tfrac{1}{2}a^2\overline{c} + \Big(q + \tfrac{1-\gamma}{\gamma}a\lambda'\Theta\rho\Big)\overline{c} + \tfrac{1}{2}a^2M\overline{c}^2 + (1-\gamma)r_1 + \tfrac{1-\gamma}{2\gamma}\lambda'\Theta\lambda\Big]x,$$

for some constant C. When either r_1 or $\lambda'\Theta\lambda$ is strictly positive, the coefficient of x is negative for sufficiently small $\overline{c}$. Combining the previous two cases, we confirm that $\mathfrak{F}[\phi]$ is bounded from above on $(0, \infty)$, hence verify Assumption 3.11.

APPENDIX A. BASIC FACTS

Direct calculations specify various functions introduced in Section 2.3.

LEMMA A.1 *The Epstein-Zin aggregator f in* (2.1)*, and its double conjugates F, G, g, defined in* (2.6), (2.10), (2.13)*, respectively, satisfy the following statements.*

(i) $f(c,u)$ is concave in c. For $c, (1-\gamma)u > 0$,

$$-f_u(c,u) = \delta(1-\theta)c^{1-\frac{1}{\psi}}((1-\gamma)u)^{-\frac{1}{\theta}} + \delta\theta,$$

$$f_{uu}(c,u) = \delta\frac{\gamma\psi - 1}{\psi}c^{1-\frac{1}{\psi}}((1-\gamma)u)^{-\frac{1}{\theta}-1}.$$

Therefore $f(c,u)$ is convex in u if and only if $\gamma\psi \geq 1$.

(ii) When $\gamma\psi \neq 1$,

$$F(c,\nu) = \delta^\theta \frac{c^{1-\gamma}}{1-\gamma}\left(\frac{\delta\theta-\nu}{\theta-1}\right)^{1-\theta}, \quad \textit{for } c>0,\ \tfrac{\delta\theta-\nu}{\theta-1}>0,$$

$$F_{cc}(c,\nu) = -\delta^\theta \gamma c^{-\gamma-1}\left(\frac{\delta\theta-\nu}{\theta-1}\right)^{1-\theta},$$

$$F_{\nu\nu}(c,\nu) = \delta^\theta \frac{\psi}{1-\gamma\psi} c^{1-\gamma}\left(\frac{\delta\theta-\nu}{\theta-1}\right)^{-1-\theta}.$$

Therefore $F(c,\nu)$ is concave in c. Moreover $F(c,\nu)$ is concave in ν if and only if $\gamma\psi > 1$.

(iii) When $\gamma\psi \neq 1$,

$$G(d,\nu) = \delta^{\frac{\theta}{\gamma}} \frac{\gamma}{1-\gamma} d^{\frac{\gamma-1}{\gamma}}\left(\frac{\delta\theta-\nu}{\theta-1}\right)^{\frac{1-\theta}{\gamma}}, \quad \textit{for } d>0,\ \tfrac{\delta\theta-\nu}{\theta-1}>0,$$

$$G_{dd}(d,\nu) = \delta^{\frac{\delta}{\gamma}} \frac{1}{\gamma} d^{-\frac{1}{\gamma}-1}\left(\frac{\delta\theta-\nu}{\theta-1}\right)^{\frac{1-\theta}{\gamma}},$$

$$G_{\nu\nu}(d,\nu) = \delta^{\frac{\theta}{\gamma}} \frac{1}{\gamma(1-\gamma\psi)} d^{\frac{\gamma-1}{\gamma}}\left(\frac{\delta\theta-\nu}{\theta-1}\right)^{\frac{1-\theta}{\gamma}-2}.$$

Therefore $G(d,\nu)$ is convex in d. Moreover $G(d,\nu)$ is concave in ν if and only if $\gamma\psi > 1$.

(iv)

$$g(d,v) = \delta^\psi \frac{d^{1-\psi}}{\psi-1}((1-\gamma)v)^{1-\frac{\gamma\psi}{\theta}} - \delta\theta v, \quad d>0, (1-\gamma)v>0,$$

$$-g_v(d,v) = \delta^\psi (1-\theta) d^{1-\psi}((1-\gamma)v)^{-\frac{\gamma\psi}{\theta}} + \delta\theta,$$

$$g_{vv}(d,v) = \delta^\psi \gamma(\gamma\psi-1) d^{1-\psi}((1-\gamma)v)^{-\frac{\gamma\psi}{\theta}-1}.$$

Therefore $g(d,v)$ is convex in d. Moreover $g(d,\nu)$ is convex in v if and only if $\gamma\psi \geq 1$.

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