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Article (Accepted version)
(Refereed)

Original citation:
Gapeev, Pavel V and Stoev, Yavor I. On the construction of non-affine jump-diffusion models. ISSN 0736-2994

DOI: 10.1080/07362994.2017.1333008

On the construction of non-affine jump-diffusion models

Pavel V. Gapeev* Yavor I. Stoev†

We describe a method for construction of jump analogues of certain one-dimensional diffusion processes satisfying solvable stochastic differential equations. The method is based on the reduction of the original stochastic differential equations to the ones with linear diffusion coefficients, which are reducible to the associated ordinary differential equations, by using the appropriate integrating factor processes. The analogues are constructed by means of adding the jump components linearly into the reduced stochastic differential equations. We illustrate the method by constructing jump analogues of several diffusion processes and expand the notion of market price of risk to the resulting non-affine jump-diffusion models.

1 Introduction

Stochastic differential equations play an important role in the theory of stochastic processes and are commonly used to describe the dynamics of assets of random nature in various models of insurance and finance. The standard methods based on Picard iterations are used for the construction of pathwise solutions of such equations in the case of regular drift and diffusion coefficients. Alternatively, one can study certain classes of the so-called *solvable* stochastic differential equations which can either admit explicit solutions or at least be reduced to the corresponding first-order ordinary differential equations. The former class was considered in Gard [12; Chapter IV], where closed-form strong solutions were obtained to stochastic differential equations with linear coefficients, by introducing the appropriate integrating factor processes. The latter class was studied in Øksendal [23; Chapter V], where the equations with general drift and linear diffusion coefficients were reduced to the ordinary differential form. The extensions of the Ornstein-Uhlenbeck processes to the case of driving Lévy processes was proposed in Barndorff-Nielsen [1]. The general class of solvable stochastic differential equations was expanded in [10] to the case of jump-diffusion processes driven by Wiener processes and Poisson random measures of finite intensities. The tractability of the resulting analytic solutions of the constructed equations was shown in İyigünler, Çağlar, and Ünal [16], by analysing the accuracy of the numerical approximations obtained from the appropriate discretisation schemes. The Laplace transforms of the first exit times from two-sided intervals for certain one-dimensional jump-diffusion processes satisfying solvable stochastic differential equations were recently computed in [11].

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Mathematics Subject Classification 2010: Primary 60J75, 60H10, 91G30. Secondary 60J60, 60G51, 60G44.

Key words and phrases: Stochastic differential equations, jump-diffusion processes, non-affine processes, market price of risk.

It is known that exponential Lévy processes are widely used for the description of the price dynamics of risky assets in models of financial markets (see, e.g. Cont and Tankov [3] for an extensive overview containing various theoretical and numerical issues). Along with the classical models based on compound Poisson processes, modern examples include exponential variance gamma processes, normal inverse Gaussian processes, and hyperbolic processes (see, e.g. Madan and Seneta [22], Barndorff-Nielsen [1], and Eberlein and Keller [9] for the description of the corresponding models). An introduction to jump-diffusion modelling for asset prices and the term structure of interest rates was provided in Runggaldier [24] in the context of the pricing and hedging problems. The definition and a complete characterisation of the class of regular affine processes was carried out in Duffie, Filipović, and Schachermayer [8], which laid the foundations for a wide range of financial applications. The popularity of the affine class can be explained by the fact that the logarithms of the characteristic functions of the transition distributions of such time-homogeneous Markov processes represent affine functions of the initial states. The analytical tractability of the affine models is implied from the properties of the coefficients defining these affine relationships, which solve an associated family of ordinary differential equations. These features made the affine processes widely applicable for the description of the dynamics of term structures of interest rates, the models of credit risk and stochastic volatility, as well as the pricing of contingent claims by means of Fourier transforms (see, e.g. Duffie et al. [8; Chapter XIII], Duffie [7], Kallsen [18], and Kallsen, Muhle-Karbe, and Voß [19] and the references therein, respectively).

Despite an obvious recent focus on the affine and more general polynomial processes (see, e.g. Cuchiero, Keller-Ressel, and Teichmann [6] for the definition of the latter processes and their applications), some alternative models have attracted a considerable attention in the financial mathematics literature. Such non-affine examples include the constant elasticity of variance (CEV) and the related SABR models for local and stochastic volatility introduced in Cox [4] and Hagan et al. [15], respectively (see also the latter reference for model-dependent calibration methods). An overview of various continuous diffusion models of stochastic interest rates was provided in Shiryaev [25; Chapter III, Section 4]. In the present paper, we develop the method for construction of non-affine jump analogues of certain diffusion processes in the case of driving Wiener processes and Poisson random measures of infinite intensity. We also describe how the notion of market price of risk, or relative risk, can be extended to the constructed non-affine jump-diffusion models in relation with the pricing of derivative securities.

The paper is structured as follows. In Section 2, we first apply the method of [12; Chapter IV] to obtain explicit solutions to linear jump-diffusion stochastic differential equations driven by a Wiener process and a Poisson random measure of infinite intensity. Then, we follow [23; Chapter V, Example 5.16] to reduce the equations with general drift and linear diffusion and jump coefficients to the corresponding first-order ordinary differential equations that are solvable in the pathwise sense. In Section 3, we extend the class of solvable stochastic differential equations to the reducible ones by means of applying smooth invertible transformations. We present sufficient conditions for the reducibility of the general stochastic differential equations to the solvable ones. We also construct jump analogues of continuous diffusions and illustrate our results on several examples of non-affine jump-diffusion processes. In Section 4, we discuss the extension of the notion of market price of risk, or relative risk, for the constructed jump-diffusion models, which stays in accordance with the same notions for the appropriate continuous diffusion models of financial markets.

2 Solvable stochastic differential equations

In this section, we describe a class of stochastic differential equations which can either be solved explicitly or reduced to ordinary differential equations, by means of integrating factor processes. For this purpose, we suppose that on a complete probability space $(\Omega, \mathcal{F}, P)$ there exists a standard

Wiener process $W = (W_t)_{t \geq 0}$ and a homogeneous Poisson random measure $\mu(dt, dv)$ on $(\mathbb{R}_+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}))$ with the intensity (compensator) measure $\nu(dt, dv) = dtF(dv)$ (see, e.g. [17; Chapter II, Definition 1.20]), where $F(dv)$ is a positive σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $F(\{0\}) = 0$, and W is assumed to be independent of $\mu(dt, dv)$.

2.1 The setting. Let us consider the stochastic differential equation

$$\begin{aligned} dX_t = {} & \beta(t, X_t)\, dt + \gamma(t, X_t)\, dW_t \\ & + \int h(\delta(t, X_{t-}, v))\, (\mu - \nu)(dt, dv) + \int \overline{h}(\delta(t, X_{t-}, v))\, \mu(dt, dv), \end{aligned} \tag{2.1}$$

where $h(x) = x I_{\{|x| \leq 1\}}$ with $I_{\{\cdot\}}$ as the indicator function, $\overline{h}(x) = x - h(x)$, $x \in \mathbb{R}$, and $\beta(t, x)$, $\gamma(t, x) > 0$, and $\delta(t, x, v)$ are continuous functions on $\mathbb{R}_+ \times \mathbb{R}$ and $\mathbb{R}_+ \times \mathbb{R}^2$, respectively. Assume that, for any $n \in \mathbb{N}$, there exist a constant $C_n > 0$ and a function $\rho_n(v)$ with $\int \rho_n^2(v) F(dv) < \infty$ such that the equalities

$$|\beta(t, x) - \beta(t, y)| \leq C_n\, |x - y|, \quad |\beta(t, x)| \leq C_n\, (1 + |x|), \tag{2.2}$$
$$|h(\delta(t, x, v)) - h(\delta(t, y, v))| \leq \rho_n(v)\, |x - y|, \tag{2.3}$$
$$|h(\delta(t, x, v))| \leq \rho_n(v)\, (1 + |x|), \tag{2.4}$$
$$|\overline{h}(\delta(t, x, v)) - \overline{h}(\delta(t, y, v))| \leq \rho_n^2(v)\, |x - y|, \tag{2.5}$$
$$|\overline{h}(\delta(t, x, v))| \leq (\rho_n^2(v) \wedge \rho_n^4(v))\, (1 + |x|), \tag{2.6}$$

are satisfied, for all $0 \leq t \leq n$ and $x, y, v \in \mathbb{R}$. We additionally assume that

$$\gamma(t, x) = \gamma_0(t) + \gamma_1(t)\, x \quad \text{and} \quad \delta(t, x, v) = \delta_0(t, v) + \delta_1(t, v)\, x \tag{2.7}$$

holds, where $\gamma_i(t)$ and $\delta_i(t, v)$ for $i = 0, 1$ are continuous functions such that $\delta_1(t, v) > -1$, for all $t \geq 0$ and $x, v \in \mathbb{R}$. These conditions guarantee the existence of a unique strong solution $X = (X_t)_{t \geq 0}$ to (2.1) for a given $X_0 \in \mathbb{R}$ (see, e.g. [17; Chapter III, Theorem 2.32]). Finally, the equation in (2.1) takes the form

$$\begin{aligned} dX_t = {} & \beta(t, X_t)\, dt + (\gamma_0(t) + \gamma_1(t) X_t)\, dW_t \\ & + \int h(\delta_0(t, v) + \delta_1(t, v) X_{t-})\, (\mu - \nu)(dt, dv) + \int \overline{h}(\delta_0(t, v) + \delta_1(t, v) X_{t-})\, \mu(dt, dv). \end{aligned} \tag{2.8}$$

2.2 The case of affine coefficients. Following the arguments in [12; Chapter IV], we see that when

$$\beta(t, x) = \beta_0(t) + \beta_1(t)\, x \tag{2.9}$$

holds, for all $t \geq 0$ and $x \in \mathbb{R}$, the stochastic differential equation (2.8) for X can be solved explicitly, and X represents a polynomial process (see, e.g. [6] for the definition and applications of this concept). Let us assume that the condition

$$\int_0^t \int \bigg(\frac{\delta_1^2(s, v) I_{\{|\delta(s, x, v)| \leq 1\}}}{1 + |\delta_1(s, v)|} + \big| \ln(1 + \delta_1(s, v)) - \delta_1(s, v)\, I_{\{|\delta(s, x, v)| \leq 1\}} \big| \bigg) F(dv)\, ds < \infty \tag{2.10}$$

holds, for all $t \geq 0$ and $x \in \mathbb{R}$. Therefore, the integrating factor process $Z = (Z_t)_{t \geq 0}$ given by

$$Z_t = \exp\bigg(\int_0^t \frac{\gamma_1^2(s)}{2}\, ds - \int_0^t \gamma_1(s)\, dW_s - \int_0^t \int \delta_1(s,v)\, I_{\{|\delta(s,X_{s-},v)| \leq 1\}}\, (\mu - \nu)(ds, dv) \qquad (2.11)$$
$$- \int_0^t \int \big(\ln(1 + \delta_1(s,v)) - \delta_1(s,v)\, I_{\{|\delta(s,X_{s-},v)| \leq 1\}} \big)\; \mu(ds, dv) \bigg)$$

is well-defined according to [25; Chapter VII, Section 3, Theorem 2]. Hence, applying Itô's formula (see, e.g. [17; Chapter I, Theorem 4.57]) to (2.11), we get that the process Z satisfies the equation

$$dZ_t = Z_{t-} \bigg(\gamma_1^2(t)\, dt - \gamma_1(t)\, dW_t - \int \delta_1(t,v)\, I_{\{|\delta(s,X_{s-},v)| \leq 1\}}\, (\mu - \nu)(dt, dv) \qquad (2.12)$$
$$- \int \frac{\delta_1(t,v) I_{\{|\delta(s,X_{s-},v)| > 1\}} - \delta_1^2(t,v)\, I_{\{|\delta(s,X_{s-},v)| \leq 1\}}}{1 + \delta_1(t,v)}\, \mu(dt, dv) \bigg).$$

It follows from the expressions in (2.8) and (2.9) that the process $F = (F_t)_{t \geq 0}$ defined by

$$F_t = \frac{Z_t X_t}{R_t} \quad \text{with} \quad R_t = \exp\bigg(\int_0^t \beta_1(s)\, ds \bigg) \qquad (2.13)$$

admits the representation

$$dF_t = \frac{1}{R_t} \big(Z_{t-}\, dX_t + X_{t-}\, dZ_t + d\langle Z^c, X^c \rangle_t + \Delta Z_t \Delta X_t - Z_{t-} X_{t-} \beta_1(t)\, dt \big) \qquad (2.14)$$
$$= \frac{Z_{t-}}{R_t} \bigg(\big(\beta_0(t) - \gamma_0(t)\gamma_1(t) \big)\, dt + \gamma_0(t)\, dW_t + \int \delta_0(t,v)\, I_{\{|\delta(s,X_{s-},v)| \leq 1\}}\, (\mu - \nu)(dt, dv)$$
$$+ \int \bigg(\frac{\delta_0(t,v)}{1 + \delta_1(t,v)} - \delta_0(t,v)\, I_{\{|\delta(s,X_{s-},v)| \leq 1\}} \bigg)\, \mu(dt, dv) \bigg).$$

Therefore, we may conclude from the expressions in (2.13) and (2.14) that the process $X = (X_t)_{t \geq 0}$ given by

$$X_t = \frac{R_t}{Z_t} \bigg(X_0 + \int_0^t \frac{Z_s}{R_s} \big(\beta_0(s) - \gamma_0(s)\gamma_1(s) \big)\, ds + \int_0^t \frac{Z_s}{R_s}\, \gamma_0(s)\, dW_s \qquad (2.15)$$
$$+ \int_0^t \frac{Z_{s-}}{R_s} \bigg(\int \delta_0(s,v)\, I_{\{|\delta(s,X_{s-},v)| \leq 1\}}\, (\mu - \nu)(ds, dv)$$
$$+ \int \bigg(\frac{\delta_0(s,v)}{1 + \delta_1(s,v)} - \delta_0(s,v)\, I_{\{|\delta(s,X_{s-},v)| \leq 1\}} \bigg)\, \mu(ds, dv) \bigg) \bigg)$$

provides a (unique strong) solution of the equation in (2.8) under the condition of (2.9), for a given $X_0 \in \mathbb{R}$. In this case, we call the stochastic differential equation of the form of (2.8) *solvable* in an explicit form.

2.3 The case of linear diffusion coefficients. Following the arguments in [23; Chapter V, Example 5.16], we now show that the stochastic differential equation in (2.8) can be reduced to an ordinary differential equation, if we assume that $\gamma_0(t) = \delta_0(t,v) = 0$ holds in (2.7), for all $t \geq 0$ and $v \in \mathbb{R}$. By applying the integration-by-parts formula to the process $G = (G_t)_{t \geq 0}$ given by $G_t = Z_t X_t$ and using the form of the functions $h(x)$ and $\overline{h}(x)$, and the expressions in (2.8) and

(2.12), we obtain

$$
\begin{aligned}
dG_t &= Z_{t-}\,dX_t + X_{t-}\,dZ_t + d\langle Z^c, X^c\rangle_t + \Delta Z_t\,\Delta X_t \qquad (2.16)\\
&= Z_{t-}\bigg(\beta(t,X_{t-})\,dt + \gamma_1(t)\,X_{t-}\,dW_t\\
&\qquad + \int h(\delta_1(t,v)X_{t-})\,(\mu-\nu)(dt,dv) + \int \overline{h}(\delta_1(t,v)X_{t-})\,\mu(dt,dv)\bigg)\\
&\quad + Z_{t-}X_{t-}\bigg(\gamma_1^2(t)\,dt - \gamma_1(t)\,dW_t - \int \frac{h(\delta_1(t,v)X_{t-})}{X_{t-}}\,(\mu-\nu)(dt,dv)\\
&\qquad - \int \frac{\overline{h}(\delta_1(t,v)X_{t-}) - \delta_1(t,v)h(\delta_1(t,v)X_{t-})}{(1+\delta_1(t,v))X_{t-}}\,\mu(dt,dv)\bigg)\\
&\quad - Z_{t-}X_{t-}\gamma_1^2(t)\,dt - Z_{t-}X_{t-}\int \frac{\delta_1^2(t,v)}{1+\delta_1(t,v)}\,\mu(dt,dv).
\end{aligned}
$$

Note that the all the fractions in the expression of (2.16) above are well-defined, since we use the notation $h(\delta_1(t,v)x)/x = \delta_1(t,v)$ for $x = 0$. Therefore, if $\beta(t,x)$ satisfies the conditions in (2.2), then the (unique strong) solution X of (2.8) is given by $X_t = G_t/Z_t$, where for all $\omega \in \Omega$ the process $G(\omega) = (G_t(\omega))_{t\geq 0}$ is the unique solution of the ordinary differential equation

$$dG_t(\omega) = Z_t(\omega)\,\beta(t, G_t(\omega)/Z_t(\omega))\,dt. \qquad (2.17)$$

Since the solution to the first-order ordinary differential equation in (2.17) can be obtained by means of Picard iterations, under certain regularity conditions, we further also call the stochastic differential equation of the form of (2.8) *solvable* in a closed form.

2.4 An example with a special semimartingale. Let us finally consider the stochastic differential equation in (2.1) with the truncation function $h(x) = x$, $x \in \mathbb{R}$, so that it takes the form

$$dX_t = \beta(t,X_t)\,dt + \gamma(t,X_t)\,dW_t + \int \delta(t,X_{t-},v)\,(\mu-\nu)(dt,dv). \qquad (2.18)$$

Then, the conditions of (2.3)-(2.6) can be written as

$$|\delta(t,x,v) - \delta(t,y,v)| \leq \rho_n(v)\,|x-y| \quad \text{and} \quad |\delta(t,x,v)| \leq \rho_n(v)\,(1+|x|), \qquad (2.19)$$

for all $0 \leq t \leq n$, $n \in \mathbb{N}$, and $x, y, v \in \mathbb{R}$. In this case, the equation in (2.8) takes the form

$$dX_t = \beta(t,X_t)\,dt + (\gamma_0(t) + \gamma_1(t)X_t)\,dW_t + \int (\delta_0(t,v) + \delta_1(t,v)X_{t-})\,(\mu-\nu)(dt,dv) \qquad (2.20)$$

and the condition of (2.10) can be simplified to

$$\int_0^t \int \bigg(\frac{\delta_1^2(s,v)}{1+|\delta_1(s,v)|} + \big|\ln(1+\delta_1(s,v)) - \delta_1(s,v)\big|\bigg)\,F(dv)\,ds < \infty. \qquad (2.21)$$

Thus, the integrating factor process Z from (2.11) admits the representation

$$Z_t = \exp\bigg(\int_0^t \frac{\gamma_1^2(s)}{2}\, ds - \int_0^t \gamma_1(s)\, dW_s - \int_0^t \int \delta_1(s,v)\, (\mu-\nu)(ds,dv) \qquad (2.22)$$
$$- \int_0^t \int \big(\ln(1+\delta_1(s,v)) - \delta_1(s,v)\big)\, \mu(ds,dv) \bigg).$$

Hence, the application of Itô's formula to the expression in (2.22) yields

$$dZ_t = Z_{t-} \bigg(\gamma_1^2(t)\, dt - \gamma_1(t)\, dW_t - \int \delta_1(t,v)\, (\mu-\nu)(dt,dv) + \int \frac{\delta_1^2(t,v)}{1+\delta_1(t,v)}\, \mu(dt,dv) \bigg). \qquad (2.23)$$

In a way similar to the one presented above, by using the expressions in (2.20) and (2.23), we can apply the Itô's formula to the processes F and G defined as in (2.13) and Subsection 2.3, respectively, and obtain the equations of (2.14) and (2.16). We conclude again that if $\beta(t,x)$ satisfies the conditions of (2.2), then the (unique strong) solution X of the equation in (2.18) is given by (2.15) in the setting of Subsection 2.2 and by $X_t = G_t/Z_t$ in the setting of Subsection 2.3. Note that, in this case, however, the indicator functions appearing in (2.14)-(2.15) are equal to one and $\overline{h}(x) = 0$, $x \in \mathbb{R}$, holds in (2.16).

3 Reducibility to solvable equations

In this section, we extend the class of *solvable* stochastic differential equations by means of smooth invertible transformations and provide sufficient conditions for the *reducibility* of the stochastic differential equations to the solvable ones.

3.1 The invertible transformations. Let us consider the stochastic differential equation

$$dY_t = \eta(t,Y_t)\, dt + \sigma(t,Y_t)\, dW_t \qquad (3.1)$$
$$+ \int h(\theta(t,Y_{t-},v))\, (\mu-\nu)(dt,dv) + \int \overline{h}(\theta(t,Y_{t-},v))\, \mu(dt,dv),$$

where $\eta(t,y)$, $\sigma(t,y) > 0$, and $\theta(t,y,v)$ are continuous functions on $\mathbb{R}_+ \times \mathcal{D}_Y$ and $\mathbb{R}_+ \times \mathcal{D}_Y \times \mathbb{R}$, respectively, for some open set $\mathcal{D}_Y \subseteq \mathbb{R}$. Suppose that $f(t,y)$ is an *invertible* function from the class $C^{1,2}(\mathbb{R}_+, \mathcal{D}_Y)$ in the sense that there exists a function $g(t,x)$ such that $f(t,g(t,x)) = x$ and $g(t,f(t,y)) = y$, for all $t \ge 0$, $x \in \mathcal{D}_X$, and $y \in \mathcal{D}_Y$, where $\mathcal{D}_X$ denotes the range of $f(t,y)$. Let the functions $\beta(t,x)$, $\gamma(t,x)$, and $\delta(t,x,v)$ be given by

$$\beta(t,x) = \partial_t f(t,g(t,x)) + \eta(t,g(t,x))\, \partial_y f(t,g(t,x)) + \frac{\sigma^2(t,g(t,x))}{2}\, \partial_{yy} f(t,g(t,x)), \qquad (3.2)$$
$$\gamma(t,x) = \sigma(t,g(t,x))\, \partial_y f(t,g(t,x)), \qquad (3.3)$$
$$h(\delta(t,x,v)) = h(\theta(t,g(t,x),v))\, \partial_y f(t,g(t,x)), \qquad (3.4)$$
$$\overline{h}(\delta(t,x,v)) = f(t,g(t,x)+\theta(t,g(t,x),v)) - f(t,g(t,x)) - h(\theta(t,g(t,x),v))\, \partial_y f(t,g(t,x)), \qquad (3.5)$$

for $t \ge 0$, $x \in \mathcal{D}_X$, and $v \in \mathbb{R}$, and assume that they satisfy the conditions (2.2)-(2.6), so that the equation for X in (2.1) has a (unique strong) solution with a state space $\mathcal{D}_X$, and $X_0 \in \mathcal{D}_X$. By virtue of the invertibility of the function $f(t,y)$ and an application of Itô's formula, we conclude that Y defined as $Y_t = g(t,X_t)$ is a (unique strong) solution to the equation (3.1) with a state space $\mathcal{D}_Y$ and $Y_0 = g(0,X_0) \in \mathcal{D}_Y$. Moreover, by virtue of the arguments of the previous section, when the functions $\gamma(t,x)$ and $\delta(t,x,v)$ satisfy the conditions of (2.7), the stochastic differential equation in (3.1) is reduced to the one in (2.8), which is solvable in a closed form under either the conditions of (2.9) or the assumption $\gamma_0(t) = \delta_0(t,v) = 0$, for all $t \geq 0$ and $v \in \mathbb{R}$.

On the other hand, if the equation in (3.1) has a (unique strong) solution Y with a state space $\mathcal{D}_Y$, by means of Itô's formula applied to the process $X_t = f(t, Y_t)$, we get

$$\begin{aligned} dX_t = \Big(&\partial_t f(t,Y_t) + \eta(t,Y_t)\,\partial_y f(t,Y_t) + \frac{\sigma^2(t,Y_t)}{2}\,\partial_{yy} f(t,Y_t)\Big)\,dt \\ &+ \sigma(t,Y_t)\,\partial_y f(t,Y_t)\,dW_t + \int h(\theta(t,Y_{t-},v))\,\partial_y f(t,Y_{t-})\,(\mu(dt,dv) - \nu(dt,dv)) \\ &+ \int \Big(f(t,Y_{t-} + \theta(t,Y_{t-},v)) - f(t,Y_{t-}) - h(\theta(t,Y_{t-},v))\,\partial_y f(t,Y_{t-})\Big)\,\mu(dt,dv). \end{aligned} \tag{3.6}$$

Therefore, when $f(t,y)$ solves the equations

$$\beta(t,f(t,y)) = \partial_t f(t,y) + \eta(t,y)\,\partial_y f(t,y) + \frac{\sigma^2(t,y)}{2}\,\partial_{yy} f(t,y), \tag{3.7}$$

$$\gamma_0(t) + \gamma_1(t) f(t,y) = \sigma(t,y)\,\partial_y f(t,y), \tag{3.8}$$

$$h(\delta_0(t,v) + \delta_1(t,v) f(t,y)) = h(\theta(t,y,v))\,\partial_y f(t,y), \tag{3.9}$$

$$\overline{h}(\delta_0(t,v) + \delta_1(t,v) f(t,y)) = f(t,y+\theta(t,y,v)) - f(t,y) - h(\theta(t,y,v))\,\partial_y f(t,y), \tag{3.10}$$

for some continuous functions $\beta(t,x)$, $\gamma_i(t)$, and $\delta_i(t,v)$, $i = 0,1$, and all $t \geq 0$, $x \in \mathcal{D}_X$, $y \in \mathcal{D}_Y$, and $v \in \mathbb{R}$, we obtain that the equation in (3.1) is reduced to the one of (2.8), which is solvable in a closed form under either the conditions of (2.9) or the assumption $\gamma_0(t) = \delta_0(t,v) = 0$, for all $t \geq 0$ and $v \in \mathbb{R}$. In this case, we call the stochastic differential equation in (3.1) *reducible* to a solvable equation, by means of the invertible transformation $f(t,y)$ described above (see also [12; Chapter IV], [23; Chapter V, Example 5.16], [10], and [16] for definitions of the related concepts).

Example 3.1. *(Black-Karasinski model [2].)* Suppose that in (3.1) we have $\eta(t,y) = y(\eta_0(t) + \eta_1(t)\ln y)$, $\sigma(t,y) = \sigma_0(t)y$, and $\theta(t,y,v) = 0$, for all $t \geq 0$, $y > 0$ and $v \in \mathbb{R}$. Then the function $f(t,y) = \ln y$, $y > 0$, with the inverse $g(t,x) = e^x$, $x \in \mathbb{R}$, reduces the equation in (3.1) to the equation of (2.8) with (2.9), where $\beta_0(t) = \eta_0(t) - \sigma_0^2(t)/2$, $\beta_1(t) = \eta_1(t)$, $\gamma_0(t) = \sigma_0(t)$, $\gamma_1(t) = \delta_i(t,v) = 0$, $i = 0,1$, for all $t \geq 0$ and $v \in \mathbb{R}$.

Example 3.2. *(Stochastic population model [23; Chapter V, Example 5.15].)* Suppose that in (3.1) we have $\eta(t,y) = \eta_0(t)y(\eta_1(t) - y)$, $\eta_0(t) > 0$, $\eta_1(t) > 0$, $\sigma(t,y) = \sigma_0(t)y$, and $\theta(t,y,v) = 0$, for all $t \geq 0$, $y > 0$ and $v \in \mathbb{R}$. Then the function $f(t,y) = 1/y$, $y > 0$, with the inverse $g(t,x) = 1/x$, $x > 0$, reduces the equation in (3.1) to the equation (2.8) with (2.9), where $\beta_0(t) = \eta_0(t)$, $\beta_1(t) = \sigma_0^2(t) - \eta_0(t)\eta_1(t)$, $\gamma_1(t) = -\sigma_0(t)$, $\gamma_0(t) = \delta_i(t,v) = 0$, $i = 0,1$, for all $t \geq 0$ and $v \in \mathbb{R}$.

Remark 3.3. Observe that, in Examples 3.1 and 3.2, the function $\eta(t,y)$ does not satisfy the second part of the conditions of (2.2), but we see that the equation in (3.1) has a unique solution, since it is reducible to the linear equation in (2.8) with (2.9).

3.2 The reducibility criterion. Let us now describe the invertible transformations $f(t,y)$ which reduce the stochastic differential equation in (3.1) to the solvable one in (2.8), in the *time-homogeneous case*. Suppose that (3.1) has a (unique strong) solution Y, where $\eta(t,y) = \eta(y)$, $\sigma(t,y) = \sigma(y)$, $\theta(t,y,v) = \theta(y,v)$, and $f(t,y) = f(y)$, $g(t,x) = g(x)$, for all $t \geq 0$, $x \in \mathcal{D}_X$, $y \in \mathcal{D}_Y$, and $v \in \mathbb{R}$. Assume that $\eta(y)$, $\sigma(y) > 0$, and $\theta(y,v)$ are twice continuously differentiable functions, and define

$$r(y) = \int^y \frac{dz}{\sigma(z)}, \quad p(y) = \frac{\eta(y)}{\sigma(y)} - \frac{\sigma'(y)}{2}, \quad \text{and} \quad q(y,v) = e^{r(y+\theta(y,v))-r(y)}, \tag{3.11}$$

for all $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$. We are now ready to state the reducibility assertions for jump-diffusion processes solving the equation in (3.1).

Theorem 3.4. *The equation in* (3.1) *is reducible to the one of* (2.8)*, where the appropriate invertible transformation* $f(y)$ *is given by*

$$f(y) = c\, e^{\gamma_1 r(y)} - \frac{\gamma_0}{\gamma_1}, \tag{3.12}$$

for all $y \in \mathcal{D}_Y$ *and some constant* $\gamma_0 \in \mathbb{R}$*, if the following conditions are satisfied:*

(i) either the equality

$$(q\partial_y q + \sigma(\partial_y q)^2)(y,v) = (q\partial_y \sigma \partial_y q - \sigma(\partial_y q)^2 + \sigma \partial_{yy} q)(y,v) = 0 \tag{3.13}$$

or the equality

$$\left(\frac{q\partial_y \sigma \partial_y q - \sigma(\partial_y q)^2 + \sigma \partial_{yy} q}{q\partial_y q + \sigma(\partial_y q)^2}\right)(y,v) = c_1 \tag{3.14}$$

holds, for some constant $c_1 \in \mathbb{R}$ *and all* $y \in \mathcal{D}_Y$*,* $v \in \mathbb{R}$*;*

(ii) the conditions

$$|\theta(y,v)| > 1 \quad \textit{if and only if} \quad \left|c\left(e^{\gamma_1 r(y+\theta(y,v))} - e^{\gamma_1 r(y)}\right)\right| > 1, \tag{3.15}$$

$$|\theta(y,v)| \le 1 \quad \textit{if and only if} \quad e^{\gamma_1 (r(y+\theta(y,v))-r(y))} = \gamma_1 \frac{\theta(y,v)}{\sigma(y)} + 1 \tag{3.16}$$

hold, for all $y \in \mathcal{D}_Y$ *and* $v \in \mathbb{R}$*, and some constants* $c \in \mathbb{R}$ *and* $\gamma_1 \neq 0$*.*

The solution of the equation in (2.8) *is given by the expression in* (2.15) *if, in addition, the following condition is satisfied:*

(iii) either the equality $p'(y) = 0$ *or the equalities*

$$\left(\frac{(\sigma p')'}{p'}\right)(y) = c_2 \tag{3.17}$$

and

$$\frac{(\sigma p')'}{p'} = \frac{q\partial_y \sigma \partial_y q - \sigma(\partial_y q)^2 + \sigma \partial_{yy} q}{q\partial_y q + \sigma(\partial_y q)^2} \quad \textit{with} \quad (3.14) \tag{3.18}$$

hold, for some constant $c_2 \in \mathbb{R}$*, and all* $y \in \mathcal{D}_Y$ *and* $v \in \mathbb{R}$*.*

On the other hand, solving the stochastic differential equation in (2.8) *can be reduced to solving the ordinary differential equation in* (2.17) *if the equality* $(\partial_y q)(y,v) = 0$ *holds, for all* $y \in \mathcal{D}_Y$ *and* $v \in \mathbb{R}$*.*

Proof. In order to prove the reducibility of the equation in (3.1) to the one of (2.8), we need to check whether the equalities in (3.7)-(3.10) are satisfied for some $\beta(t,x) = \beta(x)$, $\gamma_i(t) = \gamma_i$, $\delta_i(t,v) = \delta_i(v)$, $i = 0,1$, and $f(t,y) = f(y)$, for all $t \ge 0$, $y \in \mathcal{D}_Y$, and $v \in \mathbb{R}$.

By using the notations of (3.11) and the fact that $\sigma(y) > 0$ for $y \in \mathcal{D}_Y$, we obtain that the function $f(y)$ given by (3.12) is invertible. It can be shown by means of direct calculations that the equality in (3.8) is satisfied. Then, by summing up the equations in (3.9) and (3.10), instead of checking the equality in (3.10), we can verify whether

$$f(y+\theta(y,v)) - f(y) = \delta_0(v) + \delta_1(v)\, f(y) \tag{3.19}$$

holds. It follows by substituting the expressions of (3.12) with the notation of (3.11) for $f(y)$ that the equality in (3.19) is equivalent to

$$\big(q^{\gamma_1}(y,v) - (1+\delta_1(v))\big)\, e^{\gamma_1 r(y)} = \frac{\gamma_1\delta_0(v) - \gamma_0\delta_1(v)}{c\gamma_1}. \tag{3.20}$$

Then, by differentiating the expression in (3.20), we see that it should be verified whether the equality

$$\left(q^{\gamma_1}(y,v) - (1+\delta_1(v)) + \frac{\sigma(y)}{\gamma_1}\partial_y q^{\gamma_1}(y,v)\right)\frac{\gamma_1}{\sigma(y)}\, e^{\gamma_1 r(y)} = 0 \tag{3.21}$$

holds, while after multiplying both parts of the expression in (3.21) by $e^{-\gamma_1 r(y)}\sigma(y)$ and differentiating again, we get that the equality

$$\gamma_1\, \partial_y q^{\gamma_1}(y,v) + \partial_y(\sigma\partial_y q^{\gamma_1})(y,v) = 0 \tag{3.22}$$

needs to be verified. Applying the chain rule and dividing by $\gamma_1 q^{\gamma_1-2}$, we get that the equality in (3.22) is equivalent to

$$\gamma_1\, (q\partial_y q + \sigma(\partial_y q)^2)(y,v) + (q\partial_y\sigma\partial_y q - \sigma(\partial_y q)^2 + \sigma\partial_{yy}q)(y,v) = 0. \tag{3.23}$$

Hence, the equality in (3.22) can be verified by means of either the equality in (3.13) or

$$\gamma_0 = 0 \quad \text{and} \quad \gamma_1 = -\left(\frac{q\partial_y\sigma\partial_y q - \sigma(\partial_y q)^2 + \sigma\partial_{yy}q}{q\partial_y q + \sigma(\partial_y q)^2}\right)(y,v), \tag{3.24}$$

combined with the one of (3.14). By choosing

$$\delta_1(v) = q^{\gamma_1}(y,v) - 1 + \frac{\sigma(y)}{\gamma_1}\,\partial_y q^{\gamma_1}(y,v), \tag{3.25}$$

we get that the equality (3.21) is also verified. Thus, when we set $\gamma_0 = 0$ and

$$\delta_0(v) = \big(q^{\gamma_1}(y,v) - (1+\delta_1(v))\big)\, c\, e^{\gamma_1 r(y)}, \tag{3.26}$$

we obtain that the equality in (3.19) holds.

Let us now check whether the equality in (3.9) is satisfied. For this purpose, we define the auxiliary sets

$$\Theta_0 = \{(y,v) \in \mathcal{D}_Y \times \mathbb{R} \colon |\theta(y,v)| = 0\}, \tag{3.27}$$
$$\Theta_1 = \{(y,v) \in \mathcal{D}_Y \times \mathbb{R} \colon |\theta(y,v)| > 1\}, \tag{3.28}$$
$$\Delta_0 = \{(y,v) \in \mathcal{D}_Y \times \mathbb{R} \colon |\delta(f(y),v)| = 0\}, \tag{3.29}$$
$$\Delta_1 = \{(y,v) \in \mathcal{D}_Y \times \mathbb{R} \colon |\delta(f(y),v)| > 1\}, \tag{3.30}$$

and note that from the invertibility of $f(y)$ and the equality in (3.19) we have $\Theta_0 = \Delta_0$. It follows from the equality in (3.9) that we should verify whether $\Theta_1 \subseteq \Delta_1$ holds, but we have $f'(y) = 0$ for $y \in \Delta_1 \setminus \Theta_1$, that contradicts the invertibility of $f(y)$. Therefore, we need to verify whether

$\Theta_1 = \Delta_1$ holds, but the former represents the condition of (3.15), by virtue of the equality in (3.19). Then, substituting the expression in (3.19) into the equality of (3.9), we also need to verify whether

$$f(y + \theta(y,v)) - f(y) = \theta(y,v)\, f'(y) \tag{3.31}$$

holds, for $y \in (\mathcal{D}_Y \times \mathbb{R}) \setminus (\Delta_0 \cup \Delta_1)$, but the latter equality is equivalent to the condition of (3.16). Thus, the conditions of (3.15)-(3.16) are equivalent to the one of (3.9). Finally, the equality (3.7) is satisfied when we choose $\beta(x)$ as in (3.2), for $x \in \mathcal{D}_X$.

Assuming additionally that the condition of (iii) holds, let us now check whether the equality in (3.7) is satisfied with $\beta(x)$ of the form (2.9), for some constants $\beta_0, \beta_1 \in \mathbb{R}$. Observe that when the expressions in (3.17)-(3.18) are satisfied, we can set

$$\gamma_0 = 0 \quad \text{and} \quad \gamma_1 = -\bigg(\frac{(\sigma p')'}{p'} \bigg)(y) \tag{3.32}$$

and note that if the expression in (3.14) holds then γ_0 and γ_1 agree with the ones from (3.24). Substituting the expression from (3.12) with (3.11) for $f(y)$ into (3.7) and using the assumption of (2.9), we need to verify whether

$$\bigg(\gamma_1\, p(y) + \frac{\gamma_1^2}{2} - \beta_1 \bigg)\, e^{\gamma_1 r(y)} = \frac{\gamma_1 \beta_0 - \gamma_0 \beta_1}{c \gamma_1} \tag{3.33}$$

holds. It follows by differentiating the expression in (3.33) and using the notations of (3.11) that the equality

$$\bigg(\gamma_1\, p(y) + \frac{\gamma_1^2}{2} - \beta_1 + \sigma(y)\, p'(y) \bigg)\, \frac{\gamma_1}{\sigma(y)}\, e^{\gamma_1 r(y)} = 0 \tag{3.34}$$

needs to be verified, and multiplying both parts of (3.34) by $e^{-\gamma_1 r(y)} \sigma(y)/\gamma_1$ and differentiating again, we see that the equality

$$\gamma_1\, p'(y) + (\sigma p')'(y) = 0 \tag{3.35}$$

should also be satisfied. Hence, the equality in (3.35) is satisfied under the condition $p'(y) = 0$ or the ones of (3.17)-(3.18) with (3.32). It follows that the equality in (3.34) holds when we set

$$\beta_1 = \gamma_1\, p(y) + \frac{\gamma_1^2}{2} + \sigma(y)\, p'(y). \tag{3.36}$$

Thus, the equality in (3.33) is verified when we set $\gamma_0 = 0$ and

$$\beta_0 = \bigg(\gamma_1\, p(y) + \frac{\gamma_1^2}{2} - \beta_1 \bigg)\, c\, e^{\gamma_1 r(y)}. \tag{3.37}$$

We may therefore conclude that the equality in (3.7) holds with $\beta(x)$ of the form (2.9), and we can solve the equation in (2.8) by the expression of (2.15).

On the other hand, when the equality $(\partial_y q)(y,v) = 0$ holds, for all $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$, it follows from the expressions in (3.25)-(3.26) that $\delta_0(v) = 0$ holds, so that we can set $\gamma_0 = 0$ and reduce the equation in (2.8) to the ordinary differential equation of (2.17). □

Theorem 3.5. *The equation in* (3.1) *is reducible to the one of* (2.8) *with* $\gamma_1 = 0$, *where the appropriate invertible transformation* $f(y)$ *is given by*

$$f(y) = \gamma_0\, r(y) + c, \tag{3.38}$$

for all $y \in \mathcal{D}_Y$ *and some constant* $c \in \mathbb{R}$, *if the following conditions are satisfied:*

(iv) the equality

$$\left(\frac{\sigma\partial_y q}{q}\right)(y,v) = c_3(v) \tag{3.39}$$

holds, for some function $c_3(v)$, and all $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$;
(v) the conditions

$$|\theta(y,v)| > 1 \quad \textit{if and only if} \quad \big|\gamma_0 \left(r(y+\theta(y,v)) - r(y)\right)\big| > 1, \tag{3.40}$$

$$|\theta(y,v)| \le 1 \quad \textit{if and only if} \quad r(y+\theta(y,v)) - r(y) = \frac{\theta(y,v)}{\sigma(y)} \tag{3.41}$$

hold, for some $\gamma_0 \neq 0$, $y \in \mathcal{D}_Y$, and $v \in \mathbb{R}$.
The solution of (2.8) is given by (2.15) if, in addition, the equality $(\sigma p')'(y) = 0$ holds, for all $y \in \mathcal{D}_Y$.

Proof. By using the notations of (3.11) and the assumption that $\sigma(y) > 0$ holds, for $y \in \mathcal{D}_Y$, we obtain that the function $f(y)$ given by (3.38) is invertible. Direct calculations show that $f(y)$ satisfies the equality in (3.8). It follows by substituting the expression of (3.38) with (3.11) for $f(y)$ into (3.19) that we can equivalently check whether

$$\big(\ln q(y,v) - \delta_1(v)\, r(y)\big)\, \gamma_0 = \delta_0(v) + \delta_1(v)\, c \tag{3.42}$$

holds. Then, differentiating the equality in (3.42) and multiplying both parts of the resulting expression by $\sigma(y)$, we see that we can verify whether

$$\left(\frac{\sigma\partial_y q}{q}\right)(y,v) = \delta_1(v) \tag{3.43}$$

holds. It follows from the expression in (3.39) that the equality in (3.43) is satisfied when we set

$$\delta_1(v) = \left(\frac{\sigma\partial_y q}{q}\right)(y,v), \tag{3.44}$$

for all $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$. Hence, the equality in (3.42) is verified when we choose

$$\delta_0(v) = \big(\ln q(y,v) - \delta_1(v)\, r(y)\big)\, \gamma_0 - \delta_1(v)\, c, \tag{3.45}$$

for some $c \in \mathbb{R}$. By means of the arguments similar to the ones used in Theorem 3.4, the conditions of (3.40)-(3.41) are equivalent to the ones of (3.9). Again, the equality in (3.7) holds when we choose $\beta(x)$ as in (3.2), for $x \in \mathcal{D}_X$.

Finally, assuming additionally that the equality $(\sigma p')'(y) = 0$ holds, for all $y \in \mathcal{D}_Y$, let us verify whether the equality in (3.7) is satisfied with $\beta(x)$ of the form (2.9), for some constants $\beta_0, \beta_1 \in \mathbb{R}$. By means of substituting the expression of (3.38) with (3.11) for $f(y)$ into the one of (3.7) with (2.9), it follows that we can equivalently verify whether

$$\big(p(y) - \beta_1\, r(y)\big)\, \gamma_0 = \beta_0 + c\, \beta_1 \tag{3.46}$$

holds, for some constant $c \in \mathbb{R}$. Then, by differentiating the equality in (3.46), applying the notations of (3.11) and multiplying both parts of the resulting expression by $\sigma(y)$, we see that it should be verified whether the equality

$$\sigma(y)\, p'(y) - \beta_1 = 0 \tag{3.47}$$

holds. Hence, by using the equality $(\sigma p')'(y) = 0$, we get that the equality in (3.47) is satisfied when we set

$$\beta_1 = \sigma(y)\, p'(y), \tag{3.48}$$

for all $y \in \mathcal{D}_Y$. Thus, the equality in (3.46) is verified when we set

$$\beta_0 = c\,\beta_1 - \big(p(y) - \beta_1\, r(y)\big)\,\gamma_0. \tag{3.49}$$

We may therefore conclude that the equality in (3.7) holds with $\beta(x)$ of the form (2.9) and any $\gamma_0 \neq 0$, so that we can solve the equation in (2.8) by the expression of (2.15). $\square$

Remark 3.6. It follows from the proof presented above that if the truncation function $h(x)$ is non-zero, that is, if the equation in (3.9) is not trivially satisfied, then the process Y should have a diffusion coefficient $\sigma(y)$ which satisfies either the condition of (3.16) or (3.41). This is relevant only in the case of infinite jump intensity, because the condition (3.9) is always satisfied by putting $h(x) \equiv 0$ for the finite jump intensity case. We also note that the transformation equations of (3.12) and (3.38) are similar to the equations of (3.10)-(3.12) in [10] for a single Poisson random measure and the equations of (8) and (16) in [16] for multiple Poisson random measures in the time-inhomogeneous setting. However, in the present paper, the conditions of (ii) and (v) of Theorems 3.4-3.5 reflect the fact that we work in the infinite jump intensity case.

Example 3.7. *(Cox-Ingersoll-Ross model I [5].)* Suppose that in (3.1) we have $\eta(y) = \eta_0 + \eta_1 y$, $\sigma(y) = \sigma_0\sqrt{y}$, $\eta_0 \geq \sigma_0^2/2$, $\eta_1 \neq 0$, and $\theta(y, v) = 0$, for all $y > 0$ and $v \in \mathbb{R}$. Then the function $f(y) = \exp(2\sqrt{y})$, $y > 0$, with the inverse $g(x) = (\ln x/2)^2$, $x > 1$, reduces the equation in (3.1) to the one of (2.8), where $\beta(x) = x(2\eta_0 + \eta_1 \ln^2 x/2 + \sigma_0^2(\ln x - 1)/2)/\ln x$, $\gamma_1 = \sigma_0$, and $\gamma_0 = \delta_0(v) = \delta_1(v) = 0$, for all $x > 1$ and $v \in \mathbb{R}$.

Example 3.8. *(Cox-Ingersoll-Ross model II [5].)* Suppose that in (3.1) we have $\eta(y) = \eta_0 y(\eta_1 - y)$, $\sigma(y) = \sigma_0\sqrt{y^3}$ and $\theta(y, v) = 0$, for all $y > 0$ and $v \in \mathbb{R}$, where η_0, $\eta_1 \in \mathbb{R}$, and $\sigma_0 > 0$. Then the function $f(y) = \exp(-2/\sqrt{y})$, $y > 0$, with the inverse $g(x) = 4/\ln^2 x$, $x \in (0,1)$, reduces the equation in (3.1) to the one of (2.8), where $\beta(x) = -\eta_0 x(\eta_1 \ln x - 4/\ln x)/2 + \sigma_0^2 x(1 + 3/\ln x)/2$, $\gamma_1 = \sigma_0$, and $\gamma_0 = \delta_0(v) = \delta_1(v) = 0$, for all $x \in (0,1)$ and $v \in \mathbb{R}$.

Example 3.9. *(Constant elasticity of variance model [4] and [15].)* Suppose that in (3.1) we have $\eta(y) = \eta_1 y$, $\sigma(y) = \sigma_0 y^\alpha$ and $\theta(y, v) = 0$, for all $y > 0$ and $v \in \mathbb{R}$, where $\eta_1 \in \mathbb{R}$, σ_0, and $\alpha > 0$. In the case when $\alpha = 1$, the function $f(y) = y$, $y > 0$, with the inverse $g(x) = x$, $x > 0$, reduces the equation in (3.1) to the one of (2.8), where $\beta(x) = x\eta_1$, $\gamma_1 = \sigma_0$ and $\gamma_0 = \delta_0(v) = \delta_1(v) = 0$, for all $x > 0$ and $v \in \mathbb{R}$. In the case when $\alpha \in (0,1)$, the function $f(y) = \exp(y^{1-\alpha}/(1-\alpha))$, $y > 0$, with the inverse $g(x) = (\ln(x)(1-\alpha))^{1/(1-\alpha)}$, $x > 1$, reduces the equation in (3.1) to the one of (2.8), where $\beta(x) = \eta_1(1-\alpha)x\ln x + \sigma_0^2 x(1 - \alpha/((1-\alpha)\ln x))/2$, $\gamma_1 = \sigma_0$, and $\gamma_0 = \delta_0(v) = \delta_1(v) = 0$, for all $x > 1$ and $v \in \mathbb{R}$. The case $\alpha > 1$ yields the same reduced equation as the case $\alpha \in (0,1)$ does, but with the same $\beta(x)$ defined for $x \in (0,1)$.

Example 3.10. *(Shiryaev filtering model [21; Chapter IX].)* Suppose that in (3.1) we have $\eta(y) = \eta_0(1-y)$, $\sigma(y) = \sigma_0 y(1-y)$ and $\theta(y, v) = 0$, for all $y \in (0,1)$ and $v \in \mathbb{R}$. Then the function $f(y) = y/(1-y)$, $y \in (0,1)$, with the inverse $g(x) = x/(1+x)$, $x > 0$, reduces the equation in (3.1) to the one of (2.8), where $\beta(x) = \eta_0(1+x) + \sigma_0^2 x^2/(1+x)$, $\gamma_1 = \sigma_0$, and $\gamma_0 = \delta_0(v) = \delta_1(v) = 0$, for all $x > 0$ and $v \in \mathbb{R}$.

3.3 The jump analogues of some diffusions. In the rest of this section, we will construct *jump analogues* of several diffusion processes, by adding the jump components in the models of their solvable counterparts. To this purpose, we will use the Wiener process $W = (W_t)_{t\geq 0}$ and the Poisson random measure $\mu(dt, dv)$ with the compensator $\nu(dt, dv) = dtF(dv)$ existing on the probability space $(\Omega, \mathcal{F}, P)$.

Let $Y = (Y_t)_{t\geq 0}$ be a continuous process with a state space $\mathcal{D}_Y$ solving the stochastic differential equation (3.1) with $\theta(t, y, v) = 0$ for all $t \geq 0$, $y \in \mathcal{D}_Y$, and $v \in \mathbb{R}$. Suppose that there exists an invertible transformation $f(t, y) \in C^{1,2}(\mathbb{R}_+, \mathcal{D}_Y)$ satisfying (3.7)-(3.10) and such that the process $X = (X_t)_{t\geq 0}$, $X_t = f(t, Y_t)$, solves the equation in (2.8) with $\delta_i(t, v) = 0$, for $i = 0, 1$, $t \geq 0$, and $v \in \mathbb{R}$. Let us take a continuous function $\widehat{\delta}(t, x, v) = \widehat{\delta}_0(t, v) + \widehat{\delta}_1(t, v)x$ such that $\widehat{\delta}_1(t, v) > -1$ holds and the expression in (2.10) is satisfied with $\delta(t, x, v)$ replaced by $\widehat{\delta}(t, x, v)$. Assume also that

$$\widehat{\delta}_i(t, v) \neq 0 \quad \text{if and only if} \quad \gamma_i(t) \neq 0 \tag{3.50}$$

holds, for $i = 0, 1$, and all $t \geq 0$ and $v \in \mathbb{R}$. Consider the stochastic differential equation

$$\begin{aligned} d\widehat{X}_t = \; & \beta(t, \widehat{X}_t)\, dt + (\gamma_0(t) + \gamma_1(t)\widehat{X}_t)\, dW_t \\ & + \int h(\widehat{\delta}_0(t, v) + \widehat{\delta}_1(t, v)\widehat{X}_{t-})\, (\mu(dt, dv) - \nu(dt, dv)) + \int \overline{h}(\widehat{\delta}_0(t, v) + \widehat{\delta}_1(t, v)\widehat{X}_{t-})\, \mu(dt, dv), \end{aligned} \tag{3.51}$$

where $\beta(t, x)$ satisfies either the condition of (2.9) or $\gamma_0(t) = \widehat{\delta}_0(t, v) = 0$ holds, for all $t \geq 0$ and $v \in \mathbb{R}$, and assume that its (unique strong) solution $\widehat{X} = (\widehat{X}_t)_{t\geq 0}$ has the state space $\mathcal{D}_X$. Then, according to the arguments of Section 2, we conclude that equation in (3.51) is solvable in a closed form, and applying the inverse transformation $g(t, x)$, for $t \geq 0$ and $x \in \mathcal{D}_X$, to the solution $\widehat{X}$, we obtain that the process $\widehat{Y}_t = g(t, \widehat{X}_t)$ solves the equation

$$\begin{aligned} d\widehat{Y}_t = \; & \eta(t, \widehat{Y}_t)\, dt + \sigma(t, \widehat{Y}_t)\, dW_t \\ & + \int \widehat{\theta}_0(t, \widehat{Y}_{t-}, v)\, (\mu(dt, dv) - \nu(dt, dv)) + \int \widehat{\theta}_1(t, \widehat{Y}_{t-}, v)\, \mu(dt, dv), \end{aligned} \tag{3.52}$$

with

$$\widehat{\theta}_0(t, y, v) = h(\widehat{\delta}_0(t, v) + \widehat{\delta}_1(t, v)f(t, y))\partial_x g(t, f(t, y)), \tag{3.53}$$

$$\widehat{\theta}_1(t, y, v) = g(t, \widehat{\delta}_0(t, v) + (1 + \widehat{\delta}_1(t, v))f(t, y)) - g(t, f(t, y)) - \widehat{\theta}_0(t, y, v), \tag{3.54}$$

for $t \geq 0$, $y \in \mathcal{D}_Y$, and $v \in \mathbb{R}$. We will call such process $\widehat{Y} = (\widehat{Y}_t)_{t\geq 0}$ a *jump analogue* of the diffusion process $Y = (Y_t)_{t\geq 0}$ (see [10; Section 4]). Note that when $h(x) = 0$, $x \in \mathbb{R}$, holds the jump analogue $\widehat{Y}$ also solves the equation of the form of (3.1).

Remark 3.11. Let us now introduce the *pure jump analogue* $\widetilde{Y} = (\widetilde{Y}_t)_{t\geq 0}$ of the given $Y = (Y_t)_{t\geq 0}$, by setting $\sigma(t, y) = 0$ in (3.52), for all $t \geq 0$ and $y \in \mathcal{D}_Y$. Such a process $\widetilde{Y}$ can be defined as a (unique strong) solution of the stochastic differential equation

$$d\widetilde{Y}_t = \eta(t, \widetilde{Y}_t)\, dt + \int \widehat{\theta}_0(t, \widetilde{Y}_{t-}, v)\, (\mu(dt, dv) - \nu(dt, dv)) + \int \widehat{\theta}_1(t, \widetilde{Y}_{t-}, v)\, \mu(dt, dv), \tag{3.55}$$

with $\widehat{\theta}_i(t, y, v)$, $i = 0, 1$, given by (3.53)-(3.54).

Let us now give some examples of jump analogues of diffusion processes presented above. We assume throughout that the truncation function is $h(x) = 0$, $x \in \mathbb{R}$, and therefore $\widehat{\theta}_0(t, y, v) = 0$ holds, for $t \geq 0$, $y \in \mathcal{D}_Y$, and $v \in \mathbb{R}$.

Example 3.12. *(Extended Black-Karasinski model.)* Suppose that in (3.52) we have the same $\eta(t, y)$ and $\sigma(t, y)$ as in Example 3.1. Then, for a jump analogue in (3.54) we can take $\widehat{\delta}_1(t, v) = 0$, and thus $\widehat{\theta}_1(t, y, v) = y(\exp(\widehat{\delta}_0(t, v)) - 1)$, for all $t \geq 0$, $y > 0$, and $v \in \mathbb{R}$.

Example 3.13. *(Extended stochastic population model.)* Suppose that in (3.52) we have the same $\eta(t,y)$ and $\sigma(t,y)$ as in Example 3.2. Then, for a jump analogue in (3.54) we can take $\widehat{\delta}_0(t,v)=0$, and thus $\widehat{\theta}_1(t,y,v)=-y(\widehat{\delta}_1(t,v)/(1+\widehat{\delta}_1(t,v)))$, for all $t\geq 0$, $y>0$, and $v\in\mathbb{R}$.

Example 3.14. *(Extended Cox-Ingersoll-Ross model I.)* Suppose that in (3.52) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 3.7. Then, for a jump analogue in (3.54) we can take $\widehat{\theta}_1(y,v)=\sqrt{y}\ln(1+\widehat{\delta}_1(v))+\ln^2(1+\widehat{\delta}_1(v))/4$, for all $y>0$ and $v\in\mathbb{R}$.

Example 3.15. *(Extended Cox-Ingersoll-Ross model II.)* Suppose that in (3.52) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 3.8. Then, for a jump analogue in (3.54) we can take $\widehat{\theta}_1(y,v)=y\sqrt{y}\ln\sqrt{1+\widehat{\delta}_1(v)}(2-\sqrt{y}\ln\sqrt{1+\widehat{\delta}_1(v)})/(\sqrt{y}\ln\sqrt{1+\widehat{\delta}_1(v)}-1)^2$, for all $y>0$ and $v\in\mathbb{R}$.

Example 3.16. *(Extended constant elasticity of variance model.)* Suppose that in (3.52) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 3.9. In the case when $\alpha=1$, for a jump analogue in (3.54), we can take $\widehat{\theta}_1(y,v)=\widehat{\delta}_0(v)+\widehat{\delta}_1(v)y$, for all $y>0$ and $v\in\mathbb{R}$. In the cases when $\alpha\in(0,1)$ or $\alpha>1$, for a jump analogue in (3.54), we can put $\widehat{\delta}_0(v)=0$ and $\widehat{\theta}_1(y,v)=(y^{1-\alpha}+(1-\alpha)\ln^{1-\alpha}(1+\widehat{\delta}_1(v)))^{1/(1-\alpha)}-y$, for all $y>0$ and $v\in\mathbb{R}$.

Example 3.17. *(Extended Shiryaev filtering model.)* Suppose that in (3.52) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 3.10. Then, for a jump analogue in (3.54) we can take $\widehat{\theta}_1(y,v)=y(1-y)\widehat{\delta}_1(v)/(1+y\widehat{\delta}_1(v))$, for all $y\in(0,1)$ and $v\in\mathbb{R}$ (see, e.g. [21; Chapter XIX]).

4 Market price of risk

In this section, we expand the notion of *market price of risk*, or *relative risk*, to the constructed jump-diffusion models driven by natural exponential families of Lévy processes, in respect to their applications to financial markets.

4.1 The relative risk representations. In the setting of the previous sections, suppose that there exists a process $S=(S_t)_{t\geq 0}$ defined as

$$S_t=\exp\Big(\beta_1\,t+\gamma_1\,W_t+\int_0^t\int v\,(\mu-\nu)(du,dv)\Big), \tag{4.1}$$

and thus, solving the stochastic differential equation

$$dS_t=\overline{\beta}_1\,S_t\,dt+\gamma_1\,S_t\,dW_t+S_{t-}\int\Big(e^v-1\Big)\,(\mu-\nu)(dt,dv) \tag{4.2}$$

with

$$\overline{\beta}_1=\beta_1+\frac{\gamma_1^2}{2}+\int\Big(e^v-1-v\Big)\,F(dv), \tag{4.3}$$

where $\beta_1\in\mathbb{R}$, $\gamma_1>0$, the compensator of the measure $\mu(dt,dv)$ has the form $\nu(dt,dv)=dtF(dv)$ with respect to the probability measure P, and the condition

$$\int\Big((v^2\wedge|v|)+e^v\,I_{\{|v|>1\}}\Big)\,F(dv)<\infty \tag{4.4}$$

holds. Then, $\ln S=(\ln S_t)_{t\geq 0}$ forms a Lévy process with the triplet of characteristics $(\beta_1,\gamma_1^2,F(dv))$ with respect to the truncation function $h(x)=x$, $x\in\mathbb{R}$.

Let $(P_\lambda)_{\lambda\in\Lambda}$ be a parametric set of probability measures which are locally absolutely continuous with respect to $P_0 \equiv P$ on the filtration $(\mathcal{F}_t)_{t\ge 0}$, where $\Lambda \subseteq \mathbb{R}$ is an open subset. Assume that the probability measure P_λ admits the density

$$\frac{dP_\lambda}{dP_0}\bigg|_{\mathcal{F}_t} = \exp\Big(\lambda \,\ln S_t - K(\lambda)\, t\Big), \tag{4.5}$$

with respect to the dominating measure $P_0 \equiv P$, where the cumulant function $K(\lambda)$ is given by

$$K(\lambda) = \lambda\beta_1 + \frac{\lambda^2\gamma_1^2}{2} + \int \Big(e^{\lambda v} - 1 - \lambda v\Big)\, F(dv), \tag{4.6}$$

and the condition

$$\int \Big((v^2 \wedge |v|) + I_{\{|v|>1\}}\Big)\, e^{\lambda v}\, F(dv) < \infty \tag{4.7}$$

holds, for any $\lambda \in \Lambda$. In this case, the set $(P_\lambda)_{\lambda\in\Lambda}$ forms a *natural exponential family* generated by the Lévy process $\ln S$ (see, e.g. [25; Chapter X, Section 2] and [20; Chapter II]). We further assume that the characteristics of the triplet $(\beta_1(\lambda), \gamma_1^2, F(dx;\lambda))$ of the process $\ln S$ with respect to P_λ satisfy the conditions

$$\beta_1(\lambda') = \beta_1(\lambda) + (\lambda' - \lambda)\, \gamma_1^2 + \int v\, \Big(e^{(\lambda'-\lambda)v} - 1\Big)\, F(dv;\lambda), \tag{4.8}$$

$$\frac{F(dv;\lambda')}{F(dv;\lambda)} = e^{(\lambda'-\lambda)v}, \quad \text{and} \quad \int \Big(e^{(\lambda'-\lambda)v/2} - 1\Big)^2\, F(dv;\lambda) < \infty, \tag{4.9}$$

for any $\lambda, \lambda' \in \Lambda$. Hence, it follows from Girsanov's theorem (see, e.g. [17; Chapter III, Theorem 5.34]) that the probability measures P_λ and $P_{\lambda'}$ are locally equivalent on the filtration $(\mathcal{F}_t)_{t\ge0}$ and the density process takes the form

$$\frac{dP_{\lambda'}}{dP_\lambda}\bigg|_{\mathcal{F}_t} = \exp\Big((\lambda' - \lambda)\, \ln S_t - \big(K(\lambda') - K(\lambda)\big)\, t\Big), \tag{4.10}$$

so that any measure P_λ, $\lambda \in \Lambda$, can be taken as dominating.

Suppose that the process $S = (S_t)_{t\ge0}$ expresses the dynamics of the price of a risky asset (e.g. a stock) in a model of a financial market in which the riskless asset (e.g. a bank account) has a constant value. Assume that there exists $\lambda_* \in \Lambda$ such that S is a (local) martingale on the filtration $(\mathcal{F}_t)_{t\ge0}$ under the probability measure P_{λ_*}, for some $\lambda_* \in \Lambda$. Thus, it follows from the structure of the density of the probability measures in (4.5) and (4.10) that λ_* represents a root of the arithmetic equation

$$K(\lambda+1) - K(\lambda) \equiv \beta_1 + \frac{\gamma_1^2}{2} - \int v\, F(dv) + \lambda\, \gamma_1^2 + \int \Big(e^v - 1\Big)\, F(dv;\lambda) = 0, \tag{4.11}$$

whenever it exists (see, e.g. [13]-[14] and [25; Chapter VII, Section 3]). The value λ_* is called the *market price of risk*, or *relative risk*, for the model of financial market with the risky asset price S. The associated with λ_* probability measure P_{λ_*} is called the risk-neutral measure for the process S and is used for the no-arbitrage pricing of derivative securities in the related models of financial markets. Thus, the process S admits the (local martingale) representation

$$dS_t = \gamma_1\, S_t\, dW_t^* + S_{t-} \int \Big(e^v - 1\Big)\, (\mu - \nu)(dt, dv; \lambda_*), \tag{4.12}$$

where the process $W^* = (W_t^*)_{t \ge 0}$ and the measure $\nu(dt, dv; \lambda_*)$ defined by

$$W_t^* = W_t - \lambda_* \, t \quad \text{and} \quad \nu(dt, dv; \lambda_*) = dt \, F(dv; \lambda_*) \equiv dt \, e^{\lambda_* v} \, F(dv) \tag{4.13}$$

are the standard Brownian motion and the compensator of the Poisson random measure $\mu(dt, dv)$ with respect to P_{λ_*}, respectively.

Suppose that there exists a strictly positive and twice continuously differentiable functions $g(s)$ and its inverse $f(q)$ such as the ones considered in the previous section. Then, define the process $Q = (Q_t)_{t \ge 0}$ by $Q_t = g(S_t)$, and assume that the function

$$\overline{\eta}(q; \lambda_*) = \int \Big(g(f(q)(e^v - 1)) - q - (e^v - 1) \, f(q) \, g'(f(q)) \Big) \, F(dv; \lambda_*) \tag{4.14}$$

is finite, for all $q > 0$. Hence, by means of the Itô's formula, we get that the process Q satisfies the stochastic differential equation

$$dQ_t = \eta(Q_t; \lambda_*) \, dt + \sigma(Q_t) \, dW_t^* + \int \theta(Q_{t-}, v) \, (\mu - \nu)(dt, dv; \lambda_*), \tag{4.15}$$

where the functions $\eta(q; \lambda_*)$, $\sigma(q)$, and $\theta(q, v; \lambda_*)$ are given by

$$\eta(q; \lambda_*) = \frac{\gamma_1^2 f^2(q)}{2} \, g''(f(q)) + \overline{\eta}(q; \lambda_*), \tag{4.16}$$

$$\sigma(q) = \gamma_1 \, f(q) \, g'(f(q)), \quad \text{and} \quad \theta(q, v) = g(f(q)(e^v - 1)) - q, \tag{4.17}$$

for all $q > 0$ and $v \in \mathbb{R}$, and the function $\overline{\eta}(q; \lambda_*)$ is defined in (4.14). The process Q can express the price dynamics of another risky asset (e.g. an interest rate) in the considered financial market model. We also observe from the expressions in (4.12)-(4.13) and (4.15)-(4.17) that the processes S and Q have the structure of coefficients under the probability measure P_{λ_*} which is similar to the one of the processes X and Y from (2.18) and (3.1) with (3.7)-(3.10), and $\beta(t, x) = \gamma_0(t) = \delta_0(t, v) = 0$ and $\delta_1(t, v) = e^v - 1$, for all $t \ge 0$, $x \in \mathcal{D}_X$, and $v \in \mathbb{R}$.

Remark 4.1. Note that, for any process Q satisfying the stochastic differential equation in (4.15) with the coefficients of the form of (4.16)-(4.17) and (4.14), we can construct a process S by setting $S_t = f(Q_t)$, which satisfies the stochastic differential equation in (4.12) with (4.13). We can therefore expand the notion of *relative risk* λ_* defined for the process S by the equation in (4.11) on the process Q such that $Q_t = g(S_t)$, for all $t \ge 0$. In this case, the payoff of a derivative of the form $H(Q_t)$, for some measurable function $H(q)$, $q \in \mathcal{D}_Q$, can be represented in the form $H(g(S_t))$, for each $t \ge 0$. In other words, the process S can be considered as the price of an auxiliary underlying risky asset, which is a (local) martingale under the probability measure P_{λ_*}. In this respect, we can regard the expression in (4.15) as the *risk-neutral representation* for Q under the probability measure P_{λ_*}, where the value λ_* is chosen by the market. Such an expansion of this notion stays in accordance with the other ones considered in the literature and is related to the several particular examples of continuous models from the previous section. One can also say that this expansion defines the appropriate risk-neutral probability measure which can be used for the no-arbitrage pricing of derivatives on the associated financial assets with the price processes satisfying solvable stochastic differential equations.

4.2 Some natural exponential families. We conclude the section by referring several examples of *natural exponential families* of Lévy processes.

Example 4.2. Suppose that the process $\ln S$ has the triplet $(-\lambda, 1, 0)$ with respect to the measure P_λ, $\lambda \in \Lambda = \mathbb{R}$, that is, $\ln S$ is a *Brownian motion* with the local drift rate $(-\lambda) \in \mathbb{R}$ and the marginal density function $p_t^\lambda(x) = \exp(-(x+\lambda)^2/(2t))/\sqrt{2\pi t}$, $t > 0$, $x \in \mathbb{R}$. Then $(P_\lambda)_{\lambda\in\Lambda}$ is a natural exponential family with the dominating measure P_0 and the cumulant function $K(\lambda) = \lambda^2/2$, $\lambda \in \Lambda = \mathbb{R}$.

Example 4.3. Suppose that the process $\ln S$ has the triplet $(e^\lambda, 0, I_{\{x=1\}}e^{\lambda x}dx)$ with respect to the measure P_λ, $\lambda \in \Lambda = \mathbb{R}$, that is, $\ln S$ is a *Poisson process* of intensity e^λ, $\lambda \in \mathbb{R}$, so that $P_\lambda(\ln S_t = n) = (e^\lambda t)^n \exp(-e^\lambda t)/(n!)$, $t > 0$, $n \in \mathbb{N}$. Then $(P_\lambda)_{\lambda\in\Lambda}$ is a natural exponential family with the dominating measure P_0 and the cumulant function $K(\lambda) = \exp(e^\lambda) - 1$, $\lambda \in \Lambda = \mathbb{R}$.

Example 4.4. Suppose that the process $\ln S$ admits the representation $\ln S_t = \sum_{j=1}^{N_t} \Xi_j$, where $N = (N_t)_{t\geq 0}$ is a Poisson process of intensity $(-\eta/\lambda)$ and $(\Xi_j)_{j\in\mathbb{N}}$ is a sequence of independent (also of N) identically distributed random variables with probability density function $p_\Xi^\lambda(x) = -\lambda e^{\lambda x}$, $x > 0$, with respect to the measure P_λ, $\lambda \in \Lambda = (-\infty, 0)$. The process $\ln S$ is a *compound Poisson process with exponential jumps*, where the value $(-\eta/\lambda)$ plays the role of the jump intensity (see, e.g. [25; Chapter III, Section 1]). In this case, the process $\ln S$ has the triplet $(\eta/\lambda^2, 0, I_{\{x>0\}}\eta e^{\lambda x}dx)$ with respect to the probability measure P_λ, $\lambda \in \Lambda$. Hence $(P_\lambda)_{\lambda\in\Lambda}$ is a natural exponential family with the cumulant function $K(\lambda) = \eta/(2\lambda)$, $\lambda \in \Lambda = (-\infty, 0)$.

Example 4.5. Suppose that the process $\ln S$ admits the representation $\ln S_t = \sum_{j=1}^{N_t} \Xi_j$, where $N = (N_t)_{t\geq 0}$ is a Poisson process of intensity $\eta e^{\lambda^2/2}$ and $(\Xi_j)_{j\in\mathbb{N}}$ is a sequence of independent (also of N) identically distributed random variables with probability density function $p_\Xi^\lambda(x) = (1/\sqrt{2\pi})e^{-(x-\lambda)^2/2}$, $x \in \mathbb{R}$, with respect to the measure P_λ, $\lambda \in \Lambda = \mathbb{R}$. The process $\ln S$ is a *compound Poisson process with normal jumps*, where the value $\eta e^{\lambda^2/2}$ plays the role of the jump intensity (see, e.g. [25; Chapter III, Section 1]). In this case, the process $\ln S$ has the triplet $(\eta e^{\lambda^2/2}, 0, (\eta e^{\lambda^2/2}/\sqrt{2\pi})e^{-(x-\lambda)^2/2}dx)$ with respect to the probability measure P_λ, $\lambda \in \Lambda$. Hence $(P_\lambda)_{\lambda\in\Lambda}$ is a natural exponential family with the dominating measure P_0 and the cumulant function $K(\lambda) = \eta(e^{\lambda^2/2} - 1)$, $\lambda \in \Lambda = \mathbb{R}$.

Example 4.6. Suppose that the process $\ln S$ has the triplet $(-1/\lambda, 0, I_{\{x>0\}} (e^{\lambda x}/x)dx)$ with respect to the measure P_λ, $\lambda \in \Lambda = (-\infty, 0)$, so that it is a *gamma process* with parameter $(-\lambda)$ and the marginal density function $p_t^\lambda(x) = x^{t-1}\exp(x/\lambda)/((-\lambda)^t\Gamma(t))$, $t > 0$, $x > 0$ (see, e.g. [25; Chapter III, Section 1]). Then $(P_\lambda)_{\lambda\in\Lambda}$ is a natural exponential family with the cumulant function $K(\lambda) = \ln(-\lambda)$, $\lambda \in \Lambda = (-\infty, 0)$.

Example 4.7. Suppose that the process $\ln S$ has the triplet $(\delta/\sqrt{(-2\lambda)}, 0, (\delta I_{\{x>0\}}e^{\lambda x}/\sqrt{2\pi x^3})dx)$, $\delta > 0$, with respect to the measure P_λ, $\lambda \in \Lambda = (-\infty, 0)$, so that $\ln S$ is an *inverse Gaussian process* with one of the parameters $\sqrt{(-2\lambda)}$, and the density of the random variable $\ln S_1$ has the form $p_1^\lambda(x) = \delta e^{\delta\sqrt{(-2\lambda)}}\exp(-(\delta^2 x^{-1} - 2\lambda x)/2)/\sqrt{2\pi x^3}$, $x > 0$. Then $(P_\lambda)_{\lambda\in\Lambda}$ is a natural exponential family with the cumulant function $K(\lambda) = \delta\sqrt{(-2\lambda)}$, $\lambda \in \Lambda = (-\infty, 0)$.

Example 4.8. Suppose that the process $\ln S$ has the triplet $(\mu + \delta\lambda/\sqrt{\alpha^2 - \lambda^2}, 0, \alpha\delta K_1(\alpha|x|)e^{\lambda x}/(\pi|x|)dx)$, $\alpha > 0$, $0 \leq |\lambda| < \alpha$, $\mu \in \mathbb{R}$, $\delta > 0$, with respect to the measure P_λ, $\lambda \in \Lambda = (-\alpha, \alpha)$, where $K_1(x)$ is the modified Bessel function of the third kind with index 1. The process $\ln S$ is a *normal inverse Gaussian process*, where λ is one of the parameters and the marginal density is $p_1^\lambda(x) = \alpha e^{\delta\sqrt{\alpha^2-\lambda^2}} K_1(\alpha\delta\sqrt{1 + (x-\mu)^2/\delta^2})e^{\lambda(x-\mu)}/(\pi\sqrt{1 + (x-\mu)^2/\delta^2})$ (see, e.g. [1] or [25; Chapter III, Section 1]). Then $(P_\lambda)_{\lambda\in\Lambda}$ is a natural exponential family generated by $\ln S$ with the dominating measure P_0 and the cumulant function $K(\lambda) = \delta(\alpha - \sqrt{\alpha^2 - \lambda^2}) + \mu\lambda$, $\lambda \in \Lambda = (-\alpha, \alpha)$.

Example 4.9. Suppose that the process $\ln S$ has the triplet $(\mu + \delta\lambda K_2(\delta\alpha)/(\sqrt{\alpha^2 - \lambda^2}K_1(\delta\alpha)),\ 0, f_0(x)e^{\lambda x}dx)$, $\alpha > 0$, $0 \leq |\lambda| < \alpha$, $\mu \in \mathbb{R}$, $\delta > 0$, with respect to the measure P_λ, $\lambda \in \Lambda = (-\alpha, \alpha)$. Here, we have the density of the compensator

$$f_0(x) = \frac{1}{\pi^2 |x|} \int_0^\infty \frac{\exp(-|x|\sqrt{2y+\alpha^2})}{y(J_1^2(\delta\sqrt{2}y) + Y_1^2(\delta\sqrt{2}y))}\, dy + \frac{\exp(-|x|)}{|x|}, \tag{4.18}$$

where J_1, Y_1 are Bessel function of the first and second kind, respectively, and K_2 is the modified Bessel function of the third kind with index 2. The process $\ln S$ is a *hyperbolic process*, where λ is one of the parameters of the marginal density $p_1^\lambda(x) = \sqrt{\alpha^2 - \lambda^2} \exp(-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \lambda(x-\mu))/(2\alpha\delta K_1(\delta\sqrt{\alpha^2-\lambda^2}))$ (see, e.g. [9] or [25; Chapter III, Section 1]). Then $(P_\lambda)_{\lambda\in\Lambda}$ is a natural exponential family generated by $\ln S$ with the dominating measure P_0 and the cumulant function $K(\lambda) = \ln(\alpha K_1(\delta\sqrt{\alpha^2-\lambda^2})/(K_1(\delta\alpha)\sqrt{\alpha^2-\lambda^2})) + \mu\lambda$, $\lambda \in \Lambda = (-\alpha, \alpha)$.

Acknowledgments. The authors are grateful to the anonymous Referee for their useful comments. The paper was essentially written when the first author was visiting the Department of Mathematical Stochastics of the University of Freiburg in Germany. The hospitality and financial support from the Alexander von Humboldt Foundation through the fellowship for experienced researchers is gratefully acknowledged. This research was also supported by a Small Grant from the Suntory and Toyota International Centres for Economics and Related Disciplines (STICERD) at the London School of Economics and Political Science.

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