# A note on introducing a "zeroline" of welfare as an escape route from Arrow's theorem

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# A Note on Introducing a 'Zero-Line' of Welfare as an Escape-Route from Arrow's Theorem*

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**Abstract.** Since Sen's insightful analysis of Arrow's Impossibility Theorem (Sen, 1970/1979), Arrow's theorem is often interpreted as a consequence of the exclusion of interpersonal information from Arrow's framework. Interpersonal comparability of *either* welfare levels *or* welfare units is known to be *sufficient* for circumventing Arrow's impossibility result (e.g. Sen, 1970/1979, 1982; Roberts, 1980; d'Aspremont, 1985). But it is less well known whether one of these types of comparability is also *necessary* or whether Arrow's conditions can already be satisfied in much narrower informational frameworks. This note explores such a framework: the assumption of (ONC+0), ordinal measurability of welfare with the additional measurability of a 'zero-line', is shown to point towards new, albeit limited, escape-routes from Arrow's theorem. Some existence and classification results are established, using the condition that social orderings be transitive as well as the condition that social orderings be quasi-transitive.

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## 1. Introduction

Over the last two or three decades, a great range of different assumptions about the measurability and interpersonal comparability of welfare have been analysed with regard to their potential for providing escape-routes from Arrow's famous impossibility theorem (see, amongst many others, Sen 1970/1979, 1982; Roberts, 1980; d'Aspremont, 1985). In this context, Roberts (1980) and particularly Blackorby & Donaldson (1982) have investigated the implications of ratio-scale measurability with full interpersonal comparability, an informational framework that attaches interpersonal significance to a 'zero-line' of welfare as well as enabling interpersonal comparisons of welfare levels and units.

Ratio-scale measurability with full interpersonal comparability can be seen as a *rich* informational framework which attaches interpersonal significance to a 'zero-line' of welfare. The present note is concerned with an extremely *narrow* informational assumption that still admits the measurability of a 'zero-line': namely, the assumption that welfare is only ordinally measurable, that neither welfare levels nor welfare units are interpersonally comparable, but that a 'zero-line' of welfare is interpersonally significant. This framework will be called (ONC+0) and defined formally below.

The idea of introducing a 'zero-line' of welfare whilst disallowing other kinds of interpersonal comparisons has been studied much less extensively than richer informational assumptions. However, there are some notable exceptions. In a paper on distributionally sensitive cost-benefit analysis, Blackorby and Donaldson (1987) have proposed a method of social evaluation on the basis of 'welfare ratios' defined in terms of the ratio of household income to a suitable poverty line, requiring no interpersonal comparisons except those provided by the relevant poverty lines. Tsui & Weymark (1997), Naumova (1998) and Yanovskaya (1998) all address the informational framework of ratio-scale measurability without interpersonal comparability of welfare levels or units, an informational framework that is still less narrow than the one discussed in the present note. Gibbard, Hylland and Weymark (1987) have explored the implications of introducing a fixed feasible alternative for Arrow's theorem. While the welfare-level generated by this fixed alternative could in principle be interpreted as the 'zero-line' or 'norm-line', such an interpretation would require us to make two contestable assumptions: first, the assumption that there always exists a fixed alternative generating a 'zero' or 'norm' level of welfare, and, second, the assumption that the fixed alternative generating this welfare-level is the same for all individuals and profiles. The present approach, on the other hand, requires no such assumptions. Tungodden (1998), finally, has proposed the idea of introducing 'independent norm levels' within an ordinalist framework by partitioning the set of persons into two or more subsets and assuming interpersonal comparability of welfare levels between members of different subsets, but not between members of the same subset. In this framework, Tungodden explores 'head count' social choice rules paralleling the ones discussed in the context of theorem **3.1.** below.

Some of the results to be stated below (in particular theorems **3.1.** and **3.3.**) can be interpreted and alternatively proved as implications of Gibbard, Hylland, & Weymark's results (1987) or as implications of Tungodden's results (1998); in the former case, by adding a new "auxiliary element" to the set of alternatives, to be interpreted as a fixed feasible alternative, and identifying the 'zero-line' with each person's welfare-level under that new alternative; and in the latter case, by partitioning the set of persons into (at most) two subsets.

Before we can explore the escape-routes from Arrow's theorem provided by the assumption of (ONC+0) (section **3.**), it will be requisite to survey Arrow's theorem and some related results (section **2.**), as these results will serve as a 'reference frame' not only for assessing the results concerning (ONC+0), but also for proving some of them.

## 2. Arrow's Theorem and Related Results

Let $N = \{1, 2, ..., n\}$ be a set of persons and $X$ a set of alternative social states ($n \geq 2$, and $|X| \geq 3$). Following Sen's social-choice-theoretic formalism (Sen, 1970/1979, 1982), we shall assume that, to each person $i \in N$, there corresponds a *personal welfare function* $W_i : X \rightarrow \mathbb{R}$, mapping each social state $x \in X$ to a real number that indicates the 'level of welfare' of person $i$ in social state $x$. A *profile of personal welfare functions* $\{W_i\}$ is an assignment of one such function to each person in $N$. Let $\mathbf{W}^n$ denote the set of all logically possible such profiles. A *social welfare functional* (SWFL) is a function $F$ that aggregates each profile $\{W_i\}$ in a given domain to a social ordering $R$ on the set $X$, where $R$ is reflexive, connected and, unless stated otherwise, transitive. In what follows, the letters $P$ and $I$ will be used to denote, respectively, the strong ordering and the indifference relation induced by $R$: for all $x_1, x_2 \in X$,

$x_1 P x_2$ if and only if $x_1 R x_2$ and not $x_2 R x_1$, and
$x_1 I x_2$ if and only if $x_1 R x_2$ and $x_2 R x_1$.

Different assumptions about the measurability and interpersonal comparability of welfare are usually stated by referring to the class of those transformations up to which a profile $\{W_i\}$ is taken to be unique and with respect to which an acceptable SWFL should therefore be invariant.

The informational framework corresponding to Arrow's original result is the following. A transformation $\phi$ on (a subset of) the real numbers is *positive monotonic* if, for all $t_1$ and $t_2$ in the domain of $\phi$, $t_1<t_2$ implies $\phi(t_1)<\phi(t_2)$.

**ORDINAL MEASURABILITY, NO INTERPERSONAL COMPARABILITY (ONC).** For any $\{W_i\}$ and $\{W^*_i\}$ in the domain of $F$, $F(\{W_i\}) = F(\{W^*_i\})$ if, for each $i$, $W^*_i = \phi_i(W_i)$, where $\{\phi_i\}$ is some $n$-tuple of positive monotonic transformations.

Note that (ONC) precludes not only interpersonal comparisons of welfare levels and units, but also the identification of an interpersonally significant 'zero' or 'norm' line, because no such line would be invariant under all of the transformations admitted by (ONC).

In terms of Sen's SWFL-based formalism, the conditions of Arrow's theorem are as follows:

**UNIVERSAL DOMAIN (U).** The domain of $F$ is the set of all logically possible profiles of personal welfare functions.

**WEAK PARETO PRINCIPLE (P).** Let $\{W_i\}$ be any profile in the domain of $F$, and let $R = F(\{W_i\})$. For any $x_1, x_2 \in X$, we have $x_1 P x_2$ whenever, for all $i \in N$, $W_i(x_1) > W_i(x_2)$.

**INDEPENDENCE OF IRRELEVANT ALTERNATIVES (I).** Let $\{W_i\}$ and $\{W^*_i\}$ be any profiles in the domain of $F$, and let $R = F(\{W_i\})$ and $R^* = F(\{W^*_i\})$. For any $x_1, x_2 \in X$, if, for all $i \in N$, $W_i(x_1) = W^*_i(x_1)$ and $W_i(x_2) = W^*_i(x_2)$, $x_1 R x_2$ if and only if $x_1 R^* x_2$.

**NON-DICTATORSHIP (D).** $F$ must not be dictatorial: there must not exist an $i \in N$ such that, for all $\{W_i\}$ in the domain of $F$ and any $x_1, x_2 \in X$, $W_i(x_1) > W_i(x_2)$ implies $x_1 P x_2$, where $R = F(\{W_i\})$.

Now Arrow's impossibility theorem states that, in the pure ordinalist framework of (ONC), these four conditions are not simultaneously satisfiable:

**THEOREM 2.1.** (Arrow, 1951/1963, Sen 1970/1979) Any SWFL $F$ satisfying (ONC), (U), (P) and (I) must violate (D).

If we also impose a stronger version of the Pareto principle, a lexicographic extension of the dictatorial rule emerges.

**DEFINITION 2.2.** For any subset $S$ of $N$, define the relations $R_S(\{W_i\})$, $P_S(\{W_i\})$ and $I_S(\{W_i\})$

induced by a profile $\{W_i\}$ as follows: for any $x_1, x_2 \in X$,

$x_1 R_S(\{W_i\}) x_2$ if and only if, for all $i \in S$, $W_i(x_1) \geq W_i(x_2)$;

$x_1 P_S(\{W_i\}) x_2$ if and only if $x_1 R_S(\{W_i\}) x_2$ and not $x_2 R_S(\{W_i\}) x_1$; and

$x_1 I_S(\{W_i\}) x_2$ if and only if $x_1 R_S(\{W_i\}) x_2$ and $x_2 R_S(\{W_i\}) x_1$.

**STRONG PARETO PRINCIPLE (SP)**[1]**.** Let $\{W_i\}$ be any profile in the domain of $F$, and let $R = F(\{W_i\})$. For any $x_1, x_2 \in X$, if $x_1 R_N(\{W_i\}) x_2$, then $x_1 R x_2$; if $x_1 P_N(\{W_i\}) x_2$, then $x_1 P x_2$.

**THEOREM 2.3.** (d'Aspremont, 1985) Suppose $F$ satisfies (ONC), (U), (I) and (SP). Then $F$ is a lexicographic dictatorship, i.e. there exists a permutation $\sigma$ of $N$ such that, for any $\{W_i\} \in \mathbf{W}^n$ and any $x_1, x_2 \in X$,

$$x_1 P x_2 \text{ if and only if } \quad W_{\sigma(i)}(x_1) > W_{\sigma(i)}(x_2) \text{ for some } i \in N$$
$$\text{and } \quad W_{\sigma(j)}(x_1) = W_{\sigma(j)}(x_2) \text{ for all } j < i,$$

where $R = F(\{W_i\})$.

In section **3.**, we shall see that theorems **2.1.** and **2.3.** are logically dependent upon the absence of an interpersonally significant 'zero-line' of welfare, for, if we introduce such a 'zero-line' by adopting (ONC+0), the class of SWFLs satisfying (U), (P) and (I) is not restricted to dictatorial rules.

It is well known that, if we relax the requirement that the ordering $R$ generated by a SWFL be transitive and demand only that $R$ be quasi-transitive -- i.e. $P$ must be transitive, but $I$ need not --, Arrow's original impossibility result can be circumvented even under the assumption of (ONC):

**THEOREM 2.4.** There exist SWFLs generating quasi-transitive social orderings which satisfy (ONC), (U), (P), (I) and (D).

**Proof.** The following SWFL satisfies the required properties: for each $\{W_i\}$, define $F(\{W_i\})$ to be the ordering $R$ such that, for all $x_1, x_2 \in X$, $x_1 R x_2$ if and only if it is not the case that $x_2 P_N(\{W_i\}) x_1$. **Q.E.D.**

However, the possibility result of theorem **2.4.** is not very robust: it is also well known that any SWFL generating quasi-transitive social orderings which satisfies Arrow's conditions

[1]In fact, the present version of (SP) is the conjunction of what is often called the 'strong Pareto principle' and the condition of 'Pareto indifference'.

must violate the following two conditions:

**NON-OLIGARCHY (O).** $F$ must not be oligarchic: there must not exist an $M \subseteq N$ such that, for all $\{W_i\}$ in the domain of $F$ and any $x_1, x_2 \in X$, (i) $x_1 R x_2$ whenever $W_i(x_1) > W_i(x_2)$ for *some* $i \in M$, and (ii) $x_1 P x_2$ whenever $W_i(x_1) > W_i(x_2)$ for *all* $i \in M$, where $R = F(\{W_i\})$.

**POSITIVE RESPONSIVENESS (PR).** Let $\{W_i\}$ and $\{W^*_i\}$ be any profiles in the domain of $F$, $x_1$, $x_2 \in$ X, and $j \in N$ such that, for all $i \in N$, with $i \neq j$, $W_i = W^*_i$, and *either* $[W_j(x_1) < W_j(x_2)$ and $W^*_j(x_1) \geq W^*_j(x_2)]$ *or* $[W_j(x_1) = W_j(x_2)$ and $W^*_j(x_1) > W^*_j(x_2)]$. Then $x_1 R x_2$ implies $x_1 P^* x_2$, where $R = F(\{W_i\})$ and $R^* = F(\{W^*_i\})$.

**THEOREM 2.5.** (Sen, 1982, p. 167) Any SWFL generating quasi-transitive social orderings which satisfies (ONC), (U), (P), (I) and (D) must violate (PR), if $n \geq 3$.

**THEOREM 2.6.** (Gibbard[2]) Any SWFL generating quasi-transitive social orderings which satisfies (ONC), (U), (P) and (I) must violate (O)[3].

If we once again strengthen the Pareto condition, a lexicographic extention of an oligarchy emerges:

**THEOREM 2.7.** (Guha, 1972) Suppose $F$ is a SWFL generating quasi-transitive social orderings which satisfies (ONC), (U), (SP) and (I). Then $F$ is a lexicographic oligarchy, i.e. $N$ can be partitioned into $S_1, S_2, ..., S_k$ (for some $k$) such that, for any $\{W_i\} \in \mathbf{W}^n$ and any $x_1$, $x_2 \in X$,

$$xPy \text{ if and only if } \quad x_1 P_{S_i}(\{W_i\}) x_2 \text{ for some } i \in \{1, 2, ..., k\}$$
$$\text{and} \quad x_1 I_{S_j}(\{W_i\}) x_2 \text{ for all } j < i.$$

where $R = F(\{W_i\})$.[4]

As shown in section **3.**, theorems **2.5.**, **2.6.** and **2.7.** are also logically dependent upon the absence of an interpersonally significant 'zero-line' of welfare: in the informational framework of (ONC+0), the class of SWFLs generating quasi-transitive social orderings

[2]This result was proved by Allan Gibbard in an unpublished paper in 1969; see Sen (1982, pp. 166 / 167). Note that we here restate theorems **2.5.**, **2.6.** and **2.7.** in the framework of social welfare functionals rather than social welfare functions.
[3]Note that a dictatorship violates (O) and that (O) therefore implies (D).
[4]Note that, although Guha uses the strong Pareto principle, his definition of an oligarchy is derived from the context of the weak Pareto principle (as in condition (O)). Here we restate Guha's result using the 'strong' definition of an oligarchy.

which satisfy (U), (P) and (I) is not restricted to oligarchic rules, and, in this class, only those rules which satisfy the condition of anonymity (to be defined below) in addition to (D) violate (PR).

## 3. Introducing a 'Zero-Line' of Welfare

The present informational framework can be defined as follows. A transformation $\phi$ on (a subset of) the real numbers is *sign-preserving* if, for all $t$ in the domain of $\phi$, $sign(\phi(t)) = sign(t)$, where, for each $t \in R$,

$$sign(t) = \begin{cases} -1 & \text{if } t<0 \\ 0 & \text{if } t=0 \\ 1 & \text{if } t>0. \end{cases}$$

**ORDINAL MEASURABILITY, MEASURABILITY OF A 'ZERO-LINE', NO INTERPERSONAL COMPARABILITY OF WELFARE LEVELS OR UNITS (ONC+0).** For any $\{W_i\}$ and $\{W^*_i\}$ in the domain of $F$, $F(\{W_i\}) = F(\{W^*_i\})$ if, for each $i$, $W^*_i = \phi_i(W_i)$, where $\{\phi_i\}$ is some n-tuple of positive monotonic and sign-preserving transformations.

(ONC+0) can be obtained by conjoining (ONC) with the condition that only those transformations are admissible that leave a 'zero-line' of welfare invariant. For each person $i$ and each alternative $x$, $W_i(x) < 0$, $W_i(x) = 0$, and $W_i(x) > 0$ are interpreted to mean that person $i$'s welfare-level is, respectively, below, exactly on, and above the 'zero-line'.

To operationalize (ONC+0), one could, conceivably, identify, for each person $i$ (or household), a bundle of goods, or a bundle of basic *functionings*, $y(i)$, which are (*judged* to be) so fundamental to person $i$'s well-being that their lack would (be *judged* to) imply impoverished living conditions for person $i$. Let us call this bundle of goods or functionings, $y(i)$, person $i$'s *'poverty' consumption bundle* (Blackorby & Donaldson, 1987). The bundle $y(i)$ could, for instance, include a certain minimal standard of nutrition, shelter, health care provision, education, and so on. One could then *define*, for each person $i$, the welfare-level generated by $y(i)$ to be equal to 0 and thus obtain, by stipulation, an interpersonally significant 'zero-line', i.e.

$$W_1(y(1)) = W_2(y(2)) = ... = W_n(y(n)) = 0 \qquad (*)$$

(this is a version of a proposal by Blackorby & Donaldson, 1987; for a discussion of the concept of *functionings*, see, for example, Sen, 1992). Note that this proposal is neutral on the question of whether 'poverty' consumption bundles are the same for all persons (households) or whether, in view of variations in needs and circumstances, different persons (households) require different 'poverty' consumption bundles. The following assumption is now sufficient to obtain (ONC+0): (*) holds, and each person's welfare is ordinally measurable; in particular, it is always possible to compare person *i*'s welfare-level in a given social state $x$ with person *i*'s welfare-level generated by $y(i)$ (i.e. by the social state in which person $i$ 'consumes' the 'poverty' consumption bundle $y(i)$). No interpersonal information over and above that provided by (*) is required. Of course, 'poverty' consumption bundles and 'poverty-lines' constitute rather crude welfare indicators (for further discussion, see also Sen, 1982, part V; Sen, 1997), but they should here be viewed from the perspective of trying to identify a *narrow* informational framework for social choice.

Although we use the concept of a 'zero-line' in this note, in essence none of the present results depends upon identifying the distinguished 'norm-line' with 0, and the results of the present note can be generalized to accommodate a special case of Tungodden's idea of 'independent norm levels' (Tungodden, 1998), namely the idea of a single 'norm-line'. It should become obvious that, with minor adjustments in the following theorems and proofs, (ONC+0) could be replaced with the new condition (ONC+$a$) for a fixed $a \in \mathrm{R}$:

**ORDINAL MEASURABILITY, MEASURABILITY OF A 'NORM-LINE', NO INTERPERSONAL COMPARABILITY OF WELFARE LEVELS OR UNITS (ONC+*a*).** For any $\{W_i\}$ and $\{W^*_i\}$ in the domain of $F$, $F(\{W_i\}) = F(\{W^*_i\})$ if, for each $i$, $W^*_i = \phi_i(W_i)$, where $\{\phi_i\}$ is some $n$-tuple of positive monotonic transformations such that, for each $i$ and all $t$, $t<a$ implies $\phi_i(t)<a$, $t=a$ implies $\phi_i(t)=a$, and $t>a$ implies $\phi_i(t)>a$.

It is even conceivable that different 'norm-lines' are assigned to different persons and that, formally, for each person $i$, a person-specific $a_i \in \mathrm{R}$ is substituted for the fixed $a \in \mathrm{R}$ in the definition of (ONC+$a$). But, whilst (ONC+0) and (ONC+$a$) are fully compatible with the condition of anonymity, to be defined below, it would be harder to reconcile the introduction of person-specific 'norm-lines' with this condition.

For reasons of mathematical simplicity, however, we shall use (ONC+0) throughout the present note.

We first observe that (ONC+0) is sufficient for circumventing the original Arrowian impossibility result (theorem **2.1.**):

**THEOREM 3.1.** There exist SWFLs satisfying (ONC+0), (U), (P), (I) and (D).

To prove this result, define, for each $\{W_i\}$ and each $x\in X$, $N^+(\{W_i(x)\}) = |\{i\in X : W_i(x)>0\}|$, $N^0(\{W_i(x)\}) = |\{i\in X : W_i(x)=0\}|$, $N^-(\{W_i(x)\}) = |\{i\in X : W_i(x)<0\}|$, interpreted as the number of people whose welfare level is, respectively, above, on, and below the fixed 'zero-line'. Now different versions of the 'head-count' rule, often used for the measurement of poverty (for discussion, see Sen, 1982, part V), supplemented with suitable tie-breaking rules can be seen to satisfy all of Arrow's conditions in the new informational framework:

**Proof.** Consider the following SWFL: for each $\{W_i\}$, let $F(\{W_i\})$ be the ordering $R$ such that, for all $x_1, x_2\in X$,

$$
\begin{aligned}
x_1Rx_2 \text{ if and only if } \quad & N^-(\{W_i(x_1)\}) < N^-(\{W_i(x_2)\}) \\
\text{or} \quad & [N^-(\{W_i(x_1)\}) = N^-(\{W_i(x_2)\}) \text{ and } N^0(\{W_i(x_1)\}) < N^0(\{W_i(x_2)\})] \\
\text{or} \quad & [N^-(\{W_i(x_1)\}) = N^-(\{W_i(x_2)\}) \text{ and } N^0(\{W_i(x_1)\}) = N^0(\{W_i(x_2)\}) \\
& \quad \text{and } [W_{\sigma(i)}(x_1) > W_{\sigma(i)}(x_2) \text{ for some } i\in N \\
& \qquad \text{and } W_{\sigma(j)}(x_1) = W_{\sigma(j)}(x_2) \text{ for all } j<i]] \\
\text{or} \quad & [N^-(\{W_i(x_1)\}) = N^-(\{W_i(x_2)\}) \text{ and } N^0(\{W_i(x_1)\}) = N^0(\{W_i(x_2)\}) \\
& \qquad \text{and } W_i(x_1) = W_i(x_2) \text{ for all } i\in N],
\end{aligned}
$$

where $\sigma$ is a fixed permutation of $N$. Since $N^-(\{W_i(.)\})$, $N^0(\{W_i(.)\})$, $N^+(\{W_i(.)\})$ and the rankings of the tie-breaking lexicographic dictatorship are all invariant under the admissible transformations of (ONC+0), $F$ satisfies (ONC+0). It is also easily seen that $F$ generates transitive social orderings and that it satisfies (U), (P) (in fact, it satisfies (SP)) and (I). Here we confine ourselves to showing that $F$ is not dictatorial. Assume, for a contradiction, that person $k$ is a dictator. By (U), we may consider $\{W_i\}$ and $x_1, x_2\in X$ such that $W_k(x_1)>W_k(x_2)>0$ and, for all $i\neq k$, $W_i(x_1)<0<W_i(x_2)$. Let $R = F(\{W_i\})$. Since $k$ is the dictator, we have $x_1Px_2$. But since $n\geq 2$, $N^-(\{W_i(x_2)\}) = 0 < 1 \leq n-1 = N^-(\{W_i(x_1)\})$, whence, by definition of $F$, $x_2Px_1$, a contradiction. Therefore $F$ satisfies (D). **Q.E.D.**

Informally, $F$ is the following rule: first minimize the number of people whose welfare-level is below the 'zero-line'; if there are ties, minimize the number of people whose welfare-level is on the 'zero-line'; if there are still ties, install a lexicographic dictatorship to break these ties.

We have a considerable degree of freedom in designing a version of the 'head-count' rule satisfying the conditions of theorem **3.1.**. Depending on how frequently a person's welfare-level lies *exactly* on the 'zero-line', the above defined SWFL differs from the SWFL that maximizes, firstly, the number of people whose welfare-level is above the 'zero-line' and, secondly, the number of people whose welfare-level is on the 'zero-line', before finally using a lexicographic dictatorship for breaking ties. But we have a more substantial degree of freedom in specifying how a lexicographic dictatorship for breaking ties should be installed. In the SWFL defined above, this tie-breaking lexicographic dictatorship, formally represented by the permutation $\sigma$, is completely fixed. However, given the informational resources of (ONC+0), it is also possible to make the the-breaking lexicographic dictatorship dependent upon the number of people below, on, or above the 'zero-line'; it is, for example, perfectly compatible with the conditions of theorem **3.1.** to define a different $\sigma_m$ for each $m = N^{-}(\{W_i(x_1)\}) = N^{-}(\{W_i(x_2)\})$.

An easy example shows that, unfortunately, one of the most attractive ideas on how the tie-breaking lexicographic dictatorship could be made sensitive to the pattern of persons below, on, or above the 'zero-line' does not work. If two alternatives $x_1$ and $x_2$ tie with respect to the 'head-count' criterion, it may seem desirable to break this tie by giving lexicographic priority to the welfare of those persons whose welfare levels are below the 'zero-line' under *both* $x_1$ *and* $x_2$. However, such a rule will sometimes produce cyclical social orderings: Consider the following profile for $N = \{1, 2, 3\}$ and $X = \{x_1, x_2, x_3\}$.

$$W_1(x_1) > 0 > W_1(x_2) > W_1(x_3)$$
$$W_2(x_2) > 0 > W_2(x_3) > W_2(x_1)$$
$$W_3(x_3) > 0 > W_3(x_1) > W_3(x_2)$$

Clearly, each pair of alternatives in $X$ ties with respect to the 'head-count' criterion. Now suppose that we define a tie-breaking rule which, for a given pair of tied alternatives, gives lexicographic priority to the welfare of those persons whose welfare-level is below the 'zero-line' under *both* of these alternatives. In the present example, there exists a unique person of this description for each pair of alternatives. Thus we must have $x_1 P x_2$ (tie broken by person 3), $x_2 P x_3$ (tie broken by person 1) and $x_3 P x_1$ (tie broken by person 2), a cycle.

Is some type of lexicographic dictatorship the only way of breaking ties? The following classification theorem provides an answer to this question with respect to an important class of ties.

**THEOREM 3.2.** Suppose $F$ satisfies (ONC+0), (U), (I) and (SP). Then, for each $(\delta_1, \delta_2, ..., \delta_n) \in \{-1, 0, 1\}^n$, there exists a permutation $\sigma$ of $N$ with the following property: for any $\{W_i\} \in \mathbf{W}^n$ and any $x_1, x_2 \in X$ such that, for each $i$, $sign(W_i(x_1)) = sign(W_i(x_2)) = \delta_i$,

$$x_1 P x_2 \text{ if and only if } \quad W_{\sigma(i)}(x_1) > W_{\sigma(i)}(x_2) \text{ for some } i \in N$$
$$\text{and} \quad W_{\sigma(j)}(x_1) = W_{\sigma(j)}(x_2) \text{ for all } j < i,$$

where $R = F(\{W_i\})$.

**Proof.** Suppose $F$ satisfies (ONC+0), (U), (I) and (SP). Let $(\delta_1, \delta_2, ..., \delta_n) \in \{-1, 0, 1\}^n$ be given.

In step (1), we will use $F$ to construct a SWFL, $G : \mathbf{W}^m \rightarrow$ *set of transitive social orderings*, that satisfies the conditions of theorem **2.3.**. In step (2), theorem **2.3.** will then enable us to determine the structure of $G$; and in step (3), we will infer the structure of $F$ from the structure of $G$.

**(1)** Partition $N$ into $\{i_1, i_2, ..., i_m\}$ and $\{j_1, j_2, ..., j_{n-m}\}$ such that, for all $i$ in the former set, $\delta_i \neq 0$, and, for all $i$ in the latter set, $\delta_i = 0$. Construct $G : \mathbf{W}^m \rightarrow$ *set of transitive social orderings* as follows: for each $\{W_i\} \in \mathbf{W}^m$, define $G(\{W_1, W_2, ..., W_m\}) = F(\{W^*_1, W^*_2, ..., W^*_n\})$, where, for each $i$,

$$W^*_i = \begin{cases} exp(W_j), \text{ with } j \text{ such that } i_j = i & \text{if } \delta_i = 1 \\ 0 & \text{if } \delta_i = 0 \\ -exp(-W_j), \text{ with } j \text{ such that } i_j = i & \text{if } \delta_i = -1. \end{cases}$$

Since $F$ satisfies (U) and (I), $G$ clearly satisfies (U) and (I) too. To see that $G$ satisfies (SP), consider any $\{W_i\} \in \mathbf{W}^m$, and let $\{W^*_i\} \in \mathbf{W}^n$ be the above defined profile such that $R = F(\{W^*_i\}) = G(\{W_i\})$. Note that, for all $i \in \{1, 2, ..., m\}$, $W^*_i$ is either $exp(W_j)$ or $-exp(-W_j)$ for some $j \in \{1, 2, ..., m\}$ *or* 0. Now suppose that $x_1, x_2 \in X$ are alternatives such that, for all $i \in \{1, 2, ..., m\}$, $W_i(x_1) \geq W_i(x_2)$. By the definition of $\{W^*_i\}$, it then follows that $W^*_i(x_1) \geq W^*_i(x_2)$ for all $i \in N$, and since $F$ satisfies (SP), $x_1 R x_2$. Moreover, if in addition $W_k(x_1) > W_k(x_2)$ for some $k \in \{1, 2, ..., m\}$, the corresponding $W^*_i$, with $i = i_k$, equals either $exp(W_k)$ or $-exp(-W_k)$, whence $W^*_i(x_1) > W^*_i(x_2)$, and hence $x_1 P x_2$; and so $G$ satisfies (SP) too.

We shall now prove that $G$ satisfies (ONC). For any $\{W_i\} \in \mathbf{W}^m$ and any $m$-tuple of positive monotonic transformations $\{\phi_i\}$, first define an $n$-tuple of positive monotonic and sign-preserving transformations $\{\phi^*_i\}$ as follows: for each $i \in \{1, 2, ..., n\}$,

$$\phi^*_i(t) = \begin{cases} \text{if } \delta_i = 1: \begin{cases} exp(\phi_j(ln(t))), \text{ with } j \text{ such that } i_j = \text{i} & \text{for } t>0 \\ t & \text{for } t\leq 0 \end{cases} \\ \text{if } \delta_i = 0: t \\ \text{if } \delta_i = -1: \begin{cases} t & \text{for } t\geq 0 \\ -exp(-\phi_j(-ln(-t))), \text{ with } j \text{ such that } i_j = i & \text{for } t<0. \end{cases} \end{cases}$$

Now

$$\begin{aligned} G(\{W_i\}) &= F(\{W^*_i\}) && \text{(with } \{W^*_i\} \text{ as defined above)} \\ &= F(\{\phi^*_i(W^*_i)\}) && \text{(since } \{\phi^*_i\} \text{ is an n-tuple of positive monotonic and sign-preserving transformations, and } F \text{ satisfies (ONC+0)).} \end{aligned}$$

But, for each $i$,

$$\phi^*_i(W^*_i) = \begin{cases} \phi^*_j(exp(W_j)), \text{ with } j \text{ such that } i_j = i & \text{if } \delta_i = 1 \\ \phi^*_j(0) & \text{if } \delta_i = 0 \\ \phi^*_j(-exp(-W_j)), \text{ with } j \text{ such that } i_j = i & \text{If } \delta_i = -1 \end{cases}$$

$$= \begin{cases} exp(\phi_j(ln(exp(W_j)))), \text{ with } j \text{ such that } i_j = i & \text{if } \delta_i = 1 \\ 0 & \text{if } \delta_i = 0 \\ -exp(-\phi_j(-ln(-(-exp(-W_j))))), \text{ with } j \text{ such that } i_j = i & \text{if } \delta_i = -1 \end{cases}$$

$$= \begin{cases} exp(\phi_j(W_j)), \text{ with } j \text{ such that } i_j = i & \text{if } \delta_i = 1 \\ 0 & \text{if } \delta_i = 0 \\ -\text{exp}(-\phi_j(W_j)), \text{ with } j \text{ such that } i_j = i & \text{if } \delta_i = -1 \end{cases}$$

$$= V^*_i,$$

where $\{V^*_i\}\in\mathbf{W}^n$ is the profile corresponding to $\{V_i\} = \{\phi_i(W_i)\}\in\mathbf{W}^m$ such that $G(\{V_i\}) = F(V^*_i)$. But then $G(\{\phi_i(W_i)\}) = G(\{V_i\}) = F(\{V^*_i\}) = F(\{\phi^*_i(W^*_i)\}) = G(\{W_i\})$ as required.
**(2)** Since $G$ satisfies all the conditions of theorem **2.3.**, there exists a permutation $\sigma$ of $\{1, 2, ..., m\}$ such that, for any $\{W_i\}\in\mathbf{W}^m$ and any $x_1, x_2\in X$,

$$\begin{aligned} x_1Px_2 \text{ if and only if } \quad & W_{\sigma(i)}(x_1) > W_{\sigma(i)}(x_2) \text{ for some } i\in\{1, 2, ..., m\} \\ \text{and} \quad & W_{\sigma(j)}(x_1) = W_{\sigma(j)}(x_2) \text{ for all } j<i, \end{aligned}$$

where $R = G(\{W_i\})$.

**(3)** Define a permutation $\pi$ of $N$ as follows: for each $j\in N$,

$$\pi(j) = \begin{cases} i_{\sigma(j)} & \text{if } j\in \{1, 2, ..., m\} \\ j_{j-m} & \text{if } j\in \{m+1, m+2, ..., n\} \end{cases}$$

We will now show that, for any $\{W_i\}\in\mathbf{W}^n$ and any $x_1, x_2\in X$ such that, for each $i$, $sign(W_i(x_1)) = sign(W_i(x_2)) = \delta_i$,

$$x_1Px_2 \text{ if and only if } \quad W_{\pi(i)}(x_1) > W_{\pi(i)}(x_2) \text{ for some } i\in N$$
$$\text{and} \quad W_{\pi(j)}(x_1) = W_{\pi(j)}(x_2) \text{ for all } j<i,$$

where $R = F(\{W_i\})$.

Given any $\{W_i\}\in\mathbf{W}^n$ and any $x_1, x_2\in X$ such that, for each $i$, $sign(W_i(x_1)) = \delta_i = sign(W_i(x_2))$, first define $\{W'_i\}$ as follows: for each $i$, $W'_i(x_1) = W_i(x_1)$ and $W'_i(x_2) = W_i(x_2)$ and, for all $y\neq x_1, x_2$, $W'_i(y) = \delta_i$. Since $F$ satisfies $I$, $x_1Px_2$ if and only if $x_1P'x_2$, where $R = F(\{W_i\})$ and $R' = F(\{W'_i\})$.

Now define $\{V_i\}\in\mathbf{W}^m$ as follows: for each $j$,

$$V_j = \begin{cases} ln(W'_{i_j}) & \text{if } \delta_{i_j} = 1 \\ -ln(-W'_{i_j}) & \text{if } \delta_{i_j} = -1, \end{cases}$$

and $G(\{V_i\}) = F(\{W'_i\}) = R'$. But, by (2), we know that

$$x_1P'x_2 \text{ if and only if } \quad V_{\sigma(i)}(x_1) > V_{\sigma(i)}(x_2) \text{ for some } i\in \{1, 2, ..., m\}$$
$$\text{and} \quad V_{\sigma(j)}(x_1) = V_{\sigma(j)}(x_2) \text{ for all } j<i,$$

and, by the definition of $\{V_i\}$ and $\pi$, this holds

$$\text{if and only if } \quad W'_{\pi(i)}(x_1) > W'_{\pi(i)}(x_2) \text{ for some } i\in \{1, 2, ..., m\}$$
$$\text{and} \quad W'_{\pi(j)}(x_1) = W'_{\pi(j)}(x_2) \text{ for all } j<i.$$

But since $W'_{\pi(j)}(x_1) = 0 = W'_{\pi(j)}(x_2)$ for all $j>m$, this holds

$$\text{if and only if } \quad W'_{\pi(i)}(x_1) > W'_{\pi(i)}(x_2) \text{ for some } i\in N$$
$$\text{and} \quad W'_{\pi(j)}(x_1) = W'_{\pi(j)}(x_2) \text{ for all } j<i,$$

and, finally, since, for all $i$, $W'_i(x_1) = W_i(x_1)$ and $W'_i(x_2) = W_i(x_2)$, the desired result follows. **Q.E.D.**

Given a SWFL satisfying (ONC+0), (U), (SP) and (I), theorem **3.2.** states that whenever the same persons are above, on, and below the 'zero-line' under both $x_1$ and $x_2$ -- formally $sign(W_i(x_1)) = sign(W_i(x_2)) = \delta_i$ for all $i \in N$ and a fixed $(\delta_1, \delta_2, ..., \delta_n) \in \{-1, 0, 1\}^n$ -- the relative ranking of $x_1$ and $x_2$ is determined by a lexicographic dictatorship; if we use (P) instead of (SP), small modifications in the proof of theorem **3.2.** are sufficient to establish that a dictatorship as defined in condition (D) must determine the relative ranking of $x_1$ and $x_2$.

Given this classification result, it is also easy to see that our possibility result under (ONC+0) is rather limited. As soon as we replace the condition of non-dictatorship with the slightly stronger, but -- arguably -- equally plausible, condition of anonymity, an impossibility result reappears.

**ANONYMITY (A).** For any $\{W_i\}$ in the domain of $F$ and any permutation $\sigma$ of $N$, $F(\{W_i\}) = F(\{W_{\sigma(i)}\})$.

**THEOREM 3.3.** There exist no SWFLs satisfying (ONC+0), (U), (P), (I) and (A).

**Proof.** Assume, for a contradiction, that $F$ is a SWFL which satisfies (ONC+0), (U), (P), (I) and (A). We will construct a SWFL which satisfies the conditions of theorem **2.1.**.
Let $(\delta_1, \delta_2, ..., \delta_n) = (1, 1, ..., 1)$, and define a corresponding SWFL, $G : \mathbf{W}^m \rightarrow$ *set of transitive social orderings*, as in the proof of theorem **3.2.** (note that, for the present definition of the $\delta_i$, $m = n$). As before, $G$ satisfies (ONC), (U), (I) and -- with small modifications in the previous proof -- (P). We shall now show that $G$ also satisfies (A). Let $\sigma$ be any permutation of $N$. Then

$$\begin{aligned} G(\{W_{\sigma(i)}\}) &= F(\{exp(W_{\sigma(i)})\}) \\ &= F(\{exp(W_i)\}) \text{ (since } F \text{ satisfies (A))} \\ &= G(\{W_i\}) \text{ as required.} \end{aligned}$$

But since $G$ satisfies (A), it also satisfies (D), whence $G$ satisfies (ONC), (U), (P), (I) and (D). This contradicts theorem **2.1.**. **Q.E.D.**

In analogy to theorems **2.4.** to **2.7.**, we will now explore the consequences of moving from transitivity to quasi-transitivity and ask how robust the impossibility result of theorem **3.3.** is with regard to a relaxation of the requirement that the social orderings generated by a SWFL be transitive. Theorem **3.3.** itself fails to persist.

**THEOREM 3.4.** There exist SWFLs generating quasi-transitive social orderings which satisfy (ONC+0), (U), (P), (I) and (A).

**Proof.** The following SWFL satisfies the required properties: for each $\{W_i\}$, let $F(\{W_i\})$ be the ordering $R$ such that, for all $x_1, x_2 \in X$,

$$
\begin{aligned}
x_1 R x_2 \text{ if and only if } \quad & N^-(\{W_i(x_1)\}) < N^-(\{W_i(x_2)\}) \\
\text{or} \quad & [N^-(\{W_i(x_1)\}) = N^-(\{W_i(x_2)\}) \text{ and } N^0(\{W_i(x_1)\}) < N^0(\{W_i(x_2)\})] \\
\text{or} \quad & [N^-(\{W_i(x_1)\}) = N^-(\{W_i(x_2)\}) \text{ and } N^0(\{W_i(x_1)\}) = N^0(\{W_i(x_2)\} \\
& \text{ and not } x_2 P_N(\{W_i\}) x_1].
\end{aligned}
$$

**Q.E.D.**

Moreover, it is easily seen that condition (O), which was sufficient to reinstate an impossibility result under (ONC), does *not* rule out the SWFL defined in the proof of theorem **3.4.** (even the SWFL defined in the proof of theorem **3.1.** satisfies (O) -- in the SWFLs of both proofs no member of $N$ has a veto). Nonetheless, an oligarchy acts as the tie-breaker in this SWFL, but this oligarchy consists of all members of $N$ (implying that, *when it comes to breaking ties*, every member of $N$ has a veto -- we will return to this issue in proposition **3.6.**).

In analogy to theorem **3.2.**, we will now show that, for an important class of ties, any SWFL generating quasi-transitive social orderings which satisfies (U), (SP) and (I) must use suitable lexicographic oligarchies as tie-breakers.

**THEOREM 3.5.** Suppose $F$ is a SWFL generating quasi-transitive social orderings which satisfies (ONC+0), (U), (I) and (SP). Then, for each $(\delta_1, \delta_2, ..., \delta_n) \in \{-1, 0, 1\}^n$, there exists a partition of $N$ into $S_1, S_2, ..., S_k$ (for some $k$ dependent upon $F$ and $(\delta_1, \delta_2, ..., \delta_n)$) with the following property: for any $\{W_i\} \in \mathbf{W}^n$ and any $x_1, x_2 \in X$ such that, for each $i$, $sign(W_i(x_1)) = sign(W_i(x_2)) = \delta_i$,

$$
\begin{aligned}
x_1 P x_2 \text{ if and only if } \quad & x_1 P_{S_i}(\{W_i\}) x_2 \text{ for some } i \in \{1, 2, ..., k\} \\
\text{and} \quad & x_1 I_{S_j}(\{W_i\}) x_2 \text{ for all } j < i.
\end{aligned}
$$

where $R = F(\{W_i\})$.

**Proof.** Suppose $F$ is a SWFL generating quasi-transitive social orderings satisfying (ONC+0), (U), (I) and (SP). Let $(\delta_1, \delta_2, ..., \delta_n) \in \{-1, 0, 1\}^n$ be given.

In analogy to our proof of thereom **3.2.**, step (1) will be to use $F$ to construct a SWFL, $G$ :

$\mathbf{W}^m \rightarrow$ *set of quasi-transitive social orderings*, that satisfies the conditions of theorem **2.7.**. In step (2), we will apply theorem **2.7.** to determine the structure of $G$; in step (3), we will again infer the structure of $F$ from the structure of $G$.

**(1)** Partition $N$ into $\{i_1, i_2, ..., i_m\}$ and $\{j_1, j_2, ..., j_{n-m}\}$, and construct a SWFL, $G : \mathbf{W}^m \rightarrow$ *set of quasi-transitive social orderings*, as in the proof of theorem **3.2.** (the only difference is that, in the present case, $G$ generates quasi-transitive social orderings). Again, $G$ satisfies (ONC), (U), (I) and (SP).

**(2)** Since $G$ satisfies the conditions of theorem **2.7.**, $\{1, 2, ..., m\}$ can be partitioned into $S_1$, $S_2$, ..., $S_k$ (for some $k$) such that, for any $\{W_i\}\in\mathbf{W}^m$ and any $x_1, x_2\in X$,

$$x_1 P x_2 \text{ if and only if } \quad x_1 P_{S_i}(\{W_i\}) x_2 \text{ for some } i\in\{1, 2, ..., k\}$$
$$\text{and} \quad x_1 I_{S_j}(\{W_i\}) x_2 \text{ for all } j<i.$$

where $R = G(\{W_i\})$.

**(2)** Partition $N$ into $T_1, T_2, ..., T_{k+1}$ as follows: for each $i\in\{1, 2, ..., k\}$, $T_i = \{i_j : j\in S_i\}$, and $T_{k+1} = \{j_1, j_2, ..., j_{n-m}\}$.

We will now show that, for any $\{W_i\}\in\mathbf{W}^n$ and any $x_1, x_2\in X$ such that, for each $i$, $sign(W_i(x_1)) = sign(W_i(x_2)) = \delta_i$,

$$x_1 P x_2 \text{ if and only if } \quad x_1 P_{T_i}(\{W_i\}) x_2 \text{ for some } i\in\{1, 2, ..., k+1\}$$
$$\text{and} \quad x_1 I_{T_j}(\{W_i\}) x_2 \text{ for all } j<i.$$

where $R = F(\{W_i\})$.

Given any $\{W_i\}\in\mathbf{W}^n$ and any $x_1, x_2\in X$ such that, for each $i$, $sign(W_i(x_1)) = \delta_i = sign(W_i(x_2))$, define $\{W'_i\}$ as in the proof of theorem **3.2.** such that $x_1 P x_2$ if and only if $x_1 P' x_2$, where $R = F(\{W_i\})$ and $R' = F(\{W'_i\})$.

Now define $\{V_i\}\in\mathbf{W}^m$ as follows: for each $j$,

$$V_j = \begin{cases} ln(W'_{i_j}) & \text{if } \delta_{i_j} = 1 \\ -ln(-W'_{i_j}) & \text{if } \delta_{i_j} = -1. \end{cases}$$

Then $G(\{V_i\}) = F(\{W'_i\}) = R'$. But, by (2), we know that

$$x_1 P' x_2 \text{ if and only if } \quad x_1 P_{S_i}(\{V_i\}) x_2 \text{ for some } i\in\{1, 2, ..., k\}$$
$$\text{and} \quad x_1 I_{S_j}(\{V_i\}) x_2 \text{ for all } j<i.$$

By the definition of $\{V_i\}$ and of the $T_i$, this holds

if and only if $x_1 P_{T_i}(\{W'_i\}) x_2$ for some $i \in \{1, 2, ..., k\}$
and $x_1 I_{T_j}(\{W'_i\}) x_2$ for all $j<i$.

But since $W'_j(x_1) = 0 = W'_j(x_2)$ for all $j \in T_{k+1}$, this holds

if and only if $x_1 P_{T_i}(\{W'_i\}) x_2$ for some $i \in \{1, 2, ..., k+1\}$
and $x_1 I_{T_j}(\{W'_i\}) x_2$ for all $j<i$,

and, finally, since, for all $i$, $W'_i(x_1) = W_i(x_1)$ and $W'_i(x_2) = W_i(x_2)$, $x_1 P_{T_i}(\{W'_i\}) x_2$ if and only if $x_1 P_{T_i}(\{W_i\}) x_2$ and $x_1 I_{T_i}(\{W'_i\}) x_2$ if and only if $x_1 I_{T_i}(\{W_i\}) x_2$, and the desired result follows. **Q.E.D.**

Given a SWFL generating quasi-transitive social orderings which satisfies (ONC+0), (U), (SP) and (I), theorem **3.5.** states that whenever the same persons are above, on, and below the 'zero-line' under both $x_1$ and $x_2$ -- formally $sign(W_i(x_1)) = sign(W_i(x_2)) = \delta_i$ for each $i \in N$ and a fixed $(\delta_1, \delta_2, ..., \delta_n) \in \{-1, 0, 1\}^n$ -- the relative ranking of $x_1$ and $x_2$ is determined by a lexicographic oligarchy. Again, if we use (P) instead of (SP), small modifications in the proof of theorem **3.5.** are sufficient to establish that an oligarchy as defined in condition (O) must determine the relative ranking of $x_1$ and $x_2$.

Now it is easy to see that, if we impose condition (A), the ties covered by theorem **3.5.** must always be broken by an oligarchy consisting of the whole of $N$ (i.e. the tie-breaking lexicographic oligarchy is defined by partitioning $N$ into the singleton partition $S_1 = N$). The following result follows immediately from this observation:

**PROPOSITION 3.6.** Suppose $F$ is a SWFL generating quasi-transitive social orderings which satisfies (ONC+0), (U), (I), (SP) and (A). Then, for each $(\delta_1, \delta_2, ..., \delta_n) \in \{-1, 0, 1\}^n$, any $\{W_i\} \in \mathbf{W}^n$ and any $x_1, x_2 \in X$ such that, for each $i$, $sign(W_i(x_1)) = sign(W_i(x_2)) = \delta_i$, the following holds: if $x_1$ and $x_2$ are Pareto-incomparable (or Pareto-indifferent) (i.e. neither $x_1 P_N(\{W_i\}) x_2$ nor $x_2 P_N(\{W_i\}) x_1$), $F$ must make $x_1$ and $x_2$ socially indifferent.

Finally, if we introduce condition (PR), an impossibility result reemerges:

**THEOREM 3.7.** Under (ONC+0), any SWFL generating quasi-transitive social orderings which satisfies (U), (P), (I), (A) must violate (PR), if $n \geq 3$.

**Proof.** Assume, for a contradiction, that $F$ is a SWFL generating quasi-transitive social

orderings which satisfies (ONC+0), (U), (P), (I), (A) and (PR), where $n\geq 3$. We will construct a SWFL which satisfies the conditions of theorem **2.5.**. Define $G$, this time generating quasi-transitive social orderings, as in the proof of theorem **3.3.** (again $G$ is defined on $\mathbf{W}^n$, since $m=n$). As before, $G$ satisfies (ONC), (U), (P), (I), (A) and hence (D). We will show that $G$ satisfies (PR) too. Let $\{W_i\}$ and $\{W'_i\}\in\mathbf{W}^n$, $x_1$, $x_2\in X$ and $j\in N$ such that, for all $i\in N$, with $i\neq j$, $W_i = W'_i$, and *either* $[W_j(x_1)<W_j(x_2)$ and $W'_j(x_1)\geq W'_j(x_2)]$ *or* $[W_j(x_1)=W_j(x_2)$ and $W'_j(x_1)>W'_j(x_2)]$. This implies that, for all $i\in N$, with $i\neq j$, $exp(W_i)=exp(W'_i)$, and *either* $[exp(W_j(x_1))<exp(W_j(x_2))$ and $exp(W'_j(x_1))\geq exp(W'_j(x_2))]$ *or* $[exp(W_j(x_1))=exp(W_j(x_2))$ and $exp(W'_j(x_1))>exp(W'_j(x_2))]$. Since $G$ is defined by $G(\{W_i\}) = F(\{exp(W_i)\})$, for each $\{W_i\}$, and $F$ satisfies (PR), $x_1Rx_2$ implies $x_1P'x_2$, where $R = G(\{W_i\}) = F(\{exp(W_i)\})$ and $R' = G(\{W'_i\}) = F(\{exp(W'_i)\})$. Thus $G$ satisfies (PR) as required.
But then $G$ is a SWFL on $\mathbf{W}^n$ generating quasi-transitive social orderings which satisfies (ONC), (U), (P), (I), (D) and (PR), where $n\geq 3$. This contradicts theorem **2.5.**. **Q.E.D.**

Note that condition (A) is crucial to this result, because the SWFL of theorem **3.1.** satisfies all of (ONC+0), (U), (P), (I) and (D) as well as (PR), and certainly generates quasi-transitive social orderings (by virtue of generating transitive ones).

**4. Conclusion**

We have seen that, even in the narrow informational framework of ordinal measurability without interpersonal comparability of welfare levels or units, the introduction of a 'zero-line' of welfare points towards an escape-route from Arrow's impossibility theorem, though a limited one. In particular, we are in a position to draw the following conclusions:

(i) Different versions of the 'head-count' rule satisfy Arrow's conditions under the informational framework of (ONC+0).

(ii) If we require social orderings to be transitive, none of these rules satisfies anonymity, and, for an important class of ties, the only way of breaking these ties in accordance with Arrow's conditions is by installing a suitable (lexicographic) dictatorship, which can, however, vary depending on the number of people below, on, or above the 'zero-line'.

(iii) If we only require social orderings to be quasi-transitive, there are rules satisfying anonymity. Irrespective of whether or not we demand anonymity, the only way of breaking an important class of ties is by installing a suitable (lexicographic) oligarchy, which can again vary depending on the number of people below, on, or above the 'zero-line'. If we insist on anonymity, however, Pareto-incomparable alternatives *amongst these ties* must be made socially indifferent. Moreover, if we impose positive responsiveness in addition to anonymity, an impossibility result reemerges.

The results of the present note once again confirm Sen's important insight that Arrow's theorem should be interpreted not exclusively as a result about the impossibility of designing particular types of collective decision mechanisms, but also as a result about the inherent informational limitations of the pure ordinalist framework. Nonetheless, possibility results that use richer informational frameworks require further clarification of the operational accessibility of the relevant types of information.

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