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Article (Accepted version)
(Refereed)

Original citation:
Wong, Shiu Fung, Tong, Howell, Siu, Tak Kuen and Lu, Zudi (2017) *A new multivariate nonlinear time series model for portfolio risk measurement: the threshold copula-based TAR approach.* Journal of Time Series Analysis, 38 (2). pp. 243-265. ISSN 0143-9782

DOI: 10.1111/jtsa.12206

A New Multivariate Nonlinear Time Series Model for Portfolio Risk Measurement: The Threshold Copula-Based TAR Approach

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We propose a threshold-copula-based nonlinear time series model for evaluating quantitative risk measures for financial portfolios with a flexible structure to incorporate nonlinearities in both univariate (component) time series and their dependent structure. We incorporate different dependent structures of asset returns over different market regimes, which are manifested in their price levels. We estimate the model parameters by a two-stage maximum likelihood method. Real financial data and appropriate statistical tests are used to illustrate the efficacy of the proposed model. Simulated Results for sampling distribution of parameters estimates are given. Empirical results suggest that the proposed model leads to significant improvement of the accuracy of Value-at-Risk forecasts at the portfolio level.
Keywords: Quantitative Risk Measures; Copulas; Multivariate Nonlinear Time Series;Threshold Principle.
JEL : C10 C32 C51 G32

1 Introduction

The global financial crisis (GFC) of 2008-2009 provides a practical motivation to reappraise the current market practice of financial risk management. Indeed, some quantitative methods for risk management have been questioned. Some pricing and hedging formulae for sophisticated credit derivatives involving dependent risk such as multi-name credit default swaps (CDSs) and collateralised debt obligations (CDOs), which were developed based on Gaussian copulas, have been criticised.

What's wrong with the use of Gaussian copulas in financial markets? We believe that there are two major limitations of Gaussian copulas. Firstly, Gaussian copulas are static in nature. It fails to incorporate the situation where the dependent structure changes dynamically over time according to changing market conditions. Secondly, Gaussian copulas cannot capture "extreme" dependent structures attributed to extreme market moves. In particular, during periods of market crash or financial crisis, returns from financial assets tend to drop together. Gaussian copulas cannot describe this extreme dependent structure of asset returns. This will lead to significant underestimation of dependent risk or systemic risk and has important consequences for the stability of financial markets and the whole economy. Indeed, some of the shortcomings of Gaussian copulas may be shared with other popular copula functions used in practice. In fact, before the GFC, some researchers have already started addressing these shortcomings of Gaussian copulas or copulas in general. Unfortunately, it appears that their ideas and methods were not properly appreciated in the finance industry.

To articulate the problem arising from the static nature of copulas, the notion of dynamic copulas has been introduced with a view to modelling dependent structures in multivariate financial time series models. Patton (2002, 2009) defined the concept of conditional copulas with a view to developing a copula-based multivariate time series model. Chen and Fan (2006) developed a copula-based semi-parametric time series model. Dias and Embrechts (2003) employed a dynamic copula approach to develop a multivariate GARCH-type model for modelling high-frequency financial data. Lee et al. (2006) proposed a copula-based multivariate GARCH model with uncorrelated dependent and asymmetrically distributed innovations. These works mainly focus on the application of copulas to build multivariate conditional heteroscedastic time series models, such as the multivariate GARCH models. Some attention has been paid to model the impact of structural changes in hidden economic conditions, i.e. the regime-switching effect, on dependent structures using copulas. Chollete (2005) introduced a hidden Markov-modulated copula modelling of multivariate time series to explain the dynamic clustering of correlations attributed to transitions of turbulent and quiescent periods in international markets. However, the presence of the hidden factors apparently makes the model not easy to implement in a multivariate setting. da Silva Filho et al. (2012) modelled the dependence dynamics

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using copulas with regime switching under hidden Markov chain and use block bootstrapping in estimating covariance matrix. It appears that relatively little attention has been paid to the application of copulas to describe dynamic nonlinear dependent structures in the conditional mean functions of multivariate nonlinear time series models.

In this paper, we introduce a threshold copula-based nonlinear time series model with a view to estimating or forecasting quantitative risk measures of financial portfolios. Our aim is to provide a reasonably flexible structure to incorporate nonlinearities in both the univariate time series and their inter-dependence. The key idea of the proposed threshold copula-based approach is the threshold principle which was introduced to nonlinear time series analysis by one of us in the late 1970s. See, for example, Tong (1977, 1978, 1980, 1983) and Tong and Lim (1980). We develop a threshold copula-based multivariate self-exciting threshold autoregressive model, or in short Copula-Threshold Autoregression (Copula-TAR). This model is a natural extension of the self-exciting threshold autoregressive (SETAR) model introduced by Tong (*op. cit.*), which is possibly the oldest parametric nonlinear time series model still in active service. The Copula-TAR model has two levels of threshold structure, one for the univariate time series components and the other for the dependent structure of these components, which is modeled by the threshold copulas. The threshold structure for the univariate time series coincides with that of the univariate SETAR models, where the regimes in the threshold structure are determined by threshold parameters dividing the state space according to some past value of the univariate time series. A new ingredient is introduced to model the dependence among the time series components via some appropriate thresholding. Specifically, in the Copula-TAR approach, we allow both the functional form of the copula function and the parameters to vary across different regimes. An interesting particular case of the proposed model, namely the piecewise, or local, copula is considered. The proposed structure can address the two major limitations of Gaussian copulas and possibly other commonly used copula functions as discussed above. It can additionally incorporate the dynamic nonlinear dependent structure in the conditional mean functions of multivariate time series models. We propose a two-stage maximum likelihood estimation method. We demonstrate the effectiveness of the estimation method by simulation examples. Real financial data and appropriate statistical tests are used to illustrate the efficacy and practical utility of the proposed model in quantitative risk measurement at a portfolio level.

The rest of this paper is structured as follows. The next section presents the proposed Copula-TAR model and gives some practical examples in finance to illustrate the main idea of the model. We also discuss the rationale of the Copula-TAR model and compare it to some major existing approaches. Section 3 discusses the estimation method. Section 4 conducts simulation studies to illustrate the qualitative behaviour of the proposed model as well as to demonstrate the efficacy of the estimation method. In Section 5, using real financial data, we compute a popular risk metric in the finance and insurance industries, namely Value at Risk (VaR), based on the proposed model. Backtesting using a binomial test is conducted to evaluate the performance of the proposed model in VaR forecasts. The final section gives concluding remarks.

2 The Copula-TAR Model

In this section, we present the main idea of the threshold copula-based multivariate SETAR, (Copula-TAR), model by considering two financial time series $X := \{X_t | t = 1, 2, \cdots, T\}$ and $Y := \{Y_t | t = 1, 2, \cdots, T\}$. The generalisation to the multivariate setting follows immediately. For illustration, we first assume that the marginal distributions of the two financial time series X and Y are specified by two different self-exciting threshold autoregressive, (SETAR), models. In general, one may consider some second-generation nonlinear time series models, such as SETAR-GARCH models, as described in Tong (1990) as the marginal time series models. Conditional on this parametrisation, we introduce the dependent structure between X and Y using a bivariate self-exciting threshold copula.

Firstly, we suppose that X and Y are governed by the following two k regimes SETAR models with delay parameter being one:

$$
\begin{aligned}
X_t &= \sum_{i=1}^{k} \left(\alpha_0^{(i)} + G^{(i)}(L) X_t + \sigma^{(i)} \epsilon_t \right) I_{\{r_{i-1}^X < X_{t-1} \leq r_i^X\}} \ , \\
\epsilon_t | \mathcal{I}_{t-1}^X &\sim D_1(0,1) \ , \\
Y_t &= \sum_{i=1}^{k} \left(\beta_0^{(i)} + H^{(i)}(L) Y_t + \phi^{(i)} \eta_t \right) I_{\{r_{i-1}^Y < Y_{t-1} \leq r_i^Y\}} \ , \\
\eta_t | \mathcal{I}_{t-1}^Y &\sim D_2(0,1) \ ,
\end{aligned}
\tag{1}
$$

where

1. The composite operators $G^{(i)}$ and $H^{(i)}$ specify the autoregressive model for X and Y in regime i. Here L is the lag operator, $LX_t = X_{t-1}$, e.g. if X is of $m(i)$-th order in regime i, $G^{(i)}(L) = \alpha_1^{(i)} L + \alpha_2^{(i)} L^2 + \cdots + \alpha_{m(i)}^{(i)} L^{m(i)}$;
2. ϵ_t and η_t are random error terms following continuous distributions with zero mean and unit variance, denoted by $D_1(0,1)$ and $D_2(0,1)$, respectively;

3. $\mathcal{I}^X_{t-1}$ and $\mathcal{I}^Y_{t-1}$ are defined as the σ-fields generated by the random variables $\{X_1, X_2, \ldots, X_{t-1}\}$ and $\{Y_1, Y_2, \ldots, Y_{t-1}\}$ respectively;

4. r^X_i and r^Y_i, $i = 0, 1, \cdots, k$, are the threshold parameters for X and Y, respectively; these parameters satisfy the following constraints:

$$-\infty = r^X_0 < r^X_1 < \cdots < r^X_{k-1} < r^X_k = \infty \ ,$$
$$-\infty = r^Y_0 < r^Y_1 < \cdots < r^Y_{k-1} < r^Y_k = \infty \ ,$$

however, the numbers of regimes are not limited to be the same in X and Y;

5. I_E is the indicator function of an event E;

6. for each $i = 1, 2, \cdots, k$, $\alpha^{(i)}_0$, $\alpha^{(i)}_1$, $\ldots$, $\alpha^{(i)}_{m(i)}$ and $\sigma^{(i)}$ are the parameters in the autoregressive model for X in the i^{th}-regime, while $\beta^{(i)}_0$, $\beta^{(i)}_1$, $\ldots$, $\beta^{(i)}_{n(i)}$ and $\phi^{(i)}$ are the parameters in the autoregressive model for Y in the i^{th}-regime;

7. We assume the marginal time series of X and Y follow univariate SETAR model. Series depends on the past of another series falls into the category of vector SETAR models which is beyond of the scope of this paper.

We now present the main idea of the self-exciting threshold copula. Let $F^X_t(x|\mathcal{I}^X_{t-1})$ and $F^Y_t(y|\mathcal{I}^Y_{t-1})$ be the conditional cumulative distribution functions of X_t and Y_t given $\mathcal{I}^X_{t-1}$ and $\mathcal{I}^Y_{t-1}$, respectively. That is,

$$\begin{aligned} F^X_t(x|\mathcal{I}^X_{t-1}) &= P[X_t \le x|\mathcal{I}^X_{t-1}] \ , \\ F^Y_t(y|\mathcal{I}^Y_{t-1}) &= P[Y_t \le y|\mathcal{I}^Y_{t-1}] \ . \end{aligned}$$

Let $U_t := F^X_t(X_t|\mathcal{I}^X_{t-1})$ and $V_t := F^Y_t(Y_t|\mathcal{I}^Y_{t-1})$, for each $t = 1, 2, \cdots, T$. Note that U_t and V_t are $\mathcal{I}^X_t$-measurable and $\mathcal{I}^Y_t$-measurable respectively. Then conditional on $\mathcal{I}^X_{t-1}$ and $\mathcal{I}^Y_{t-1}$, U_t and V_t are continuous uniform random variables on $(0, 1)$ respectively. For each $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, k$, let $C^{i,j}_{\theta_{i,j}} : [0, 1]^2 \to [0, 1]$ be a bivariate copula function indexed by the parameter $\theta_{i,j}$. For each $t = 1, 2, \cdots, T$, let $\{r^U_{i,t-1}|i = 0, 1, \cdots, k\}$ and $\{r^V_{j,t-1}|j = 0, 1, \cdots, k\}$ be the sets of threshold parameters dividing the state spaces of U_{t-1} and V_{t-1}, respectively. We suppose that for each $t = 1, 2, \cdots, T$, $r^U_{i,t-1}$ and $r^V_{i,t-1}$ are $\mathcal{I}^X_{t-1}$-measurable and $\mathcal{I}^Y_{t-1}$-measurable respectively and that the threshold parameters satisfy the following constraints:

$$0 = r^U_{0,t-1} < r^U_{1,t-1} < \cdots < r^U_{k-1,t-1} < r^U_{k,t-1} = 1 \ ,$$
$$0 = r^V_{0,t-1} < r^V_{1,t-1} < \cdots < r^V_{k-1,t-1} < r^V_{k,t-1} = 1 \ .$$

Here $\mathcal{I}_t$ denotes the minimal σ-field containing both the σ-fields $\mathcal{I}^X_t$ and $\mathcal{I}^Y_t$. Note that the threshold parameters of U and V are time varying such that the regime can be matched with the original series. See Example 2.1 for a canonical threshold principle.

We define a conditional bivariate copula function $C_t(\cdot, \cdot|\mathcal{I}_{t-1})$ given $\mathcal{I}_{t-1}$ as follows:

$$C_t(u, v|\mathcal{I}_{t-1}) = \sum_{i=1}^{k}\sum_{j=1}^{k} C^{i,j}_{\theta_{i,j}}(u, v) I_{\{(U_{t-1}, V_{t-1}) \in (r^U_{i-1,t-1}, r^U_{i,t-1}) \times (r^V_{j-1,t-1}, r^V_{j,t-1})\}} \ . \tag{2}$$

We call this a self-exciting threshold copula function. The key idea is that we divide the squared region $[0, 1]^2$ into sub-rectangular regions which we call regimes, therefore the indicator function become a selector from the product space of U and V. In each regime, the conditional dependent structure between X_t and Y_t given $\mathcal{I}_{t-1}$ is described by the copula function $C^{i,j}_{\theta_{i,j}}(u, v)$. Here the delay parameter $d = 1$. In general, we can consider the pair of random variables $(U_{t-d}, V_{t-d}) \in (r^U_{i-1,t-d}, r^U_{i,t-d}) \times (r^V_{i-1,t-d}, r^V_{i,t-d})$ in the indicator function.

We now specify the probability law of the Copula-TAR model by the following conditional joint cumulative distribution function of X_t and Y_t given $\mathcal{I}_{t-1}$.

$$\begin{aligned} & F^{X,Y}_t(x, y|\mathcal{I}_{t-1}) \\ = & \sum_{i=1}^{k}\sum_{j=1}^{k} C^{i,j}_{\theta_{i,j}}(F^X_t(x|\mathcal{I}^X_{t-1}), F^Y_t(y|\mathcal{I}^Y_{t-1})) I_{\{(U_{t-1}, V_{t-1}) \in (r^U_{i-1,t-1}, r^U_{i,t-1}) \times (r^V_{j-1,t-1}, r^V_{j,t-1})\}} \ . \end{aligned} \tag{3}$$

This is based on an extended version of Sklar's theorem in a "dynamic" environment. For Sklar's theorem, one may refer to Joe (2015).

Now, we discuss some particular cases of the Copula-TAR model in the following examples. These examples are motivated from nonlinear time series modelling and financial applications.

Example 2.1 *A Canonical Threshold Principle*

Suppose, for each $i, j = 0, 1, \cdots, k$,

$$r_{it}^U := F_t^X(r_i^X|\mathcal{I}_{t-1}^X) \ , \quad r_{jt}^V := F_t^Y(r_i^Y|\mathcal{I}_{t-1}^Y) \ ,$$

so $r_{0t}^U = r_{0t}^V = 0$ *and* $r_{kt}^U = r_{kt}^V = 1$.

In this case, we consider the situation that the thresholds in the self-exciting threshold copula function come up naturally from the probability levels of the thresholds of the univariate SETAR models. In other words, the thresholds of the copula function are determined completely by those of the univariate SETAR models. That means there is no necessity to evaluate the threshold parameters of U *and* V *because the regimes of* U *and* V *are solely determined by the regimes of* X *and* Y *respectively. We call this a canonical, or natural, threshold principle. The canonical threshold principle apparently provides a neat way to partition a high dimensional space arising from a multivariate nonlinear time series.*

Example 2.2 *A Copula-TAR Extension of the Merton Structural Model for Firm Values*

We consider a multivariate extension of the Merton Structural Model for Firm Values based on the proposed Copula-TAR model in a discrete-time economy. The idea is to incorporate the tail dependence of market values of firms in structural model for credit risk. In particular, we allow different levels of the lower tail and upper tail dependence of the firms' values. Firstly, let X_t *and* Y_t *be the market values of two firms at time* t. *We assume that* $X := \{X_t|t = 1, 2, \cdots, T\}$ *and* $Y := \{Y_t|t = 1, 2, \cdots, T\}$ *follow the univariate SETAR models.*

Consider the following threshold parameters dividing the state spaces of U *and* V:

$$\begin{aligned} 0 = r_{0,t-1}^U < r_{1,t-1}^U < r_{2,t-1}^U < r_{3,t-1}^U = 1 \ , \\ 0 = r_{0,t-1}^V < r_{1,t-1}^V < r_{2,t-1}^V < r_{3,t-1}^V = 1 \ . \end{aligned}$$

We suppose that these threshold parameters are determined by the conditional distributions of X_t *and* Y_t *given* $\mathcal{I}_{t-1}$ *as in Example 2.2.*

Let $C_{\bar{\theta}_1}^{Cl}(u_1, u_2)$, $C_{\bar{\theta}_2}^{Ga}(u_1, u_2)$ *and* $C_{\bar{\theta}_3}^{Gu}(u_1, u_2)$ *be the bivariate Clayton, Gaussian and Gumbel copulas with parameters* $\bar{\theta}_1$, $\bar{\theta}_2$ *and* $\bar{\theta}_3$, *respectively. That is,*

$$C_{\bar{\theta}_1}^{Cl}(u_1, u_2) := (u_1^{-\bar{\theta}_1} + u_2^{-\bar{\theta}_1} - 1)^{-1/\bar{\theta}_1} \ ,$$

$$\begin{aligned} & C_{\bar{\theta}_2}^{Ga}(u_1, u_2) \\ := & \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi(1-\bar{\theta}_2^2)^{1/2}} \exp\Big\{ - \frac{(a_t^2 - 2\bar{\theta}_2 a_t b_t + b_t^2)}{2(1-\bar{\theta}_2^2)} \Big\} da_t db_t \ , \end{aligned}$$

and

$$C_{\bar{\theta}_3}^{Gu}(u_1, u_2) := \exp\{-[(-\ln u_1)^{\bar{\theta}_3} + (-\ln u_2)^{\bar{\theta}_3}]^{1/\bar{\theta}_3}\} \ ,$$

where $\bar{\theta}_1 \in (0, \infty)$, $\bar{\theta}_2 \in (-1, 1)$ *and* $\bar{\theta}_3 \in [1, \infty)$. $\Phi(\cdot)$ *is the cumulative distribution function of a standard normal distribution.*

Write $\mathcal{S}_3 := \{(i, j)|i, j = 1, 2, 3\}$ *and* $\bar{\mathcal{S}}_3 := \mathcal{S}_3 \backslash \{(1, 1), (3, 3)\}$. *We set* $\bar{\theta}_1 = \theta_{1,1}$, $\bar{\theta}_3 = \theta_{3,3}$, $\bar{\theta}_2 = \theta_{i,j}$, *for all* $(i, j) \in \bar{\mathcal{S}}_3$. *Then the self-exciting threshold copula function becomes:*

$$\begin{aligned} C_t(U_t, V_t|\mathcal{I}_{t-1}) \quad = \quad & C_{\bar{\theta}_1}^{Cl}(U_t, V_t) I_{\{(U_{t-1}, V_{t-1}) \in (r_{0,t-1}^U, r_{1,t-1}^U) \times (r_{0,t-1}^V, r_{1,t-1}^V)\}} \\ & + \sum_{(i,j) \in \bar{\mathcal{S}}_3} C_{\bar{\theta}_2}^{Ga}(U_t, V_t) I_{\{(U_{t-1}, V_{t-1}) \in (r_{i-1,t-1}^U, r_{i,t-1}^U) \times (r_{j-1,t-1}^V, r_{j,t-1}^V)\}} \\ & + C_{\bar{\theta}_3}^{Gu}(U_t, V_t) I_{\{(U_{t-1}, V_{t-1}) \in (r_{2,t-1}^U, r_{3,t-1}^U) \times (r_{2,t-1}^V, r_{3,t-1}^V)\}} \ . \end{aligned}$$

Consequently, in the lower tail part where the market values of the two firms are small, the lower tail dependence is described by the bivariate Clayton copula $C_{\bar{\theta}_1}^{Cl}(u_1, u_2)$. *In the upper tail part where the market values of the two firms are high, the upper tail dependence is described by the bivariate Gumbel copula. In other parts, the dependence of the market values of the two firms is described by the Gaussian copula. Of course, in practice, the market values of the firms are not directly observed. One may need to infer these values from equity prices of the firms.*

Example 2.3 *A Gaussian Mixture Copula Function*

We consider a Gaussian mixture copula function, which has a Gaussian copula function in each regime, in the state space of (U_{t-1}, V_{t-1}) *introduced via the threshold principle. The rationale is to approximate a general copula function locally using a Gaussian copula function. By noting the relationship between Gaussianity and linearity, the idea of the Gaussian mixture copula function is not unlike that of the SETAR model which gives a piecewise linear approximation to a nonlinear autoregressive model.*

In this case, we consider a set of threshold parameters satisfying the following constraints:

$$0 = r^U_{0,t-1} < r^U_{1,t-1} < \cdots < r^U_{k-1,t-1} < r^U_{k,t-1} = 1 \ ,$$
$$0 = r^V_{0,t-1} < r^V_{1,t-1} < \cdots < r^V_{k-1,t-1} < r^V_{k,t-1} = 1 \ .$$

As before, we assume that for each $i = 0, 1, \cdots, m$, $j = 0, 1, \cdots, n$ *and* $t = 1, 2, \cdots, T$, $r^U_{i,t-1}$ *and* $r^V_{j,t-1}$ *are* $\mathcal{I}^X_{t-1}$ *and* $\mathcal{I}^Y_{t-1}$*-measurable respectively.*

For each $i = 0, 1, \cdots, k$ *and* $j = 0, 1, \cdots, k$, *we consider the following Gaussian copula function* $C^{Ga}_{\theta_{i,j}}(U_t, V_t)$ *with parameter* $\theta_{i,j}$.

$$\begin{aligned} &C^{Ga}_{\theta_{i,j}}(u_1, u_2) \\ := &\int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi(1-\theta^2_{i,j})^{1/2}} \exp\Big\{ - \frac{(a_t^2 - 2\theta_{i,j} a_t b_t + b_t^2)}{2(1-\theta^2_{i,j})} \Big\} da_t db_t \ , \end{aligned}$$

Then the self-exciting threshold copula function is given by:

$$C_t(U_t, V_t | \mathcal{I}_{t-1}) \quad = \quad \sum_{i=1}^{k} \sum_{j=1}^{k} C^{Ga}_{\theta_{i,j}}(U_t, V_t) I_{\{(U_{t-1}, V_{t-1}) \in (r^U_{i-1,t-1}, r^U_{i,t-1}) \times (r^V_{j-1,t-1}, r^V_{j,t-1})\}} \ .$$

There are different approaches to extend the SETAR model to the multivariate case. Lewis and Ray (1993) adopted a multivariate adaptive regression spline fitting methodology (MARS) originated from Friedman (1991) to model multivariate nonlinear time series. The MARS may be regarded as a generalisation of a recursive partitioning (RP) strategy considered in Morgan and Sonquist (1963) and Breiman et al. (1984). The RP strategy adopts spline fitting for an additive nonlinear regression model. The main advantage of the MARS approach is that it can incorporate long-range dependence and allow the inclusion of categorical predictor series. However, models fitted by the MARS algorithm are not always easy to interpret. Moreover, being essentially a nonparametric approach, it inherits all its weaknesses, to which we shall return later.

The Copula-TAR approach provides an intuitive way to incorporate structural changes in the dependent structure of financial time series that may be attributed to financial crises or asset bubbles. Empirical studies reveal that there is a significantly higher level of correlation, (or, in general, dependence), among financial prices during the crisis period than the normal period. See, for example, Bertero and Mayer (1990), King and Wadhwani (1990), Calvo and Reinhardt (1996) and Baig and Goldfajn (1999). This structural shift in correlation is referred to as “correlation breakdown”. The linear correlation and Gaussian copulas cannot produce a sufficiently high level of dependency experienced in a crisis period, nor can they incorporate structural shifts in the dependent relationship. In contrast, we shall see that the self-exciting threshold copula function provides a flexible way to describe different levels of dependence based on regimes introduced by the threshold principle. It can also provide a flexible and convenient way to incorporate various cross-sectional dependent structures among constituent time series as well as temporally dependent structures among them. The key behind the flexibility is the convenience of separating, via thresholds, the cross-sectional dependent structure of the multivariate nonlinear time series and the temporal dependent structure of the marginal time series. Consequently, the modelling of each type of dependent structures does not impose any restriction on the other. A distinct disadvantage of the MARS approach lies in its inability to do so. Further, being nonparametric, the MARS approach cannot escape from the problem of curse of dimensionality, a problem not suffered by the Copula-TAR approach as it is a parametric approach. The statistical identification of the Copula-TAR model is not significantly different from that of the univariate SETAR model. It is quite intuitive and easy to implement, requiring no complicated recursive partitioning strategy or machine-learning procedure which may be required in the nonparametric MARS model.

There are a few models that consider separating regimes in dependent structures of multivariate time series. In a recent paper by Lai et al. (2009), a copula-threshold GARCH model was considered. Our approach with the Copula-TAR model is different from that in Lai et al. in at least two major aspects. Firstly, we attend to the threshold effects in both the conditional means and the volatilities (i.e. the conditional variances), while Lai et al. is only concerned with threshold effects in the conditional variances. Secondly, we introduce threshold effects in the copula function. Lai et al. considered threshold effects in the univariate time series only. Next, Jondeau and Rockinger (2006) uses a gridded dependence structure to model the marginal of a GARCH model with conditional skewness and kurtosis. The choices of grid / threshold of regimes is arbitrary. However in Copula-TAR, we separate the regimes in a natural and intuitive way as demonstrated in Example 2.2, which yields easily interpreted results.

We would stress that the Copula-TAR model is quite flexible in that it also covers volatility as well as the conditional mean function. A recent paper by Chan et al. (2014) justified this claim. In particular, they argued that, as a dynamic mixing of white noise, the GARCH approach is not so natural because a square-root mixing function is hard to interpret and often quite restrictive in its parameter admissibility. In contrast, the copula-TAR uses a piecewise constant mixing function as in T-CHARM in Chan et al. (2014) for which only very mild conditions are required on the model parameters.

From an economic perspective, the Copula-TAR model provides a particularly convenient way to modelling multivariate time series which partitions the economy into different regimes by the levels of observed time series. The self-exciting threshold copula function may be regarded as a more general concept than the threshold co-integration in the sense that the former incorporates the threshold effect in modelling nonlinear dependence among univariate time series and the latter only describes the threshold effect in modelling linear associations among univariate time series.

The Copula-TAR model targets data with dependent structures governed by the levels of observed data. Its regimes often have clear interpretation; for example, high/low levels of stock returns can be interpreted as bull/bear markets, different foreign exchange rate levels can be interpreted as different policy regimes, and so on. Furthermore, the use of the threshold principle provides a way to distinguish market regimes based on observed market data. The facility is provided by the fact that the number of regimes is data driven. Real data analysis in Section 5 shows that a two-regime linear marginal model provides reasonably accurate estimates of Value-at-Risk. The principle of separating regimes by data value levels can be applied to more sophisticated marginal models to describe conditional heteroscedasticity.

3 Estimation

In this section, we present two methodologies of maximum likelihood estimation based on partial and full information, respectively.

First, we introduce some notation. Define the vectors of the unknown parameters in the two marginal SETAR models and the self-exciting threshold copula function as:

$$\begin{aligned}
&\boldsymbol{\Theta}_X := (\boldsymbol{\Theta}'_{X,1}, \boldsymbol{\Theta}'_{X,2})', \text{ with} \\
&\boldsymbol{\Theta}_{X,1} = (\alpha_0^{(1)}, \alpha_1^{(1)}, \cdots, \alpha_{m(1)}^{(1)}, \cdots, \alpha_0^{(k)}, \alpha_1^{(k)}, \cdots, \alpha_{m(k)}^{(k)}, \sigma^{(1)}, \sigma^{(2)}, \cdots, \sigma^{(k)})', \\
&\boldsymbol{\Theta}_{X,2} = (r_1^X, r_2^X, \cdots, r_{k-1}^X)' \ ; \\
&\boldsymbol{\Theta}_Y := (\boldsymbol{\Theta}'_{Y,1}, \boldsymbol{\Theta}'_{Y,2})', \text{ with} \\
&\boldsymbol{\Theta}_{Y,1} = (\beta_0^{(1)}, \beta_1^{(1)}, \cdots, \beta_{n(1)}^{(1)}, \cdots, \beta_0^{(k)}, \beta_1^{(k)}, \cdots, \beta_{n(k)}^{(k)}, \phi^{(1)}, \phi^{(2)} \cdots, \phi^{(k)})', \\
&\boldsymbol{\Theta}_{Y,2} = (r_1^Y, r_2^Y, \cdots, r_{k-1}^Y)' \ ; \\
&\boldsymbol{\Theta}_C := (\theta_{1,1}, \theta_{1,2}, \cdots, \theta_{m,n})' \ .
\end{aligned} \tag{4}$$

Assume each component univariate time series to be stationary, we write the marginal conditional distributions of X_t and Y_t given $\mathcal{I}_{t-1}^X$ and $\mathcal{I}_{t-1}^Y$ and the self-exciting copula function given $\mathcal{I}_{t-1}$ as functions of the unknown parameters $\boldsymbol{\Theta}_X$, $\boldsymbol{\Theta}_Y$ and $\boldsymbol{\Theta}_C$ as follows:

$$\begin{aligned}
F_t^X(x|\mathcal{I}_{t-1}^X) &:= F_t^X(x|\mathcal{I}_{t-1}^X, \boldsymbol{\Theta}_X) \ , \\
F_t^Y(y|\mathcal{I}_{t-1}^Y) &:= F_t^Y(y|\mathcal{I}_{t-1}^Y, \boldsymbol{\Theta}_Y) \ , \\
C_t(u_t, v_t|\mathcal{I}_{t-1}) &:= C_t(u_t, v_t|\mathcal{I}_{t-1}, \boldsymbol{\Theta}_C) \ .
\end{aligned}$$

Here note that $u_t = u_t(\boldsymbol{\Theta}_X) = F_t^X(x_t|\mathcal{I}_{t-1}^X, \boldsymbol{\Theta}_X)$ and $v_t = v_t(\boldsymbol{\Theta}_Y) = F_t^Y(y_t|\mathcal{I}_{t-1}^Y, \boldsymbol{\Theta}_Y)$. For a general discussion of stationarity of nonlinear time series, see, e.g., Chan (2009). For the important special case with $m(i) = 1$, for all i, the condition $\max\{|\alpha_1^{(1)}|, |\alpha_1^{(k)}|\} < 1$ and $\max\{|\beta_1^{(1)}|, |\beta_1^{(k)}|\} < 1$ suffices. For details, we refer to Chan et al. (1985) and Lu (1998).

Suppose that F_t^X, F_t^Y and C_t are differentiable functions with respect to x, y and (u_t, v_t), respectively. Denote their derivatives as follows:

$$\begin{aligned}
f_t^X(x|\mathcal{I}_{t-1}^X, \boldsymbol{\Theta}_X) &:= \frac{\partial F_t^X}{\partial x}(x|\mathcal{I}_{t-1}^X, \boldsymbol{\Theta}_X) \ , \\
f_t^Y(y|\mathcal{I}_{t-1}^Y, \boldsymbol{\Theta}_Y) &:= \frac{\partial F_t^Y}{\partial y}(y|\mathcal{I}_{t-1}^Y, \boldsymbol{\Theta}_Y) \ , \\
c_t(u_t, v_t|\mathcal{I}_{t-1}, \boldsymbol{\Theta}_C) &:= \frac{\partial^2 C_t}{\partial u \partial v}(u_t, v_t|\mathcal{I}_{t-1}, \boldsymbol{\Theta}_C) \ .
\end{aligned}$$

Then the conditional joint density function of X_t and Y_t given $\mathcal{I}_{t-1}$ exists and is given by:

$$\begin{aligned} & f_t^{X,Y}(x_t, y_t|\mathcal{I}_{t-1}, \mathbf{\Theta}_X, \mathbf{\Theta}_Y, \mathbf{\Theta}_C) \\ = \;& c_t(F_t^X(x_t|\mathcal{I}_{t-1}^X, \mathbf{\Theta}_X), F_t^Y(y_t|\mathcal{I}_{t-1}^Y, \mathbf{\Theta}_Y)|\mathcal{I}_{t-1}, \mathbf{\Theta}_C) \times f_t^X(x_t|\mathcal{I}_{t-1}^X, \mathbf{\Theta}_X) \times f_t^Y(y_t|\mathcal{I}_{t-1}^Y, \mathbf{\Theta}_Y) \\ = \;& c_t(u_t, v_t|\mathcal{I}_{t-1}, \mathbf{\Theta}_C) \times f_t^X(x_t|\mathcal{I}_{t-1}^X, \mathbf{\Theta}_X) \times f_t^Y(y_t|\mathcal{I}_{t-1}^Y, \mathbf{\Theta}_Y) \ . \end{aligned}$$

The log-likelihood function from the t-th observation is then given by:

$$\begin{aligned} & \ln f_t^{X,Y}(x_t, y_t|\mathcal{I}_{t-1}, \mathbf{\Theta}_X, \mathbf{\Theta}_Y, \mathbf{\Theta}_C) \\ = \;& \ln c_t(u_t, v_t|\mathcal{I}_{t-1}, \mathbf{\Theta}_C) + \ln f_t^X(x_t|\mathcal{I}_{t-1}^X, \mathbf{\Theta}_X) + \ln f_t^Y(y_t|\mathcal{I}_{t-1}^Y, \mathbf{\Theta}_Y) \ . \end{aligned}$$

Let $\mathbf{\Theta} := (\mathbf{\Theta}_X, \mathbf{\Theta}_Y, \mathbf{\Theta}_C)$. Write

$$\begin{aligned} l_{x_t,y_t}(\mathbf{\Theta}) &= \ln f_t^{X,Y}(x_t, y_t|\mathcal{I}_{t-1}, \mathbf{\Theta}) \ , \\ l_{u_t,v_t}(\mathbf{\Theta}_C) &:= \ln c_t(u_t, v_t|\mathcal{I}_{t-1}, \mathbf{\Theta}_C) \ , \\ l_{x_t}(\mathbf{\Theta}_X) &:= \ln f_t^X(x_t|\mathcal{I}_{t-1}^X, \mathbf{\Theta}_X) \ , \\ l_{y_t}(\mathbf{\Theta}_Y) &:= \ln f_t^Y(y_t|\mathcal{I}_{t-1}^Y, \mathbf{\Theta}_Y) \ . \end{aligned}$$

Then

$$l_{x_t,y_t}(\mathbf{\Theta}) = l_{u_t,v_t}(\mathbf{\Theta}_C) + l_{x_t}(\mathbf{\Theta}_X) + l_{y_t}(\mathbf{\Theta}_Y) \ . \quad (5)$$

Now, suppose $\{x_1, x_2, \cdots, x_T\}$ and $\{y_1, y_2, \cdots, y_T\}$ are observations of the time series X and Y, respectively. To simplify the notation, we consider the situation that both the univariate SETAR models and the self-exciting threshold copula function have two regimes. The same principle applies to general cases with more than two regimes. However, the likelihood maximisation on the threshold parameters will become a higher dimension optimisation problem. In this case, let r^X, r^Y and r_{t-1}^U and r_{t-1}^V be the threshold parameters dividing the state spaces of X_{t-1}, Y_{t-1}, U_{t-1} and V_{t-1}, respectively.

3.1 Two-Stage Estimation

In this subsection, we discuss a two-stage estimation method to estimate the Copula-TAR model introduced in the last section. The two-stage estimation method was introduced by Joe and Xu (1996). This method is also named as inference for the margins (IFM). The main idea of the method is to separate the estimation of the marginal densities from the copula density. It is an computationally efficient estimation method for multivariate models, especially when the dimension is large. It lowers the number of parameters to be estimated in each steps by breaking a high dimensional maximum likelihood estimation problem into multiple steps. This can maintain a high degree of computation feasibility in estimation when the number of regimes is large. The IFM method is also easy to implement in practice. It is much more computation efficient than standard maximum likelihood estimation. Model selection using different statistical principles, such as Akaike's information criterion (AIC), Bayesian information criterion (BIC) or Generalised information criterion (GIC), can be used at this stage. However, attention should be paid to the penalty function that the number of parameters changes when different orders or number of regimes are chosen.[1] In our current context, we have to modify the IFM method to estimate the Copula-TAR model. Specifically, at the first stage, we use the maximum likelihood method to estimate the marginal SETAR models (e.g. Tong (1983, 1990)). We select the threshold parameters and model parameters that maximise the likelihood function. At the second stage, we again use the maximum likelihood method to select the individual copula parameters and the threshold parameters of the self-exciting threshold copula functions. We can select the copula function in each regime using the information criteria in the first stage. The discussion on which criteria should be used in selecting SETAR models and copula models is beyond the scope of this paper. In what follows, we present the main idea of the IFM method adapted to our current model set-up.

At the first estimation stage, we estimate the parameters of the univariate SETAR model for the time series X using the maximum likelihood estimation. Suppose we are estimating a SETAR model with two regimes. Firstly, we assume that the possible values taken by the threshold parameter r^X are given by $\{\tilde{r}_1^X, \tilde{r}_2^X, \cdots, \tilde{r}_l^X\}$. For each possible values of threshold parameter, $\tilde{r}_j^X$ where $j = 1, \cdots, l$, the samples are split into two sub-samples, in which one contains observations less than or equal to $\tilde{r}_j^X$, while the second sub-sample contains observations greater than $\tilde{r}_j^X$. We estimate the marginal SETAR model via maximum likelihood as:

$$\hat{\theta}_X^{(1)} = \arg\max_{\theta_X^{(1)}} \sum_{\tau=1}^{T} \ln f_{\tilde{X}}^{(1)}(x_\tau|\mathcal{I}_{\tau-1}^X, \theta_X^{(1)}) I\{x_{\tau-1} \leq \tilde{r}_j^X\} \ ,$$

[1] The problem of determining number of regime is non-standard, in that the number of independently adjusted parameters is not equal to the nominal number of parameters. This problem may be circumvented by increasing the penalty term from 2 $\times$ (the nominal number of parameters) to $K \times$ (the nominal number of parameters), where K is likely to be much greater than 2. The appropriate choice of K is beyond the scope of the present paper.

and

$$\hat{\theta}_X^{(2)} = \arg\max_{\theta_X^{(2)}} \sum_{\tau=1}^{T} \ln f_{\tilde{X}}^{(2)}(x_\tau | \mathcal{I}_{\tau-1}^X, \theta_X^{(2)}) I\{x_{\tau-1} > \tilde{r}_j^X\} ,$$

where $\theta^{(1)} = (\alpha_0^{(1)}, \alpha_1^{(1)}, \cdots, \alpha_k^{(1)}, \sigma^{(1)})$ and $\theta^{(2)} = (\alpha_0^{(2)}, \alpha_1^{(2)}, \cdots, \alpha_k^{(2)}, \sigma^{(2)})$ are the parameters of the marginal univariate SETAR model for X in the first and second regime respectively, and $f_{\tilde{X}}^{(1)}$ and $f_{\tilde{X}}^{(2)}$ are the transition probability densities of X to the next time step in the first and second regimes respectively. The density functions can be approximated by normal distributions when the sample size is sufficiently large.

We repeat the above procedure for other possible values of $\tilde{r}_j^X$'s and compute the likelihood of the estimated model. The estimate of r^X is given by the value of $\tilde{r}_j^X$ which maximises the likelihood. The estimate $\widehat{\mathbf{\Theta}}_X$ of $\mathbf{\Theta}_X$ is then given by the estimated parameters which give the maximum likelihood.

For the estimation of the univariate SETAR model for Y, we follow the same procedure as above. We now discuss the estimation of the self-exciting threshold copula in the second stage of the estimation procedure.

In the second stage, we first evaluate the *estimated* sample values of the continuous uniform random variables U_t and V_t from the observations $\{x_1, x_2, \cdots, x_T\}$ and $\{y_1, y_2, \cdots, y_T\}$ of the time series X and Y, respectively, as follows:

$$\begin{aligned} u_t &:= F_t^X(x_t | \mathcal{I}_{t-1}^X, \widehat{\mathbf{\Theta}}_X) , \\ v_t &:= F_t^Y(y_t | \mathcal{I}_{t-1}^Y, \widehat{\mathbf{\Theta}}_Y) , \quad t = 1, 2, \cdots, T , \end{aligned}$$

where $\widehat{\mathbf{\Theta}}_X$ and $\widehat{\mathbf{\Theta}}_Y$ are the estimates of $\mathbf{\Theta}_X$ and $\mathbf{\Theta}_Y$, respectively, from the first stage of the estimation procedure.

We then assume that the threshold parameters r^U and r^V can take the following possible values $\{\tilde{r}_{1,t-1}^U, \tilde{r}_{2,t-1}^U, \cdots, \tilde{r}_{p,t-1}^U\}$ and $\{\tilde{r}_{1,t-1}^V, \tilde{r}_{2,t-1}^V, \cdots, \tilde{r}_{q,t-1}^V\}$, respectively, at time $t-1$. If we use the canonical threshold principle on choosing threshold parameters of self-exciting threshold copula function, these threshold parameters are determined endogenously by the thresholds of the time series X and Y by

$$\begin{aligned} \tilde{r}_t^U &:= F_t^X(r^X | \mathcal{I}_{t-1}^X, \widehat{\mathbf{\Theta}}_X) , \quad t = 1, 2, \cdots, T-1 \\ \tilde{r}_t^V &:= F_t^Y(r^Y | \mathcal{I}_{t-1}^Y, \widehat{\mathbf{\Theta}}_Y) , \quad t = 1, 2, \cdots, T-1 \end{aligned}$$

where $\widehat{\mathbf{\Theta}}_X$ and $\widehat{\mathbf{\Theta}}_Y$ are the estimates of $\mathbf{\Theta}_X$ and $\mathbf{\Theta}_Y$, respectively, from the first stage of the estimation procedure. Otherwise, maximum likelihood method , similar to the one used in the first stage estimation of the threshold parameters of the time series X and Y but in higher dimension, can be used to determine the threshold parameters.

To illustrate the estimation method, we consider the model with only a pair of threshold parameters determined by the canonical threshold principle. The method demonstrated here can be extended to any numbers of threshold parameters in which the estimation method in each regime is the same though the computation becomes more complex. For a pair of threshold parameters, the sample space of (U_t, V_t) is split into four regimes. Then the estimates $\widehat{\theta}_{1,1}$, $\widehat{\theta}_{1,2}$, $\widehat{\theta}_{2,1}$ and $\widehat{\theta}_{2,2}$ of $\theta_{1,1}$, $\theta_{1,2}$, $\theta_{2,1}$ and $\theta_{2,2}$, respectively, are obtained from maximum likelihood as

$$\widehat{\theta}_{1,1} = \arg\max_{\theta_{1,1}} \sum_{t=1}^{T} \ln c_{1,1}(u_t, v_t | \theta_{1,1}) I\{u_t \leq \tilde{r}_t^U, v_t \leq \tilde{r}_t^V\} ,$$

$$\widehat{\theta}_{1,2} = \arg\max_{\theta_{1,2}} \sum_{t=1}^{T} \ln c_{1,2}(u_t, v_t | \theta_{1,2}) I\{u_t \leq \tilde{r}_t^U, v_t > \tilde{r}_t^V\} ,$$

$$\widehat{\theta}_{2,1} = \arg\max_{\theta_{2,1}} \sum_{t=1}^{T} \ln c_{2,1}(u_t, v_t | \theta_{2,1}) I\{u_t > \tilde{r}_t^U, v_t \leq \tilde{r}_t^V\} ,$$

and

$$\widehat{\theta}_{2,2} = \arg\max_{\theta_{2,2}} \sum_{t=1}^{T} \ln c_{2,2}(u_t, v_t | \theta_{2,2}) I\{u_t > \tilde{r}_t^U, v_t > \tilde{r}_t^V\} .$$

In practice, we do not need to evaluate $\tilde{r}_t^U$ and $\tilde{r}_t^V$ because the regime selectors, i.e. indication function, are equivalent to the regime selectors in the original series, such that,

$$I\{u_t \leq \tilde{r}_t^U, v_t \leq \tilde{r}_t^V\} = I\{X_{t-1} \leq \tilde{r}^X, Y_{t-1} \leq \tilde{r}^Y\} ,$$

$$I\{u_t \le \tilde{r}_t^U, v_t > \tilde{r}_t^V\} = I\{X_{t-1} \le \tilde{r}^X, Y_{t-1} > \tilde{r}^Y\},$$

$$I\{u_t > \tilde{r}_t^U, v_t \le \tilde{r}_t^V\} = I\{X_{t-1} > \tilde{r}^X, Y_{t-1} \le \tilde{r}^Y\},$$

and

$$I\{u_t > \tilde{r}_t^U, v_t > \tilde{r}_t^V\} = I\{X_{t-1} > \tilde{r}^X, Y_{t-1} > \tilde{r}^Y\}.$$

It is possible to extend the method to more threshold states by taking more threshold parameter values. However, the search of threshold parameters will become an optimisation problem on an extended set, e.g. in three-regime case (two threshold parameter), the "candidates" of (r_1^X, r_2^X) are selected from $\{\tilde{r}_1^X, \tilde{r}_2^X, \cdots, \tilde{r}_l^X\} \times \{\tilde{r}_1^X, \tilde{r}_2^X, \cdots, \tilde{r}_l^X\}$ where $r_1^X < r_2^X$. The estimations of copula functions are done separately in each subregion divided by threshold parameters.

Chan (1993) discusses the asymptotic properties of the two-step estimation method in which a proof for the case with two regimes is provided. When the sample size is large, the proportion of the mis-split sample size among the whole sample is small. This proportion tends to zero when the number of sample size tends to infinity. The estimators of parameters of the individual time series models are shown to be strongly consistent and asymptotically normal. The numerical results in later sections lend support to this conclusion. Asymptotic results for the case with more than one threshold are scanty if available. This will be the topic of further research to broaden the theoretical base of the estimation.

3.2 Full-Information Likelihood Estimation

The two-stage estimation method is convenient in computation. However, the estimation of $\mathbf{\Theta}_X$ and $\mathbf{\Theta}_Y$ is based on partial information about the series X and Y, respectively. Although our simulation studies to be presented in the next section will reveal that the partial information method provides accurate estimation results for the parameters, we wish to give some remarks for the full-information likelihood estimation method here for the sake of completeness.

As noticed in the above, in (5), $u_t = u_t(\mathbf{\Theta}_X)$ and $v_t = v_t(\mathbf{\Theta}_Y)$. Consequently, the full log likelihood is given by:

$$\ell(\mathbf{\Theta}) = \sum_{t=1}^{T} l_{x_t,y_t}(\mathbf{\Theta}) = \sum_{t=1}^{T} l_{u_t(\mathbf{\Theta}_X),v_t(\mathbf{\Theta}_Y)}(\mathbf{\Theta}_C) + \sum_{t=1}^{T} l_{x_t}(\mathbf{\Theta}_X) + \sum_{t=1}^{T} l_{y_t}(\mathbf{\Theta}_Y)\ . \tag{6}$$

Then the full-information-based maximum likelihood estimate is defined by:

$$\tilde{\mathbf{\Theta}} = (\tilde{\mathbf{\Theta}}_C, \tilde{\mathbf{\Theta}}_X, \tilde{\mathbf{\Theta}}_Y) = \arg\max_{\mathbf{\Theta}} \ell(\mathbf{\Theta}).$$

It is worth noting that this likelihood function cannot be optimised by the maximising algorithm which requires differentiability of the target function since the likelihood function is not differentiable with respect to the threshold parameters, namely, $\mathbf{\Theta}_{X,2}$ and $\mathbf{\Theta}_{Y,2}$. There are several heuristic algorithms which do not require differentiability of the target function. One practical approach is the differential evolution method considered in Price et al. (2005). However, the accuracy of the optimisation in the differential evolution method very much depends on the algorithm chosen.

4 Simulation Studies

This section has two aims. Firstly, we perform simulation studies for the Copula-TAR model with a view to investigating key dependent structures that can be generated from the model. We exhibit different dependent structures that can be included by the Copula-TAR model. Secondly, we investigate the accuracy of the estimation method using simulated data. For illustration, we consider different piecewise copula functions in different regimes with the canonical threshold principle. We illustrate the effect of regime switching described by the threshold principle by comparing the dependent structure from piecewise copula functions with that from the Gaussian and Archimedean copula functions.

4.1 Simulation Studies for Dependent Structures

Here we consider the dependent structure of a two-regime, self-exciting threshold copula with Gaussian and Archimedean copula functions, including Gumbel copula, Frank copula and Clayton copula. This two-regime, self-exciting threshold copula model is taken as

$$\begin{aligned}
&X_t = (\alpha_0^{(1)} + \alpha_1^{(1)} X_{t-1} + \sigma^{(1)}\epsilon_t) I_{\{X_{t-1} \le r^X\}} + (\alpha_0^{(2)} + \alpha_1^{(2)} X_{t-1} + \sigma^{(2)}\epsilon_t) I_{\{X_{t-1} > r^X\}}\ ,\\
&\epsilon_t | \mathcal{I}_{t-1}^X \sim N(0,1)\ ,\\
&Y_t = (\beta_0^{(1)} + \beta_1^{(1)} Y_{t-1} + \phi^{(1)}\eta_t) I_{\{Y_{t-1} \le r^Y\}} + (\beta_0^{(2)} + \beta_1^{(2)} Y_{t-1} + \phi^{(2)}\eta_t) I_{\{Y_{t-1} > r^Y\}}\ ,\\
&\eta_t | \mathcal{I}_{t-1}^Y \sim N(0,1)\ .
\end{aligned}$$

To perform the simulation study, we consider the following hypothetical parameter values:

$$
\begin{aligned}
X_0 = Y_0 &= 0;\\
\alpha_0^{(1)} = -0.007; \alpha_1^{(1)} &= 0.3; \sigma^{(1)} = 0.025;\\
\alpha_0^{(2)} = -0.027; \alpha_1^{(2)} &= 0.1; \sigma^{(2)} = 0.015;\\
\beta_0^{(1)} = 0.004; \beta_1^{(1)} &= 0.6; \phi^{(1)} = 0.02;\\
\beta_0^{(2)} = -0.004; \beta_1^{(2)} &= 0.2; \phi^{(2)} = 0.01;\\
\theta_{1,1} = 1.5(\text{Gumbel}); \theta_{1,2} &= 3(\text{Frank});\\
\theta_{2,1} = 1.5(\text{Clayton}); \theta_{2,2} &= 0.7(\text{Gaussian});\\
r^X = -0.025; r^Y &= -0.005.
\end{aligned}
$$

This set of parameters satisfies the conditions for the piecewise stationary AR(1) in each regime. It also satisfies the conditions for the global stationarity of the TAR models in marginal cases. We choose these values of the parameters because the magnitudes of the simulated results are similar to those observed in real financial data except for the copula functions which are chosen to describe relatively extreme dependent structures. The two regimes in the bivariate threshold autoregressive time series model divide the two-dimensional state space into four sub-regions, namely, Regime (1,1), Regime (1,2), Regime (2,1) and Regime (2,2). The representation of these four regimes is shown in Table 1. The two-regime, self-exciting threshold copula function considered in the simulation example is given as follows:

$$
\begin{aligned}
C_t(U_t, V_t|\mathcal{I}_{t-1}) &= C^{Gb}_{\theta_{1,1}}(U_t, V_t) I_{\{U_{t-1} \le r^U_{t-1}, V_{t-1} \le r^V_{t-1}\}} + C^{F}_{\theta_{1,2}}(U_t, V_t) I_{\{U_{t-1} \le r^U_{t-1}, V_{t-1} > r^V_{t-1}\}} \\
&\quad + C^{Ct}_{\theta_{2,1}}(U_t, V_t) I_{\{U_{t-1} > r^U_{t-1}, V_{t-1} \le r^V_{t-1}\}} + C^{Ga}_{\theta_{2,2}}(U_t, V_t) I_{\{U_{t-1} > r^U_{t-1}, V_{t-1} > r^V_{t-1}\}} ,
\end{aligned}
$$

where

1. (r^U_{t-1}, r^V_{t-1}) is a pair of the threshold parameters for (U, V) at time $t-1$;
2. $C^{Gb}_{\theta_{1,1}}$ is the Gumbel copula function with parameter $\theta_{1,1}$;
3. $C^{F}_{\theta_{1,2}}$ is the Frank copula function with parameter $\theta_{1,2}$;
4. $C^{Ct}_{\theta_{2,1}}$ is the Clayton copula function with parameter $\theta_{2,1}$;
5. $C^{Ga}_{\theta_{2,2}}$ is the Gaussian copula function with parameter $\theta_{2,2}$.

We simulated 1,000 bivariate data points using the aforementioned parameters. The original simulated data points of the time series and their corresponding (U, V) in different regimes are plotted on Figure 1 and 2. The XY-plots of the simulated time series and the corresponding (U, V) from the first order autoregression are showed in Figures 3 and 4 respectively. From Figures 1 and 2, we see that there is a higher level of dependence between X and Y in Regimes (2,1) and (2,2) than in Regimes (1,1) and (1,2). From Figures 3 and 4, we see that the higher level of dependence between U and V in Regimes (2,1) and (2,2) are inherited from that of X and Y in Regimes (2,1) and (2,2).

We fitted the simulated data without consideration of threshold to see whether there are any hints in regime switching. The copula functions used in estimation are Gaussian copula, Clayton copula, Gumbel copula and Frank copula. Because all copulas here have only one model parameters, we can compare the fitting performance by solely looking at the sum of log likelihood or by information criterion (AIC, BIC) approach. The corresponding log likelihoods and model parameters are reported in Table 2 and 3. We can see Gaussian copula has the highest log likelihood with parameter value 0.5069. It appears that the dependent structures implied by the copulas in different regimes cannot be explained or detected without the threshold treatment.

Regime	Representation
1, 1	$X_t \le r^X$ and $Y_t \le r^Y$
1, 2	$X_t \le r^X$ and $Y_t > r^Y$
2, 1	$X_t > r^X$ and $Y_t \le r^Y$
2, 2	$X_t > r^X$ and $Y_t > r^Y$

Table 1: Representation of regimes

Copula	Gaussian	Clayton	Gumbel	Frank
Log-like	141.9*	101.7	133.5	140.9

Table 2: Sum of log likelihood of different copula functions from simulated data (Highest value is marked with *)

Copula	Gaussian	Clayton	Gumbel	Frank
θ	0.5069	0.6821	1.4685	3.5404

Table 3: Estimated parameters of different copula functions from simulated data

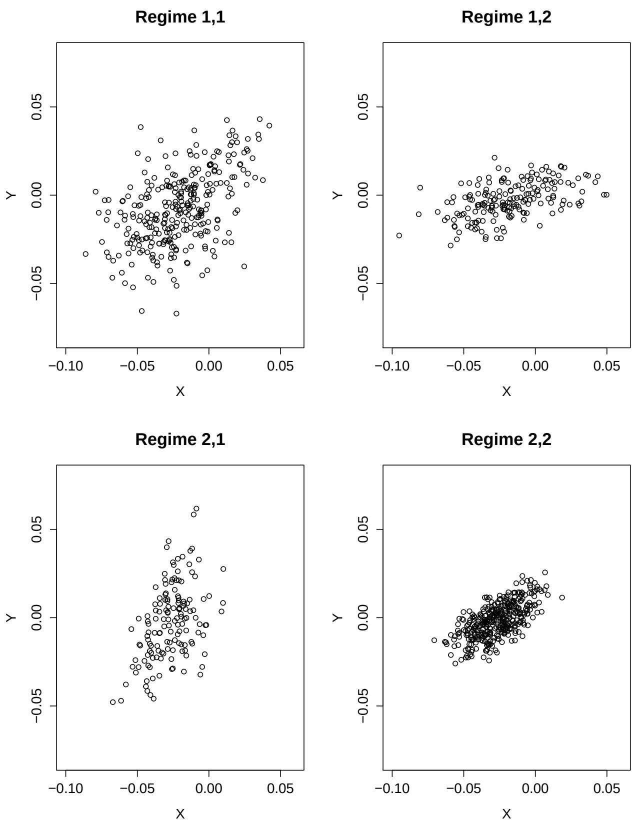

Figure 1: X and Y pairwise scatter plot of 1000 simulated points from two-regime switching threshold copula

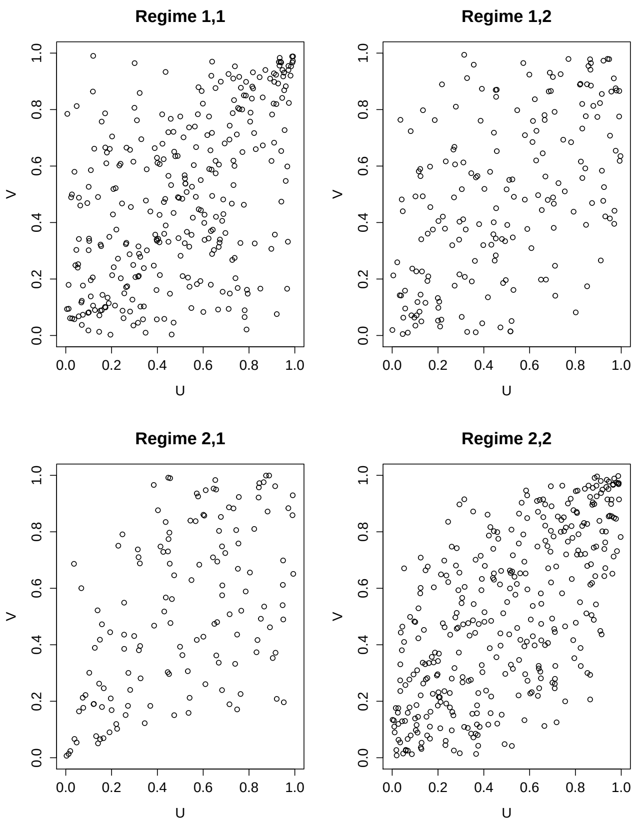

Figure 2: U and V pairwise scatter plot of 1000 simulated points from two-regime switching threshold copula

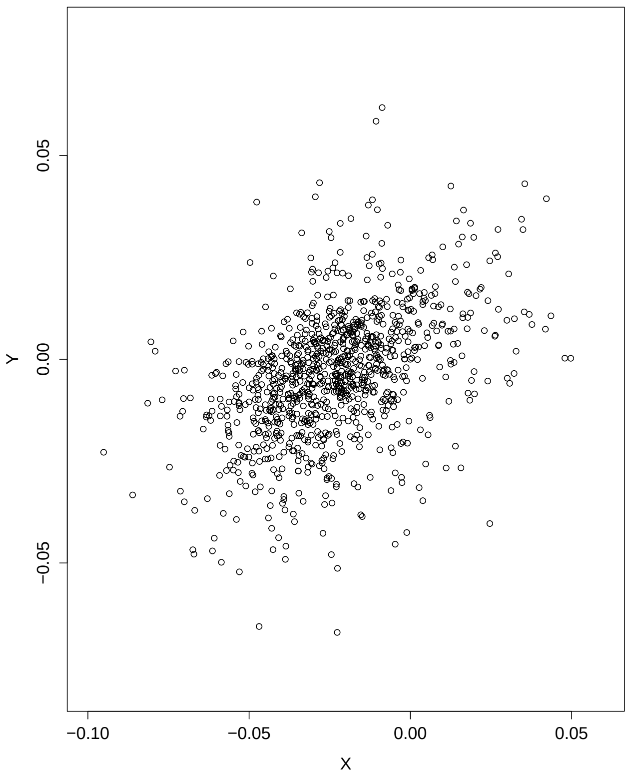

Figure 3: X and Y pairwise scatter plot of 1000 simulated points from two-regime switching threshold copula view as if no regime considered

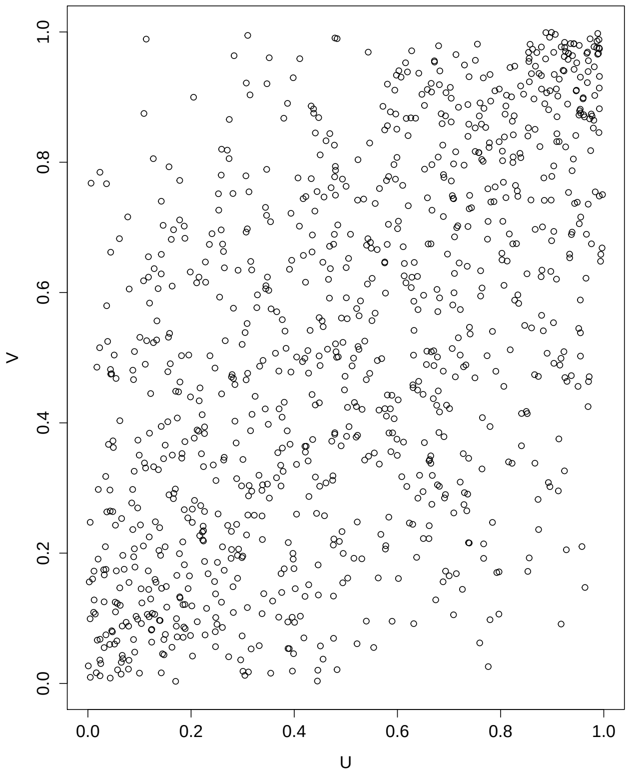

Figure 4: U and V pairwise scatter plot of 1000 simulated points from two-regime switching threshold copula view as if no regime considered

4.2 Simulation Studies for the Estimation Method

We performed the estimation method discussed in Section 3.1 to estimate the parameters of the simulated data from Section 4.1. In the first stage of estimation, we used 20 equally spaced quantiles of simulated values as the "candidate" points of the threshold parameter to fit the Copula-TAR model with first order autoregressive marginals and canonical threshold copula. The value of the threshold parameter with the largest likelihood is then chosen and used for the second stage of estimation for the copulas in different regimes. The copula function of each regime is chosen from the maximum likelihood of Gaussian copula, Clayton copula, Gumbel copula and Frank copula. The sums of log likelihood of copulas in each regime are reported in Table 4. The copula functions which agree with the simulation setup have the largest sum of log likelihood in their corresponding regime. The estimated copula parameters are reported in Table 5. Taking advantage of the availability of closed form formula in maximum likelihood estimation of marginal autoregressive model parameters, the estimation of the model parameters using the above setting takes 3.8 seconds on a Intel i7 2.2GHz CPU. (If 10000 data points are used, the computation time goes up to 20.3 seconds using the same setup.)

The XY-plots of estimated time series and the corresponding (U, V) in different regimes are shown in Figures 5 and 6 respectively. Comparing to the estimation result without regime consideration in the previous sub-section, extreme hidden dependent structures can be overlooked as simple Gaussian copula if no regime switching is considered.

Copula	Gaussian	Clayton	Gumbel	Frank
Log-like(1, 1)	52.9	24.5	59.2*	46.2
Log-like(1, 2)	27.1	21.7	21.1	27.7*
Log-like(2, 1)	23.0	31.3*	14.4	19.9
Log-like(2, 2)	105.3*	64.3	98.8	95.8

Table 4: Sum of log likelihood of different copula functions from simulated data in each regime (Highest value is marked with *)

We repeated the simulation and estimation for 1,000 times to study the estimation results and biases. The estimation results are shown in Tables 6, 7, 8 and 9. All of the model parameters in the individual time series and the copulas are estimated with a considerably high degree of accuracy. We also see that, except the two threshold

Copula	Gaussian	Clayton	Gumbel	Frank
$\theta_{1,1}$	0.5098	0.6079	1.5890	3.7012
$\theta_{1,2}$	0.4943	0.7279	1.4274	3.4768
$\theta_{2,1}$	0.5227	1.1643	1.4177	3.4313
$\theta_{2,2}$	0.6674	0.9632	1.7780	5.0386

Table 5: Estimated parameters of different copula functions from simulated data in each regime

		Estimation			
Parameter	True Value	Mean	Std. Dev.	Skewness	Kurtosis
$\alpha_0^{(1)}$	-0.007	-0.0086	0.0041	-0.0251	3.0147
$\alpha_1^{(1)}$	0.300	0.2658	0.0976	-0.1000	2.9402
$\sigma^{(1)}$	0.025	0.0249	0.0008	0.0983	3.1091
$\alpha_0^{(2)}$	-0.027	-0.0270	0.0007	-0.1267	2.9073
$\alpha_1^{(2)}$	0.100	0.0963	0.0462	0.0190	3.0773
$\sigma^{(2)}$	0.015	0.0150	0.0005	-0.1525	2.7847
r^X	-0.025	-0.0246	0.0007	-0.0207	3.7391

Table 6: Estimation of model parameters of time series X

		Estimation			
Parameter	True Value	Mean	Std. Dev.	Skewness	Kurtosis
$\beta_0^{(1)}$	0.004	0.0035	0.0017	0.1193	2.8619
$\beta_1^{(1)}$	0.600	0.5738	0.0815	-0.1736	2.7990
$\phi^{(1)}$	0.020	0.0198	0.0006	-0.0086	2.8166
$\beta_0^{(2)}$	-0.004	-0.0040	0.0006	-0.0097	2.9152
$\beta_1^{(2)}$	0.200	0.1985	0.0444	-0.1146	3.1664
$\phi^{(2)}$	0.010	0.0100	0.0003	-0.0013	2.9893
r^Y	-0.005	-0.0046	0.0006	-0.0529	2.3278

Table 7: Estimation of model parameters of time series Y

		Estimation			
Parameter	True Value	Mean	Std. Dev.	Skewness	Kurtosis
$\theta_{1,1}$(Gumbel)	1.5	1.5070	0.0758	0.2417	3.1915
$\theta_{1,2}$(Frank)	3.0	3.0567	0.5421	0.1811	3.0524
$\theta_{2,1}$(Clayton)	1.5	1.5109	0.2669	0.2475	3.0487
$\theta_{2,2}$(Gaussian)	0.7	0.6984	0.0281	-0.3341	3.3031

Table 8: Estimation of piecewise copula parameters

Regime	True Number of Samples	Estimated Number of Samples (Misclassified)
(1,1)	300	300 (0)
(1,2)	179	185 (6)
(2,1)	146	146 (0)
(2,2)	374	368 (0)

Table 9: Estimation accuracy on regime thresholds

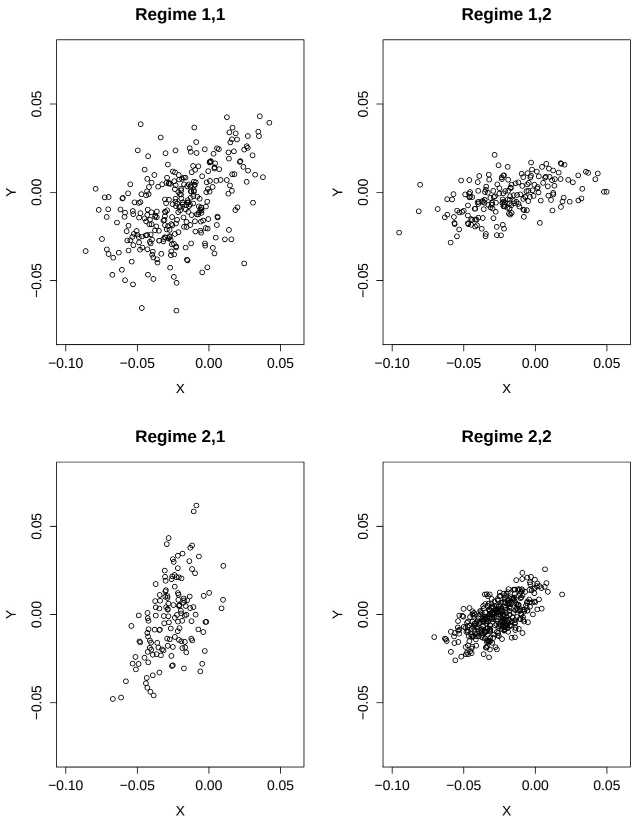

Figure 5: X and Y pairwise scatter plot of estimated points from two-regime switching threshold copula

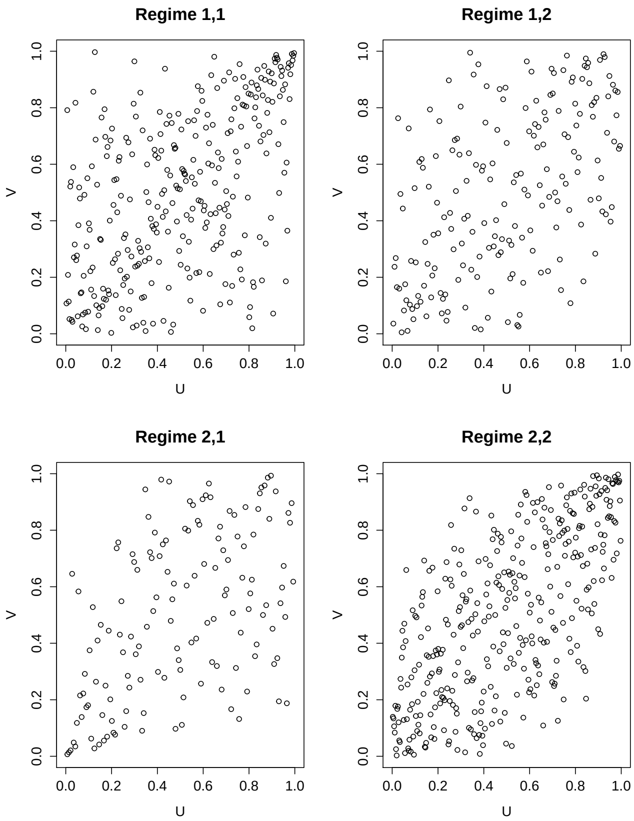

Figure 6: U and V pairwise scatter plot of estimated points from two-regime switching threshold copula

parameters, the skewness and kurtosis of each parameter are close to 0 and 3 respectively. This indicates that the sampling distributions of the estimates, except the threshold parameters, may be well approximated by a normal distribution. It is worth noting that the threshold parameters generally have non-standard limiting distribution; see Chan (1993). Nevertheless, the estimation accuracy of classification is very high that only 6 misclassifications in Regime (1,2) are found. The simulation study of the estimation method provides strong support for the proposed two-stage estimation method. It is expected that if a larger data set is used, the overall accuracy can be further improved.

5 An Application to Portfolio Risk Measurement

In this section we adopt the Copula-TAR model to forecast Value at Risk for financial portfolios using real data. We also compare the performances of VaR forecasts from baseline model (Gaussian copula) and conditional variance model (Dias and Embrechts (2003)) by backtesting based on a binomial test which is widely used in practice.

Daily close values of FTSE and S&P 500 from Apr-2007 to Apr-2009, were obtained from Yahoo! Finance. The first 300 data points, Apr-2007 to Jul-2008 is used as a training data set for model parameter estimation. The remaining 200 data points, Aug-2008 to Apr-2009, which include the period of the global financial crisis (GFC) of 2008, is used for backtesting ex-post Value at Risk forecasts. We fit the daily returns of the training samples to the following three different models:

1. Copula-TAR model with two self excited regimes AR(1) marginal time-series and four regimes in the copula (Copula-TAR);
2. Bivariate AR(1) model with a simple correlation structure (Gaussian Copula) (AR).
3. Bivariate conditional Variance model with marginal AR(1)-GARCH(1,1) and single copula function (AR-GARCH).

The copula regimes in Copula-TAR are based on the price levels of the indices. The copula functions in Copula-TAR and AR-GARCH are selected by maximum likelihood among Gaussian, Clayton, Gumbel and Frank copula each of which have only one parameter. Table 10 displays the estimation of the model parameters of different models using the training samples. The results suggest that the Copula-TAR model can separate high volatility and low volatility regimes by index levels, with the standard errors in the lower regimes being higher. It also shows high correlation between the returns of the two indices when they are both above or both below the their marginal threshold, i.e. regime (1,1) and (2,2). We can identify the time of GFC of 2008 is included in regime (2,2) which has the highest correlation among all regimes. Copula-TAR also suggests a low correlation regime in (2,1) which is much lower than the correlation when no regime is considered in AR model.

Copula-TAR	CPU time = 3.1sec						
FTSE	$\alpha_0^{(1)}$	$\alpha_1^{(1)}$	$\sigma^{(1)}$	$\alpha_0^{(2)}$	$\alpha_1^{(2)}$	$\sigma^{(2)}$	r^X
	-0.0005	-0.1994	0.0148	-0.0008	-0.0106	0.0089	6449.2
S&P 500	$\beta_0^{(1)}$	$\beta_1^{(1)}$	$\phi^{(1)}$	$\beta_0^{(2)}$	$\beta_1^{(2)}$	$\phi^{(2)}$	r^X
	-0.0003	-0.1811	0.0134	-0.0012	-0.1874	0.0091	1486.3
Copula	$\theta_{1,1}$=0.5503(Gaussian)			$\theta_{1,2}$=3.1856(Gumbel)			
	$\theta_{2,1}$=0.2409(Gaussian)			$\theta_{2,2}$=0.6064(Gaussian)			

AR	CPU time = 2.4sec						
	FTSE			S&P 500			
	α_0	α_1	σ	β_0	β_1	ϕ	θ
	-0.0007	-0.1685	0.0130	-0.0006	-0.1812	0.0120	0.5763

AR-GARCH	CPU time = 8.1sec					
	AR(1)		GARCH(1,1)			
	α_0	α_1	α_0	α_1	β_1	Copula
FTSE	-0.0007	-0.1685	0.0000	0.1452	0.8228	θ=4.62(Frank)
S&P 500	-0.0006	-0.1812	0.0000	0.0500	0.9000	

Table 10: Estimated parameters and CPU time of Copula-TAR, AR and AR-GARCH of daily returns of FTSE and S&P 500 in Apr-2007 to Jul-2008

5.1 Application on evaluating Value at Risk (VaR)

Now, we wish to apply the estimated models in financial risk measurement. Value at Risk (VaR) has emerged as a popular risk metrics in both the finance and insurance industries. Its use in financial risk management has been recommended by regulators and central bankers. Informally speaking, VaR is an estimate of the maximum loss of a risky portfolio that will incur at a certain probability level in a given time horizon. This provides a useful summary for a risky profile.

Here we estimate the 95% daily VaR using the three estimated models statically and dynamically. The portfolio we consider in the VaR estimation is constructed by FTSE and S&P 500 in equal weightings. The backtesting period covers the GFC of 2008 and the following huge drops in global financial indices.

In the static analysis, we use the estimated parameters in the previous section to calculate the 95% daily VaR of the testing samples. This means that the model parameters are estimated at the start of the testing period and kept constant through out that period. The training data window size used is 300. The size is chosen in such a way that the number of data points in each regime is larger than 40. The number of days that daily loss excesses VaR (violations) is recorded. We proceed to perform Kupiec's (1995) backtesting procedure for VaR models. This test is essentially a binomial test used to determine whether the observed frequency of violations is consistent with the frequency of expected violations predicted by the VaR model, which is 5% in our setting. The null hypothesis is that the model is "correct", meaning violations occurred in 5% of the days in the testing period. In this case, under the null hypothesis and the assumption of independence, the number of violation days follows a binomial distribution. If the p-value of the test is greater than the desired "null" significance level, we accept the model. Otherwise, we reject the the model. The number of days that the VaR is being violated and binomial test results are reported in Table 11. It can be seen that the results of Copula-TAR and AR are quite similar, while AR-GARCH shows lower numbers of violation days. Rejected results with significance level of 5% are marked with * in the table. Among all results, only FTSE from AR-GARCH is not rejected at up to the 1% significance level in the binomial test. All models are rejected at the significance level of 5% during the financial crisis period.

Next, we evaluate the model parameters dynamically. The models are estimated everyday in the training period using moving windows and the daily estimations are used to evaluate the VaR of the next day. Table 12 reports the binomial test results and the number of days on which the VaR forecast is violated. At the 5% level of significance, no results for the Copula-TAR case are rejected, while all results for the AR case are rejected. One result in AR-GARCH is not rejected. The results reveal that VaR forecasts produced by Copula-TAR are more adaptive to changing market regimes than those obtained from the other two multivariate time series models. The estimation of Copula-TAR is computationally efficient. It may not be unreasonable to believe that daily VaR forecasts for large portfolios can be performed efficiently using Copula-TAR model.

	FTSE	p-value	S&P 500	p-value	Portfolio	p-value
Copula-TAR	29*	0.0000	45*	0.0000	41*	0.0000
AR	36*	0.0000	49*	0.0000	41*	0.0000
AR-GARCH	17*	0.0121	25*	0.0000	22*	0.0002

Number of testing days = 200

Table 11: Number of days that model Value at Risk are violated and binomial test results when static model parameters are used. Results marked * are rejected at 5% significance.

	FTSE	p-value	S&P 500	p-value	Portfolio	p-value
Copula-TAR	9	0.5453	8	0.6730	6	0.8763
AR	20*	0.0012	24*	0.0000	25*	0.0000
AR-GARCH	14	0.0781	17*	0.0121	16*	0.0238

Number of testing days = 200

Table 12: Number of days that model Value at Risk are violated and binomial test results when model parameters are re-estimated daily. Results marked * are rejected at 5% significance.

6 Conclusion

We have introduced a threshold-copula-based SETAR model with a view to improving the standard Gaussian copulas in current practice of financial risk management. We have proposed a simple and practical two-step estimation method for the model. Simulation studies indicate a high level of accuracy is achieved by the two-stage estimation method. We have used the FTSE and S&P 500 daily returns data to illustrate how the proposed model can significantly improve the performance of portfolio VaR forecasts. Backtesting results based on a binomial test have revealed that the incorporation of the threshold effect in both the copula function and the individual time series has led to a significant improvement on portfolio VaR forecasts. Since FTSE and S&P 500 have different trading hours, there may be lead-lag effects between the two returns series.

While we do not claim that the proposed model can describe fully all of the dependent structures appearing in the real world, we believe that the proposed model can provide a substantial improvement on single parameter copula time series models by incorporating the risk associated with individual financial assets under different market conditions.

Threshold-copula-based SETAR model provides a new way to tackle correlation structures in regime switching multivariate data. Forecasting VaR is just one example given in this paper. We do believe the use of the model is much broader than the examples given in this paper. Further studies could be directed at diagnostics and gaining deeper insights through more real applications.

7 Acknowledgment

We thank the journal editor Professor Robert Taylor, the co-editor Doctor Granville Tunnicliffe-Wilson and two anonymous referees for their valuable insights and comments that greatly improved this paper. Lu's research was partially supported by EU's Marie Curie Career Integration Grant.

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