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ARRANGEMENTS OF HOMOTHETS OF A CONVEX BODY

MÁRTON NASZÓDI, JÁNOS PACH, AND KONRAD SWANEPOEL

ABSTRACT. Answering a question of Füredi and Loeb (1994), we show that the maximum number of pairwise intersecting homothets of a d -dimensional centrally symmetric convex body K , none of which contains the center of another in its interior, is at most $O(3^d d \log d)$. If K is not necessarily centrally symmetric and the role of its center is played by its centroid, then the above bound can be replaced by $O(3^d \binom{2d}{d} d \log d)$. We establish analogous results for the case where the center is defined as an arbitrary point in the interior of K . We also show that in the latter case, one can always find families of at least $\Omega((2/\sqrt{3})^d)$ translates of K with the above property.

1. INTRODUCTION

A *convex body* K in the d -dimensional Euclidean space \mathbb{R}^d is a compact convex set with non-empty interior, and is *o-symmetric* if $K = -K$. A (positive) *homothet* of K is a set of the form $\lambda K + v := \{\lambda k + v : k \in K\}$, where $\lambda > 0$ is the homothety ratio, and $v \in \mathbb{R}^d$ is a translation vector. We investigate arrangements of homothets of convex bodies. The starting point of our investigations is Problem 4.4 of a paper of Füredi and Loeb [FüL94]:

Is it true that for any centrally symmetric body K of dimension $d, d \geq d_0$, the number of pairwise intersecting homothetic copies of K which do not contain each other's centers is at most 2^d ?

There exist 8 such homothets of the circular disc [MM92, HJLM93] (Fig. 1). A *Minkowski arrangement* of an *o-symmetric* convex body K is defined to be a family $\{v_i + \lambda_i K\}$ of positive homothets of K such that none of the homothets contains the center of any other homothet in its interior. This notion was introduced by L. Fejes Tóth [FT65] in the context of Minkowski's fundamental theorem on the minimal determinant of a packing lattice for a symmetric convex body, and further studied in the papers [FT67, FT99, BS04], and in connection to the Besicovitch covering theorem in [FüL94]. Recently, Minkowski arrangements have been used to study a problem arising in the design of wireless networks [NSS17].

We also define a *strict Minkowski arrangement* of K to be a family $\{v_i + \lambda_i K\}$ of positive homothets of K such that none of the homothets contains the center of any other homothet. We write $\kappa(K)$ ($\kappa'(K)$) for the largest number of homothets that a pairwise intersecting (strict) Minkowski arrangement of K can have.

Thus, the question of Füredi and Loeb may be phrased as follows: *Is it true that $\kappa'(K) \leq 2^d$ for any o-symmetric convex body K in \mathbb{R}^d with d sufficiently large?* A construction of Talata [Tal05] implies that the answer to this question is negative: there

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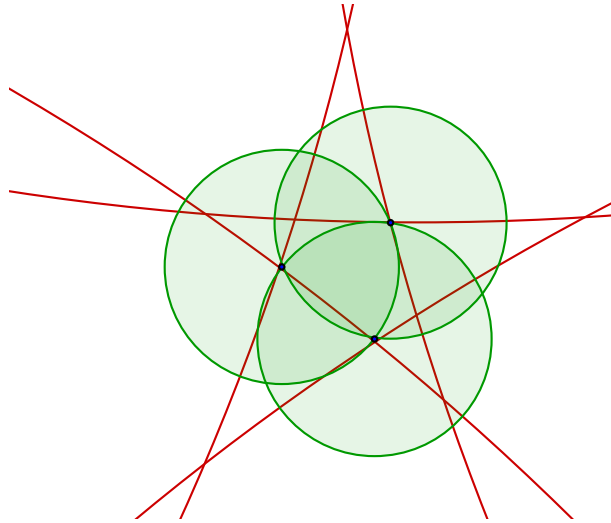


FIGURE 1. A pairwise intersecting strict Minkowski arrangement of 8 circles (after Harary et al. [HJLM93])

exists a d -dimensional convex body K such that $\kappa'(K) \geq \frac{16}{35}\sqrt{7}^d$ for all $d \geq 3$ (see Section 2). The question now becomes to find an upper bound for $\kappa'(K)$. It follows from [Fül94, Theorem 2.1] that $\kappa'(K) \leq \kappa(K) \leq 5^d$ for any o -symmetric d -dimensional convex body K . Our first result is the following improvement.

Theorem 1. *For any d -dimensional o -symmetric convex body K ,*

$$\kappa'(K) \leq \kappa(K) \leq O(3^d d \log d).$$

It is easy to see that for the d -cube C^d , $\kappa(C^d) = 3^d$, which shows that the upper bound for $\kappa(K)$ in Theorem 1 is sharp up to the $O(d \log d)$ factor. We will in fact prove a strengthening of this theorem in Theorem 5 below. However, we have no better upper bound for $\kappa'(K)$ than that for $\kappa(K)$.

Theorem 1 implies that if we have a sequence of balls B_1, B_2, \dots, B_n (of not necessarily equal radii) in a d -dimensional normed space, such that for any $1 \leq i < j \leq n$, the center of B_j is on the boundary of B_i , then $n \leq O(6^d d^2 \log d)$ (Corollary 14 in Section 4). This has recently been improved to an almost tight bound by Polyanskii [Pol17]. This result has an application to k -distance sets [Swa17].

We next consider convex bodies that are not necessarily o -symmetric, and extend the notion of Minkowski arrangement as follows. In the absence of a center, we choose a fixed reference point interior to the convex body.

Definition 1. Let K be a convex body and p a fixed point in the interior of K . A *Minkowski arrangement of K with respect to p* is a family $\{v_i + \lambda_i K\}$ of positive homothets of K with the property that $v_i + p$ is not in $v_j + \lambda_j \text{int}(K)$, for any distinct i and j . We denote the largest number of homothets that a pairwise intersecting Minkowski arrangement of K with respect to p can have by $\kappa(K, p)$.

Similarly, we define a *strict Minkowski arrangement of K with respect to p* to be a family $\{v_i + \lambda_i K\}$ of positive homothets of K such that $v_i + p \notin v_j + \lambda_j K$, for any $i \neq j$, and we write $\kappa'(K, p)$ for the largest number of homothets that a pairwise intersecting strict Minkowski arrangement of K with respect to p can have.

Thus, when K is o -symmetric, $\kappa(K) = \kappa(K, o)$ and $\kappa'(K) = \kappa'(K, o)$. For bodies that are not o -symmetric, we also need to measure in some way how far they are from being o -symmetric.

Definition 2. Let K be a convex body with p in its interior. Define $\theta(K, p)$, the *measure of asymmetry of K with respect to p* to be $\theta(K, p) := \inf\{\theta : p - K \subseteq \theta(K - p)\}$.

(Grünbaum [Grü63, Section 6.1] defines a quantity similar to θ .) Our next result generalizes Theorem 1.

Theorem 2. Let K be a convex body in \mathbb{R}^d with $p \in \text{int}(K)$. Then

$$\kappa'(K, p) \leq \kappa(K, p) \leq \left(\frac{3}{2}\right)^d \frac{\text{vol}(K - K)}{\text{vol}((K - p) \cap (p - K))} O(d(\log d + \log \theta(K, p))).$$

If c is the centroid of K then

$$\kappa'(K, c) \leq \kappa(K, c) \leq 3^d \binom{2d}{d} O(d \log d).$$

There exists a d -dimensional convex body K with centroid c such that $\kappa(K, c) \geq \sqrt{10}^d$ (detailed at the end of Section 3). We prove a strengthening of Theorem 2 in Theorem 8 below.

When K is o -symmetric, Arias-de Reyna, Ball, and Villa [AdRBV98, Theorem 1] derived a lower bound $\Omega((2/\sqrt{3})^d)$ for the strict Hadwiger number $H'(K)$ (see Definition 3 in Section 2), which implies that $\kappa'(K) = \Omega((2/\sqrt{3})^d)$. We show the same lower bound in the non-symmetric case.

Theorem 3. Let K be a convex body with $p \in \text{int}(K)$. Then $\kappa'(K, p) > c(2/\sqrt{3})^d$ for some universal constant $c > 0$.

We also prove a lower bound for a variant $h'(K)$ (see Definition 7 in Section 3) of the strict Hadwiger number $H'(K)$ for K that is not o -symmetric.

Theorem 4. Let K be a convex body in \mathbb{R}^d with $o \in \text{int}(K)$. Then for sufficiently large d ,

$$\kappa'(K, o) \geq h'(K) \geq \frac{1}{4d^2} \left(\frac{2}{\sqrt{3}}\right)^d.$$

The paper is organized as follows. In Section 2, we apply a result of Talata to give a negative answer to the question of Füredi and Loeb quoted at the beginning of the Introduction. In Section 3, we state two stronger versions of Theorem 1 (Theorems 5 and 8). The latter one, which is the main result in this paper, is valid for all (not necessarily centrally symmetric) convex bodies. It is proved in Section 4. The other two main results, Theorems 3 and 4 also hold for non-symmetric bodies. They are proved in the last section. Along the way, we obtain some facts (Lemmas 9 and 15, Theorem 16) that are useful in studying non-symmetric convex bodies in general.

2. A NEGATIVE ANSWER TO THE QUESTION OF FÜREDI AND LOEB

Let K be an o -symmetric convex body in \mathbb{R}^d . Denote the norm with unit ball K by $\|\cdot\|_K$.

Definition 3. For any convex body K , the *Hadwiger number* (resp., *strict Hadwiger number*) of K is defined as the maximum number $H(K)$ (resp., $H'(K)$) of non-overlapping (resp., disjoint) translates of K touching K .

When K is o -symmetric, $H(K)$ equals the maximum number of points v_1, \dots, v_m such that $\|v_i\|_K = 1$ for all i and $\|v_i - v_j\|_K \geq 1$ for all distinct i, j . Then $\{K\} \cup \{K + v_i : i = 1, \dots, m\}$ is a Minkowski arrangement of translates of K all intersecting in o , hence $\kappa(K) \geq H(K) + 1$. Similarly, $H'(K)$ equals the maximum number of points v_1, \dots, v_m such that $\|v_i\|_K = 1$ for all i and $\|v_i - v_j\|_K > 1$ for all distinct i, j . Thus, $\{K + v_i : i = 1, \dots, m\}$ is a strict Minkowski arrangement of translates of K all intersecting in o , hence $\kappa'(K) \geq H'(K)$. To answer the question of Füredi and Loeb in the negative, it is therefore sufficient to find an o -symmetric convex body K with $H'(K) > 2^d$. In dimension 3, we may take the Euclidean ball B^3 , for which it is well known that $H'(B^3) = 12$. For $d > 3$, we may use a result of Talata [Tal05, Lemma 3.1] according to which $H'(C^k \times K) = 2^k H'(K)$ holds for any o -symmetric convex body K , where C^k is the k -dimensional cube. In particular, $H'(B^3 \times C^{d-3}) = 3 \cdot 2^{d-1} > 2^d$ for all $d \geq 3$. In fact, Talata [Tal05, Theorem 1.3] constructed d -dimensional o -symmetric convex bodies K such that $H'(K) \geq \frac{16}{35} \sqrt{7}^d$ for all $d \geq 3$. It follows that for these bodies, $\kappa(K) \geq \kappa'(K) \geq \Omega(\sqrt{7}^d)$.

3. PACKING AND NON-SYMMETRIC NORMS

Definition 4. If K is o -symmetric, we define the *packing number* $P(K, \lambda)$ of K as the maximum number of points in the normed space with unit ball K , such that the ratio of the maximal distance to the minimal distance is at most λ . We denote the normed space with unit ball K as \mathcal{N} , and also use the notations $\kappa(\mathcal{N})$, $P(\mathcal{N}, \lambda)$, $H(\mathcal{N})$, \dots in place of $\kappa(K)$, $P(K, \lambda)$, $H(K)$, \dots

It follows from the isodiametric inequality in normed spaces (an immediate corollary to the Brunn–Minkowski Theorem [Bus47, Mel63]) that

$$(1) \quad P(\mathcal{N}, \lambda) \leq (\lambda + 1)^d$$

for any d -dimensional normed space \mathcal{N} . (See Lemma 7 below for a generalization.) Our next result strengthens Theorem 1.

Theorem 5. *Let \mathcal{N} be a d -dimensional real normed space. Then*

$$\kappa'(\mathcal{N}) \leq \kappa(\mathcal{N}) \leq P(\mathcal{N}, 2(1 + \frac{1}{d})) (d + O(1)) \log d = O(3^d d \log d).$$

Since $\kappa(C^d) \geq H(C^d) + 1 = 3^d$, which shows that the upper bound for $\kappa(\mathcal{N})$ in Theorem 5 is sharp up to the $O(d \log d)$ factor. Theorem 5 is a special case of Theorem 8 below that also deals with K that are not necessarily o -symmetric, considered next.

Definition 5. If the convex body K contains the origin in the interior, we define the (asymmetric) norm $\|\cdot\|_K : \mathbb{R}^d \rightarrow \mathbb{R}$ by $\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$.

Note that the measure of asymmetry of K with respect to o can be defined in terms of the norm:

$$\theta(K, o) = \sup\{\|x\|_K / \|-x\|_K : x \in \text{bd } K\}.$$

We need the following well-known result.

Lemma 6 (Minkowski [Min97]). *For any d -dimensional convex body K with centroid c , $\theta(K, c) \leq d$.*

We will also use the (symmetric) norm defined by the unit ball $K \cap -K$. Thus, $\|x\|_{K \cap -K} = \max\{\|x\|_K, \|-x\|_K\}$. We also need another symmetric norm.

Definition 6. For any convex body K , define its *central symmetral* to be $\frac{1}{2}(K - K)$. If $o \in \text{int}(K)$, then $P(K, \lambda)$ is defined to be the maximum number of points p_1, \dots, p_m such that

$$\frac{\max\{\|p_i - p_j\|_{\frac{1}{2}(K-K)} : 1 \leq i < j \leq m\}}{\min\{\|p_i - p_j\|_{K \cap -K} : 1 \leq i < j \leq m\}} \leq \lambda.$$

If K is o -symmetric, then the norms $\|\cdot\|_K$, $\|\cdot\|_{K \cap -K}$, and $\|\cdot\|_{\frac{1}{2}(K-K)}$ are all identical, and $P(K, \lambda)$ coincides with the definition given before.

Lemma 7. For any convex body K with o in its interior and any $\lambda > 0$,

$$P(K, \lambda) \leq (\lambda + 1)^d \frac{\text{vol}(\frac{1}{2}(K - K))}{\text{vol}(K \cap -K)}.$$

We also need to generalize the Hadwiger number to the non-symmetric case, in the following non-standard way.

Definition 7. If $o \in \text{int}(K)$, define $h(K)$ to be the maximum number of points p_1, \dots, p_m on $\text{bd } K$ such that $\|p_i - p_j\|_K \geq 1$ for all distinct $i, j = 1, \dots, m$. Similarly, we define $h'(K)$ to be the maximum number of points $p_1, \dots, p_m \in \text{bd } K$ such that $\|p_i - p_j\|_K > 1$ for all distinct $i, j = 1, \dots, m$.

If $K = -K$, then $h(K) = H(K)$ and $h'(K) = H'(K)$ (cf. Definition 3). This is not necessarily the case if K is not o -symmetric. (Note that for all convex bodies, $H(K) = H(\frac{1}{2}(K - K))$.) Generalizing our observation for the symmetric case above, if $p_1, \dots, p_m \in \text{bd } K$ satisfy $\|p_i - p_j\|_K > 1$ for all distinct i, j , then the collection $\{K - p_i : i = 1, \dots, m\}$ is a pairwise intersecting strict Minkowski arrangement of translates of K , hence $\kappa'(K, o) \geq h'(K)$. Similarly (by adding K to the collection) we have $\kappa(K, o) \geq h(K) + 1$. We can now formulate our generalization of Theorem 2.

Theorem 8. Let K be a convex body in \mathbb{R}^d with $o \in \text{int}(K)$. Then

$$\kappa'(K, o) \leq \kappa(K, o) \leq P\left(K, 2\left(1 + \frac{1}{d}\right)\right) (d + O(1)) (\log d + \log \theta(K, o)).$$

If c is the centroid of K then

$$\kappa(K, c) \leq P\left(K, 2\left(1 + \frac{1}{d}\right)\right) (2d + O(1)) \log d.$$

The proof is postponed to the next section. The following is an example of a d -dimensional convex body K with centroid c for which $\kappa(K, c)$ is much larger than in the symmetric case. Note that $\kappa(\Delta, o) = 10$, where Δ is a triangle with centroid o [FT95] (see Fig. 2). A Cartesian product of $\lfloor d/2 \rfloor$ triangles gives a d -dimensional convex body K with centroid c such that $\kappa(K, c) \geq 10^{\lfloor d/2 \rfloor} \geq \sqrt{10}^{d-1}$.

4. BOUNDING κ FROM ABOVE

Proof of Lemma 7. Let $T \subset \mathbb{R}^d$ be such that $\|x - y\|_{K \cap -K} \geq 1$ and $\|x - y\|_{\frac{1}{2}(K-K)} \leq \lambda$ for all distinct $x, y \in T$. Then $\{v + \frac{1}{2}(K \cap -K) : v \in T\}$ is a packing. Let $P = T + \frac{1}{2}(K \cap -K)$. Then $\text{vol}(P) = 2^{-d} |T| \text{vol}(K \cap -K)$ and

$$P - P = T - T + (K \cap -K) \subseteq \frac{\lambda}{2}(K - K) + \frac{1}{2}(K - K) = \frac{\lambda + 1}{2}(K - K).$$

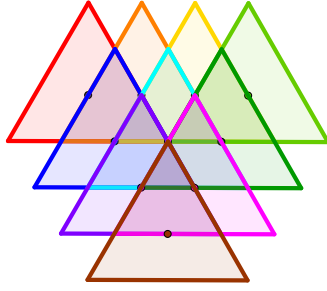


FIGURE 2. A pairwise intersecting Minkowski arrangement of 10 triangles [FT95]

By the Brunn–Minkowski inequality, $\text{vol}(P - P) \geq 2^d \text{vol}(P)$, and it follows that

$$|T| = \frac{2^d \text{vol}(P)}{\text{vol}(K \cap -K)} \leq \frac{\text{vol}(P - P)}{\text{vol}(K \cap -K)} \leq \frac{(\lambda + 1)^d \text{vol}(\frac{1}{2}(K - K))}{\text{vol}(K \cap -K)}. \quad \square$$

Before we prove Theorem 8, we first show an extension of the so-called “bow-and-arrow” inequality of [FüL94] (Corollary 10 below) to the case of an asymmetric norm.

Definition 8. For any non-zero $v \in \mathbb{R}^d$ write $v^\wedge = \frac{1}{\|v\|_K} v$ for the normalization of v with respect to $\|\cdot\|_K$.

We will only consider normalizations with respect to $\|\cdot\|_K$, never with respect to $\|\cdot\|_{K \cap -K}$ or $\|\cdot\|_{\frac{1}{2}(K - K)}$.

Lemma 9. Let K be a convex body in \mathbb{R}^d containing o in its interior. Let $a, b \in \mathbb{R}^d$ such that $\|a\|_K \geq \|b\|_K > 0$. Then

$$\|a^\wedge - b^\wedge\|_K \geq \frac{\|a - b\|_K - \|a\|_K + \|b\|_K}{\|b\|_K}.$$

Proof.

$$\begin{aligned} \|a - b\|_K &= \| \|a\|_K a^\wedge - \|b\|_K b^\wedge \|_K \\ &= \| \|b\|_K (a^\wedge - b^\wedge) + (\|a\|_K - \|b\|_K) a^\wedge \|_K \\ &\leq \|b\|_K \|a^\wedge - b^\wedge\|_K + \|a\|_K - \|b\|_K. \end{aligned} \quad \square$$

Corollary 10. For any two non-zero elements a and b of a normed space,

$$\|a^\wedge - b^\wedge\| \geq \frac{\|a - b\| - \| \|a\| - \|b\| \|}{\|b\|}.$$

Proof of Theorem 8. Consider a pairwise intersecting Minkowski arrangement $\{\lambda_i K + v_i : i = 1, \dots, m\}$. Without loss of generality, $\lambda_1 = \min_i \lambda_i = 1$ and $v_1 = o$. Given $N \in \mathbb{N}$ and $\delta > 0$, we partition the Minkowski arrangement into N subarrangements as follows. Let $I_j = \{i : \lambda_i \in [(1 + \delta)^{j-1}, (1 + \delta)^j]\}$ for each $j = 1, \dots, N$, and let $I_\infty = \{i : \lambda_i \in [(1 + \delta)^N, \infty)\}$. We bound the size of each subarrangement $\{\lambda_i K + v_i : i \in I_j\}$, $j \in \{1, \dots, N, \infty\}$, separately. Finally, we choose appropriate values for N and δ .

The next lemma bounds I_j , $j \neq \infty$, in terms of δ and K .

Lemma 11. Let K be a d -dimensional convex body with $o \in \text{int}(K)$. Let $\{v_i + \lambda_i K : i \in I\}$ be a pairwise intersecting Minkowski arrangement of positive homothets of K , with $\lambda_i \in [1, 1 + \delta)$ for each $i \in I$. Then

$$|I| \leq P(K, 2(1 + \delta)).$$

Proof. For any distinct $i, j \in I$, $(v_i + \lambda_i K) \cap (v_j + \lambda_j K) \neq \emptyset$, so there exist $x, y \in K$ such that $v_i - v_j = \lambda_j y - \lambda_i x$. Since $\lambda_i, \lambda_j \in [1, 1 + \delta]$, $o \in K$ and K is convex, $\lambda_i x, \lambda_j y \in (1 + \delta)K$. Hence, $v_i - v_j \in (1 + \delta)(K - K)$ and $\|v_i - v_j\|_{\frac{1}{2}(K-K)} \leq 2(1 + \delta)$. Since $v_i \notin v_j + \lambda_j \text{int}(K)$, it follows that $v_i - v_j \notin \text{int}(K \cap -K)$ for all distinct $i, j \in I$, which gives $\|v_i - v_j\|_{K \cap -K} \geq 1$. \square

The following lemma is used to bound I_∞ .

Lemma 12. *Let K be a d -dimensional convex body with $o \in \text{int}(K)$. Let $\{v_i + \lambda_i K : i \in I\}$ be a Minkowski arrangement of positive homothets of K with $\lambda_i \geq 1$, $(v_i + \lambda_i K) \cap -\varepsilon K \neq \emptyset$ and $o \notin v_i + \lambda_i \text{int}(K)$ for all $i \in I$. Then*

$$|I| \leq P\left(K, \frac{2}{1 - \varepsilon}\right).$$

We first consider any two homothets in the Minkowski arrangement of the previous lemma.

Lemma 13. *Let $v_1 + \lambda_1 K$ and $v_2 + \lambda_2 K$ be two positive homothets of K such that $\lambda_1, \lambda_2 \geq 1$, $v_1 \notin v_2 + \lambda_2 \text{int}(K)$, $v_2 \notin v_1 + \lambda_1 \text{int}(K)$, $o \notin v_i + \lambda_i \text{int}(K)$ and $(v_i + \lambda_i K) \cap -\varepsilon K \neq \emptyset$ ($i = 1, 2$). Then $\left\| \frac{1}{\| -v_1 \|_K} (-v_1) - \frac{1}{\| -v_2 \|_K} (-v_2) \right\|_{K \cap -K} \geq 1 - \varepsilon$.*

Proof. Since $\|\cdot\|_{K \cap -K}$ is symmetric, we may assume that $\| -v_1 \|_K \leq \| -v_2 \|_K$. Since $(v_1 + \lambda_1 K) \cap -\varepsilon K \neq \emptyset$, $v_1 + \lambda_1 x = -\varepsilon y$ for some $x, y \in K$. Therefore, $\| -v_1 \|_K \leq \lambda_1 \|x\|_K + \varepsilon \|y\|_K \leq \lambda_1 + \varepsilon$. Also, since $o \notin v_1 + \lambda_1 \text{int}(K)$, we have that $\| -v_1 \|_K \geq \lambda_1$. Similarly, $\lambda_2 \leq \| -v_2 \|_K \leq \lambda_2 + \varepsilon$, and it follows from $v_1 \notin v_2 + \lambda_2 \text{int}(K)$ that $\|v_1 - v_2\|_K \geq \lambda_2$. We apply Lemma 9 to obtain

$$\begin{aligned} \|(-v_1)^\wedge - (-v_2)^\wedge\|_{K \cap -K} &\geq \|(-v_2)^\wedge - (-v_1)^\wedge\|_K \\ &\geq \frac{\|v_1 - v_2\|_K - \| -v_2 \|_K + \| -v_1 \|_K}{\| -v_1 \|_K} \\ &\geq \frac{\lambda_2 - (\lambda_2 + \varepsilon) + \| -v_1 \|_K}{\| -v_1 \|_K} \\ &= 1 - \frac{\varepsilon}{\| -v_1 \|_K} \geq 1 - \frac{\varepsilon}{\lambda_1} \geq 1 - \varepsilon. \end{aligned} \quad \square$$

Proof of Lemma 12. For each $i \in I$, let $t_i = (-v_i)^\wedge$. Let $T := \{t_i : i \in I\}$. By Lemma 13, $\|t_i - t_j\|_{K \cap -K} \geq 1 - \varepsilon$ for all distinct $i, j \in I$. Since $T \subset \text{bd } K \subset K$, $\|t_i - t_j\|_{\frac{1}{2}(K-K)} \leq 2$. It follows that $|I| \leq P(K, 2/(1 - \varepsilon))$. \square

We now finish the proof of Theorem 8. By Lemma 11, $|I_j| \leq P(K, 2(1 + \delta))$ for $j = 1, \dots, N$, and by Lemma 12 applied to I_∞ and $\varepsilon = \theta(K, o)(1 + \delta)^{-N}$,

$$|I_\infty| \leq P\left(K, \frac{2}{1 - \theta(K, o)(1 + \delta)^{-N}}\right).$$

It follows that

$$m = \sum_{j=1}^N |I_j| + |I_\infty| \leq NP(K, 2(1 + \delta)) + P\left(K, \frac{2}{1 - \theta(K, o)(1 + \delta)^{-N}}\right).$$

We now choose

$$N := 1 + \left\lceil \frac{\log d + \log \theta(K, o)}{\log(1 + \frac{1}{d})} \right\rceil = (d + O(1))O(\log d + \log \theta(K, o))$$

and $\delta = 1/d$. Then

$$N \geq 1 + \frac{\log d + \log \theta(K, o)}{\log(1 + \delta)},$$

which implies that

$$\frac{2}{1 - \theta(K, o)(1 + \delta)^{-N}} \leq 2(1 + \delta),$$

hence

$$m \leq P(K, 2(1 + \frac{1}{d})) (N + 1) = P(K, 2(1 + \frac{1}{d})) (d + O(1))(\log d + \log \theta(K, o)).$$

The second inequality follows from the first and Lemma 6. \blacksquare

Note that Theorem 5 immediately follows from Theorem 8, and Theorem 1 from Theorem 5.

Proof of Theorem 2. The first statement follows from Theorem 8 and Lemma 7. Also, by a result of Milman and Pajor [MP00, Corollary 3] for a convex body K with centroid o , $\text{vol}(K)/\text{vol}(K \cap -K) \leq 2^d$, which, together with the Rogers–Shephard inequality [RS57] $\text{vol}(K - K)/\text{vol}(K) \leq \binom{2d}{d}$, gives the second statement. \square

We derive the following application of Theorem 1.

Corollary 14. *Let K be an o -symmetric convex body, and p_1, p_2, \dots, p_n be points in \mathbb{R}^d . Let $r_1, r_2, \dots, r_n > 0$, and assume that for any $1 \leq i < j \leq n$, we have that $p_j \in p_i + r_i \text{bd } K$. Then $n \leq O(6^d d^2 \log d)$.*

Proof. Let $D \subseteq \{1, 2, \dots, n-1\}$ be the index set of a longest decreasing subsequence of r_1, r_2, \dots, r_{n-1} . Thus, if $i, j \in D$ with $i < j$, then $r_i \geq r_j$. Then $\{p_i + r_i K : i \in D\}$ is a pairwise intersecting Minkowski arrangement, and by Theorem 1, $|D| = O(3^d d \log d)$.

Next, let $I \subseteq \{1, 2, \dots, n-1\}$ be the index set of a longest increasing subsequence of r_1, r_2, \dots, r_{n-1} , that is, if $i, j \in I$ with $i < j$, then $r_i \leq r_j$. Let $m := \min I$. By the triangle inequality we have $r_i \leq 2r_m$ for any $i \in I$. Indeed, without loss we may assume $i \neq m$, and then, since $m < i < n$, $\|p_i - p_m\|_K = \|p_n - p_m\|_K = r_m$ and $\|p_n - p_i\|_K = r_i$, from which $r_i \leq 2r_m$ follows.

We now use the same “logarithmic cut” method as in the proof of Theorem 8. Choose $N \in \mathbb{N}$. For each $k = 1, \dots, N$, let $I_k := \{i \in I : r_i/r_m \in [2^{(k-1)/N}, 2^{k/N}]\}$. Then $\|p_i - p_j\|_K \in [2^{(k-1)/N}, 2^{k/N}]$ for any distinct $i, j \in I_k$, hence

$$|I_k| \leq P(K, 2^{1/N}) \leq (1 + 2^{1/N})^d$$

by (1), and

$$|I| = \sum_{k=1}^N |I_k| \leq N(1 + 2^{1/N})^d.$$

We now choose an optimal value $N := d$ to obtain $|I| \leq d(1 + 2^{1/d})^d = O(2^d d)$.

By the Erdős–Szekeres Theorem [ES35], any sequence of real numbers for which all decreasing subsequences are of length at most s and all increasing subsequences are of length at most t , has length at most st . It follows that $n - 1 \leq |D| \cdot |I|$, hence

$$n = O(3^d d \log d) O(2^d d) = O(6^d d^2 \log d),$$

as claimed. \square

5. BOUNDING κ' FROM BELOW

In this section we prove Theorems 3 and 4, by extending a lower bound for the strict Hadwiger number $H'(K)$ by Arias-de Reyna, Ball, and Villa [AdRBV98, Theorem 1] to non-symmetric convex bodies. Earlier, Bourgain [FüL94] showed an exponential lower bound to $H'(K)$ for o -symmetric K that depends only on the dimension of K . (This argument was also independently discovered by Talata [Tal98].) The key tool used by Bourgain and Talata is Milman's Quotient of Subspace Theorem (or, in short, the QS Theorem) [Mil85].

In order to obtain a lower bound on $\kappa(K, p)$ in the non-symmetric case, it is possible to use a non-symmetric version of the QS Theorem (see Milman and Pajor [MP00] or Rudelson [Rud00]), or one may generalize the approach from [AdRBV98]. The first approach does not lead to a concrete lower bound, and we will follow the second. However, neither approach is straightforward. One obstacle is that p may not coincide with the centroid of K . To bypass this problem, we use the following topological result.

Lemma 15. *Let K be a convex body in \mathbb{R}^d . Then there is a $(d - 1)$ -dimensional linear subspace H of \mathbb{R}^d such that the centroid of the orthogonal projection of K onto H is the origin.*

Statements similar to this lemma are known (see for instance [Izm14]). The lemma itself is surely also known. However, since we could not find a reference, we include its simple proof.

Proof. For any unit vector $u \in \mathbb{S}^{d-1}$, let $f(u)$ be the centroid of the orthogonal projection of K onto u^\perp . We need to show that $f(u) = o$ for some $u \in \mathbb{S}^{d-1}$. Suppose not. Then $f^\wedge: \mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1}$ defined by $f^\wedge(u) = f(u) / \|f(u)\|_2$ is a continuous, even mapping such that $\langle u, f^\wedge(u) \rangle = 0$ for all $u \in \mathbb{S}^{d-1}$. Since f^\wedge is even, its degree is even (see for instance [Hat02, Proposition 2.30]). Also, $f^\wedge(u) \neq -u$ for all $u \in \mathbb{S}^{d-1}$. It follows that f^\wedge is homotopic to the identity map, which has degree 1, a contradiction. \square

We briefly outline how this lemma can be combined with the non-symmetric QS Theorem to obtain that $h'(K) > c^d$ for some universal constant $c > 1$. Later on in this section, we will obtain more explicit bounds (Theorems 3 and 4) using the main result of [AdRBV98].

First, the non-symmetric version of the QS Theorem ([MP00, Theorem 9] and [Rud00, Theorem 4]), combined with Lemma 15 yields that for any convex body K in \mathbb{R}^d , there is a roughly $(d/2)$ -dimensional subspace E and an origin centered ellipsoid \mathcal{E} in E , such that for an appropriate projection P of \mathbb{R}^d , we have $\mathcal{E} \subseteq P(K) \cap E \subseteq c\mathcal{E}$ with some universal constant c . By a theorem of Milman [Mil71] (see also [MS86, Section 4.3]), we can take a $C(d/2)$ -dimensional subspace E' of E such that $\mathcal{E}' := \mathcal{E} \cap E' \subseteq P(K) \cap E' \subseteq 1.1\mathcal{E}'$, for a universal constant $C > 0$. Although this is stated only for symmetric bodies K in [MS86], the proof works in the non-symmetric case as well.

Now, we can follow the proof of the symmetric case (Theorem 4.3) in [FüL94] closely. There are exponentially many points on the relative boundary of $K' := P(K) \cap E'$ such that the distance (with respect to the slightly non-symmetric norm $\|\cdot\|_{K'}$ on E') between any two points is at least 1.21. Let X be the set of these points. For every $x \in X$, choose a point $p \in \text{bd } K$ such that $P(p) = x$. These points satisfy Definition 7.

Before we prove Theorems 3 and 4, we state a non-symmetric version of [AdRBV98, Theorem 1].

Theorem 16. *Let K be a convex body in \mathbb{R}^d with $o \in \text{int}(K)$. Let μ denote the uniform (with respect to Lebesgue measure) probability measure on K . Then, for any $0 < t < \sqrt{2}$,*

$$F(t) := \mu \otimes \mu\{(x, y) \in K \times K : \|x - y\|_K \leq t\} \leq \left(\frac{t^2(4 - t^2)}{4} \right)^{d/2}.$$

The proof of Theorem 16 is virtually the same as in [AdRBV98]. We recall the first part of this proof, which is where the only (slight) difference lies. In that proof, $\mu \otimes \mu\{(x, y) \in K \times K : \|x - y\|_K \leq t\}$ is written as a threefold convolution. In the non-symmetric case it is easy to see that for any $t \geq 0$,

$$F(t) = \mu \otimes \mu\{(x, y) \in K \times K : \|x - y\|_K \leq t\} = (\chi_K * \chi_{-K} * \chi_{tK})(0),$$

where χ_A denotes the indicator function of a set A . The only difference with the symmetric case is the occurrence of χ_{-K} instead of χ_K in the right-hand side. This does not affect the rest of the proof in [AdRBV98], which is an application of a strong form of Young's inequality to this threefold convolution, and which we do not repeat.

Proof of Theorem 3. Let $m = \frac{1}{4}(2/\sqrt{3})^d$, and choose $2m$ points x_1, \dots, x_{2m} independently and uniformly from K . Then the expected number of ordered pairs (x_i, x_j) , $i \neq j$, such that $\|x - y\|_K \leq 1$, equals $2m(2m - 1)F(1)$ by linearity of expectation. This quantity is at most m by Theorem 16 and the choice of m . Thus, there exists a choice of points x_1, \dots, x_{2m} from K such that $\|x_i - x_j\|_K > 1$ for all except at most m pairs (i, j) , $i \neq j$. For each such pair, delete one of the points. We end up with m points x_1, \dots, x_m , say, such that $\|x_i - x_j\|_K > 1$ for all distinct i, j . It follows that $-x_1 + K, \dots, -x_m + K$ is a strict Minkowski arrangement. This family of translates of K is clearly pairwise intersecting, since all members contain the origin. \square

If K is o -symmetric, it follows from Corollary 10 that if $a, b \in K$ satisfy $\|a - b\|_K > 1$, then $\|a^\wedge - b^\wedge\|_K > 1$. We therefore obtain the lower bound $h'(K) = H'(K) = \Omega((2/\sqrt{3})^d)$ by normalizing the points x_i in the proof above. When K is not o -symmetric, the probabilistic argument above can be adapted to obtain the slightly worse lower bound $h'(K) = \Omega((2/\sqrt{3})^d/d^2)$ of Theorem 4. The proof is technically more involved, and the details are as follows.

Proof of Theorem 4. We first assume that o is the centroid of K . By Lemma 6, $\theta(K, o) \leq d$, hence $\|-x\|_K \leq d\|x\|_K$ for all $x \in \mathbb{R}^d$. Let $k \in \mathbb{N}$, to be fixed later, and choose k points x_1, \dots, x_k independently and uniformly from K . Let δ be such that $e^{\delta d} = (d + 4)/(d + 1)$ (thus, $\delta \sim 3/d^2$). Then the expected number of points x_i such that $\|x_i\|_K \leq 1 - \delta$ (we call these points *short*) equals

$$(1 - \delta)^d k < e^{-\delta d} k = \left(1 - \frac{3}{d + 4}\right) k.$$

We say that an ordered pair (x_i, x_j) , $i \neq j$, is *close* if $\|x_i - x_j\|_K \leq 1 + (d+1)\delta$. Then by Theorem 16, the expected number of ordered close pairs is less than

$$\begin{aligned} & k^2 \left(\frac{(1 + (d+1)\delta)^2 (4 - (1 + (d+1)\delta)^2)}{4} \right)^{d/2} \\ & < k^2 \left(\frac{\sqrt{3}}{2} \right)^d \left(1 + \frac{4}{3}(d+1)\delta \right)^{d/2} \\ & < k^2 \left(\frac{\sqrt{3}}{2} \right)^d e^{2\delta d(d+1)/3} = k^2 \left(\frac{\sqrt{3}}{2} \right)^d \left(\frac{d+4}{d+1} \right)^{2(d+1)/3} \\ & < e^2 k^2 \left(\frac{\sqrt{3}}{2} \right)^d. \end{aligned}$$

Thus, if we delete each short point, as well as one member of each close pair, then the expected number m of points left is at least

$$m \geq k \left(\frac{3}{d+4} - \left(\frac{\sqrt{3}}{2} \right)^d e^2 k \right).$$

To maximize this quadratic expression in k , we set $k = \frac{1}{2}(2/\sqrt{3})^d \frac{3}{e^2(d+4)}$. Thus, there exist at least

$$m \geq \left(\frac{2}{\sqrt{3}} \right)^d \frac{9}{4e^2(d+4)^2}$$

points $x_1, \dots, x_m \in K$ such that $\|x_i\| > 1 - \delta$ for each i and $\|x_i - x_j\|_K > 1 + (d+1)\delta$ for each pair of distinct i, j . Normalize these points to obtain

$$x_1^\wedge = \frac{1}{\|x_1\|_K} x_1, \dots, x_m^\wedge = \frac{1}{\|x_m\|_K} x_m \in \text{bd } K.$$

Note that $\|x_i^\wedge - x_i\|_K < \delta$, hence $\|x_i - x_i^\wedge\|_K < d\delta$. By the triangle inequality, for distinct i, j we have

$$\begin{aligned} \|x_i^\wedge - x_j^\wedge\|_K & \geq \|x_i - x_j\|_K - \|x_i - x_i^\wedge\|_K - \|x_j^\wedge - x_j\|_K \\ & > 1 + (d+1)\delta - d\delta - \delta = 1. \end{aligned}$$

Therefore, the requirements of Definition 7 are satisfied.

Next, we reduce the case when o is an arbitrary point in $\text{int}(K)$ to the case where o is the centroid of K . By Lemma 15, there is an orthogonal projection $\pi: \mathbb{R}^d \rightarrow H$ where H is a linear subspace of dimension $d-1$ such that $\pi(K)$ has centroid o . By what was shown above, there exist $m \geq (2/\sqrt{3})^{d-1} 9/(4e^2(d+3)^2)$ points $y_1, \dots, y_m \in \text{bd } \pi(K)$ such that $\|y_i - y_j\|_{\pi(K)} > 1$ for all distinct i, j . For each i , choose $p_i \in \pi^{-1}(y_i) \in K$. Since $\|x\|_K \geq \|\pi(x)\|_{\pi(K)}$ for all $x \in \mathbb{R}^d$, we obtain that $p_1, \dots, p_m \in \text{bd } K$ satisfy $\|p_i - p_j\|_K \geq \|y_i - y_j\|_{\pi(K)} > 1$ for all distinct i, j . This finishes the proof of Theorem 4 in the general case. \square

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