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Heterogeneous Risk/Loss Aversion in Complete Information All-pay Auctions

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Abstract

We extend previous theoretical work on n-player complete information all-pay auctions to incorporate heterogeneous risk- and loss-averse utility functions. We provide sufficient and necessary conditions for the existence of equilibria with a given set of active players with any strictly increasing utility functions and characterize the players' equilibrium mixed strategies. Assuming that players can be ordered by their risk aversion (player a is more risk-averse than player b, if whenever player b prefers a certain payment over a given lottery, so does player a), we find that in equilibrium, the more risk-averse players either bid higher than the less risk-averse players and win with higher ex-ante probability – or they drop out. Furthermore, while each player's expected bid decreases with the other players' risk aversion, her expected bid increases with her own risk aversion. Thus, increasing a player's risk aversion creates two opposing effects on total expected bid. A sufficient condition for the total expected bid to decrease with a player's risk aversion is that this player is relatively more risk-averse compared to the rest of the players. Our findings have important implications for the literature on gender differences in competitiveness and for gender diversity in firms that use personnel contests for promotions.

Keywords: All-pay auction, Risk aversion, Loss aversion
JEL: D44, D72, D81

1. Introduction

Sunk cost contests, where effort is unrecoverable, are pervasive (Frank and Cook, 2010). All-pay auctions are those where the winners need only exert slightly higher effort to take all. Indeed, all-pay auctions theory has been used to study many types of sunk cost contest and tournaments, e.g., rent seeking contest and lobbying (Baye et al., 1993; Ellingsen, 1991; Hillman and Riley, 1989), election campaigns (Che and Gale, 1998), R&D races (Dasgupta et al., 1982), curved grades (Andreoni and Brownback, 2014), college admission (Hickman, 2014), and job promotion (Rosen, 1986). In these contests, the risk of lost effort, opportunities, or resources to individuals can be significant. Furthermore, even contests between organizations, like firms, can involve significant loss to individuals to the extent that decisions are made by CEOs and managers who care about the consequences of those decisions on their own welfare through mechanisms such as options in compensation packages (Bertrand, 2009), and of course, in promotions and dismissals based upon relative performance. Heterogeneous risk aversion (e.g., as indicated by gender) could thus have a significant influence on behavior.

Evidence is accumulating of a gender difference in risk aversion, where women are usually found to be more risk-averse than men (Charness and Gneezy, 2012; Borghans et al., 2009; Croson and Gneezy, 2009). This gender difference emerges even before adolescence (Khachatryan et al., 2015). A gender difference in risk attitude and its interactions with all-pay auction incentives in the business world can help explain the paucity of women among top executives (Bertrand, 2009), particularly in entrepreneurial settings (Coates et al., 2009). However, despite the importance of observable differences in attitude towards risk in such contests, the modeling of all-pay auction incentives has generally been limited to risk neutral players or to specific tractable utility functional forms.

In order to fill the gap in the theory of all-pay auctions, we extend Baye, Kovenock, and De Vries's (1996) n-player, complete information all-pay auction model to incorporate heterogeneous risk-averse players. We provide sufficient and necessary conditions for an equilibrium with a given set of active players to exist and more importantly, closed-form solutions to the equilibrium strategies for any strictly increasing utility functions. Assuming that players can be ordered by their risk aversions (player a is more risk-averse than player

b, if whenever player b prefers a certain payment over a given lottery, so does player a), we derive novel comparative statics for equilibria in which active players randomize continuously from 0 to the common value of the prize.

We find that, in equilibrium, the more risk-averse players either bid higher than the less risk-averse players (in terms of first-order stochastic dominance of their mixed strategy cumulative distribution) and win with higher ex-ante probability – or they drop out. When players are homogeneous in their risk aversion, the total expected bid decreases with their risk aversion. We also find, in the heterogeneous risk aversion case, that while each player's expected bid decreases with the other players' risk aversion, her expected bid increases with her own risk aversion. Thus, increasing a player's risk aversion creates two opposing effects on total expected bid. A sufficient condition for the total expected bid to decrease with a player's risk aversion is that this player is relatively more risk-averse compared to the rest of the players.

Our findings have important implications for the gender differences in competitiveness literature. With only two risk aversion types of players, e.g., men and women, we show that the total expected bid decreases monotonically with the share of the more risk-averse players, when the difference between the two types is not too large. Moreover, our findings suggest the possibility that if women are more risk-averse than men, they can simultaneously work harder than men and decrease everyone's effort in the firm in personnel contests that have an all-pay auction structure. In these contests, if men and women are not too different in their levels of risk aversions, then a higher share of women may lead to increased odds of a specific woman dropping out. We discuss the specific results in the gender differences in competitiveness literature that these findings can help explain after the main results.

1.1. Literature review

As is always the case with equilibria of complete information all-pay auctions with more than two players, there is no uniqueness of the equilibrium. In fact, there is a continuum of equilibria as in Baye et al. (1996); Siegel (2009); Barut and Kovenock (1998), and Hillman and Samet (1987). Some of the active players may be randomizing over a sub-interval of the form $[b, v]$. As is the case with risk neutral players, these b's are arbitrary. Varying the

b's and the set of active players generates the continuum of equilibria. We generalize Baye et al. (1996) equilibrium strategies for risk/loss-averse players. We also generalize Chen et al. (2015) from two players with heterogeneous risk aversion to n players.

Siegel (2009) studies a very general environment that also allows for risk-averse players. However, his "power condition" for a generic contest does not hold in our environment. The power of a player (in our setting) is defined as her utility at zero. The power condition states that only one of the players has a power of zero. However, if we assume that the utility of each player is zero at zero, then all players have a power of zero, and the condition is violated. If we do not make this assumption, then our model does not comply with Assumption 2 in Siegel (2009). In either case, we cannot use his results to identify all the equilibria of our all-pay auction. When players are homogenous in their risk aversion, we can conclude from Corollary 3 in Siegel (2009) that all the equilibria are of the form that we find in this paper. Hillman and Samet (1987) also solve for an equilibrium with risk-averse players when all players are homogenous in their risk aversion. They characterize the unique symmetric equilibrium in mixed strategies and show that in the presence of risk aversion, rent dissipation is incomplete. Our results characterize all the equilibria for homogenous risk aversion players.

For incomplete information (private value) all-pay contests, Fibich et al. (2006) show that risk aversion has different effects on different types of players. Low value types bid lower and high value types bid higher than they would bid in the risk neutral case. Moreover, they show, as we do, that the seller's expected payoff in the risk-averse case may be either higher or lower than in the risk neutral case.

Parreiras and Rubinchik (2010) analyze pure strategy equilibrium with heterogeneous risk-averse players, also in an incomplete information setting. They allow for heterogeneity in both the risk preferences and the supports of the distributions from which the private values are drawn. Our model can be thought of as a limiting case of their model when all supports contain only one point (the same for all players). Thus, our results often echo theirs. First, Parreiras and Rubinchik (2010) characterize conditions for a given set of players to be active on a given support, as we do. Moreover, they show that with at least three heterogeneously risk-averse contestants, some might drop out either partially or completely.

This is identical to our mixed strategy equilibria of the complete information (common value) case where some players may randomize on a sub-interval of the interval from zero to the common value, and some players may drop out completely. They also show, as we do, that more risk-averse players are more aggressive in their bidding in terms of first-order stochastic dominance of the bid function, at least in some neighborhoods of zero and of the highest bid. Most importantly, they show that with two types of risk aversions, if the risk-averse players are sufficiently risk-averse, then in any equilibrium, at least one contestant uses a discontinuous strategy: a mass of low valuation types drops out from the contest, high valuation types place high bids, but no types place low bids. This gives rise to bid bifurcation, where only bids at zero or bids above a threshold are observed for some players. Analogously, in our setting a player may randomize continuously on a sub-interval $[b, v]$ for some $b > 0$ (where v is the common value) and put a positive mass on bidding zero. The main difference between the papers is that in our complete information setting, we are able to analytically derive the bid functions while Parreiras and Rubinchik (2010) characterize only some of its properties. This allows us to derive results on the behavior of players with different risk attitudes and comparative statics on different aspects of the model.

Klose and Schweinzer (2014) analyze incomplete information all-pay auctions with symmetric variance-averse players. They assume a specific utility function which is increasing in the player's expected payoff and decreasing in the variance of the payoff. They characterize the equilibrium bid functions and show that variance aversion is a sufficient assumption to predict that high-valuation players increase their bids relative to the risk-neutral case while low types decrease their bid.

2. The model

There are m players who have a common valuation, $v_1 = \cdots = v_m = v$ for the prize[1]. Denote by M the set of players. Players compete in an all-pay auction for one prize by submitting a bid (exerting an effort): x_i. The vector of bids is denoted $(x_1, x_2, \ldots, x_m)$. The

[1]Our model can be trivially extended to the case in which one player has a higher valuation, while all other players have the same lower valuation. However, when there are finite many possible valuations, the interaction between valuation and risk attitude significantly complicates the model. We leave this for future work.

payoff function in an all-pay auction is given by:

$$\pi_i(x_1, x_2, \ldots, x_m) = \begin{cases} -x_i & \text{if} \quad \exists j, \quad x_j > x_i \\ v - x_i & \text{if} \quad x_j < x_i \quad \text{for all } j \end{cases}.$$

Moreover, there exists some tie-breaking rule to determine the winner in case there is more than one player with the highest bid. Any tie-breaking rule is applicable in our model. We assume that players are risk/loss-averse with strictly increasing utility functions which we denote by $U_1(x), U_2(x), \ldots, U_m(x)$. These utilities are common knowledge and potentially different from each other. We discuss two cases separately: 1) risk-averse and 2) loss-averse. For case 1), we assume only continuity and concavity of the strictly increasing utility functions. For case 2), we assume that the utility functions take the following form:

$$U_i(x) = \begin{cases} g_i(x) & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ l_i(x) & \text{if} \quad x < 0 \end{cases}, \tag{1}$$

where the utility from gains, $g_i(x)$ is a strictly increasing concave function while the utility from losses, $l_i(x)$ is a convex function, and both are continuous in their domains.

In this paper, we focus on mixed strategy equilibria. In any such equilibria, any active player (a player who bids a positive amount with positive probability) i is indifferent between all the bids in her equilibrium support. Formally, that means,

$$\rho U_i(v - x) + (1 - \rho)U_i(-x) = U_i(CE_i(x, b_{-i})), \tag{2}$$

where x is in the support of the player's equilibrium strategy, ρ denotes the probability that player i wins when she bids x, and $CE_i(x, b_{-i})$ is the certainty equivalent of bidding x given the other players bid b_{-i}. We can rewrite equation (2) as:

$$\Pr(i \text{ wins}|x, b_{-i}) = \rho = \frac{U_i(CE_i) - U_i(-x)}{U_i(v - x) - U_i(-x)}.$$

We define $K_{U_i}(x)$ to facilitate the analysis of the mixed strategy equilibria:

$$K_{U_i}(x) = \frac{U_i(0) - U_i(-x)}{U_i(v-x) - U_i(-x)}.$$

Note that $K_{U_i}(x)$ is strictly increasing in x with $K_{U_i}(0) = 0$ and $K_{U_i}(v) = 1$. In our analysis of equilibria, we examine equilibria with the same structure as in Baye et al. (1996) in which all players have v in the support of their equilibrium mixed strategy and no player has an atom at v. In this case, bidding v yields a certain payoff of zero. Therefore, each player is indifferent between any bid in her support and a certain payoff of zero. The equilibrium probability of winning that makes player i indifferent between bidding $x > 0$ and receiving a certain payoff of zero is equal to $K_{U_i}(x)$. We sometimes abuse notation and write $K_{U_i}(x)$ as $K_i(x)$, and we refer to $K_i(x)$ as player i's "contest risk preference" for reasons below.

In what follows, we exploit the following important property of $K_{U_i}(x)$. Its magnitude depends only on player i's risk attitude and not on any other players' risk attitude or bids. As is usually the case with mixed-strategy equilibria, since $K_{U_i}(x)$ denotes the probability that player i wins by submitting a positive bid x, this probability is pinned down by the preferences of player i alone. We show that $K_{U_i}(x)$ is monotonic in player i's risk aversion in the lemma below. All proofs are in the Appendix. We first define increasing risk aversion and increasing loss aversion.

Definition 1 *A concave utility function $U(\cdot)$ represents a more risk-averse player than the concave utility function $\tilde{U}(\cdot)$, if for any lottery l over a set of prizes Z, the lottery's certainty equivalent is smaller under U than under $\tilde{U}$. In that case, we say that the risk aversion of the player increases from $\tilde{U}$ to U.*

Definition 2 *For a player with a utility function $U(\cdot)$ of the form (1), a convex loss-averse function $l(\cdot)$ represents a more loss-averse player than the convex function $\tilde{l}(\cdot)$, if $l(x) < \tilde{l}(x)$ for all $x < 0$.*

Lemma 1 *If $U_i(x)$ is concave, then for any $x \in (0, v)$, $K_i(x)$ increases with player i's risk aversion, i.e., if U_i represents a more risk-averse player than $\tilde{U}_i$, then $K_{U_i}(x) > K_{\tilde{U}_i}(x)$ for any $x \in (0, v)$. When players are both risk- and loss-averse, as described above by the utility* *function of the form (1), then for any x, $K_i(x)$ increases with player i's loss aversion.*

Note that Lemma 1 above also suggests that the $K(x)$ function of different players never cross if the players can be ordered by their risk or loss aversions. The following is an example of the contest risk preference function $K(x)$ when the player has CARA utility function:

Example 1 *If a player has CARA utility function: $U_i(c) = 1 - e^{-\beta_i c}$ and $v = 1$, then $K_i(x) = \frac{1-e^{-\beta_i x}}{1-e^{-\beta_i}}$. In this case, player i is more risk-averse than player j iff $\beta_i > \beta_j$. In figure 1, we plot $K_i(x)$ for $\beta = 1$ (black solid), 2 (green dotted), and 3 (red dashed).*

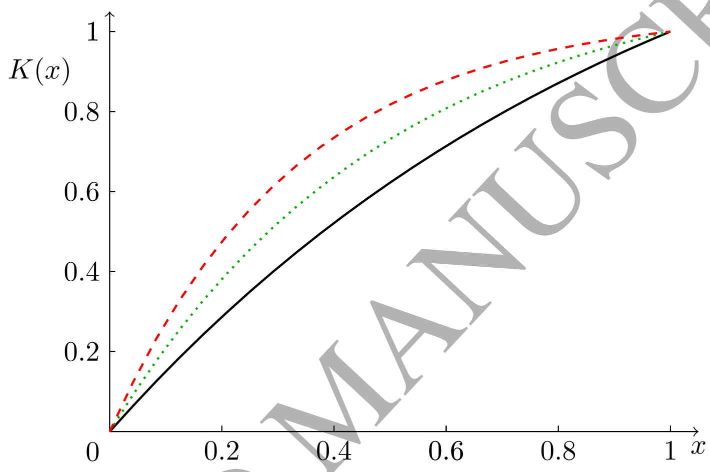

Figure 1: Contest risk preference function $K(x)$ increases with risk aversion.

In fact, the sufficient and necessary conditions for the existence of equilibrium and the closed-form expressions for the mixed strategies that we provide in the next section, rely only on the assumption that the utility functions are strictly increasing, i.e., utility functions do not have to be rankable by their certainty equivalent. The rest of the results apply to any utility function that is rankable by certainty equivalent, irrespective of whether the utility function is risk-averse, loss-averse, or even risk seeking. All the results with risk-loving players can be derived analogously, as long as the more risk-loving utilities have a higher certainty equivalent than the less risk-loving utilities for every lottery. We focus only on risk- and loss-averse utilities due to their ubiquity in the literature.

3. Equilibrium

In this section, we characterize the sufficient and necessary conditions for the existence of an equilibrium with a given structure and a given set of active players, and then, we characterize the mixed strategies in these equilibria. We also highlight some interesting features of the equilibria in the Subsection 3.2. In discussing these features, for simplicity, we focus only on the equilibria in which all active players randomize on the entire interval $[0, v]$.

3.1. Existence and closed-form solution

Our first proposition, Proposition 1 provides the necessary and sufficient conditions for the existence of an equilibrium in which a given subset of players is active. We start by defining an active player.

Definition 3 *A player is active when she bids zero with a probability strictly less than* 1. *A player is inactive when she bids zero with probability* 1.

Denote the set of active players by $B \subseteq \{1, \ldots, m\}$. For convenience and without loss of generality, we assign $i = 1, 2, ..., |B|$ as the index for the active players. Denote by $G_i(x)$ the CDF of player i's mixed strategy and by $\alpha_i(0)$ the mass player i puts on the bid zero. The equilibrium of the all-pay auction, has the following structure (as with risk neutral players):

1. Players $i = 1, 2, \ldots, h$, where $2 \leqslant h \leqslant |B|$, randomize continuously over $[0, v]$ and have $\alpha_i(0) = 0$;
2. Players $i = h + 1, h + 2, \ldots, |B|$ randomize continuously over $[b_i, v]$ and put a mass of $\alpha_i(0)$ at zero, with $0 = b_h < b_{h+1} \leqslant b_{h+2} \leqslant \ldots \leqslant b_{|B|} \leqslant b_{|B|+1} = v$;
3. Players $j = |B| + 1, \ldots, m$ are inactive, i.e., $\alpha_j(0) = 1$.

Baye et al. (1996) showed that there is a continuum of equilibria, as the parameters $b_{h+1}, b_{h+2}, \ldots, b_{|B|}$ can be chosen arbitrarily, and these determine the size of $\alpha_i(0)$. We now characterize the equilibrium strategy with the same structure but with heterogeneous risk/loss-averse players.

In our setting, this implies that player i's ($\forall i \in B$) expected utility must be equal to $U_i(0)$ when bidding in his equilibrium support. Note that only players $i \leqslant t$, for $t = h, \ldots, |B|$

randomize in the interval $[b_t, b_{t+1}]$. Thus, for player $i \leqslant t$ and $\forall x \in [b_t, b_{t+1}]$, we have in equilibrium:

$$\left(\prod_{l\neq i,1\leqslant l\leqslant t} G_l(x) \prod_{t<l\leqslant |B|} \alpha_l(0)\right) U_i(v-x)+\left(1-\prod_{l\neq i,1\leqslant l\leqslant t} G_l(x) \prod_{t<l\leqslant |B|} \alpha_l(0)\right) U_i(-x) = U_i(0). \quad (3)$$

Recall that $G_l(x)$ is the probability player $l \neq i$ bids lower than x, and $\alpha_l(0)$ is the probability that player l bids zero, so $\prod_{l\neq i,1\leqslant l\leqslant t} G_l(x) \prod_{t<l\leqslant |B|} \alpha_l(0)$ is player i's probability of winning when bidding x. Thus

$$\prod_{l\neq i,1\leqslant l\leqslant t} G_l(x) \prod_{t<l\leqslant |B|} \alpha_l(0) = K_i(x).$$

We solve this system of t equations and get the equilibrium strategy of player $i \leqslant t$:

$$G_i(x) = \left(\frac{\prod_{l\neq i,1\leqslant l\leqslant t} K_l(x)}{\prod_{t<l\leqslant |B|} \alpha_l(0)}\right)^{\frac{1}{t-1}} K_i(x)^{-\frac{t-2}{t-1}}, \quad (4)$$

where $i \leqslant t$ and $x \in [b_t, b_{t+1}]$. Recall that for player t we have $\alpha_t(0) = G_t(b_t)$. By a similar procedure, we can characterize the entire equilibrium:

- For $x \in [b_{|B|}, v]$

$$G_i(x) = \left(\prod_{1\leqslant l\leqslant |B|} K_l(x)\right)^{\frac{1}{|B|-1}} K_i(x)^{-1}, \quad (5)$$

  where $i = 1, \ldots, |B|$.

- For $t = h+1, h+2, \ldots, |B|-1$, we have for $x \in [b_t, b_{t+1}]$

$$G_i(x) = \left(\frac{\prod_{1\leqslant l\leqslant t} K_l(x)}{\prod_{t<l\leqslant |B|} \alpha_l(0)}\right)^{\frac{1}{t-1}} K_i(x)^{-1}, \quad (6)$$

  where $i = 1, 2, \ldots, t$; and

$$G_k(x) = \alpha_k(0), \quad (7)$$

where $k = t+1, \ldots, |B|$.

- Finally, for $x \in [0, b_{h+1}]$

$$G_i(x) = \left(\frac{\prod_{1 \leqslant l \leqslant h} K_l(x)}{\prod_{h < l \leqslant |B|} \alpha_l(0)} \right)^{\frac{1}{h-1}} K_i(x)^{-1}, \tag{8}$$

where $i = 1, 2, \ldots, h$; and

$$G_k(x) = \alpha_k(0), \tag{9}$$

where $k = h+1, \ldots, m$.

There are three sets of constraints to ensure that the above strategy profile is indeed an equilibrium. First, the players who do not bid some $x \in [b_t, b_{t+1}]$ in equilibrium must find it unprofitable to do so; Second, the players who bid $x \in [b_t, b_{t+1}]$ in equilibrium must have a feasible CDF at x, i.e., between zero and one; Third, all the CDFs must be non-decreasing in x.

To find the first set of constraints, we first calculate the probability of winning of a player j, for $t < j \leqslant |B|$ who deviates to a bid $x \in [b_t, b_{t+1}]$:

$$\begin{aligned}\prod_{1 \leqslant i \leqslant t} G_i(x) \prod_{t < l \leqslant |B|, l \neq j} \alpha_l(0) &= \prod_{i \leqslant t} \left(\left(\frac{\prod_{1 \leqslant l \leqslant t} K_l(x)}{\prod_{t < l \leqslant |B|} \alpha_l(0)} \right)^{\frac{1}{t-1}} K_i(x)^{-1} \right) \frac{\prod_{t < l \leqslant |B|} \alpha_l(0)}{\alpha_j(0)} \\ &= \left(\frac{\prod_{1 \leqslant l \leqslant t} K_l(x)}{\prod_{t < l \leqslant |B|} \alpha_l(0)} \right)^{\frac{1}{t-1}} \frac{1}{\alpha_j(0)}.\end{aligned}$$

Thus, we need

$$\prod_{1 \leqslant i \leqslant t} G_i(x) \prod_{t < l \leqslant |B|, l \neq j} \alpha_l(0) \leqslant K_j(x)$$

or

$$\frac{\prod_{1 \leqslant l \leqslant t} K_l(x)}{\prod_{t < l \leqslant |B|} \alpha_l(0)} \leqslant (\alpha_j(0) K_j(x))^{t-1} \tag{10}$$

to guarantee that player j finds it unprofitable to bid $x \in [b_t, b_{t+1}]$, as $K_j(x)$ is the required probability of winning for player j to be indifferent between bidding x and receiving a payoff

0 for sure. Note that if j is inactive, $|B| < j$, then $\alpha_j(0) = 1$. We refer to this set of constraints as the *Incentive Constraints*.

We find the second set of constraints by restricting the $G_i(x)$ given by (6) to satisfy $G_i(x) \leqslant 1$ for all $i \leqslant t$ and $x \in [b_t, b_{t+1}]$[2]:

$$\frac{\prod_{1\leqslant l\leqslant t} K_l(x)}{\prod_{t<l\leqslant |B|} \alpha_l(0)} \leqslant K_i\left(x\right)^{t-1}. \tag{11}$$

We refer to this set of constraints as the *Feasibility Constraints*.

Combining the two sets of constraints, (10) and (11), we have:

$$\frac{\prod_{1\leqslant l\leqslant t} K_l(x)}{\prod_{t<l\leqslant |B|} \alpha_l(0)} \leqslant \min\left\{\min_{1\leqslant i\leqslant t} K_i\left(x\right)^{t-1}, \min_{t<j\leqslant m} \left[\alpha_j(0) K_j(x)\right]^{t-1}\right\}, \tag{12}$$

for all $t = h, h+1, \ldots, |B|$, and for all $x \in [b_t, b_{t+1}]$.

The third set of constraints are derived by ensuring that the first-order derivatives of $G_i(x)$ for all $x \in [b_t, b_{t+1}]$ are non-negative in the interval. Since

$$\frac{dG_i\left(x\right)}{dx} = \left(\frac{\prod_{1\leqslant l\leqslant t} K_l(x)}{\prod_{t<l\leqslant |B|} \alpha_l(0)}\right)^{\frac{1}{t-1}} \frac{1}{(t-1) K_i\left(x\right)} \left(\sum_{l=1, l\neq i}^{t} \frac{K_l'\left(x\right)}{K_l\left(x\right)} - (t-2) \frac{K_i'\left(x\right)}{K_i\left(x\right)}\right),$$

the constraint is

$$\sum_{l=1}^{t} \frac{K_l'\left(x\right)}{K_l\left(x\right)} \geqslant (t-1) \frac{K_i'\left(x\right)}{K_i\left(x\right)}, \tag{13}$$

for all $t = h, h+1, \ldots, |B|$, $i = 1, ..., t$ and $x \in [b_t, b_{t+1}]$.

Proposition 1 *An equilibrium with the strategy profile (5)-(9) exists if and only if the conditions (12) and (13) are satisfied.*

The condition (13) restricts the level of risk/loss aversion of the players who bid in the interval $[b_t, b_{t+1}]$, whereas condition (12) imposes restrictions on both active and inactive players. Intuitively, for an equilibrium to exist, the active players cannot be too averse to risk/loss. See the following example for the importance of these conditions.

[2] All $G_i(x)$ are no less than zero since all $K(x)$ functions are non-negative by definition.

Example 2 *Assume there are three players with CARA utility functions* $U_i(x) = 1 - e^{-\beta_i x}$ *and a valuation* $v = 1$ *for* $i = 1, 2, 3$*. Assume that* $\beta_3 = 1$*,* $\beta_2 = 2$*,* $\beta_1 = 10$*. Then, there exists no equilibrium in which all three players are active and all randomize continuously on* $[0, 1]$*, since the condition is violated on* $[0.13035, 1]$*. Specifically, we have* $G_3(x) = (K_1(x)K_2(x))^{\frac{1}{2}} K_3(x)^{-\frac{1}{2}} = \left(\frac{1-e^{-2x}}{1-e^{-2}} \frac{1-e^{-10x}}{1-e^{-10}}\right)^{\frac{1}{2}} \left(\frac{1-e^{-x}}{1-e^{-1}}\right)^{-\frac{1}{2}}$*, which is larger than* 1 *for* $x \geqslant 0.13035$*. However, there exists an equilibrium in which only players* 2 *and* 3 *are active and they randomize continuously on the interval* $[0, 1]$ *according to the following strategies:* $G_2(x) = K_3(x) = \frac{1-e^{-x}}{1-e^{-1}}$ *and* $G_3(x) = K_2(x) = \frac{1-e^{-2x}}{1-e^{-2}}$*, since then, all the conditions hold.*

Similar to the risk neutral setting, there is also a continuum of equilibria when players are heterogeneously risk/loss-averse. Unlike in the risk neutral setting, players may stay inactive due to the their higher risk/loss aversion. This fact does not restrict the power of our theory in making predictions either for empirical or experimental data, since in reality we can generally observe the number of active players, especially if players play over multiple rounds.

Unlike the risk neutral setting, even if we focus only on the equilibria in which all active players randomize in the full support $[0, v]$, there always exists multiple equilibria[3] (see Corollary 1 below).

Corollary 1 *Suppose all players can be ranked by their risk aversions:* $K_1(x) \geqslant K_2(x) \geqslant \ldots \geqslant K_m(x)$ *for all* $x \in [0, v]$*, then there always exists equilibria in which only players* i *and* m *are active, where* $i \in \{1, 2, ..., m-1\}$ *and randomize continuously on* $[0, v]$*.*

This corollary follows directly from Proposition 1. Condition (12) is automatically satisfied since there are only two active players and one of them is the least risk-averse player, m. Condition (13) is satisfied due to the fact that all contest risk preference functions $K(x)$ are increasing in x.

[3] In the risk neutral setting, there exists a unique symmetric equilibrium in which all players fully randomize within $[0, v]$. See Baye et al. (1996) for details.

3.2. Some features of equilibria

Here we focus on equilibria in which all active players randomize continuously on $[0, v]$. Assume that $B = \{1, ..., |B|\}$ is the active set. Then, the conditions for such an equilibrium to exist are:

$$\prod_{1 \leqslant l \leqslant |B|} K_l(x) \leqslant \min_{1 \leqslant i \leqslant m} K_i(x)^{|B|-1},$$

for all $x \in [0, v]$ and

$$\sum_{l=1}^{|B|} \frac{K_l'(x)}{K_l(x)} \geqslant (|B| - 1) \frac{K_i'(x)}{K_i(x)},$$

for all $i = 1, ..., |B|$ and $x \in [0, v]$. In this case, player i's, $i = 1, ..., |B|$, equilibrium strategy is given by

$$G_i(x) = \left(\prod_{1 \leqslant l \leqslant |B|} K_l(x) \right)^{\frac{1}{|B|-1}} K_i(x)^{-1}. \tag{14}$$

In real life competitions, it is not uncommon for participants to differ in observable characteristics like gender, ethnicity, culture...etc. It is then important to examine whether risk attitudes associated with these characteristics help to explain differences in competitive behavior. For example, women are under-represented in the elites of many competitive industries (Bertrand, 2009), yet women are also more likely to achieve academic success (Angrist et al., 2009; DiPrete and Buchmann, 2013; Fortin et al., 2015)[4]. Importantly, our results below are consistent with this empirical evidence.

Corollary 2 *If there exists an equilibrium where the set of all active players, B, randomize continuously on the interval $[0, v]$, and these players can be ranked by their risk aversions: $K_1(x) \geqslant K_2(x) \geqslant \ldots \geqslant K_{|B|}(x)$ for all $x \in [0, v]$, then the cumulative distribution function of player s's strategy first-order stochastically dominates that of player t for every $t > s$. In particular, the players' expected bids have the same ranking as their levels of risk aversion.*

Corollary 2 suggests that more risk-averse players bid higher in expectation than less risk-averse players among all active players. Given that the more risk-averse a player is, the higher she bids conditional on her being active in equilibrium, one may expect that her

[4]See detailed discussion and literature review in Section 5.

probability of winning is also higher. Corollary 3 indicates that this conjecture is generally but not always true.

Corollary 3 *Assume an equilibrium where all active players (in the set B) randomize continuously on the interval $[0, v]$. For any two active players, s, $t \in B$, player s's probability of winning is higher or equal to that of player t if $K_t(x)$ dominates $K_s(x)$ in terms of the reverse hazard rate, i.e., $\frac{K_t'(x)}{K_t(x)} \geqslant \frac{K_s'(x)}{K_s(x)}$ for all $x \in [0, v]$.*

Note that $K_s(x)$, $K_t(x)$ are also the joint cumulative distributions of opponents' bids that players s and t are competing against (i.e., $\Pi_{l \in B, l \neq s} G_l(x) = K_s(x)$), respectively. Corollary 3 suggests that the more risk-averse player s is, the more likely she is to win the contest compared to player t, if in player t's view (as measured by $K_t(x)$) the contest is sufficiently more competitive (i.e., dominates in terms of reverse hazard rate) than in player s's view (as measured by $K_s(x)$). Example 3 below suggests that the CARA utility functions satisfy the condition given in Corollary 3. In other words, if players have CARA utility functions, then the more risk-averse a player is, the more likely she is to win ex-ante.

Example 3 *Suppose player i ($\forall i \in B$) has CARA utility function $U_i(x) = 1 - e^{-\beta_i x}$ and $v = 1$, thus, she has $K_i(x) = \frac{1 - e^{-\beta_i x}}{1 - e^{-\beta_i}}$. Take the derivative of $K_i'(x)/K_i(x)$ w.r.t. β_i, we have*

$$\frac{\partial \left(\frac{K_i'(x)}{K_i(x)} \right)}{\partial \beta_i} = -\frac{e^{-\beta_i x}}{(e^{-\beta_i x} - 1)^2} \left(e^{-\beta_i x} + \beta_i x - 1 \right) < 0.$$

The above derivative is negative because $\left(e^{-\beta_i x} + \beta_i x - 1 \right)$ increases in x, its derivative w.r.t. x is positive for $\forall x \in (0, 1]$, and it is equal to 0 when $x = 0$.

Interestingly, the more risk-averse players not only bid higher and win with higher probability, they are also more likely to drop out in the following sense.

Corollary 4 *Assume an equilibrium where all active players (in the set B) randomize continuously on the interval $[0, v]$. If for some $i \in B$ and $j \notin B$, we have $K_i(x) \geqslant K_j(x)$ for all $x \in [0, v]$, then the existence of the equilibrium with the set B of active players implies the existence of another equilibrium with the set $\widetilde{B}$ of active players who randomize continuously on the interval $[0, v]$, where $\widetilde{B} = (B \cup \{j\}) \setminus \{i\}$.*

Corollary 4 suggests that the conditions for the existence of the equilibrium in which a relatively more risk-averse player bids actively is sufficient for the existence of the equilibrium in which a less risk-averse player bids actively, holding all other active and inactive players constant, but the opposite is not necessarily true. This implies that mere differences in risk attitudes can result in different non-entry/drop out decisions, without having heterogeneity in valuations or incomplete information. The player with the higher risk aversion may not participate in the competition because she finds that the potential return from bidding any positive amount does not sufficiently compensate her for the risk.

One implication of this finding is that the well established gender difference in risk aversion alone may be sufficient to explain differences in participation rates found in gender differences in competitiveness experiments (Niederle and Vesterlund, 2007)[5], without the need to hypothesize gender differences in competitiveness, confidence, or other characteristics.

A question naturally follows: are the players who drop out always those who are more risk-averse than the active ones? The answer is not necessarily. Example 4 suggests there might exist equilibria in which players with the intermediate risk aversion drops out.

Example 4 *Assume there are three players with CARA utility functions* $U_i(x) = 1 - e^{-\beta_i x}$ *and a valuation* $v = 1$ *for* $i = 1, 2, 3$. *Assume also that* $\beta_1 = 2, \beta_2 = 1, \beta_3 = \frac{1}{2}$. *Then, there exists an equilibrium in which only players* 1 *and* 3 *are active, while player* 2 *is inactive. The conditions for this equilibrium to exist are:*

$$K_1(x)K_3(x) = \frac{1 - e^{-2x}}{1 - e^{-2}} \frac{1 - e^{-\frac{1}{2}x}}{1 - e^{-\frac{1}{2}}} \leqslant \min\{K_1(x), K_2(x), K_3(x)\} = K_3(x) = \frac{1 - e^{-\frac{1}{2}x}}{1 - e^{-\frac{1}{2}}}$$

and that the $G_i(x)$ *functions are strictly increasing for players* 1 *and* 3, *which follows directly from* $K_1(x)$ *and* $K_3(x)$ *being increasing functions. Thus, there exists an equilibrium in which the most and the least risk-averse players are active while the player with the intermediary risk aversion is inactive.*

[5] These papers examine entry into what are in effect all-pay auctions to measure gender differences in competitiveness. See Niederle (2014) for a recent survey.

4. Comparative statics

We now discuss the effect of increasing players' risk aversion on their expected bids. Our results in this section are derived for the equilibrium in which all active players randomize continuously in $[0, v]$. Subsection 4.1 shows that if players are homogeneous in their risk attitude, then increasing all players' risk aversion decreases the total expected bid. Subsection 4.2, in contrast, shows that in the case with heterogeneous risk aversion, each player's expected bid increases with her own risk aversion, though it still decreases with other active players' risk aversions.

4.1. Homogeneous risk aversion

The equilibrium strategy with homogeneous risk aversion is a special case of the equilibrium strategy with heterogeneous risk aversion derived above. When players are homogeneous in their risk aversion, we can generalize the proof of Baye et al. (1996) and prove that all equilibria are of the form described in Proposition 1[6]. In this case, there is a unique symmetric equilibrium in which all players randomize continuously on $[0, v]$, as well as many asymmetric equilibria. We, again, focus on equilibria in which all active players randomize on the interval $[0, v]$. We first examine how the players' behavior change when they all become more risk-averse.

Proposition 2 *Assume all players are homogeneous, then there exists an equilibrium where a set B of players randomize continuously on the interval $[0, v]$, and all other players are inactive. If all the players' risk aversion increases homogeneously, then their bids in the equilibrium in which the same set B of players randomize continuously on the interval $[0, v]$ decrease in terms of first-order stochastic dominance.*

We illustrate Proposition 2 with the following example.

Example 5 *Assume there are three players, each with the CARA utility function: $U_i(x) = 1 - e^{-\beta x}$ and valuation $v = 1$. Figure 2 shows the unique symmetric equilibrium strategy when $\beta = 1$ (black solid), 5 (green dotted), and 10 (red dashed). It is clear that as all players*

[6] The details of this proof are left out of this paper but can be given by the authors upon request.

become more risk-averse, the distribution function of their bids decreases in terms of first-order stochastic dominance, i.e., the probability that they bid below x for any $x \in [0, v]$ is higher when they are more risk-averse. The total expected bid decreases from 0.812 *to* 0.357 *and then to* 0.184.

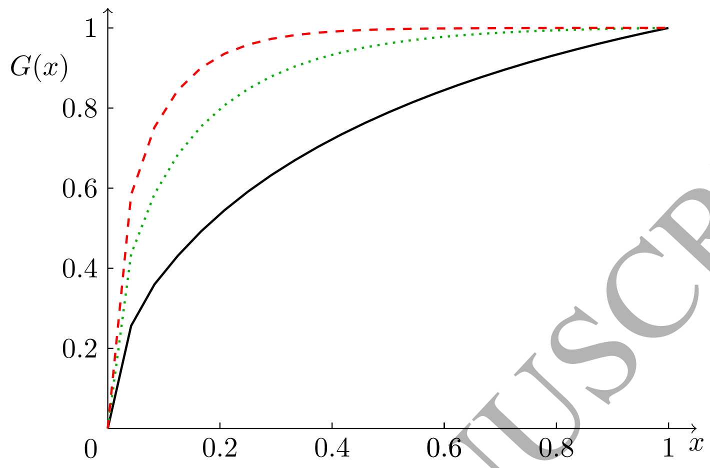

Figure 2: Homogeneously increasing all players' risk aversion decreases $G(x)$.

In the complete information all-pay auction with homogenous risk-averse players, the total expected bid, therefore, decreases in the players' risk aversion. As all players become more risk-averse, they require better odds of winning in order to be compensated for the same risk. To maintain each others' indifference conditions, as required by equilibrium, all players bid lower in the sense of first-order stochastic dominance, in order to compensate each other for the greater disutility of risk.

4.2. Heterogeneous risk aversion

In this section, we first show in Proposition 3 that each player's expected bid increases with her own risk aversion, but decreases with other active players' risk aversions. Given this result, we characterize the sufficient condition for the total expected bid to decrease when the more risk-averse player becomes even more risk-averse in Proposition 4. We again assume that players can be ordered according to their risk aversion. Without loss of generality, let $K_1(x) \geqslant K_2(x) \geqslant \ldots \geqslant K_m(x)$ for all $x \in [0, v]$, so that player 1 is the most risk-averse player.

Proposition 3 *Assume there exists an equilibrium where a set B of players randomize continuously on the interval $[0, v]$ and all other players are inactive. Assume, furthermore, that the level of risk aversion for some player $i \in B$ has increased, i.e., $K_i(x)$ changes to $\tilde{K}_i(x) \geqslant K_i(x)$ for every $x \in [0, v]$. Assume that after this change, there still exists an equilibrium where the set B of players randomize continuously on the interval $[0, v]$, and all other players are inactive. Then, the expected bid of player i increases with her level of risk aversion, while the expected bid of player k for $k \in B, k \neq i$ decreases with player i's level of risk aversion.*

The following example illustrates this result.

Example 6 *Assume there are three players with CARA utility functions $U_i = 1 - e^{-\beta_i c}$ with $\beta_1 = 1, \beta_2 = 0.5, \beta_3 = 0.1$ and valuation $v = 1$ for $i = 1, 2, 3$. Then, player 1 is the most risk averse and $K_1(x) \geqslant K_2(x) \geqslant K_3(x)$ for all $x \in [0, 1]$. In the equilibrium in which all three players are active and randomize continuously on the interval $[0, 1]$, the equilibrium strategies of the players (the CDFs of their mixed strategies) are given in the left panel of figure 3: $G_1(x)$ (black) $\leqslant G_2(x)$ (green) $\leqslant G_3(x)$ (red). Assume now that β_1 changes to $\tilde{\beta}_1 = 1.2$, then the players strategies change to the dashed lines as in the right panel of figure 3. It can be seen that player 1's mixed strategy (black) increases to $\widetilde{G}_1(x) \leqslant G_1(x)$ while players 2's (green) and 3's (red) mixed strategy decreases to $\widetilde{G}_2(x) \geqslant G_2(x)$ and $\widetilde{G}_3(x) \geqslant G_3(x)$ in the sense of first-order stochastic dominance, respectively.*

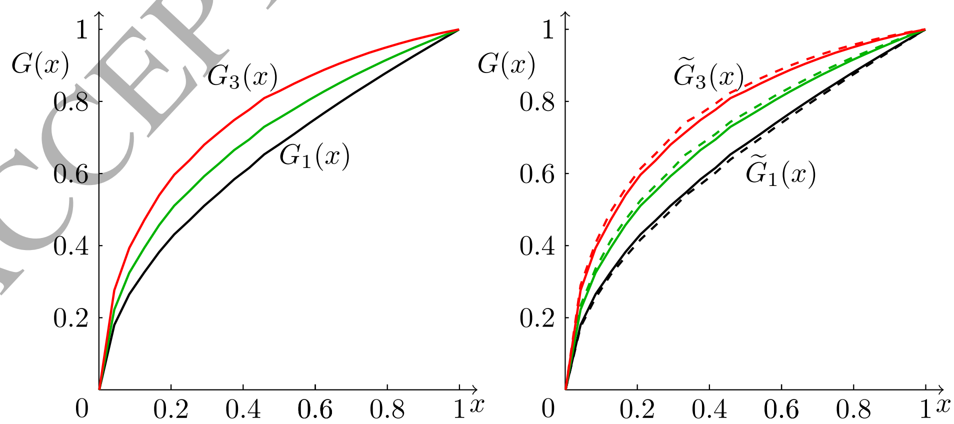

Figure 3: A player's bid increases with her own risk aversion and decreases with other players' risk aversions.

Next, we discuss the effect of a change in the risk attitude of an active player on the total expected bid in equilibrium. We first interpret the intuition behind Proposition 3. In a mixed strategy equilibrium, any active player t is made indifferent between any of his bids by the strategies of the other players. When player t becomes even slightly more risk-averse, the other players have to lower their bids to ensure player t stays indifferent (in order for an equilibrium with the same set of active players to continue to exist). Thus, by equilibrium strategy (4), increasing t's risk aversion has two effects on total expected bid, fixing the same set of active players:

1. Player t bids higher, since the CDF of her new equilibrium strategy first-order stochastically dominates the CDF of her original equilibrium strategy, before she became more risk-averse;
2. The rest of the players bid lower when $K_t(x)$ increases, since their CDFs decrease in the sense of first-order stochastic dominance.

The net effect on total expected bids is not obvious. We provide in Proposition 4 a sufficient condition for the total expected bid to decrease when one player's risk aversion increases, assuming the equilibrium with the same set of active players still exists after the increase of the player's risk aversion.

Proposition 4 *Assume an equilibrium with a set B of active players who randomize continuously on the interval $[0, v]$. For an active player i, if*

$$K_i(x) \geqslant \frac{|B| - 2}{\sum_{l \in B, l \neq i} K_l(x)^{-1}}, \tag{15}$$

for all $x \in [0, v]$, then the total expected bid decreases in i's risk aversion.

Note that the r.h.s. of (15) can be rewritten as the harmonic mean of the $K(x)$ functions of the rest of the active players multiplied by a constant:

$$\frac{|B| - 2}{\sum_{l \in B, l \neq i} K_l(x)^{-1}} = \frac{|B| - 1}{\sum_{l \in B, l \neq i} K_l(x)^{-1}} \frac{|B| - 2}{|B| - 1}.$$

Thus, condition (15) requires that player i be sufficiently risk averse compared to the rest of the active players to guarantee that an increase in her risk aversion decreases the total expected bid. See example 7 for an illustration of Proposition 4.

Example 7 *Assume there are three players $B = \{1, 2, 3\}$ who have CARA utility functions with $\beta_1 = 2$, $\beta_2 = 1$, $\beta_3 = \frac{1}{2}$, and valuation $v = 1$. Then, $K_1(x) = \frac{1-e^{-2x}}{1-e^{-2}}$, $K_2(x) = \frac{1-e^{-x}}{1-e^{-1}}$, and $K_3(x) = \frac{1-e^{-\frac{1}{2}x}}{1-e^{-\frac{1}{2}}}$. Note that the condition (15) for $i = 1$ is satisfied:*

$$K_1(x) = \frac{1 - e^{-2x}}{1 - e^{-2}} \geqslant \left(\frac{1 - e^{-\frac{1}{2}}}{1 - e^{-\frac{1}{2}x}} + \frac{1 - e^{-1}}{1 - e^{-x}} \right)^{-1} = \left(K_2(x)^{-1} + K_3(x)^{-1} \right)^{-1}$$

and the total expected bid in the equilibrium in which all three players are active and randomize continuously on $[0, 1]$ is:

$$3 - \int_0^1 \left(K_2(x)K_3(x)\right)^{\frac{1}{2}} \left(K_1(x)\right)^{-\frac{1}{2}} dx - \int_0^1 \left(K_1(x)K_3(x)\right)^{\frac{1}{2}} \left(K_2(x)\right)^{-\frac{1}{2}} dx$$
$$- \int_0^1 \left(K_1(x)K_2(x)\right)^{\frac{1}{2}} \left(K_3(x)\right)^{-\frac{1}{2}} dx = 0.779.$$

Assume now that we increase player 1's risk aversion to $\beta_1 = 3$, then the total expected bid decreases to 0.723.

Many real-life competitions are composed of participants with evidently different risk attitudes, e.g., mixed gender contests. We now analyze how the risk attitude composition of contests with two different risk types affects participation. Formally, assume there are two sets of contestants in the competition: type 1 players with the contest risk preference $K_1(x)$, and type 2 players with the contest risk preference $K_2(x)$. There are m players in total. Let n be the number of type 1 players, with $n < m$. All players have the same valuation v for the prize and $K_1(x) > K_2(x)$ for all $x \in (0, v)$. Type 1 players are thus more risk-averse than type 2 players.

Corollary 5 *For any $\frac{1}{m} \leqslant \frac{n}{m} \leqslant 1$, there exists an equilibrium in which all players randomize continuously on the interval $[0, v]$ if and only if*

$$K_1(x)^m \leqslant K_2(x)^{m-1}, \tag{16}$$

for all $x \in [0, v]$.

Corollary 5 establishes that there always exists an equilibrium with all players active for any $\frac{n}{m} \in [0,1]$, when the inequality (16) is satisfied, i.e., when the more risk averse players are not too risk-averse compared to the less risk-averse players. Given that such an equilibrium exists, Corollary 6 then provides a sufficient condition for the total expected bid in such an equilibrium to decrease with $\frac{n}{m}$ (the share of the more risk-averse players).

Corollary 6 *When there are two risk-averse types of players, with* $K_1(x) > K_2(x)$*, for* $x \in (0, v)$*, assume condition (16) in Corollary 5 is satisfied. Then the total expected bid is monotonically decreasing with the share of the* $K_1(x)$ *players,* $\frac{n}{m}$*, if*

$$K_2(x) \geqslant \frac{m-2}{m-1} K_1(x). \tag{17}$$

Corollary 6 explicates the transition in terms of total revenue from the case where all players are homogeneously less risk-averse to the case where all players are homogeneously more risk-averse. As the share of the more risk-averse players increases, the total expected bid decreases monotonically. According to (17), this is true if the two types of players are not too different, as

$$K_1(x) \geqslant K_2(x) \geqslant \frac{m-2}{m-1} K_1(x)$$

has to hold.

5. Discussion

5.1. Asymmetric valuation

The literature on complete information all-pay auctions with risk neutral players has traditionally allowed for heterogeneity in valuations of the prize. Here, we compare heterogeneity in risk aversion to heterogeneity of valuation. Assume two sets of valuations such that $v_h = v_1 = ... = v_n > v_{n+1} = ... = v_m = v_l$ and compare to a common value v but two sets of risk-averse players, i.e., $K_1(x) = ... = K_n(x) > K_{n+1}(x) = ... = K_m(x)$ for all $x \in [0, v]$. All the results for the risk neutral case are found in Baye et al. (1996).

First, observe that if $n \geq 2$, only the players with the high valuation are active in any equilibrium with risk neutral players and two different valuations. However, in our setting both the more risk-averse players and the less risk-averse players may be active in equilibrium. This phenomena is also true with more types of players. For $v_1 > v_2 > v_3 \geqslant ... \geqslant v_m$, in any equilibrium only players 1 and 2 are active while in the risk averse case with $K_1(x) > K_2(x) > K_3(x) \geqslant ... \geqslant K_m(x)$ all players may be active.

Moreover, if $n = 1$ then in the risk neutral setting, there exists a continuum of equilibria. In any equilibrium, bidder 1 randomizes continuously on the interval $[0, v_l]$. Each bidder $i, i \in \{2, ..., m\}$, employs a strategy G_i with support contained in $[0, v_l]$ that has an atom $\alpha_i(0)$ at 0. The size of the atom may differ across bidders, but $\Pi_{i=2}^{m} \alpha_i(0) = \frac{v_h - v_l}{v_l}$. Each G_i is characterized by a number $b_i \geqslant 0$, where $b_i = 0$ for at least one $i \neq 1$, such that $G_i(x) = \alpha_i(0)$, for all $x \in [0, b_i]$ and bidder i randomizes continuously on $(b_i, v_l]$. Furthermore, when two or more bidders in the set $\{2, ..., m\}$ randomize continuously on a common interval, their CDFs are identical on that interval. Finally, in any equilibrium bidder 1 earns an expected payoff of $v_h - v_l$, while each of the bidders $2, ..., m$ earns an expected payoff of zero.

In the risk-averse case, we examine equilibria with the feature that all players randomize on an interval contained in $[0, v]$ and may have an atom at zero. This is similar to the risk neutral case where they all randomize on an interval contained in $[0, v_l]$ and may have an atom at zero. However, in our setting all bidders have a payoff of zero in any equilibrium, including the less risk-averse players (the "strong" players). Moreover, weak bidders (more risk-averse) do not necessarily have an atom at zero, while in the risk neutral case, some of them must have an atom so that $\Pi_{i=2}^{m} \alpha_i(0) = \frac{v_h - v_l}{v_l}$. In the risk-averse setting, it is also true that when two or more bidders with the same risk aversion randomize continuously on a common interval, their CDFs are identical on that interval.

We can also compare the effects of changes in players' preferences (i.e., valuation for the prize in the risk neutral case, and risk aversion in the risk-averse case) on the expected total effort. Given an equilibrium of the risk neutral all-pay auction with valuations v_l and v_h, and assuming that all bidders randomize on the same interval (starting from the same b_i) before and after the increase, an increase in v_l necessarily increases the expected total effort. However, as we show in Proposition 4, increasing the risk aversion of a more risk-averse player may either decrease or increase the expected total effort depending on how much more risk-averse she is than the rest of the players. Therefore, heterogeneous risk aversion gives rise to different predictions on the behavior of the players. These can be empirically tested[7].

5.2. Gender difference

Our findings suggest the possibility that the higher risk aversion of women and girls can simultaneously lead them to avoid participating in all-pay auction type incentives while bidding higher and having a higher probability of winning than men and boys, when they do participate. Heterogeneous risk aversion, therefore, could be an important factor in explaining many of the stylized facts about gender differences in competitiveness, including women's greater reluctance to enter contests with all-pay auction incentives, like elections, unless they have a good chance of winning (Fulton et al., 2006), girls' greater willingness to exert effort in preparation (Duckworth and Seligman, 2006) and their higher odds of success in academic contests (Angrist et al., 2009; DiPrete and Buchmann, 2013; Fortin et al., 2015), and women's greater reluctance to enter laboratory contests (Niederle and Vesterlund, 2007), where they cannot assure themselves success through extra pre-experiment preparation. Moreover, at present, the experimental literature has largely taken for granted that heterogeneous risk aversion is similar to homogeneous in all-pay auction settings in depressing efforts, e.g., that the gender gap in competitiveness decreases when risk aversion is controlled for (Niederle, 2014). Our contribution is to show that this assumption is not warranted and indeed heterogeneous risk aversion is consistent with many different (and perhaps unexpected) forms of behavior.

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[7]See Chen et al. (2015) for the joint effect of changes in risk aversion and valuation on bidding behavior in the two-player case.

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6. Appendix

Proof of Lemma 1

If player i becomes more risk-averse (from $\tilde{U}_i$ to U_i), then the certainty equivalent of winning $v - x$, with probability $K_{\tilde{U}_i}(x)$, and $-x$, with probability $\left(1 - K_{\tilde{U}_i}(x)\right)$, is less than zero, which is the certainty equivalent of this gamble with a utility function $\tilde{U}_i$. Therefore, the player can be restored to indifference between winning zero for sure and the gamble only if the probability of winning the larger prize $v - x$ increases. Thus, $K_{U_i}(x) > K_{\tilde{U}_i}(x)$. For loss-averse players, we rewrite their $K_{U_i}(x)$ function as

$$K_{U_i}(x) = \frac{-l_i(-x)}{g_i(v-x) - l_i(-x)} = 1 - \frac{g_i(v-x)}{g_i(v-x) - l_i(-x)}.$$

Thus, when player i gets more loss-averse, $l_i(-x)$ gets smaller and $K_{U_i}(x)$ increases.

Proof of Corollary 2

If player s is more risk-averse than player t, where $s, t \in B$, then we have $K_s(x) \geqslant K_t(x)$ for $x \in [0, v]$. Based on the equilibrium strategy given in (4), the difference in the distributions of their mixed strategies is:

$$G_s(x) - G_t(x) = \left(\prod_{1 \leqslant l \leqslant |B|} K_l(x) \right)^{\frac{1}{|B|-1}} \left(K_s(x)^{-1} - K_t(x)^{-1} \right) \leqslant 0.$$

Thus, player s's expected bid is higher than player t's and the cumulative distribution function of player s first-order stochastically dominates the cumulative distribution function of player t. Therefore, the ranking of expected bids is the same as the ranking of risk aversion.

Proof of Corollary 3

The expected probability of winning for player s is given by (note that $K_s(v) = K_t(v) = 1$):

$$\int_0^v K_s(x) dG_s(x) = 1 - \int_0^v G_s(x) dK_s(x).$$

For player t:

$$\int_0^v K_t(x) dG_t(x) = 1 - \int_0^v G_t(x) dK_t(x).$$

Thus, the difference between the probabilities of winning is:

$$\begin{aligned}
&\int_0^v K_s(x)dG_s(x) - \int_0^v K_t(x)dG_t(x) \\
&= \int_0^v [G_t(x)dK_t(x) - G_s(x)dK_s(x)] \\
&= \int_0^v \left(\prod_{l\in B} K_l(x)\right)^{\frac{1}{|B|-1}} \left[\frac{dK_t(x)}{K_t(x)} - \frac{dK_s(x)}{K_s(x)}\right]. \qquad (18)
\end{aligned}$$

Therefore, the difference is non-negative if

$$\frac{dK_t(x)}{K_t(x)} - \frac{dK_s(x)}{K_s(x)} = \frac{K_t'(x)}{K_t(x)} - \frac{K_s'(x)}{K_s(x)} \geqslant 0,$$

for all $x \in [0, v]$[8].

Proof of Corollary 4

To prove the corollary, consider the condition for the former equilibrium with the set B of active players to exist, i.e.,

$$\prod_{l\in B} K_l(x) \leqslant \min\{K_j(x)^{|B|-1}, K_i(x)^{|B|-1}\}.$$

By the assumption that $K_i(x) \geqslant K_j(x)$, it must be true that

$$K_j(x) \prod_{l\in B, l\neq i} K_l(x) \leqslant \prod_{l\in B} K_l(x) \leqslant \min\{K_j(x)^{|B|-1}, K_i(x)^{|B|-1}\},$$

which is the sufficient and necessary condition for the latter equilibrium with the set $\widetilde{B}$ to exist.

[8]The reverse hazard rate dominance is not implied by the fact that $K_t(x)$ first-order stochastically dominates $K_s(x)$. In fact, the reverse hazard rate dominance implies first-order stochastic dominance. However, it is easy to show that the reverse hazard rate dominance condition is equivalent to first-order stochastic dominance for CARA and CRRA utility functions. Appendix B in Krishna (2009) provides a useful introduction to stochastic dominance.

Proof of Proposition 2

When players are homogeneous, we have $K_1(x) = K_2(x) = ... = K_m(x)$. By the equilibrium strategy given in (4), the strategy under homogeneous risk aversion is specified by

$$G_i(x) = K(x)^{\frac{1}{|B|-1}},$$

where $i \in B$. It is then obvious that any active player i's bid is decreased in the sense of first-order stochastic dominance when all players become more risk-averse. The total expected bid can be calculated as

$$R = \sum_{i=1}^{|B|} R_i = \sum_{i=1}^{|B|} \int_0^v x dG_i(x),$$

where

$$R_i = \int_0^v x dG_i(x) = v - \int_0^v G_i(x) dx$$

is the expected bid of any player i. The second equality follows from integration by parts. Since $G_i(x)$ for $i \in B$ is increased, R_i is decreased, and thus, the total expected bid R is decreased when $K(x)$ increases for $x \in (0, v)$.

Proof of Proposition 3

An active player i's expected bid is given by

$$R_i = \int_0^v x dG_i(x),$$

where in equilibrium we have (from (4))

$$G_i(x) = \left(\prod_{l \in B, l \neq i} K_l(x) \right)^{\frac{1}{|B|-1}} K_i(x)^{-\frac{|B|-2}{|B|-1}}.$$

Therefore, when $K_i(x)$ changes to $\tilde{K}_i(x) \geqslant K_i(x)$, then $G_i(x)$ decreases for every $x \in (0, v)$, and therefore, R_i increases. Moreover, for any other active players $k \in B$, $k \neq i$ we have

$$G_k(x) = \left(\prod_{l \in B, l \neq i,k} K_l(x) \right)^{\frac{1}{|B|-1}} K_k(x)^{-\frac{|B|-2}{|B|-1}} K_i(x)^{\frac{1}{|B|-1}}.$$

Therefore, when $K_i(x)$ changes to $\tilde{K}_i(x) \geqslant K_i(x)$, then $G_k(x)$ increases for every $x \in (0, v)$, and therefore, R_k decreases.

Proof of Proposition 4

As each player j's expected bid can be written as $R_j = \int_0^v x dG_j(x)$, we can write the total expected bid R as:

$$R = \sum_{j \in B} R_j = |B| v - \int_0^v \sum_{j \in B} G_j(x) dx.$$

Rewrite the second term:

$$\int_0^v \sum_{j \in B} G_j(x) dx = \int_0^v (\prod_{l \in B} K_l(x))^{\frac{1}{|B|-1}} \sum_{j \in B} K_j(x)^{-1} dx. \tag{19}$$

Thus, the marginal effect of an increase of $K_i(x)$ for every given $x \in (0, v)$ on (19) can be written as:

$$\int_0^v \frac{d\left(\sum_{j \in B} G_j(x)\right)}{dK_i(x)} dx$$
$$= \int_0^v \frac{1}{|B|-1} \left(\prod_{l \in B} K_l(x) \right)^{\frac{1}{|B|-1}} K_i(x)^{-1} \left(\sum_{l \in B, l \neq i} K_l(x)^{-1} - (|B|-2) K_i(x)^{-1} \right) dx.$$

This expression is positive if $\sum_{l \in B, l \neq i} K_l(x)^{-1} - (|B|-2) K_i(x)^{-1} \geqslant 0$ for all $x \in [0, v]$, which is condition (15). Therefore, the marginal effect on R is negative if the condition (15) is satisfied.

Proof of Corollary 5

Based on Proposition 1, the sufficient and necessary conditions in the current context are:

$$K_1(x)^n K_2(x)^{m-n} \leqslant K_2\,(x)^{m-1}\,,$$

for all $x \in [0, v]$. Rewrite

$$\left(\frac{K_1(x)}{K_2(x)}\right)^n \leqslant K_2\,(x)^{-1}\,,$$

by log transformation,

$$\frac{n}{m} \leqslant \frac{\ln K_2(x)^{-1}}{m\,(\ln K_1(x) - \ln K_2(x))}. \tag{20}$$

Let the r.h.s. of inequality (20) be no less than one:

$$\frac{1}{m}\frac{\ln K_2(x)^{-1}}{\ln K_1(x) - \ln K_2(x)} \geqslant 1.$$

After rearranging, we have

$$K_1(x)^m \leqslant K_2(x)^{m-1}.$$

Therefore, whenever $K_1(x)^m \leqslant K_2(x)^{m-1}$ for all x, we have that for all $\frac{n}{m}$, there is an equilibrium in which all players are active.

Proof of Corollary 6

Let $\mu = \frac{n}{m} \in \{0, \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}\}$ be the current share of $K_1(x)$ players. Substitute a $K_2(x)$ player with a $K_1(x)$ player. Then, by Proposition 4, the total expected bid decreases if

$$K_2(x) \geqslant \frac{m-2}{\frac{\mu m}{K_1(x)} + \frac{m-1-\mu m}{K_2(x)}}. \tag{21}$$

It can be verified that the r.h.s. of the above inequality is increasing with μ, and thus, is less than $\frac{m-2}{m-1}K_1(x)$. Therefore, condition (17) is sufficient for condition (21), and we have proved that increasing μ decreases total expected bid.