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Highlights

- Halting probabilities.
- Ratio of convergence.
- Restricted halting probabilities.
- Left computably enumerable reals.
- Approximations to Martin-Loef random reals.

Differences of halting probabilities *

George Barmpalias

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June 12, 2017

Abstract. We study the differences of Martin-Löf random left-c.e. reals and show that for each pair of such reals α, β there exists a unique number $r > 0$ such that $q\alpha - \beta$ is a Martin-Löf random left-c.e. real for each positive rational $q > r$ and a Martin-Löf random right-c.e. real for each positive rational $q < r$. Based on this result we develop a theory of differences of halting probabilities, which answers a number of questions about Martin-Löf random left-c.e. reals, including one of the few remaining open problems from the list of open questions in algorithmic randomness [MN06]. The halting probability of a prefix-free machine M restricted to a set X is the probability that the machine halts and outputs an element of X . Becher, Figueira, Grigorieff, and Miller asked whether $\Omega_U(X)$ is Martin-Löf random when U is universal and X is a Π_1^0 set. We apply our theory of differences of halting probabilities to give a positive answer.

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1 Introduction

Perhaps the most recognizable algorithmically random number is Chaitin's Ω number. This is the probability that a universal prefix-free machine U halts when we feed the input with successive bits (zeros and ones) until a computation converges, and is usually denoted by Ω_U . The fact that the underlying machine has a prefix-free domain allows for this number to be defined with the following rather simple formula:

$$\Omega_U = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}. \quad (1)$$

Prefix-free machines are Turing machines whose domain is a prefix-free subset of the finite binary strings, so that they operate instantaneous codes (i.e. uniquely decodable without out-of-band markers). Such machines are essentially equivalent to self-delimiting machines, i.e. Turing machines with a one-way input and one-way output tape, such that a convergent computation on a finite binary input σ cannot be extended to a different computation on an input which extends σ . Chaitin [Cha75] used these machines in order to give a definition of randomness for infinite sequences in terms of incompressibility, and showed that for each universal prefix-free machine U the number Ω_U is algorithmically random, in the sense that its binary expansion is a random sequence according to the definition of Martin-Löf [ML66]. The probability that U halts and outputs a string in a set X of binary strings is:

$$\Omega_U(X) = \sum_{U(\sigma)\downarrow \in X} 2^{-|\sigma|}. \quad (2)$$

Chaitin [Cha04] observed that if X is a computably enumerable set, then $\Omega_U(X)$ is Martin-Löf random. The question as to whether there is a set X such that its complement is computably enumerable (i.e. X is a Π_1^0 set) and $\Omega_U(X)$ is *not* Martin-Löf random was asked and discussed by a number of authors [BG05, BFGM06, FSW06], occasionally along with partial solutions. It is also one of the last remaining problems (Question 8.10) in the list of open problems in algorithmic randomness by Miller and Nies [MN06]. There is a considerable background and an original motivation surrounding this question of restricted halting probability, which we defer to Section 1.3 in order to focus on our present contribution.

Problem 1.1 (Question 8.10 in Miller and Nies [MN06]). *If U is a universal machine and $X \neq \emptyset$ is a Π_1^0 set, is the probability $\Omega_U(X)$ always a Martin-Löf random number?*

This open problem was the starting point for our investigations, which led to the study of more general questions concerning the differences of Ω numbers, and revealed a missing theory which is complementary to the well developed theory of halting probabilities (see Downey and Hirschfeldt [DH10, Chapter 9] for an overview). Before we present our solution and, perhaps more interestingly, the intriguing theory of differences of halting probabilities that it inspired, we make our discussion precise by giving a formal definition of universality. Informally, a Turing machine is universal if it can simulate any other Turing machine.

Definition 1.2 (Universal prefix-free machines). *Given an effective list (M_e) of all prefix-free machines, a prefix-free machine U is universal if there exists a computable function $e \mapsto \sigma_e$ such that, for all $\tau \in 2^{<\omega}$, $U(\sigma_e * \tau) \simeq M_e(\tau)$.*

As usual, the symbol \simeq denotes that either both sides of the relation are defined and equal, or both sides are undefined. In Kolmogorov complexity theory, universal machines are sometimes called *universal by adjunction* in order to distinguish them from a wider class of machines, the *optimal prefix-free machines*.

The latter class consists of the prefix-free machines with respect to which the Kolmogorov complexity function is minimal modulo an additive constant, within the class of all prefix-free machines. In their attempt at Problem 1.1, Figueira, Stephan, and Wu [FSW06] constructed a special optimal prefix-free machine U and a Π_1^0 set X such that $\Omega_U(X)$ is not Martin-Löf random. However their machine is not universal and, as will become clear in the following, this approach has little to do with a solution to the problem. Becher, Figueira, Grigorieff, and Miller showed in [BG05] that, given a universal prefix-free machine U , there is a Δ_2^0 set X such that $\Omega_U(X)$ is not Martin-Löf random.

1.1 Solution to the problem of restricted halting probability

Despite these negative results, we give a rather surprising positive answer to Problem 1.1. Recall that a real is left-c.e. if it has a computably enumerable (in short, c.e.) left Dedekind cut or, equivalently, if it is the limit of a computable increasing sequence of rationals. Moreover we point out that left-c.e. reals can be seen as halting probabilities of prefix-free machines, if one considers the Kraft-Chaitin algorithm for constructing prefix-free machines (see e.g. [DH10, Section 2.6]). Conversely, every halting probability of a prefix-free machine is a left-c.e. real, so the two classes of reals coincide.

Theorem 1.3. *If U is a universal prefix-free machine and X is a nonempty Π_1^0 set, the number $\Omega_U(X)$ is a Martin-Löf random left-c.e. real.*

Let us illustrate the novelty of this result with an example. Given a Σ_1^0 set X , there is a machine which enumerates X , and since the convergent computations of a prefix-free machine U are also computably enumerable, it is not surprising that $\Omega_U(X)$ has a computably enumerable Dedekind cut. As the enumeration of X progresses, more programs halt on elements of X , and the probability of this happening according to (2) can be approximated by a computable increasing sequence of rationals. If, however, X is a Π_1^0 set, such as the non-theorems of Peano arithmetic, there is no apparent way to approximate $\Omega_U(X)$ without overestimating its value. Indeed, a correct approximation would have to account for the programs that converge to the non-theorems of Peano arithmetic. As one enumerates the theorems of Peano arithmetic, the non-computability of this set means that there is no way in general to be sure that a given statement will not be enumerated at a later stage, at which point any previous consideration of programs converging to that statement will have introduced a possible overestimation of $\Omega_U(X)$. Despite this, not only does there exist a way to approximate $\Omega_U(X)$ without ever overestimating it, but any computable approximation to it essentially already has this property:

Given any universal prefix-free machine U , any Π_1^0 set $X \neq \emptyset$ and any computable sequence of rationals (α_s) converging to $\Omega_U(X)$ we have $\alpha_s < \Omega_U(X)$ for all but finitely many s . (3)

This fact will be derived from Theorem 1.3 and the properties of Martin-Löf random left-c.e. reals. So what does Problem 1.1 have to do with differences of halting probabilities? It is not hard to see that $\Omega_U(X)$ can be written as $\Omega_U - \Omega_U(\bar{X})$, where \bar{X} denotes the complement of X . Moreover if X is Π_1^0 then \bar{X} is c.e. so $\Omega_U(\bar{X})$ is a Martin-Löf random left-c.e. real. By central results from the theory of Martin-Löf random left-c.e. reals (see Section 1.2), Martin-Löf random left-c.e. reals are exactly the halting probabilities of universal prefix-free machines. Therefore there exists a universal prefix-free machine V such that $\Omega_V = \Omega_U(\bar{X})$ and hence, $\Omega_U(X)$ is the difference $\Omega_U - \Omega_V$ of two halting probabilities of universal prefix-free machines. So the question as to whether $\Omega_U(X)$ is Martin-Löf random has much to do with what happens if we subtract two Martin-Löf random left-c.e. reals.

Reals that are differences of left-c.e. reals (i.e. of the form $\alpha - \beta$ for α, β left-c.e.) are often called d.c.e. reals. There has been considerable work on this class of reals [ASWZ00, Rai05, Ng06, DWZ04, DWZ04]. Moreover the theory of Martin-Löf random left-c.e. reals is well understood, initially with [Sol75, CHKW01, KS01] and on a deeper level with [DHN02]. Despite this considerable body of work, the basic question as to what happens when we subtract two Martin-Löf random left-c.e. reals has remained largely untouched. It turns out that the solution of Problem 1.1 crucially depends on the answer to this question, which we present in the next section.

1.2 A theory of differences of universal halting probabilities

The d.c.e. reals form a field under the usual addition and multiplication, as was demonstrated by Ambos-Spies, Weihrauch, and Zheng [ASWZ00]. Raichev [Rai05] and Ng [Ng06] showed that this field is real-closed. Downey, Wu and Zheng [DWZ04] studied the Turing degrees of d.c.e. reals. Clearly d.c.e. reals are Δ_2^0 since they can be computably approximated. Downey, Wu and Zheng [DWZ04] showed that every real which is truth-table reducible to the halting problem is Turing equivalent to a d.c.e. real. However they also showed that there are Δ_2^0 degrees which do not contain any d.c.e. reals. In this strong sense, d.c.e. reals form a strict subclass of the Δ_2^0 reals.

Definition 1.4 (Approximations). *A left-c.e. approximation to a real α is a computable sequence of rationals (α_s) which converges to α and such that $\alpha_s \leq \alpha$ for almost all s . A d.c.e. approximation is a sequence of the form $(\alpha_s - \beta_s)$ where (α_s) and (β_s) are bounded increasing sequences of rationals.*

A real is called right-c.e. if its *right* Dedekind cut is c.e. or, equivalently, if it is the limit of a decreasing computable sequence of rationals. A right-c.e. approximation is defined similarly to a left-c.e. approximation but with the inequality reversed. For example, a d.c.e. approximation $(\alpha_s - \beta_s)$ to $\alpha - \beta$ is left-c.e. if for almost all s we have $\alpha_s - \beta_s < \alpha - \beta$, and is right-c.e. if for almost all s we have $\alpha_s - \beta_s > \alpha - \beta$. A real is called *properly d.c.e.* if it is d.c.e. and is neither a left-c.e. nor a right-c.e. real. Rettinger and Zheng [RZ05] (also see [DH10, Theorem 9.2.4]) proved that there are no properly d.c.e. Martin-Löf random reals.

Every Martin-Löf random d.c.e. real is a left-c.e. or a right-c.e. real. In fact, any d.c.e. approximation to a Martin-Löf random real is either left-c.e. or right-c.e. (4)

An initial reaction to this result might be that there is not much to say about d.c.e. reals and Martin-Löf randomness. This impression quickly fades, however, once rather basic questions are asked regarding differences of Martin-Löf random left-c.e. reals, which cannot be answered by the existing theory of Martin-Löf random left-c.e. reals. Problem 1.1 is such a question, and is part of the more general question as to what properties the differences of Martin-Löf random left-c.e. reals have. The latter question becomes even more interesting once we recall the maximality properties of Martin-Löf random left-c.e. reals inside the class of left-c.e. reals, which were established largely on the basis of the cumulative work of Solovay [Sol75], Calude, Hertling, Khoussainov and Wang [CHKW01] and Kučera and Slaman [KS01], showing that the Martin-Löf random left-c.e. reals are exactly the halting probabilities of universal prefix-free machines. Downey, Hirschfeldt and Nies [DHN02] proved that given any two Martin-Löf random left-c.e. reals α, β , if q is a sufficiently large positive rational number then $q\alpha - \beta$ is a Martin-Löf random left-c.e. real. Similarly, if q is a sufficiently small positive rational number then $q\alpha - \beta$ is a Martin-Löf random right-c.e. real. This result demonstrates the universality of Martin-Löf random left-c.e. reals, in the sense that they can ‘absorb’ any other left-c.e. real which is appropriately scaled. In other words, they remain left-c.e. and Martin-Löf random even when we subtract a left-c.e. real from them, provided that the latter is appropriately scaled.

But what happens in-between these appropriate values of q ? Is there a non-trivial interval of rationals q such that $q\alpha - \beta$ is neither a left-c.e. nor a right-c.e. real, and hence not a Martin-Löf random real? Given two left-c.e. reals α, β define:

$$\mathcal{D}(\alpha, \beta) = \inf \left\{ q \in \mathbb{Q}^+ \mid q\alpha - \beta \text{ is a left-c.e. real} \right\} \quad (5)$$

$$\mathcal{D}^*(\alpha, \beta) = \sup \left\{ q \in \mathbb{Q}^+ \mid q\alpha - \beta \text{ is a right-c.e. real} \right\}. \quad (6)$$

As we discussed above, if α, β are Martin-Löf random then both $\mathcal{D}(\alpha, \beta)$ and $\mathcal{D}^*(\alpha, \beta)$ are positive and finite. Moreover $q\alpha - \beta$ is a left-c.e. real for each rational $q > \mathcal{D}(\alpha, \beta)$ and a right-c.e. real for each positive rational $q < \mathcal{D}^*(\alpha, \beta)$.

Theorem 1.5. *Given any Martin-Löf random left-c.e. reals α, β , we have $\mathcal{D}(\alpha, \beta) = \mathcal{D}^*(\alpha, \beta)$.*

This is remarkable! As q ranges from 0 to infinity, there is a unique point where $q\alpha - \beta$ undergoes a sudden transition from being a left-c.e. real to being a right-c.e. real (note that this point of transition will not generally be β/α since $\mathcal{D}(\alpha, \beta)$ and $\mathcal{D}^*(\alpha, \beta)$ are unaffected by the addition or subtraction of rational values to α and β , so that $\mathcal{D}(\alpha, \beta) = \mathcal{D}(\alpha + \frac{1}{2}, \beta)$, for example). Moreover, as will become clear in the following, $q\alpha - \beta$ remains Martin-Löf random either side of the transition point, but loses its randomness precisely at the point of transition. We can use this result in order to get a complete answer to the question concerning differences of Martin-Löf random left-c.e. reals.

Theorem 1.6. *Let α, β be Martin-Löf random left-c.e. reals. Then*

- (a) *if $\mathcal{D}(\alpha, \beta) < 1$ then $\alpha - \beta$ is a Martin-Löf random left-c.e. real;*
- (b) *if $\mathcal{D}(\alpha, \beta) = 1$ then $\alpha - \beta$ is not Martin-Löf random;*
- (c) *if $\mathcal{D}(\alpha, \beta) > 1$ then $\alpha - \beta$ is a Martin-Löf random right-c.e. real.*

The proof of Theorem 1.5 actually gives another remarkable new result regarding the approximations to Martin-Löf random left-c.e. reals. Given any two Martin-Löf random left-c.e. reals α, β and any monotone left-c.e. computable approximations $(\alpha_s), (\beta_s)$ converging to α and β respectively, the ratio of the rates of convergence has a unique limit, namely $\mathcal{D}(\alpha, \beta) > 0$.

Theorem 1.7. *Let α, β be Martin-Löf random left-c.e. reals and let $(\alpha_s), (\beta_s)$ be any monotone left-c.e. computable approximations to α and β respectively. Then:*

$$\lim_s \left(\frac{\alpha - \alpha_s}{\beta - \beta_s} \right) \text{ exists and is equal to } \mathcal{D}(\alpha, \beta).$$

In interpreting this theorem, note that $\mathcal{D}(\alpha, \beta)$ depends only on α and β , and is independent of the particular choice for $(\alpha_s), (\beta_s)$. Finally, we note that if α, β, γ are Martin-Löf random left-c.e. reals and q is a positive rational then:

- (1) $\mathcal{D}(\alpha + \beta, \gamma) = \mathcal{D}(\alpha, \gamma) + \mathcal{D}(\beta, \gamma)$
- (2) $\mathcal{D}(\alpha, q \cdot \beta) = 1/q \cdot \mathcal{D}(\alpha, \beta)$ and $\mathcal{D}(q \cdot \alpha, \beta) = q \cdot \mathcal{D}(\alpha, \beta)$.
- (3) $\mathcal{D}(\alpha, \beta) \cdot \mathcal{D}(\beta, \alpha) = 1$.

These linearity properties of the operator \mathcal{D} are a direct consequence of Theorem 1.7.

Miller [Mil17] gives a beautiful account of the results of this section, along with alternative proofs and extensions to the d.c.e. reals.

1.3 Background

Some familiarity with the basic concepts of algorithmic information theory and the basic methods of computability theory would be helpful for the reader. For such background we refer to one of the monographs [LV97, DH10, Nie09], where the latter two are more focused on computability theory aspects of algorithmic randomness. In this section we lay out some preliminary material which is directly relevant to the present work and which we avoided in the introduction for the sake of clarity of presentation. We start with a brief discussion of some facts from the theory of left-c.e. reals. The results we mention in Section 1.3.1 are directly relevant to our work, and will be used throughout our analysis. We continue with some definitions of algorithmic randomness in Section 1.3.2. We define Martin-Löf randomness in two equivalent ways, namely in terms of Martin-Löf tests and in terms of Solovay tests. Both of these notions will be used in our analysis; the argument of Section 4 uses a Martin-Löf test while Section 3 uses a Solovay test. Finally in Section 1.3.3 we include a brief discussion of the original motivation behind Problem 1.1, which we omitted in the introduction.

1.3.1 Completeness in the Solovay degrees of left-c.e. reals

Solovay [Sol75] defined a reducibility on the left-c.e. reals as a measure of the hardness of approximation. Given left-c.e. reals α, β we say that α is Solovay reducible to β if there exists a partial computable function φ and a constant c such that for all rationals $q < \beta$, $\varphi(q)$ is defined and $0 \leq \alpha - \varphi(q) \leq c \cdot (\beta - q)$. This condition is equivalent to requiring that there are left-c.e. monotone approximations $(\alpha_s), (\beta_s)$ to α, β respectively and a constant c such that $\alpha - \alpha_s \leq c \cdot (\beta - \beta_s)$ for all s . Downey, Hirschfeldt and Nies [DHN02] showed that α is Solovay reducible to β if and only if there exists a left-c.e. real γ and a rational constant c such that $\beta = c \cdot \alpha + \gamma$. In the same paper it was shown that α is Solovay reducible to β if and only if there are left-c.e. monotone approximations $(\alpha_s), (\beta_s)$ to α, β respectively and a constant c such that $\alpha_{s+1} - \alpha_s < c \cdot (\beta_{s+1} - \beta_s)$ for all s . Solovay's work, combined with the work of Calude, Hertling, Khossainov and Wang [CHKW01] and Kučera and Slaman [KS01], showed that the Martin-Löf random left-c.e. reals are exactly the complete left-c.e. reals with respect to Solovay reducibility. Moreover, this work also implies that the Martin-Löf random left-c.e. reals are exactly the halting probabilities of universal prefix-free machines.

More specifically, Calude, Hertling, Khossainov and Wang [CHKW01] showed that if α, β are left-c.e. reals, α is Martin-Löf random and is Solovay reducible to β then β is Martin-Löf random. Kučera and Slaman showed that every Martin-Löf random left-c.e. real is complete with respect to Solovay reducibility. Downey, Hirschfeldt and Nies [DHN02] also showed that addition is a join operation in the Solovay degrees and that the Martin-Löf random left-c.e. reals are join-irreducible. In other words, if α is a Martin-Löf random left-c.e. real and β is another left-c.e. real then $\alpha + \beta$ is a Martin-Löf random left-c.e. real; conversely, if α, β are left-c.e. reals and $\alpha + \beta$ is Martin-Löf random, then at least one of α, β is Martin-Löf random. The first of these results had already been proved in Demuth [Dem75].

The starting point of the present work was to consider the size of the constant c in Solovay reductions.

In particular, any two Martin-Löf random left-c.e. reals are Solovay equivalent, but how small can the constants in the associated reductions be? If one examines the proofs in the work mentioned above, one finds that the infimum of the constants c that are possible in a reduction between two left-c.e. reals α, β , is the same for all of the above characterisations of Solovay reducibility. This observation indicated that this infimum is a robust characteristic of the pair α, β – a fact that we eventually proved with Theorem 1.5 and Theorem 1.7.

1.3.2 Martin-Löf and Solovay tests for randomness

Algorithmic randomness for real numbers was originally defined by Martin-Löf in [ML66] in terms of effective statistical tests. A Martin-Löf test is a uniformly c.e. sequence of Σ_1^0 classes of reals (U_i) such that the Lebesgue measure of U_i is bounded by 2^{-i} . In other words, a Martin-Löf test is a uniform sequence of effectively open sets of reals whose measure decreases uniformly. Effectively open sets of reals are often represented as c.e. sets of finite strings, while finite strings in turn can be viewed as basic open sets in the Cantor space (a string σ represents the set of reals with prefix σ). A point on the real line can be regarded as a member of the Cantor space (consisting of the infinite binary sequences) by identifying it with its binary expansion. We say that a real α is Martin-Löf random if $\alpha \notin \bigcap_i U_i$ for all Martin-Löf tests (U_i) .

Solovay [Sol75] devised an equivalent way to test Martin-Löf randomness. A Solovay test is a uniformly c.e. sequence of Σ_1^0 classes I_j such that the sum of the measures $\mu(I_j)$ (summing over all $j \in \mathbb{N}$) is finite. Solovay showed that a real α is Martin-Löf random if and only if, for every Solovay test (I_j) there are only finitely many j with $\alpha \in I_j$.

1.3.3 Historical context of the Problem 1.1

Some researchers have been looking to obtain other algorithmically random numbers, perhaps more random than the halting probability, through an examination of the stochastic behavior of a Turing machine. Here the stochasticity refers to some form of probability measure on the space of inputs, which can be seen as the outcomes of an experiment, like a repeated coin-toss. The strength of algorithmic randomness can be calibrated through the various recursion-theoretic hierarchies. The standard notion of algorithmic randomness is Martin-Löf randomness, also called 1-randomness, which can be relativized to the n th iteration of the halting problem, giving n -randomness. Becher and Chaitin [BDC01] considered a prefix-free model for infinite computations, which was introduced by Chaitin in [Cha76] and whose study was suggested in [Cha01, Chapter 6]. They showed that the probability that a universal prefix-free machine for infinite computations outputs finitely many symbols is 2-random.

According to [BFGM06], in 2002 Grigorieff suggested the study of the halting probability of a prefix-free machine restricted to a set X of outputs. He considered the number (2), which is the probability that a standard universal prefix-free machine U halts with output a member of the set X . He conjectured that the *higher the arithmetical complexity of X is, the more random the number $\Omega_U(X)$ becomes*. If this were true, then this would be a neat way to obtain highly random numbers as halting probabilities in the standard prefix-free model for computation. Versions of of this conjecture were successfully pursued for Chaitin's infinite computation model of [Cha76], by Becher and Grigorieff in [BG05, BG09]. However, as far as the usual prefix-free machines are concerned, the conjecture proved largely false after Miller in 2004 produced a Δ_2^0 set X such that $\Omega_U(X)$ is a rational number (see [BG05] for the announcement of this result and

[BFGM06] for a proof). Becher, Figueira, Grigorieff, and Miller [BFGM06] continued with a large number of negative results, showing that $\Omega_U(X)$ is often less random than expected, given the complexity of X . In this same paper there is also a positive result, showing that $\Omega_U(X)$ is 1-random when X is Σ_n^0 -complete or Π_n^0 -complete for some $n > 1$. Nevertheless, they also show that for each $n > 1$ the number $\Omega_U(X)$ is not n -random for any Σ_n^0 or Π_n^0 set X . On a positive note, Chaitin [Cha04] had already noticed that if X is a Σ_1^0 set, then $\Omega_U(X)$ is Martin-Löf random. The case when X is Π_1^0 is the basis of Problem 1.1 and has remained open. Figueira, Stephan, and Wu [FSW06] constructed an optimal (in terms of Kolmogorov complexity) but not universal prefix-free machine U and a Π_1^0 set X such that $\Omega_U(X)$ is not Martin-Löf random. They also suggested some strategies for giving a negative answer to Problem 1.1. In the present work, we see that Problem 1.1 has a surprising positive answer.

2 Overview

In this section we reduce all of the results in this paper to two technical lemmas, which we prove in Sections 3 and 4 respectively. This style of presentation should make the content of this work readily accessible. In Section 2.1 we derive the results of Section 1.2 from Lemma 2.1, whose proof is a delicate technical argument which is deferred to Section 3. Then in Section 2.3 we give the solution of Problem 1.1, using the results of Section 1.2 and Lemma 2.2. The proof latter lemma is a *decanter argument*, which has some things in common with the original decanter argument by Downey, Hirschfeldt, Nies and Stephan [DHNS03] and is given in Section 4.

2.1 Proof of Theorem 1.5 and Theorem 1.7

Downey, Hirschfeldt and Nies [DHN02] showed that, given any two left-c.e. reals α, β , the number $\mathcal{D}(\alpha, \beta)$ is the limit infimum of $(\alpha - \alpha_s)/(\beta - \beta_s)$ with respect to all left-c.e. increasing approximations $(\alpha_s), (\beta_s)$ to α, β respectively. Hence for Theorem 1.7 it suffices to show that in the case where α, β are Martin-Löf random, given left-c.e. increasing approximations $(\alpha_s), (\beta_s)$ to α, β respectively, the limit of $(\alpha - \alpha_s)/(\beta - \beta_s)$ exists. By the density of the rationals in the real line, this statement follows from the following special case, which is proved in Section 3.

Lemma 2.1. *Suppose α and β are c.e. reals, that α is Martin-Löf random and that $p \in \mathbb{Q}$ with $p > 1$. Suppose given left-c.e. approximations (α_s) and (β_s) to α and β respectively. Then the following two conditions cannot both hold:*

1. *There exist infinitely many s with $p(\alpha - \alpha_s) < (\beta - \beta_s)$;*
2. *There exist infinitely many s with $(\beta - \beta_s) < (\alpha - \alpha_s)$.*

Next, we show how to derive Theorem 1.5 from Lemma 2.1. Suppose that α and β are c.e. reals and that α is Martin-Löf random. In order to establish Theorem 1.5, it suffices to show that if $p, q \in \mathbb{Q}^+$ with $p > q$, then either $p\alpha - \beta$ is left-c.e. or else $q\alpha - \beta$ is a right-c.e. real.¹ In fact it suffices to establish this for the case $q = 1$. Then, for general $p > q$ in \mathbb{Q}^+ , if $q\alpha - \beta$ is not right c.e., neither is $\alpha - (\beta/q)$, which means that

¹Indeed, if Theorem 1.5 did not hold, then $\mathcal{D}(\alpha, \beta) > \mathcal{D}^*(\alpha, \beta)$. If we take $q < p$ between $\mathcal{D}^*(\alpha, \beta)$ and $\mathcal{D}(\alpha, \beta)$ then we would have that $p\alpha - \beta$ is not left-c.e. and $q\alpha - \beta$ is not a right-c.e. real.

$(p/q)\alpha - (\beta/q)$ and thus $p\alpha - \beta$ must be a left-c.e. real. So to prove 1.5, it suffices to establish the following fact.

(7)

If α and β are c.e. reals and α is Martin-Löf random, then for $p \in \mathbb{Q}$ with $p > 1$, either $p\alpha - \beta$ is left-c.e., or else $\alpha - \beta$ is a right-c.e. real.

Now we show how to obtain (7) from Lemma 2.1. Consider left-c.e. approximations (α_s) and (β_s) to α and β such that α is Martin-Löf random. There are then two cases to consider.

Case 1. There do not exist infinitely many s with $p(\alpha - \alpha_s) < (\beta - \beta_s)$. If there exists an s with $p(\alpha - \alpha_s) = (\beta - \beta_s)$ then $p\alpha - \beta$ is rational and so is clearly a left-c.e. real. Otherwise, we can find s_0 such that $p(\alpha - \alpha_s) > (\beta - \beta_s)$ for all $s \geq s_0$. In order to define a left-c.e. approximation (γ_s) to $p\alpha - \beta$, define $\gamma_0 = p\alpha_{s_0} - \beta_{s_0}$. Given γ_i and s_i (such that $p(\alpha - \alpha_{s_i}) > (\beta - \beta_{s_i})$), find $s_{i+1} > s_i$ such that $p(\alpha_{s_{i+1}} - \alpha_{s_i}) > (\beta_{s_{i+1}} - \beta_{s_i})$, and define $\gamma_{i+1} = p\alpha_{s_{i+1}} - \beta_{s_{i+1}}$.

Case 2. There do exist infinitely many s with $p(\alpha - \alpha_s) < (\beta - \beta_s)$. In this case Lemma 2.1 tells us that there cannot exist infinitely many s with $(\beta - \beta_s) < (\alpha - \alpha_s)$. If there exists an s with $(\beta - \beta_s) = (\alpha - \alpha_s)$ then $\alpha - \beta$ is rational and so is a right-c.e. real. Otherwise, let s_0 be such that $(\alpha - \alpha_s) < (\beta - \beta_s)$ for all $s \geq s_0$. In order to define a right c.e. approximation (δ_s) to $\alpha - \beta$, define $\delta_0 = \alpha_{s_0} - \beta_{s_0}$. Given δ_i and s_i (such that $(\alpha - \alpha_{s_i}) < (\beta - \beta_{s_i})$), find $s_{i+1} > s_i$ such that $(\alpha_{s_{i+1}} - \alpha_{s_i}) < (\beta_{s_{i+1}} - \beta_{s_i})$, and define $\delta_{i+1} = \alpha_{s_{i+1}} - \beta_{s_{i+1}}$.

We have reduced Theorem 1.5 and Theorem 1.7 to Lemma 2.1, whose proof is given in Section 3.

2.2 Proof of Theorem 1.6

For clause (a) suppose that $\mathcal{D}(\alpha, \beta) < 1$. Then there exists a rational number $q < 1$ such that $q\alpha - \beta$ is a left-c.e. real. Recall that Demuth [Dem75] showed that the sum of a Martin-Löf random left-c.e. real and any other left-c.e. real is Martin-Löf random. Therefore $\alpha - \beta$ is Martin-Löf random as the sum of the Martin-Löf random left-c.e. real $(1 - q)\alpha$ and the left-c.e. real $q\alpha - \beta$.

For clause (c) the argument is similar. If $\mathcal{D}(\alpha, \beta) > 1$ then by Theorem 1.5 we have $\mathcal{D}^*(\alpha, \beta) > 1$. So there exists $p > 1$ such that $p\alpha - \beta$ is a right-c.e. real. Recall that Demuth [Dem75] showed that the sum of a Martin-Löf random right-c.e. real and any other right-c.e. real is Martin-Löf random. Hence $\alpha - \beta$ is a Martin-Löf random right-c.e. real as the sum of the Martin-Löf random right-c.e. real $(1 - p)\alpha$ and the right-c.e. real $p\alpha - \beta$.

For clause (b), it suffices to prove the contrapositive. So assume that $\alpha - \beta$ is Martin-Löf random. Then by (4), $\alpha - \beta$ is either a left-c.e. real or a right-c.e. real. Without loss of generality, assume that it is a Martin-Löf random left-c.e. real. Then by Downey, Hirschfeldt and Nies [DHN02] there exists a sufficiently small but positive rational q such that $(\alpha - \beta) - q\alpha$ is a Martin-Löf random left-c.e. real. But this means that $\mathcal{D}(\alpha, \beta) \leq 1 - q$ so $\mathcal{D}(\alpha, \beta) < 1$. In the case where $\alpha - \beta$ is a Martin-Löf random right-c.e. real, a similar argument gives that $\mathcal{D}(\alpha, \beta) > 1$. In any case, $\mathcal{D}(\alpha, \beta) \neq 1$, which concludes the proof.

2.3 Proof of Theorem 1.3

Let us briefly note that (3) is a direct consequence of Theorem 1.3 and (4).² We are going to reduce Theorem 1.3 to the following two lemmas, of which the first one has a self-contained proof presented in Section 4, and the latter one is based on the results of Section 1.2.

Lemma 2.2. *If X is a Π_1^0 set and $\Omega_U(X)$ is a right-c.e. real then $\Omega_U(X)$ is not Martin-Löf random.*

The following lemma can be proved using Theorem 1.6, and is of independent interest.

Lemma 2.3. *If δ is a d.c.e. real which is not Martin-Löf random and α is a Martin-Löf random left-c.e. real then $\alpha + \delta$ is a Martin-Löf random left-c.e. real.*

Proof. Let $\delta = \delta_0 - \delta_1$. We can suppose that δ_0 and δ_1 are Martin-Löf random (otherwise add a random left-c.e. real to both). Since δ is not Martin-Löf random we have $\mathcal{D}(\delta_0, \delta_1) = 1$, by Theorem 1.6. Since α is a Martin-Löf random left-c.e. real, by Downey, Hirschfeldt and Nies [DHN02] (see the discussion in Section 1.3.1) there exists some c and another left-c.e. real β such that $\alpha = 2^{-c} \cdot \delta_0 + \beta$. So $\alpha + \delta = \beta + (1 + 2^{-c}) \cdot \delta_0 - \delta_1$. But since $\mathcal{D}(\delta_0, \delta_1) = 1$ the real $(1 + 2^{-c}) \cdot \delta_0 - \delta_1$ is left-c.e. and Martin-Löf random (again by [DHN02]). So $\alpha + \delta$ is also left-c.e. and Martin-Löf random, which completes the proof. \square

We are now ready to prove Theorem 1.3, using Lemma 2.2 and Lemma 2.3. Let $x \in X$ and consider $Y = X - \{x\}$ which is a Π_1^0 set. Then $\Omega_U(X) = \Omega_U(Y) + \Omega_U(\{x\})$. Note that $\Omega_U(\{x\})$ is a Martin-Löf random left-c.e. real. By Lemma 2.2 the number $\Omega_U(Y)$ is one of the following:

- (a) a right-c.e. real which is not Martin-Löf random;
- (b) a proper d.c.e. real (hence, by (4), not Martin-Löf random);
- (c) a left-c.e. real.

In the first two cases $\Omega_U(Y)$ is a d.c.e. real which is not Martin-Löf random, so by Lemma 2.3 and the fact that $\Omega_U(\{x\})$ is a Martin-Löf random left-c.e. real we have that $\Omega_U(X)$ is a Martin-Löf random left-c.e. real. In the last case $\Omega_U(Y)$ is a left-c.e. real so $\Omega_U(X)$ is again a Martin-Löf random left-c.e. real as the sum of a left-c.e. real and a random left-c.e. real, which concludes the proof.

3 Proof of Lemma 2.1

Towards a contradiction, suppose that $\alpha, \beta, p, (\alpha_s)$ and (β_s) are as in the statement of the Lemma 2.1, and that conditions (1) and (2) both hold. We shall show how to enumerate a Solovay test establishing that $(p-1)\alpha$ is not random. It will be convenient to make use of the following notation.

Definition 3.1. *For any given $s > 0$, let $\alpha_s^* = \alpha_s - \alpha_{s-1}$. For $s_1 \geq s_0$ define $\alpha_{s_0, s_1}^* = \alpha_{s_1} - \alpha_{s_0}$, and $\alpha_{s_0, \infty}^* = \alpha - \alpha_{s_0}$. Also define β_s^* , β_{s_0, s_1}^* and $\beta_{s_0, \infty}^*$ similarly.*

²If (3) did not hold, then either there are infinitely many terms of the sequence (α_s) on both sides of $\Omega_U(X)$, or almost all terms are larger than $\Omega_U(X)$. In the first case, by (4) we get that $\Omega_U(X)$ is not Martin-Löf random, which contradicts Theorem 1.3. In the second case we get that $\Omega_U(X)$ is right-c.e. and since by Theorem 1.3 it is also left-c.e. it has to be computable. But then again this contradicts Theorem 1.3.

From the fact that condition (1) holds, we can actually deduce that there exist infinitely many s for which a stronger condition holds:

$$(*1) \text{ For all } s' > s, p\alpha_{s,s'}^* < \beta_{s,s'}^*.$$

In order to see this, let s_0 be such that $p\alpha_{s_0,\infty}^* < \beta_{s_0,\infty}^*$. If (*1) holds for s_0 then we have found a stage $\geq s_0$ which satisfies (*1). Otherwise, let $s \geq s_0$ be minimal such that, for all $s' > s$, $p\alpha_{s_0,s'}^* < \beta_{s_0,s'}^*$. This means that $p\alpha_{s_0,s}^* \geq \beta_{s_0,s}^*$. If $s' > s$ and $p\alpha_{s,s'}^* \geq \beta_{s,s'}^*$, then $p\alpha_{s_0,s'}^* = p(\alpha_{s_0,s}^* + \alpha_{s,s'}^*) \geq \beta_{s_0,s}^* + \beta_{s,s'}^* = \beta_{s_0,s'}^*$, a contradiction. So s is a stage $\geq s_0$ which satisfies (*1). In each case, we can conclude that there exists a stage $\geq s_0$ which satisfies (*1).

From (2) we can similarly deduce that there are infinitely many s for which the following stronger condition holds:

$$(*2) \text{ For all } s' > s, \beta_{s,s'}^* < \alpha_{s,s'}^*.$$

In order to see this, let s_0 be such that $\beta_{s_0,\infty}^* < \alpha_{s_0,\infty}^*$. If (*2) holds for s_0 then we have found a stage $\geq s_0$ which satisfies (*2). Otherwise, let $s \geq s_0$ be minimal such that, for all $s' > s$, $\beta_{s_0,s'}^* < \alpha_{s_0,s'}^*$. This means that $\beta_{s_0,s}^* \geq \alpha_{s_0,s}^*$. If $s' > s$ and $\beta_{s,s'}^* \geq \alpha_{s,s'}^*$, then $\alpha_{s_0,s'}^* = \alpha_{s_0,s}^* + \alpha_{s,s'}^* \leq \beta_{s_0,s}^* + \beta_{s,s'}^* = \beta_{s_0,s'}^*$, a contradiction. So s is a stage $\geq s_0$ which satisfies (*2). In each case, we can conclude that there exists a stage $\geq s_0$ which satisfies (*2).

Without loss of generality we may suppose that $s = 0$ is one of those stages for which (*1) holds (otherwise begin the enumeration at some later stage).

3.1 Proof idea

The key to the proof lies in the following basic observation. Suppose that $s_1 > 0$ is the least stage for which (*2) holds. Let $\beta^1 = \beta_{0,s_1}^*$, $\beta^2 = \beta_{s_1,\infty}^*$ and $\alpha^1 = \alpha_{0,s_1}^*$, $\alpha^2 = \alpha_{s_1,\infty}^*$ (where 1 and 2 are used as indices rather than to denote exponentiation). Then we have:

$$\beta^1 + \beta^2 > p(\alpha^1 + \alpha^2), \quad \beta^2 < \alpha^2, \tag{8}$$

which gives,

$$\beta^1 - \alpha^1 > (p-1)(\alpha^1 + \alpha^2). \tag{9}$$

In order to begin enumerating a Solovay test which will be failed by $(p-1)\alpha$, (9) suggests that we could start by enumerating intervals in the following fashion. Let $a_1 = (p-1)\alpha_0$ and $b_1 = a_1 + (\beta_1^* - \alpha_1^*)$. At stage 1 we enumerate the interval $[a_1, b_1]$. At later stages s , we could consider the last interval $[a_i, b_i]$ enumerated, and look to see whether $a_1 + (\beta_{0,s}^* - \alpha_{0,s}^*) > b_i$. If so, then we could define $a_{i+1} = b_i$, $b_{i+1} = a_1 + (\beta_{0,s}^* - \alpha_{0,s}^*)$ and enumerate this into our Solovay test. What happens if we proceed in this way? The first thing to observe is that, by the end of stage s_1 (recall that (*2) holds for s_1), (9) ensures $(p-1)\alpha$ meets at least one element of the test enumerated thus far. What then happens after stage s_1 ? The fact that s_1 satisfies (*2) means that at every stage $s > s_1$ we will find that $a_1 + (\beta_{0,s}^* - \alpha_{0,s}^*) = a_1 + (\beta_{0,s_1}^* - \alpha_{0,s_1}^*) + (\beta_{s_1,s}^* - \alpha_{s_1,s}^*) < a_1 + (\beta_{0,s_1}^* - \alpha_{0,s_1}^*)$. So the rules described so far mean that we will not enumerate any further elements into our Solovay test attempting to capture our first initial segment of $(p-1)\alpha$ after stage s_1 . This is good news, because to do so would be wasteful – we have already captured one initial segment of $(p-1)\alpha$, and subsequent enumerations into the test should be aimed at capturing another initial segment.

During the construction of our Solovay test we shall place movable markers $x_0 < y_0 < x_1 < y_1 < \dots$ on stages, which are our present guesses as to stages which satisfy the conditions (*1) and (*2). We shall have $x_0 = 0$, since we have assumed that $s = 0$ satisfies (*1). If $y_0 \downarrow (> 0)$ then our present guess is that this is a stage which satisfies (*2), while x_1 may be placed on a stage which we believe satisfies (*1) again, and so on. Suppose that at some stage s , we have $y_0 = s_1$ and $x_1 \downarrow > y_0$. Then at stages $s' > x_1$ we shall enumerate elements into our Solovay test of the form $[a'_1, b'_1], [a'_2, b'_2], \dots$ but now beginning with $a'_1 = (p-1)\alpha_{x_1}$. Now for the sake of argument, suppose that s_1 *does not* in fact satisfy (*2) – that it merely appeared to be a stage satisfying this property, up until some later stage $s > x_1$. At this point it might initially seem that we have a difficulty, because the measure we have enumerated into the sequence $[a'_1, b'_1], [a'_2, b'_2], \dots$ should actually be measure we enumerate into the first sequence $[a_1, b_1], [a_2, b_2], \dots$ attempting to capture our first initial segment of $(p-1)\alpha$. Thus it may seem that we are threatened with having to enumerate certain measure into our Solovay test twice. The fact that x_1 appeared to satisfy (*2) up until stage s , however, means that this is not a problem. We have $\beta_{s_1, s-1}^* - \alpha_{s_1, s-1}^* < 0$, while $\beta_{s_1, s}^* - \alpha_{s_1, s}^* \geq 0$. One way to look at this is to see that, for some r with $0 < r \leq 1$ we have $(\beta_{s_1, s-1}^* - \alpha_{s_1, s-1}^*) + r(\beta_s^* - \alpha_s^*) = 0$, and that enumerating the measure $\beta_{s_1, s}^* - \alpha_{s_1, s}^*$ into our Solovay test actually only requires us to enumerate the new measure $(1-r)(\beta_s^* - \alpha_s^*)$.

At stage $s+1$ of the construction, we shall say that x_i *requires moving* if $x_i \downarrow$ and $\beta_{x_i, s+1}^* \leq p\alpha_{x_i, s+1}^*$. We say that y_i *requires moving* if $y_i \downarrow$ and $\beta_{y_i, s+1}^* \geq \alpha_{y_i, s+1}^*$. When we enumerate an interval into the Solovay test, it will always be the case that we associate that element of the Solovay test with one of the markers x_i . If the marker x_i is subsequently made undefined (because it or a marker of higher priority requires moving), then we cannot change the fact that elements of the Solovay test already associated with x_i have been enumerated, but we no longer consider them to be associated with x_i , i.e. the list of intervals associated with x_i is initialised.

3.2 Construction

At stage 0, place the marker x_0 on 0.

At stage $s+1$, let z be the least number on which a marker is placed and which requires moving, or if there exists no such number, then leave z undefined. There are three cases to consider.

- (1) $z \downarrow = x_i$ for some i . In this case, make $x_{i'}$ and $y_{i'}$ undefined for all $i' \geq i$, and go to the next stage.
- (2) $z \downarrow = y_i$ for some i . Let r be such that $(\beta_{y_i, s}^* - \alpha_{y_i, s}^*) + r(\beta_{s+1}^* - \alpha_{s+1}^*) = 0$. Let $[a, b]$ be the most recently enumerated interval associated with x_i . Enumerate the interval $[b, b + (1-r)(\beta_{s+1}^* - \alpha_{s+1}^*)]$ into the Solovay test, and associate it with x_i . Make y_i undefined, and make all $x_{i'}$ and $y_{i'}$ undefined for $i' > i$. Go to the next stage.
- (3) $z \uparrow$. In this case, let w be the greatest number on which a marker is placed. There are then two subcases.
 - (a) $w = x_i$ for some i . If $\beta_{s+1}^* \geq \alpha_{s+1}^*$, then let $[a, b]$ be the most recently enumerated interval associated with x_i , or let $b = (p-1)\alpha_0$ if there exists no such, and enumerate the interval $[b, b + (\beta_{s+1}^* - \alpha_{s+1}^*)]$ into the Solovay test, associating it with x_i . If $\beta_{s+1}^* < \alpha_{s+1}^*$ then place the marker y_i on s . In either case, go to the next stage.

- (b) $w = y_i$ for some i . In this case, if $\beta_{s+1}^* > p\alpha_{s+1}^*$ proceed as follows, and otherwise proceed immediately to the next stage. Place the marker x_{i+1} on s . Enumerate the interval

$$[(p-1)\alpha_s, (p-1)\alpha_s + (\beta_{s+1}^* - \alpha_{s+1}^*)]$$

into the Solovay test, and associate it with x_{i+1} . Go to the next stage.

3.3 Verification

Since the total measure enumerated into the Solovay test is bounded by β , it must be finite. It is also clear that if a marker is placed at some stage s and never subsequently moved, then s must satisfy (*1) if the marker is x_i for some i , and s must satisfy (*2) if the marker is y_i for some i .

In order to see that each marker is eventually permanently placed, suppose inductively that $x_0, y_0, \dots, x_i, y_i$ are all correctly placed at some stage $y_i + 1$ (meaning that the marker y_i has just been placed correctly). Let $s > y_i$ be the least such that (*1) holds for s , and consider what happens at stage $s + 1$. If the marker x_{i+1} is already placed on a number $< s$, then it must be the case that $\beta_{x_{i+1},s}^* > p\alpha_{x_{i+1},s}^*$, otherwise the marker x_{i+1} would have been moved prior to stage $s + 1$. But then, for any $s' > s$, $\beta_{x_{i+1},s'}^* = \beta_{x_{i+1},s}^* + \beta_{s,s'}^* > p(\alpha_{x_{i+1},s}^* + \alpha_{s,s'}^*) = p\alpha_{x_{i+1},s'}^*$, and so x_{i+1} satisfies (*1), contradicting the minimality of s . Thus x_{i+1} cannot be placed at the start of stage $s + 1$, and will be placed on s at this stage.

Now suppose that x_0, y_0, \dots, x_i have been placed correctly, let s be the least $> x_i$ satisfying (*2), and consider what happens at stage $s + 1$. If the marker y_i is already placed on a number $< s$, then it must be the case that $\beta_{y_i,s}^* < \alpha_{y_i,s}^*$, otherwise the marker y_i moved have been moved prior to stage $s + 1$. But then, for any $s' > s$, $\beta_{y_i,s'}^* = \beta_{y_i,s}^* + \beta_{s,s'}^* < \alpha_{y_i,s}^* + \alpha_{s,s'}^* = \alpha_{y_i,s'}^*$, and so y_i satisfies (*2), contradicting the minimality of s . Thus y_i cannot be placed at the start of stage $s + 1$, and will be placed on s at this stage.

Finally we note that once each x_i is correctly placed, an interval to which $(p-1)\alpha$ belongs is subsequently enumerated into the Solovay test and associated with x_i . This follows because, letting y_i take its final value, intervals are enumerated into the test and associated with x_i covering the entire interval $[(p-1)\alpha_{x_i}, (p-1)\alpha_{x_i} + (\beta_{x_i,y_i}^* - \alpha_{x_i,y_i}^*)]$. It follows from (9) that $(p-1)\alpha$ belongs to this interval.

4 Proof of Lemma 2.2

Let X be a Π_1^0 set. Given the existence of effective bijections between \mathbb{N} and $2^{<\omega}$, it does not matter whether we consider X as a subset of \mathbb{N} or $2^{<\omega}$, and so for convenience we shall think of X as a set of natural numbers. We must show that if $\Omega_U(X)$ is right-c.e. then it cannot be Martin-Löf random. If $\mathbb{N} - X$ is finite then $\Omega_U(X)$ is left-c.e., so if it was right-c.e. then it would be computable and so not Martin-Löf random. We may therefore assume that $\mathbb{N} - X$ is infinite. If $\Omega_U(X)$ is rational then it is clearly not Martin-Löf random, as required. So we may assume that $\Omega_U(X)$ is not rational. Let (X_s) be a Π_1^0 approximation to X and consider the sequence of rationals $(\Omega_U(X)[s])$ which converges to $\Omega_U(X)$. If all but finitely many terms of $(\Omega_U(X)[s])$ lie to the left of the limit $\Omega_U(X)$ then $\Omega_U(X)$ is a left-c.e. real, so if right-c.e. it would be computable and so not Martin-Löf random. If there are infinitely many terms of $(\Omega_U(X)[s])$ on either side of the limit $\Omega_U(X)$ then $\Omega_U(X)$ is not Martin-Löf random by (4), as required. So it remains to consider the case where all but finitely many terms of $(\Omega_U(X)[s])$ lie to the right of the limit $\Omega_U(X)$. In this case, since

$\Omega_U(X)$ is not rational, there exists an increasing sequence of stages (s_i) such that $\Omega_U(X)[s_{i+1}] < \Omega_U(X)[s_i]$ for all i . In particular, the terms $\Omega_U(X)[s_i]$ form a decreasing sequence which converges to $\Omega_U(X)$. Define $\omega_i = \Omega_U(X)[s_i]$ for each i . It remains to show that in this case $\Omega_U(X)$ is not Martin-Löf random. We will do this by constructing a Martin-Löf test (U_i) such that $\Omega_U(X) \in U_i$ for all i .

4.1 Description of the decanter

At each stage s_{i+1} we consider the *weight* (i.e. measure in the domain of Ω_U) which has *come in*, as well as the weight which has *gone out*, since stage s_i . The weight that has come in consists of the weight added into Ω_U during the stages in $(s_i, s_{i+1}]$, on numbers in $X_{s_{i+1}}$. So the weight that has come in is $\sum 2^{-|\sigma|}$, where the sum is over all σ for which we have seen new computations $U(\sigma) \downarrow = n$ for $n \in X_{s_{i+1}}$. The weight that goes out during the same interval, consists of the weight accumulated in Ω_U on numbers in $X_{s_i} - X_{s_{i+1}}$, i.e. $\Omega_U(\{n\})[s_i]$ for numbers n that were removed from the Π_1^0 approximation to X during the stages in $(s_i, s_{i+1}]$. Note that $\omega_i - \omega_{i+1}$ is exactly the difference between the weight that has gone out and the weight that has come in. In particular, *the weight that comes in is at most the weight that goes out*.

The initial weight $\Omega_U(X)[s_0]$, as well as the weight that comes in at later stages s_{i+1} , will be dynamically split into *quanta* (namely rational amounts that add up to the total weight in question) and placed onto various levels of a *decanter* whose levels extend downward to infinity. Each quantum will be part of $\Omega_U(\{n\})$ for some $n \in \mathbb{N}$, and in this case we shall say that it has label n . The weight in $\Omega_U(\{n\})[s_i]$ will, however, be split into potentially many different quanta, which are distributed at various levels of the decanter at stage s_i . The basic rule for the creation of quanta corresponding to additional weight appearing in Ω_U , is that each quantum has a unique label, i.e. it does not correspond to weight which is partly in $\Omega_U(\{n\})$ for some n and partly in $\Omega_U(\{m\})$ for some $m \neq n$.

At stage s_0 , for each $n \in X_{s_0}$ such that $\Omega_U(\{n\})[s_0] > 0$ we place a quantum of weight $\Omega_U(\{n\})[s_0]$ on the zero level of the decanter, which we call C_0 . Note that, by standard conventions, for each n such that $\Omega_U(\{n\})[s_0] > 0$ we have $n < s_0$, so this is a finite and effective operation and only finitely many quanta are added into C_0 . Each of the following stages s_{i+1} consists of two operations (to be defined precisely later):

- (a) *Purge* the quanta that have labels in $X_{s_i} - X_{s_{i+1}}$.
- (b) Split the weight that has come in into quanta, and distribute these quanta to the various levels of the decanter.

These two operations will be carried out so as to create the effect of quanta flowing down the levels of the decanter, possibly with some loss of weight as they travel from each level to the one below. Although this is the most convenient way to look at the construction, the reader should not forget that, in reality, when a quantum (or a part of it) flows from one level to the next one, the quantum in the next level corresponds to different weight in Ω_U than the quantum at the previous level (and will have a different label). In other words, there is a resistance to quanta flowing from level to level, due to the fact that each flow corresponds to additional weight in Ω_U . This is the crucial observation upon which we will build a Martin-Löf test for $\Omega_U(X)$.

4.2 Construction of the decanter flow

At each stage s_{i+1} we consider the numbers in $X_{s_i} - X_{s_{i+1}}$ and *purge* all the quanta in the decanter which have their label in $X_{s_i} - X_{s_{i+1}}$. According to the basic rule for the creation of quanta described above, each quantum has a unique label so there is no ambiguity in this selection. Each of these quanta can be thought of as a pair (n, δ) , where n is the label and δ is the weight of the quantum. To say that a quantum sitting at level i is purged, simply means that we no longer consider that quantum to sit at level i .

The basic idea now is that we wish to place quanta corresponding to the weight that has come in during the interval $(s_i, s_{i+1}]$, into various levels of the decanter. We wish to do this in such a way that each quantum placed can be thought of replacing or *flowing from* one of the quanta (or part of one of the quanta) just purged, being placed in the decanter one level below the quantum it flows from. The fact that the weight coming in is at most the weight going out means that the purged quanta will have sufficient weight for us to be able to carry out such a procedure. Let the weight that has come in be given as a finite list v_1, \dots, v_r where each v_i is a pair (n_i, δ_i) such that $\delta_i \neq 0$ is the new weight that has appeared in $\Omega_U(\{n_i\})$. Similarly enumerate the quanta just purged as u_1, \dots, u_l where each u_i is a pair (m_i, ρ_i) which has just been purged from level ℓ_i . Define $k_1 = 1$ and $x_1 = 0$. We define and place the new quanta in r steps. At step t with $1 \leq t \leq r$, we are given k_t which is the least i such that u_i is still available for use in creating new quanta, as well as x_t which is the weight of u_{k_t} which has already been used. We create new quanta corresponding to v_t as follows. Redefine ρ_{k_t} to be $\rho_{k_t} - x_t$ (to account for the weight from this quantum already used). Let k_{t+1} be the least such that $\sum_{i=k_t}^{k_{t+1}} \rho_i \geq \delta_t$ and let x_{t+1} be such that $(\sum_{i \in [k_t, k_{t+1})} \rho_i) + x_{t+1} = \delta_t$. For each $i \in [k_t, k_{t+1})$ place a quantum (n_t, ρ_i) at level $C_{\ell_{i+1}}$ and say that this quantum *flows from* u_i . Also place a quantum (n_t, x_{t+1}) at level $C_{\ell_{k_{t+1}+1}}$ and say that this quantum flows from $u_{k_{t+1}}$.

Once step r is completed and all of the new quanta have been created and placed at stage s_{i+1} , there remains a set of quanta just purged, which were not used in creating new quanta at this stage. More precisely, consider k_{r+1} and x_{r+1} (according to the notation above). Redefine $u_{k_{r+1}} = (m_{k_{r+1}}, \rho_{k_{r+1}} - x_{r+1})$. We consider the quanta $u_{k_{r+1}}, \dots, u_l$ to be *evicted* at stage s_{i+1} . This concludes the description of the construction.

4.3 Properties of the decanter flow

As we explained above, the construction can be visualised as a dynamic flow of quanta through a decanter with a countably infinite number of levels C_i , extending downwards and starting from level C_0 .

Lemma 4.1 (Finiteness of throughput). *The number of quanta ever enumerated into each level is finite.*

Proof. We use induction on the levels of the decanter. By construction at each stage only finitely many quanta are enumerated into each level. In fact, no enumerations into C_0 occur except at stage s_0 . Inductively assume that there is a stage u_0 after which no enumeration occurs in $C_i, i < m$. After stage u_0 there are only finitely many stages where some quantum in $C_i, i < m$ is purged. So there exists a stage $u_1 > u_0$ after which no quantum in $C_i, i < m$ is purged. By construction, this means that no enumeration of quanta into C_m will occur after stage u_1 . This concludes the induction step and the proof of the lemma. \square

Lemma 4.2 (Weight bound of total throughput). *Let $k \in \mathbb{N}$. The sum of the weights of all quanta ever enumerated into C_k , is less than $1/(k+1)$.*

Proof. This is trivially true for $k = 0$, because $\Omega_U < 1$. For $k = 1$ note that every quantum that is enumerated into C_1 at some stage s_{i+1} flows from part of a quantum in C_0 of equal weight that has just been purged during the first step of stage s_{i+1} . Moreover the weight in Ω_U that enters C_0 is disjoint from the weight that enters C_1 (i.e. corresponds to different convergent computations for U). This means that the total weight entering C_1 can be at most $\Omega_U/2 < \frac{1}{2}$. Similarly, the quanta entering C_2 flow from quanta of equal weight in C_1 , but once again the weight in Ω_U enumerated into C_2 is disjoint from the weight in Ω_U enumerated into C_0 and C_1 . So if q is the total weight of quanta enumerated into C_2 , then $3q \leq \Omega_U$, which means that $q < 1/3$. In the same way, for each k , the weight of the quanta enumerated into C_k is less than $1/(k+1)$. \square

4.4 The Martin-Löf test

The basic idea is that the i th member U_i of the Martin-Löf test will be defined according to the enumeration of quanta into C_i . The upper bound of Lemma 4.2 will be instrumental in giving an upper bound for the measure of U_i . The enumeration of U_i can be split into different cycles, such that during each cycle there is no eviction of quanta from $C_j, j < i$. Upon such an eviction we regard the current cycle as *injured*, we terminate it (preventing any further enumeration of intervals into U_i this very stage) and then we begin a new cycle at the next stage. By Lemma 4.1 there will be finitely many injuries of U_i , and the enumeration of U_i will have a final cycle which runs indefinitely. When counting the measure of reals enumerated into U_i , of course we also need to consider the reals enumerated into U_i during cycles that are later injured. Each cycle of U_i has a parameter δ_i which is the length of the intervals enumerated into U_i in response to the approximation (ω_s) . For the k -th cycle of U_i we define $\delta_i = 2^{-i-k-2}$. A stage is called i -valid if no eviction of quanta occurs in $C_j, j < i$.

Construction of the Martin-Löf test. At each i -valid stage s we enumerate the interval $(\omega_s - \delta_i[s], \omega_s)$ into U_i . At each stage which is not i -valid, we say that U_i is injured, we redefine $\delta_i := \delta_i/2$ and go to the next stage. This completes the definition of U_i .

Properties of the Martin-Löf test. Note that the intervals enumerated into U_i are typically overlapping. Since U_i is injured only finitely many times, δ_i reaches a limit, meaning that $\Omega_U(X)$ belongs to co-finitely many of the intervals enumerated into U_i . It remains to obtain an appropriate upper bound for the measure of U_i . Let $U_i(k)$ consist of the intervals enumerated into U_i during its k th cycle. Then the measure of $U_i(k)$ is the measure of the initial interval, which is 2^{-i-k-2} , plus the weight of all the quanta evicted during the stages of the k th cycle. Note that these are all quanta that have passed through C_i , and which cannot be evicted again at later stages or during later cycles (a quantum can only be evicted once, and once evicted no quanta flow from it). By Lemma 4.2 the total weight of these evicted quanta over all cycles is bounded by $1/(i+1)$. We can conclude that the weight of U_i is bounded above by:

$$\sum_j 2^{-i-j-2} + 1/(i+1) < 2/(i+1).$$

Defining $V_i = U_{2^{i+1}}$ we have that (V_i) is a Martin-Löf test such that $\cap_i V_i$ contains $\Omega_U(X)$. So $\Omega_U(X)$ is not Martin-Löf random, which concludes the proof of Lemma 2.2.

References

- [ASWZ00] Klaus Ambos-Spies, Klaus Weihrauch, and Xizhong Zheng. Weakly computable real numbers. *J. Complexity*, 16(4):676–690, 2000.
- [BDC01] Verónica Becher, Sergio Daicz, and Gregory Chaitin. A highly random number. In C.S. Calude, M.J. Dinneen, and S. Sburlan, editors, *Combinatorics, Computability and Logic*, Discrete Mathematics and Theoretical Computer Science, pages 55–68. Springer London, 2001.
- [BFGM06] Verónica Becher, Santiago Figueira, Serge Grigorieff, and Joseph S. Miller. Randomness and halting probabilities. *J. Symb. Log.*, 71(4):1411–1430, 2006.
- [BG05] Verónica Becher and Serge Grigorieff. Random reals and possibly infinite computations. Part I: Randomness in \emptyset' . *J. Symb. Log.*, 70(3):891–913, 2005.
- [BG07] Verónica Becher and Serge Grigorieff. Random reals à la Chaitin with or without prefix-freeness. *Theor. Comput. Sci.*, 385(1-3):193–201, 2007.
- [BG09] Verónica Becher and Serge Grigorieff. From index sets to randomness in $\emptyset^{(n)}$: random reals and possibly infinite computations. Part II. *J. Symb. Log.*, 74(1):124–156, 2009.
- [Cha75] Gregory J. Chaitin. A theory of program size formally identical to information theory. *J. Assoc. Comput. Mach.*, 22:329–340, 1975.
- [Cha76] Gregory J. Chaitin. Algorithmic entropy of sets. *Computers & Mathematics with Applications*, 2(3-4):233–245, 1976.
- [Cha01] Gregory J. Chaitin. *Exploring RANDOMNESS*. Discrete Mathematics and Theoretical Computer Science. Springer London, 2001.
- [Cha04] Gregory J. Chaitin. *Algorithmic Information Theory*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2004.
- [CHKW01] Christian Calude, Peter Hertling, Bakhadyr Khoussainov, and Yongge Wang. Recursively enumerable reals and Chaitin Ω numbers. *Theoret. Comput. Sci.*, 255(1-2):125–149, 2001.
- [Dem75] Oswald Demuth. On constructive pseudonumbers. *Commentationes Mathematicae Universitatis Carolinae*, 16:315–331, 1975. In Russian.
- [DH10] Rodney G. Downey and Denis Hirshfeldt. *Algorithmic Randomness and Complexity*. Springer, 2010.
- [DHN02] Rodney G. Downey, Denis R. Hirschfeldt, and André Nies. Randomness, computability, and density. *SIAM Journal of Computation*, 31:1169–1183, 2002.
- [DHNS03] Rodney G. Downey, Denis R. Hirschfeldt, André Nies, and Frank Stephan. Trivial reals. In *Proceedings of the 7th and 8th Asian Logic Conferences*, pages 103–131, Singapore, 2003. Singapore University Press.
- [DWZ04] Rodney G. Downey, Guohua Wu, and Xizhong Zheng. Degrees of d.c.e. reals. *Math. Log. Q.*, 50(4–5):345–350, 2004.

- [FSW06] Santiago Figueira, Frank Stephan, and Guohua Wu. Randomness and universal machines. *J. Complexity*, 22(6):738–751, 2006.
- [KS01] Antonín Kučera and Theodore Slaman. Randomness and recursive enumerability. *SIAM J. Comput.*, 31(1):199–211, 2001.
- [LV97] Ming Li and Paul Vitányi. *An introduction to Kolmogorov complexity and its applications*. Graduate Texts in Computer Science. Springer-Verlag, New York, second edition, 1997.
- [Mil17] Joseph S. Miller. On work of Barmpalias and Lewis-Pye: a derivation on the d.c.e. reals. In *Proceedings of the International Symposium on Computability and Complexity (in honour of Rod Downey’s 60th birthday)*, volume 10010 of *Lecture Notes in Comput. Sci.*, Berlin, 2017. Springer.
- [ML66] Per Martin-Löf. The definition of random sequences. *Information and Control*, 9:602–619, 1966.
- [MN06] Joseph S. Miller and André Nies. Randomness and computability: open questions. *Bull. Symbolic Logic*, 12(3):390–410, 2006.
- [Ng06] Keng-Meng Ng. *Some Properties of d.c.e. Reals and their Degrees*. M.Sc. thesis, National University of Singapore, 2006.
- [Nie09] André Nies. *Computability and Randomness*. Oxford University Press, 444 pp., 2009.
- [Rai05] Alexander Raichev. Relative randomness and real closed fields. *J. Symbolic Logic*, 70(1):319–330, 2005.
- [RZ05] Robert Rettinger and Xizhong Zheng. Solovay reducibility on d-c.e real numbers. In *Computing and Combinatorics: 11th Annual International Conference, COCOON 2005 Kunming, China, August 16–19, 2005 Proceedings*, pages 359–368, Berlin, Heidelberg, 2005. Springer Berlin Heidelberg.
- [Sol75] Robert Solovay. Handwritten manuscript related to Chaitin’s work. IBM Thomas J. Watson Research Center, Yorktown Heights, NY, USA, 215 pages, 1975.