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Long-Term Optimal Investment in Matrix Valued Factor Models*

Scott Robertson† and Hao Xing‡

Abstract. Long horizon optimal investment problems are studied in a factor model with matrix valued state variables. Explicit parameter restrictions are obtained under which, for an isoelastic investor, the finite horizon value function and optimal strategy converge to their long-run counterparts as the investment horizon approaches infinity. Additionally, portfolio turnpikes are obtained in which finite horizon optimal strategies for general utility functions converge to the long-run optimal strategy for isoelastic utility. By using results on large time behavior of semilinear partial differential equations, our analysis extends, to a nonaffine setting, affine models where the Wishart process drives investment opportunities.

Key words. portfolio choice, long-run, risk sensitive control, portfolio turnpike, Wishart process

AMS subject classifications. 91G10, 60J60, 49J20, 35K58

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1. Introduction.

When investment opportunities are stochastic and the market is incomplete, optimal strategies in portfolio choice problems rarely admit explicit forms. The primary difficulty is that hedging demand depends implicitly upon the investment horizon. This motivates the approximation of optimal policies, and one useful approximation occurs in the limit of a long investment horizon. Long horizon analysis removes dependence on the investment horizon, but still maintains the relationship between investor preferences, underlying economic factors, and dynamic asset demand. Long-run approximations typically take two forms: first, the *long-run optimal investment* or *risk sensitive control* problem identifies growth optimal policies for isoelastic utilities; second, the *portfolio turnpike* problem connects optimal policies for general utilities with those for a corresponding isoelastic utility.

In this article, in addition to studying the aforementioned two formulations of long-run problems, we are particularly concerned with connecting the finite horizon and long-run problems. Specifically, our goal is to prove convergence of finite horizon optimal policies to their long-run counterparts. Importance of this convergence has been emphasized by Buraschi, Porchia, and Trojani in [8]: "How do both optimal investment in risky assets and covariance hedging demand vary with respect to the investment horizon? This question is important for life-cycle decisions as well as for pension fund managers." From a theoretical point of

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view, confirmation of the long-run convergence is necessary to validate long-run analysis, and though heuristics often indicate convergence, from a mathematically rigorous standpoint it is not a priori clear that the long-run policies arise as the limit of finite horizon policies.

For isoelastic utilities, the risk sensitive control (or long-run optimal investment) problem has been addressed by many authors from different aspects: see [5, 6, 4, 19, 20, 39, 18, 44, 14, 26, 29]. When specified to a Markovian framework, these studies typically identify and analyze an ergodic Hamilton–Jacobi–Bellman (HJB) equation. The ergodic equation is usually obtained via a heuristic argument, where one first derives the finite horizon HJB equation, and then conjectures that for long horizons, the logarithm of the value function decomposes into a spatial component $\hat{v}$ and a growth rate component $\hat{\lambda}$. The HJB equation then arises by substituting $\hat{\lambda}T + \hat{v}(\cdot)$ into the finite horizon HJB equation.

The above derivation indicates that finite horizon and long-run problems are parallel. Of primary importance is to connect these two problems. Indeed, let $v(T, \cdot)$ be the logarithm of the value function for the problem with an investment horizon T. As T approaches infinity, does $v(T, \cdot) - \hat{\lambda}T - \hat{v}(\cdot)$ converge? If so, in what sense? Does the optimal strategy for the finite horizon problem converge to its long-run analogue? Affirmative answers to these questions verify the intuition underpinning the study of the risk sensitive controls and provide consistency between the finite and infinite horizon problems.

For general utility functions, portfolio turnpikes provide a useful approximation for optimal policies. Qualitatively, turnpike theorems state that in a growing market, when the investment horizon is far in the future, the optimal trading strategy of a generic utility is close to the optimal trading strategy of its isoelastic counterpart. Turnpike theorems were first investigated in [43] for utilities with affine risk tolerance and have since been extensively studied: in particular we mention [40, 49, 28, 32, 11, 34, 31, 17, 15]. Turnpike theorems suggest that for long horizons, a generic utility investor may use the associated optimal isoelastic strategy. In [25] it is shown that indeed, in the long run, an investor with generic utility can employ the associated optimal isoelastic portfolio with minimal loss of utility.

For the risk sensitive control and turnpike approximations, we summarize the desired relationship between the finite and long horizon problems in Statements 2.5 and 2.8, respectively. Verification of these statements confirms the convergence of value functions, optimal wealth processes, and optimal investment strategies for finite horizon problems to their long-run analogue. Each of Statements 2.5 and 2.8 have been proved in [24] in a univariate state variable model where, additionally, the hedgeable and unhedgeable shocks have constant correlation. The present paper extends these results to a matrix valued factor model setting while also allowing for stochastic correlations.

In this article, we work with a multiasset factor model where the state variable, as an autonomous diffusion, takes values in the space of positive definite matrices, thus generalizing the Wishart process [7]. We choose to work with matrix valued factor models for two reasons. First it is documented in [8] that matrix valued diffusions, used to model the covariance matrices of asset returns, can reproduce several empirical features and provide a natural framework to study the implications of stochastic correlation on portfolio choice. Second, as argued in [16], affine models on the canonical state space fail badly at forecasting future bond yields. Therefore matrix valued processes provide an alternative state variable for interest rates. Due to these reasons, matrix valued processes have been applied not only to option pricing [22, 23, 12, 13] but also to portfolio optimization. For portfolio optimization, beyond [8], in [30] the isoelastic problem is solved in the Wishart case via a matrix Riccati differential equation, [2] considers logarithmic utility, and [47] discusses indifference pricing.

In contrast to the aforementioned results, which exploit the affine structure of the Wishart process, our results rely upon large time asymptotic analysis of partial differential equations with quadratic nonlinearities in the gradient. Using the framework developed in [48], we are able to consider general, nonaffine, matrix valued state variable models. Moreover, stochastic correlation between the state variable and risky assets can be treated, whereas a special (constant) correlation structure is needed to ensure the affine structure. However, having only the analytic results in [48] is not enough. To link these analytic results to portfolio optimization problems, a verification result is essential. To this end, our Proposition 3.6 extends [30, Theorem 3.1] to the standard class of nonnegative wealth processes. In fact, the framework introduced in [48] can also be used to study models with vector valued factor models; see [48, section 3.1]. However, we do not pursue this direction in the present article, and instead use matrix valued models to illustrate our methodology.

We introduce explicit restrictions on the model parameters (cf. Assumption 3.1), under which Statements 2.5 and 2.8 hold for the general class of matrix valued factor models. This confirms numerical experiments in [8], which showed the hedging demand converges to a steady-state level when the remaining time to maturity is long. For isoelastic utility (i.e., when x^p/p, $0 \neq p < 1$), our parameter restrictions are mild when $p < 0$ and sharp when $0 < p < 1$. In particular, when our parameter restrictions (as well as a technical boundary case: see Example 3.2) are violated, the portfolio optimization problem with $0 < p < 1$ becomes ill-posed for horizons beyond some critical value, hence the associated long-run problem is ill-posed as well.

Our analysis, when applied to affine models, also yields new insight into the multivariate setting. Indeed, we construct a multivariate example (Example 3.12) where the affine nature of the value function and optimal policy depends entirely upon the dimension of the state variable in comparison with the number of assets. This phenomenon happens due to the noncommutative property of matrix product and hence does not appear in univariate models.

After the model and Statements 2.5 and 2.8 are introduced in section 2, the main results are presented in section 3. The key parameter restrictions are presented in Assumption 3.1 and are illustrated by three examples. The main convergence results are then presented: see Theorem 3.8 for the long-run limit results and Theorem 3.10 for the turnpike results. All proofs are deferred to appendices. Finally, we summarize several notations used throughout the paper:

- $\mathbb{M}^{d\times k}$ is the space of $d \times k$ matrices with $\mathbb{M}^d := \mathbb{M}^{d\times d}$. For $x \in \mathbb{M}^{d\times k}$, x' is the transpose of x. For $x \in \mathbb{M}^d$, $\mathrm{Tr}\,(x)$ is the trace of x and $\|x\| = \sqrt{\mathrm{Tr}(x'x)}$. $\mathbb{1}_d$ is the identity matrix in $\mathbb{M}^d$ and 1_d the d-dimensional vector with each component 1.
- $\mathbb{S}^d$ is the space of $d \times d$ symmetric matrices, and $\mathbb{S}^d_{++}$ the subset of positive definite matrices. For $x \in \mathbb{S}^d_{++}$, $\sqrt{x}$ is the unique $y \in \mathbb{S}^d_{++}$ such that $y^2 = x$. For $x, y \in \mathbb{S}^d_{++}$, $x \geq y$ when $x - y$ is positive semidefinite.
- For $E \subset \mathbb{M}^{d\times k}$, $F \subset \mathbb{M}^{m\times n}$, and $\gamma \in (0, 1]$, $C^{\ell,\gamma}(E; F)$ is the space of ℓ times continuously differentiable functions from E to F whose derivatives of order up to ℓ is locally Hölder continuous with exponent γ.

2. Setup. Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ be a filtered probability space with $(\mathcal{F}_t)_{t\geq 0}$ a right-continuous filtration. Following the treatment in [24], all N-negligible sets are included into $\mathcal{F}_0$.[1] Consider a financial model with one risk-free asset S^0 and n risky assets $(S^1, \ldots, S^n)$. Investment opportunities are driven by an $\mathbb{S}^d_{++}$-valued state variable X, which is described below.

2.1. A $\mathbb{S}^d_{++}$-valued state variable. Let $B = (B^{ij})_{i,j=1,\ldots d}$ be an $\mathbb{M}^d$-valued standard Brownian motion on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$. The state variable X is an extension of the Wishart process (see [7]). More precisely, X has dynamics

$$dX_t = b(X_t)dt + \sqrt{X_t}dB_t\Lambda' + \Lambda dB_t'\sqrt{X_t}', \qquad X_0 \in \mathbb{S}^d_{++}. \tag{2.1}$$

Here, $b \in C^{1,\gamma}(\mathbb{S}^d_{++}; \mathbb{S}^d)$ is a given function and $\Lambda \in \mathbb{M}^d$ is constant. When b is a linear function of X, we recover the Wishart process; see Example 2.2 below. For general b, we require that (2.1) admits a unique strong solution taking values in $\mathbb{S}^d_{++}$, i.e., $\mathbb{P}^x\left[X_t \in \mathbb{S}^d_{++} \text{ for all } t \geq 0\right] = 1$, for all $x \in \mathbb{S}^d_{++}$, where $\mathbb{P}^x$ is the probability such that $X_0 = x$ a.s. To enforce this requirement through restrictions upon b and Λ, the results as well as notation of [42] are used. Namely, given $\delta \in \mathbb{R}$, define $H_\delta : \mathbb{S}^d_{++} \to \mathbb{R}$ via

$$H_\delta(x; b) := \operatorname{Tr}\left(\left(b(x) - (1 + \delta + d)\Lambda\Lambda'\right) x^{-1}\right), \qquad x \in \mathbb{S}^d_{++}. \tag{2.2}$$

Here, we have explicitly identified the drift function b in H_δ, since in what follows H_δ will be used with various b. To understand H_δ, note that Itô's formula implies the drift of $\log(\det(X_t))$ is $H_0(X_t; b)$. Thus, the following assumption ensures that (2.1) admits a unique $\mathbb{S}^d_{++}$-valued strong solution $(X_t)_{t\in[0,\infty)}$; cf. [42, Theorem 3.4].

Assumption 2.1. *$\Lambda\Lambda' > 0$, b is locally Lipschitz and of linear growth, and $\inf_{x\in\mathbb{S}^d_{++}} H_0(x; b) > -\infty$.*

Note that (2.1) is shorthand for the following system:

$$dX^{ij}_t = b_{ij}(X_t)dt + \sum_{k,l=1}^{d} \left(\sqrt{X_t}\right)_{ik} dB^{kl}_t \Lambda_{jl} + \sum_{k,l=1}^{d} \left(\sqrt{X_t}\right)_{jk} dB^{kl}_t \Lambda_{il}, \qquad i, j = 1, \ldots, d.$$

For $i, j = 1, \ldots, d$, define $a^{ij} : \mathbb{S}^d_{++} \to \mathbb{M}^d$ by

$$a^{ij}_{kl}(x) := (\sqrt{x})_{ik}\Lambda_{jl} + (\sqrt{x})_{jk}\Lambda_{il}, \qquad k, l = 1, \ldots, d, \; x \in \mathbb{S}^d_{++}.$$

Then the above system takes the form

$$dX^{ij}_t = b_{ij}(X_t)dt + \operatorname{Tr}\left(a^{ij}(X_t)dB_t'\right), \tag{2.3}$$

[1] A subset A of Ω is N-negligible if there exists a sequence $(B_n)_{n\geq 0}$ of subsets of Ω such that for all $n \geq 0$, $B_n \in \mathcal{F}_n$, $\mathbb{P}[B_n] = 0$, and $A \subset \cup_{n\geq 0} B_n$. This notion is introduced in [3, Definition 1.3.23] and [45]. Such completion ensures, for all $T \geq 0$, the space $(\Omega, \mathcal{F}_T, (\mathcal{F})_{0\leq t\leq T}, \mathbb{P})$ satisfies the usual conditions. Hence all references below on finite horizon problems with completed filtration can be used in this paper.

and a direct calculation using $\Lambda\Lambda' > 0$ shows that for any $x \in \mathbb{S}^d_{++}$ and $\theta \in \mathbb{S}^d$,

$$\sum_{i,j,k,l=1}^{d} \theta_{ij} \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right)(x)\theta_{kl} = 4\mathrm{Tr}\left(x\theta\Lambda\Lambda'\theta\right) \geq c(x)\left\|\theta\right\|^2 \tag{2.4}$$

for some constant $c(x) > 0$. Therefore volatility of X is nondegenerate in the interior of $\mathbb{S}^d_{++}$.

Example 2.2. The primary example to keep in mind is when X is the Wishart process (cf. [7]):

$$dX_t = \left(LL' + KX_t + X_tK'\right)dt + \sqrt{X_t}dB_t\Lambda' + \Lambda dB_t'\sqrt{X_t}, \tag{2.5}$$

where $K, L \in \mathbb{M}^d$. Then Assumption 2.1 is satisfied when

$$LL' \geq (d+1)\Lambda\Lambda' > 0. \tag{2.6}$$

Indeed, calculation shows that $H_0(x;b) = \mathrm{Tr}\left((LL' - (d+1)\Lambda\Lambda')x^{-1}\right) + 2\mathrm{Tr}\,(K)$. Thus, (2.6) implies $H_0(x;b) \geq 2\mathrm{Tr}\,(K)$ on $\mathbb{S}^d_{++}$, then Assumption 2.1 holds.

2.2. The financial model. Having fixed notation and established well-posedness for the state variable, we may now define the financial model. As mentioned above, there is one risk-free asset S^0 and n risky assets $(S^1, \ldots, S^n)$ whose dynamics are given by

$$\frac{dS_t^0}{S_t^0} = r(X_t)dt, \qquad S_0^0 = 1, \tag{2.7}$$

$$\frac{dS_t^i}{S_t^i} = \left(r(X_t) + \mu_i(X_t)\right)dt + \sum_{j=1}^{m}\sigma_{ij}(X_t)dZ_t^j, \qquad S_0^i > 0, \qquad i = 1, \ldots, n. \tag{2.8}$$

Here, $r \in C^\gamma(\mathbb{S}^d_{++};\mathbb{R})$, $\mu \in C^{1,\gamma}(\mathbb{S}^d_{++};\mathbb{R}^n)$, $\sigma \in C^{2,\gamma}(\mathbb{S}^d_{++};\mathbb{M}^{n\times m})$, and $Z = (Z^1, \ldots, Z^m)$ is an $\mathbb{R}^m$ valued Brownian motion. That σ is of full rank as well as the existence of *mean variance ratio*, i.e., $\nu : \mathbb{S}^d_{++} \to \mathbb{R}^n$ such that $\mu = \sigma\sigma'\nu$ on $\mathbb{S}^d_{++}$, are ensured by the following assumption.

Assumption 2.3.

(i) *When* $m \geq n$, $\Sigma(x) := \sigma\sigma'(x) > 0$ *for* $x \in \mathbb{S}^d_{++}$. *Then* $\nu := \Sigma^{-1}\mu$.

(ii) *When* $m < n$, $\sigma'\sigma(x) > 0$ *for* $x \in \mathbb{S}^d_{++}$ *and there exists* $\nu \in C^{1,\gamma}(\mathbb{S}^d_{++};\mathbb{R}^n)$ *such that* $\mu = \Sigma\nu$.

When $m = n$, σ can be chosen as $\sqrt{\Sigma}$ without loss of generality, so that the assumptions in (i) and (ii) coincide.

To allow for potentially stochastic instantaneous correlations between asset returns and the state variable, we define Z in terms of the Brownian motion B which drives X and an independent $\mathbb{R}^m$-valued Brownian motion W. Specifically, let $C \in C^{2,\gamma}(\mathbb{S}^d_{++};\mathbb{M}^{m\times d})$ and $\rho \in C^{2,\gamma}(\mathbb{S}^d_{++};\mathbb{R}^d)$ be such that the following holds.

Assumption 2.4. *$\rho'\rho(x)CC'(x) \leq \mathbb{1}_m$ for each $x \in \mathbb{S}^d_{++}$.*

Set $D := \sqrt{\mathbb{1}_m - \rho'\rho CC'} \in C^{2,\gamma}(\mathbb{S}^d_{++}; \mathbb{S}^d)$. We then define Z by

$$Z^j_t := \sum_{k,l=1}^d \int_0^t C_{jk}(X_u)dB^{kl}_u \rho_l(X_u) + \sum_{k=1}^m \int_0^t D_{jk}(X_u)dW^k_u, \qquad t \geq 0, j = 1, \ldots, m. \tag{2.9}$$

By construction, Z is an m-dimensional Brownian motion. Furthermore, the instantaneous correlation between Z and B is $d\langle Z^j, B^{kl}\rangle_t = C_{jk}(X_t)\rho_l(X_t)dt$, for $1 \leq j \leq m, 1 \leq k, l \leq d$. In particular, when $m = d$, $C = \mathbb{1}_d$ and $\rho \in \mathbb{R}^d$ is constant, $d\langle Z^i, B^{jl}\rangle_t = \delta_{ij}\rho_l dt$, where $\delta_{ij} = 1$ for $i = j$ and 0 otherwise. Then Assumption 2.4 reduces to $\rho'\rho \leq 1$ on $\mathbb{S}^d_{++}$. This particular correlation structure is assumed in [8, 30, 2, 47]. Here, the matrix C introduces a general correlation structure and allows for dependence upon the state variable X.

2.3. The optimal investment problem. Consider an investor whose preferences are described by a utility function $U : \mathbb{R}_+ \to \mathbb{R}$ which is strictly increasing, strictly concave, and continuously differentiable and satisfies the Inada conditions $U'(0) = \infty$ and $U'(\infty) = 0$. In particular, we pay special attention to utilities with constant relative risk aversion (CRRA) $U(x) = x^p/p$ for $0 \neq p < 1$.

Starting from initial capital, this investor trades in the market until a time horizon $T \in \mathbb{R}_+$. She puts a proportion of her wealth $(\pi_t)_{t\leq T}$ into the risky assets and the remaining into the risk-free asset. Given her strategy π, the price dynamics in (2.7) and (2.8) imply that the wealth process $\mathcal{W}^\pi$ has dynamics

$$\frac{d\mathcal{W}^\pi_t}{\mathcal{W}^\pi_t} = (r(X_t) + \pi'_t\Sigma(X_t)\nu(X_t))dt + \pi'_t\sigma(X_t)dZ_t. \tag{2.10}$$

The set of *admissible* strategies are those π which are $\mathbb{F}$-adapted and such that $\mathbb{P}^x[\mathcal{W}^\pi_t > 0 \text{ for all } t \leq T] = 1$ for all $x \in \mathbb{S}^d_{++}$. Let us recall the notion of *supermartingale deflator*. A positive supermartingale M, starting from $M_0 = 1$, is a supermartingale deflator if $M\mathcal{W}^\pi$ is a supermartingale for any admissible strategy π. In (A.1) below, a class of supermartingale deflators is constructed, whose presence excludes arbitrage opportunities; cf. [36]. The investor seeks to maximize the expected utility of her terminal wealth at T by choosing admissible strategies, i.e.,

$$\mathbb{E}\left[U(\mathcal{W}^\pi_T)\right] \to \text{Max}. \tag{2.11}$$

In the remainder of this section, we will focus on the optimal investment problem for CRRA utilities and derive the associated HJB equation via a heuristic argument. To this end, define the (reduced) value function v via

(2.12)
$$\sup_{\pi \text{ admissible}} \mathbb{E}\left[\frac{1}{p}(\mathcal{W}^\pi_T)^p \middle| \mathcal{W}_t = w, X_t = x\right] = \frac{1}{p}w^p e^{v(T-t,x)}, \qquad 0 \leq t \leq T, w > 0, x \in \mathbb{S}^d_{++}.$$

Next, set L as the infinitesimal generator of (2.3):

$$L := \frac{1}{2}\sum_{i,j,k,l=1}^{d} \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D^2_{(ij),(kl)} + \sum_{i,j=1}^{d} b_{ij} D_{(ij)}, \tag{2.13}$$

where $D_{(ij)} = \partial_{x^{ij}}$ and $D^2_{(ij),(kl)} = \partial^2_{x^{ij}x^{kl}}$. The standard dynamic programming argument yields the following HJB equation for v:

$$\begin{aligned} \partial_t v = Lv + \frac{1}{2}\sum_{i,j,k,l=1}^{d} D_{(ij)}v\,\mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D_{(kl)}v + p\,r \\ + \sup_{\pi}\left\{p\pi'\left(\Sigma\nu + \sum_{i,j=1}^{d}\sigma C a^{ij}\rho D_{(ij)}v\right) + \frac{1}{2}p(p-1)\pi'\Sigma\pi\right\}, \quad t>0, x\in\mathbb{S}^d_{++}, \\ 0 = v(0,x), \quad x\in\mathbb{S}^d_{++}. \end{aligned} \tag{2.14}$$

The optimizer π in the previous equation can be obtained pointwise and is given by

$$\pi(t,x;v) := \begin{cases} \dfrac{1}{1-p}\Sigma^{-1}\left(\Sigma\nu + \sum_{i,j=1}^{d}\sigma C a^{ij}\rho D_{(ij)}v\right)(t,x), & m>n, \\ \dfrac{1}{1-p}\sigma(\sigma'\sigma)^{-1}\left(\sigma'\nu + \sum_{i,j=1}^{d} C a^{ij}\rho D_{(ij)}v\right)(t,x), & m\le n, \end{cases} \quad t>0, x\in\mathbb{S}^d_{++}. \tag{2.15}$$

Define $q := p/(p-1)$ as the conjugate of p and the function $\Theta : \mathbb{S}^d_{++} \to \mathbb{S}^d_{++}$ via

$$\Theta(x) := \begin{cases} \sigma'\Sigma^{-1}\sigma(x), & m>n, \\ \mathbb{1}_m, & m\le n, \end{cases} \qquad x\in\mathbb{S}^d_{++}. \tag{2.16}$$

Plugging the formula for π in (2.15) into (2.14), a lengthy calculation yields the following semilinear Cauchy problem for v:

$$\begin{aligned} v_t(t,x) &= \mathfrak{F}[v](t,x), \qquad 0<t, x\in\mathbb{S}^d_{++}, \\ v(0,x) &= 0, \qquad x\in\mathbb{S}^d_{++}. \end{aligned} \tag{2.17}$$

Here, the differential operator $\mathfrak{F}$ is defined as

$$\mathfrak{F} := \frac{1}{2}\sum_{i,j,k,l=1}^{d} A_{(ij),(kl)} D^2_{(ij),(kl)} + \sum_{i,j=1}^{d}\bar{b}_{ij}D_{(ij)} + \frac{1}{2}\sum_{i,j,k,l=1}^{d} D_{(ij)}\bar{A}_{(ij),(kl)}D_{(kl)} + V \tag{2.18}$$

with

$$\begin{aligned} A_{(ij),(kl)}(x) &:= \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right)(x), \\ \bar{A}_{(ij),(kl)}(x) &:= \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right)(x) - q\rho'\left(a^{ij}\right)'C'\Theta C a^{kl}\rho(x), \\ \bar{b}_{ij}(x) &:= b_{ij}(x) - q\nu'\sigma C a^{ij}\rho(x), \\ V(x) &:= pr(x) - \frac{1}{2}q\nu'\Sigma\nu(x), \qquad i,j,k,l = 1,\dots,d, x\in\mathbb{S}^d_{++}. \end{aligned} \tag{2.19}$$

Note that π in (2.15) and $\mathfrak{F}$ in (2.18) take different forms depending on $m > n$ or $m \leq n$ (with the two forms coinciding at $m = n$) and that using the definition of L from (2.13) we have

$$\mathfrak{F} = L - q \sum_{i,j=1}^{d} \nu' \sigma C a^{ij} \rho D_{(ij)} + \frac{1}{2} \sum_{i,j,k,l=1}^{d} D_{(ij)} \bar{A}_{(ij),(kl)} D_{(kl)} + V. \tag{2.20}$$

In section 3 well-posedness of (2.17) is proved under appropriate parameter assumptions, and it is shown that the solution v, with appropriate growth constraint, to (2.17) is the reduced value function in (2.12). Moreover the optimal strategy for (2.12) is given by

$$\pi_t^T := \pi(T - t, X_t; v), \qquad 0 \leq t \leq T, \tag{2.21}$$

for $\pi(\cdot, \cdot; v)$ from (2.15). We want to emphasize that we need (2.17) and its associated portfolio optimization problem to be well-posed for *all* time horizons $T > 0$. Our parameter restriction will achieve this goal. In particular, when $0 < p < 1$, violation of our parameter restriction may lead to an infinite value function when the investment horizon is beyond some critical level; see Example 3.2 below. In this case, the long-run problem introduced below is ill-posed.

2.4. Long horizon convergence. The long-run behavior for a CRRA investor is closely related to the ergodic analog of (2.17), given by

$$\lambda = \mathfrak{F}[v](x), \qquad x \in \mathbb{S}^d_{++}. \tag{2.22}$$

A solution to (2.22) is defined as a pair (λ, v) where $\lambda \in \mathbb{R}$ and $v \in C^2(\mathbb{S}^d_{++}; \mathbb{R})$ which satisfy (2.22). In particular we are interested in the *smallest* λ such that (2.22) admits a solution. Note that by definition of $\mathfrak{F}$, solutions v to (2.22) are defined up to additive constants.

When the state variable lies in $E \subseteq \mathbb{R}^d$, under appropriate restrictions (cf. [33, 26]), there exists a smallest $\hat{\lambda}$ such that (2.22) has a solution $(\hat{\lambda}, \hat{v})$ and such that the candidate reduced long-run value function, accounting for the growth rate, is $\hat{\lambda} T + \hat{v}(x)$. The candidate long-run optimal strategy is

$$\hat{\pi}_t := \pi(X_t; \hat{v}), \qquad t \geq 0, \tag{2.23}$$

where $\pi(\cdot; \hat{v})$ comes from (2.15) with v replaced by $\hat{v}$ which does not have a time argument. Here, when the state variable is matrix valued, Proposition 3.7 below establishes the existence of such $(\hat{\lambda}, \hat{v})$.

Comparing the finite and long horizon problems, we are interested in proving the following claim.

Statement 2.5 (long horizon convergence).

(i) *Define* $h(T, x) := v(T, x) - \hat{\lambda} T - \hat{v}(x)$ *for* $T \geq 0$ *and* $x \in \mathbb{S}^d_{++}$. *Then*

$$h(T, \cdot) \to C \quad \textit{and} \quad \nabla h(T, \cdot) \to 0 \quad \textit{in } C\left(\mathbb{S}^d_{++}\right) \quad \textit{as } T \to \infty.$$

Here C *is a constant,* $\nabla = (D_{(ij)})_{1 \leq i,j \leq d}$ *is the gradient operator, and convergence in* $C(\mathbb{S}^d_{++})$ *stands for locally uniformly convergence in* $\mathbb{S}^d_{++}$.

(ii) *As functions of $x \in \mathbb{S}^d_{++}$ the finite horizon strategies converge to the long-run counterpart, i.e.,*

$$\lim_{T\to\infty} \pi(T,\cdot;v) = \pi(\cdot;\hat{v}) \quad \text{in } C\left(\mathbb{S}^d_{++}\right).$$

(iii) *Let π^T and $\hat{\pi}$ be as in (2.21) and (2.23). Let $\mathcal{W}^T$ and $\hat{\mathcal{W}}$ be the wealth processes employing π^T and $\hat{\pi}$, respectively, starting with initial capital w. Then for all $x \in \mathbb{S}^d_{++}$ and all $t \geq 0$,*

$$\mathbb{P}^x - \lim_{T\to\infty} \sup_{0\leq u\leq t} \left| \frac{\mathcal{W}^T_u}{\hat{\mathcal{W}}_u} - 1 \right| = 0, \tag{2.24}$$

$$\mathbb{P}^x - \lim_{T\to\infty} \int_0^t \left(\pi^T_u - \hat{\pi}_u\right)' \Sigma(X_u) \left(\pi^T_u - \hat{\pi}_u\right) du = 0. \tag{2.25}$$

Here $\mathbb{P}^x - \lim$ stands for convergence in probability $\mathbb{P}^x$.

In Statement 2.5, (i) claims that the reduced value function for the finite horizon problem converges to its infinite horizon counterpart; moreover (ii) indicates that the finite horizon optimal strategy also converges, in feedback form, to a myopic long-run limit. In addition to these analytic results, (iii) states convergence in probabilistic terms: that is, the ratio between optimal wealth processes and distance between optimal strategies, when measured in a finite time window $[0,t]$, converges to zero in probability. Under appropriate parameter assumptions, Statement 2.5 is proved in [24] when the state variable is $\mathbb{R}$-valued and has constant correlation with risky assets. In section 3 below, we significantly extend this result to verify Statement 2.5 in the matrix setting.

2.5. Turnpike theorems. Consider two investors: the first one has a general utility function U which satisfies conditions at the beginning of section 2.3; the second investor has a CRRA utility $U(x) = x^p/p$ for $0 \neq p < 1$.[2] The two investors are connected through the ratio of their marginal utilities as follows.

Assumption 2.6. *With $\mathfrak{R}(x) := U'(x)/x^{p-1}$ it holds that*

$$\lim_{x\uparrow\infty} \mathfrak{R}(x) = 1. \tag{2.26}$$

Assumption 2.6 ensures that preferences of the two investors are similar for large wealths. The next assumption ensures that the market described in section 2.2 is growing over time.

Assumption 2.7. *For $r(x)$ as in (2.7) there exist constants $0 < \underline{r} < \bar{r}$ such that $\underline{r} \leq r(x) \leq \bar{r}$ for all $x \in \mathbb{S}^d_{++}$.*

For the investor with general utility U, set $\pi^{1,T}$ as the optimal strategy of (2.11) and $\mathcal{W}^{1,T}$ as the associated optimal wealth process starting from initial wealth w. We are interested in proving the turnpike theorem.

[2] The logarithmic utility case is excluded here, since [24, Proposition 2.5] already shows that turnpike theorems hold in a general semimartingale setting including the current case.

Statement 2.8 (turnpike theorem). *For all $x \in \mathbb{S}^d_{++}$ and all $t \geq 0$,*

$$\mathbb{P}^x - \lim_{T\to\infty} \sup_{u\leq t} \left| \frac{\mathcal{W}^{1,T}_u}{\hat{\mathcal{W}}_u} - 1 \right| = 0, \tag{2.27}$$

$$\mathbb{P}^x - \lim_{T\to\infty} \int_0^t \left(\pi^{1,T}_u - \hat{\pi}_u\right)' \Sigma(X_u) \left(\pi^{1,T}_u - \hat{\pi}_u\right) du = 0, \tag{2.28}$$

where $\hat{\pi}$ from (2.23) *and $\hat{\mathcal{W}}$ is the wealth process starting from w following $\hat{\pi}$.*

The first convergence above states that the ratio, when measured in a finite time window, of the optimal wealth process for the generic investor and the long-run wealth process for the CRRA investor is uniformly close to one in probability as the horizon becomes large. The message behind the second convergence is that, as the horizon becomes long, the optimal investment strategy for the generic utility investor approaches the long-run limit strategy of the CRRA investor. Such a result is called an "explicit" turnpike using the terminology of [24].

3. Main results.

3.1. Parameter restriction. To establish Statements 2.5 and 2.8 for models introduced in section 2.2, we introduce a key set of conditions on the model coefficients. In these conditions, c is a universal positive constant which may be different from line to line.

Assumption 3.1. *For some integer n_0 the following hold:*

(1) $\|\bar{b}(x)\| + |r(x)| + \nu'\Sigma\nu(x) \leq c(1 + \|x\|)$ *for* $\|x\| \geq n_0$.

(2) *There exists $\beta \in \mathbb{R}$ so that*

$$Tr\left(\bar{b}(x)'x\right) \leq -\beta\|x\|^2 + c \quad \textit{for } \|x\| \geq n_0.$$

(3) *There exists a matrix $M > (1+d)\Lambda\Lambda'$ such that*

$$\bar{b}(x) \geq M - cx \quad \textit{for } x \in \mathbb{S}^d_{++}.$$

(4) *When $p < 0$,*

   (a) *r is bounded from below on $\mathbb{S}^d_{++}$, and*

$$\sup_{\|x\|\leq n_0} \frac{r(x)}{1 + Tr(x^{-1})} < \infty, \qquad \sup_{\|x\|\leq n_0} \frac{\nu'\Sigma\nu(x)}{1 + Tr(x^{-1})} < \infty,$$

   (b)

$$\textit{either} \quad \beta > 0 \quad \textit{or} \quad \liminf_{\|x\|\to\infty} \frac{\nu'\Sigma\nu(x)}{\|x\|} > 0 \quad \textit{or} \quad \liminf_{\|x\|\to\infty} \frac{r(x)}{\|x\|} > 0,$$

   where β comes from item (2).

(5) *When $0 < p < 1$,*

   (a) *r and $\nu'\Sigma\nu$ are bounded from above on $\|x\| \leq n_0$, and*

$$\inf_{\|x\|\leq n_0} \frac{r(x)}{1 + Tr(x^{-1})} > -\infty.$$

(b)

$$\beta > 0 \quad \textit{and} \quad \beta^2 > 8\sqrt{d}\, Tr\left(\Lambda\Lambda'\right)(1 - q\lambda_C^2\lambda_\rho^2)\chi,$$

where β comes from item (2), λ_C^2 is the supremum of eigenvalues of CC' on $\mathbb{S}^d_{++}$, $\lambda_\rho^2 = \sup_{x\in\mathbb{S}^d_{++}} \rho'\rho(x)$, and χ is the smallest $\bar{\chi}$ such that

$$pr(x) - \tfrac{1}{2}q\nu'\Sigma\nu(x) \le \bar{\chi}\|x\| + c, \quad \|x\| \ge n_0, \quad \textit{for some constant } c.$$

In particular, when r and $\nu'\Sigma\nu$ are bounded from above on $\|x\| \ge n_0$, then β is only required to be positive.

Let us illustrate the parameter restriction in Assumption 3.1 via several examples. The first example shows that Assumption 3.1 is mild when $p < 0$ and is sharp when $0 < p < 1$.

Example 3.2 (Heston model). The simplest example is where the state variable follows a one-dimensional state variable suggested by Heston. The model, studied by [38] and [41], is specified as

$$dS_t = S_t\big((r + \nu X_t)dt + \sqrt{X_t}dZ_t\big),$$
$$dX_t = b(\ell - X_t)dt + a\sqrt{X_t}dB_t,$$

where $r \in \mathbb{R}$, $\ell \ge 0$, $b > 0$, $a, \nu \ne 0$, and B, Z are one-dimensional Brownian motion with instantaneous constant correlation ρ. Assumption 3.1 is specified as

(1) $b\ell > a^2/2$,

(2) when $0 < p < 1$, $b + q\nu a\rho > 0$ and $(b + q\nu a\rho)^2 > -q\nu^2 a^2(1 - q\rho^2)$.

Item (1) is the standard Feller's condition making sure that X does not hit zero in finite time. When $p < 0$, no additional condition is needed. When $0 < p < 1$, item (2) is exactly Case 1 in [35, *Theorem* 3.2]. Except a boundary case, where $b + q\nu a\rho > 0$ and the second inequality in item (2) above is an equality (cf. Case 2 in [35, *Theorem* 3.2]), item 1 in Remark of [35, *Page* 467] shows that the condition in item (2) above is almost the necessary and sufficient condition for well-posedness of the utility maximization problem for *all* finite time horizons. In other words, if the condition in item (2) and the boundary case are violated, there exists a critical time horizon T_∞ beyond which the value for the utility maximization problem is infinite. In such a case, both the long-run problem and the convergence problem are ill-posed.

The next example, motivated by [9], illustrates the flexibility of Assumption 3.1 when the state variable X is small.

Example 3.3 (inverse Heston model). Consider the model

$$dS_t = S_t\Big(\frac{r}{X_t} + \nu_0 + \frac{\nu_1}{X_t}\Big)dt + \frac{1}{\sqrt{X_t}}dZ_t,$$
$$dX_t = b(\ell - X_t)dt + a\sqrt{X_t}dB_t,$$

where $r, \ell \ge 0$, $b > 0$, $a, \nu_0 \ne 0, \nu_1 \in \mathbb{R}$, and B, Z are one-dimensional Brownian motion with instantaneous constant correlation ρ. In this model, interest rate varies with the state variable and the mean variance ratio is an affine function $\nu_0 x + \nu_1$. In particular, when $\nu_1 > 0$, the sharp ratio of stock $\nu_0\sqrt{X}+\frac{\nu_1}{\sqrt{X}}$ increases when the interest rate increases. This is empirically relevant because a high interest rate usually diverts investment away from the equity market. Therefore in equilibrium stocks need to have a higher sharp ratio to attract investors. We focus on $p<0$ in this example. In this case, Assumption 3.1 only requires $b\ell > a^2/2$, which ensures $X>0$, and hence is necessary for finite interest rate and excess return.

The canonical multivariate example is when X is a Wishart process as in Example 2.2. This model has been studied in [8, 30, 2, 47].

Example 3.4. Following the aforementioned literature, we consider the case where the dimension of X is the same as the dimension of the Brownian motion Z, i.e., $m=d$, and C in (2.4) is an identity matrix. Consider the following model:

$$\frac{dS_t^i}{S_t^i} = (r(X_t)+\mu(X_t))dt + \sigma\sqrt{X_t}dZ_t, \quad i=1,\dots,n,$$

where $r(x)=r_0+\mathrm{Tr}\,(r_1x)$ for some $r_0\in\mathbb{R}$ and $r_1\in\mathbb{M}^d$, $\mu(x)=\sigma x\sigma'\nu$ for some $\sigma\in\mathbb{M}^{n\times d}$ and $\nu\in\mathbb{R}^n$, and X follows the dynamics (2.5) with $d\langle Z^i, B^{jl}\rangle_t=\delta_{ij}\rho_l dt$ for some $\rho\in\mathbb{R}^d$. In this case, Assumption 3.1 specifies to

(1) $LL' > (d+1)\Lambda\Lambda'$,

(2) when $p<0$, r_1 satisfies $r_1+r_1'\geq 0$, and

$$\text{either}\quad r_1+r_1'>0 \quad\text{or}\quad \sigma'\nu\nu'\sigma>0, \quad\text{or}\quad (-K+q\Lambda\rho\nu'\sigma)+(-K+q\Lambda\rho\nu'\sigma)'>0,$$

(3) when $0<p<1$,

$$(-K+q\Lambda\rho\nu'\sigma)+(-K+q\Lambda\rho\nu'\sigma)'>\epsilon\mathbb{1}_d,$$

where ϵ is a positive constant such that

$$\epsilon^2 = 4\sqrt{d}\mathrm{Tr}\left(\Lambda\Lambda'\right)(1-q\rho'\rho)\|p(r_1+r_1')-q\sigma'\nu\nu'\sigma'\|.$$

In the previous parameter restrictions, item (1) is slightly stronger than the well-posedness condition (2.6). The restriction in the $p<0$ case is mild. It asks that either $r(x)$ or $\nu'\Sigma\nu(x)$ has linear growth, or a Wishart process $\overline{X}$ with dynamics

$$d\overline{X}_t = \left(LL' - \overline{KX_t} - X_t\overline{K}\right)' dt + \sqrt{\overline{X}_t}dB_t\Lambda' + \Lambda dB_t'\sqrt{\overline{X}_t},$$

with $\overline{K}(x):=-K+q\Lambda\rho\nu'\sigma$, is mean-reverting. When $0<p<1$, item (3) above requires that the force of mean-revision to be sufficiently strong. Note that this condition is sharp, since it includes Example 3.2 when $d=1$.

3.2. Main results. After introducing all parameter restrictions, we come back to the long horizon portfolio optimization problem. First, Lemma A.1 below shows that the current assumptions (Assumptions 2.1, 2.3, 2.4, and 3.1) verify [48, Assumptions 3.4–3.6]. Therefore, [48, Propositions 2.5 and 2.7 and Theorem 3.9] can be used to study the long horizon problem. To this end, let us first verify the heuristic argument in section 2.3.

Proposition 3.5. *Let Assumptions* 2.1, 2.3, 2.4, *and* 3.1 *hold. Then there exists a unique solution* $v \in C^{1,2}((0,\infty)\times\mathbb{S}^d_{++})\cap C([0,\infty)\times\mathbb{S}^d_{++})$ *to* (2.17) *such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{S}^d_{++}} (v(t,x)-\phi_0(x)) < \infty \qquad \textit{for each } T\geq 0,$$

where $\phi_0(x) = -\underline{c}\log(\det(x)) + \bar{c}\|x\|\eta(\|x\|) + C$ *for some positive constants* $\underline{c}, \bar{c}, C$ *and a cutoff function* η *such that* $\eta(r) = 1$ *for* $r > n_0 + 2$ *and* $\eta(r) = 0$ *for* $r < n_0 + 1$.

In order to connect solutions v to (2.17) with the optimization problem (2.12), we utilize the following verification result, whose proof is technically challenging and given in Appendix A.

Proposition 3.6. *Let Assumptions* 2.1, 2.3, 2.4, *and* 3.1 *hold. For* v *in Proposition* 3.5 *and any* $T > 0$, (2.12) *holds and* π^T *from* (2.21) *is the optimal strategy for* (2.12).

On the other hand, for ergodic problem (2.22), [48, Proposition 2.3 and Lemma 5.3] yield the following.

Proposition 3.7. *Let Assumptions* 2.1, 2.3, 2.4, *and* 3.1 *hold. There exists* $(\hat{\lambda}, \hat{v})$ *solving* (2.22) *such that* $\hat{v}$ *is unique (up to an additive constant) and* $\hat{\lambda}$ *is the smallest* λ *such that there exists a corresponding* v *solving* (2.22).

We are now ready to state our first main result, whose proof is presented in Appendix B.

Theorem 3.8. *Let Assumptions* 2.1, 2.3, 2.4, *and* 3.1 *hold. Then the long horizon results in Statement* 2.5 *hold.*

To state the portfolio turnpike result, we need to make an additional assumption, which is a mild strengthening of Assumption 2.4.

Assumption 3.9. *For* ρ *and* C *in Assumption* 2.4, $\rho'\rho CC'(x) < \mathbb{1}_m$ *for all* $x \in \mathbb{S}^d_{++}$.

Under the previous assumption, it is possible to construct not only supermartingale deflators (cf. (A.1) below) but also equivalent local martingale measures $\mathbb{Q}^T$, for all $T > 0$; i.e., $\mathbb{Q}^T$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ and $e^{-\int_0^\cdot r(X_u)\,du}S$ is a $\mathbb{Q}^T$ local martingale on $[0,T]$. This is needed to utilize duality results in [37] which establish the existence of an optimal strategy to (2.11) for the generic utility U. On the other hand, when $\rho'\rho CC' \equiv \mathbb{1}_m$. The market model is complete and the turnpike result has been established in [17].

We are now ready to state the following turnpike result.

Theorem 3.10. *Let Assumptions* 2.1, 2.3, 3.1, *and* 3.9 *hold. Furthermore, assume that the utility function* U *satisfies Assumption* 2.6 *and the interest rate function satisfies Assumption* 2.7. *Then the turnpike theorems in Statement* 2.8 *hold.*

3.3. An example with nonexponentially affine value function. In the Heston model, the value functions are exponentially affine in the state variable; cf. [38, 41, 35]. However, for the Wishart model, as a multivariate generalization of Heston model, whether the value function is exponentially affine depends on the dimension d of the Wishart process being less than or equal to n, the number of risky assets. When $d \leq n$, the value function is exponentially affine. However, when $d > n$, the agent's reduced value function for the long-run problem may *not* be affine, hence the long-run optimal value function may not be exponentially affine, and thus the optimal strategy $\hat{\pi}$ may not be affine. This is due to the noncommutative property of the matrix product, and we give an example below.

To streamline the presentation, we assume that $p < 0$, $r_1 + r_1' > 0$, $LL' > (d+1)\Lambda\Lambda' > 0$, and σ is of full rank. Hence all assumptions of Proposition 3.7, in particular, parameter restrictions in Example 3.4, are satisfied. Therefore a solution $(\hat{\lambda}, \hat{v})$ to (2.22) exists. Consider a candidate solution given by

$$\hat{v}(x) = \mathrm{Tr}\left(\hat{M}x\right), \qquad \hat{M} = \hat{M}'.^{3} \tag{3.1}$$

First we present the result when $d \leq n$.

Proposition 3.11. *Assume $d \leq n$ and consider the following matrix Riccati equation in M:*

$$0 = 2M\Lambda(1 - q\rho\rho')\Lambda' M + (K - q\Lambda\rho\nu'\sigma)' M + M(K - q\Lambda\rho\nu'\sigma) + \frac{1}{2}\left(p(r_1 + r_1') - q\sigma'\nu\nu'\sigma\right). \tag{3.2}$$

There exists a unique $\hat{M} \in \mathbb{S}^d$ solving (3.2) such that $(\hat{\lambda}, \hat{v})$, with $\hat{\lambda} = Tr(LL'\hat{M}) + pr_0$ and $\hat{v}(x) = Tr(\hat{M}x)$, solves (2.22), and $\hat{\lambda}$ is the smallest λ with accompanying v solving (2.22).

It is easy to calculate the solution of (3.2). In most mathematical software, it only requires one comment. However, in order to solve the finite horizon problem, one needs to solve a matrix Ricatti ODE. Therefore, long-run analysis reduces computational complexity considerably in this case.

We next present an example in the $d > n$ case showing that solutions $(\hat{\lambda}, \hat{v})$ to (2.22) *cannot* be of the affine form in (3.1).

Example 3.12. Take $n = 1, d = 2$ and

$$\begin{aligned} &\Lambda = \mathbb{1}_2, \quad L = \ell\mathbb{1}_2 \text{ for } \ell > \sqrt{3}, \quad K = \mathbb{1}_2, \quad C = \mathbb{1}_2, \\ &\sigma = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \nu = \nu \in \mathbb{R}, \quad \rho = \rho\begin{pmatrix} 1 & 1 \end{pmatrix}' \text{ for } 0 < 2\rho^2 < 1, \\ &r_0 > 0, \quad r_1 = r_1\mathbb{1}_2 \text{ for } r_1 > 0. \end{aligned} \tag{3.3}$$

Consider functions v as in (3.1). Writing the generic element $X \in \mathbb{S}^d_{++}$ and the matrix M as

$$X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}, \quad x, z > 0, y^2 < xz, \qquad M = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_3 \end{pmatrix}, \tag{3.4}$$

we have that $\Sigma(X) = \sigma X\sigma' = x > 0$ so that Assumption 2.3 holds. Furthermore, $LL' - 3\Lambda\Lambda' = (\ell^2 - 3)\mathbb{1}_2 > 0$ and for $p < 0$, $-p(r_1 + r_1') + q\sigma'\nu\nu'\sigma(x) \geq -2pr_1\mathbb{1}_2 > 0$. Thus, the assumptions in Example 3.4 hold. A lengthy calculation shows that (cf. Lemma C.2 in Appendix C)

$$\begin{aligned} \mathfrak{F}[v] = x&\left(2(M_1^2 + M_2^2) - 2q\rho^2(M_1 + M_2)^2 + 2M_1 - 2q\rho\nu(M_1 + M_2) + pr_1 - \frac{1}{2}q\nu^2\right) \\ &+ y\left(4M_2(M_2 + M_3) - 4q\rho^2(M_1 + M_2)(M_2 + M_3) + 4M_2 - 2q\rho\nu(M_2 + M_3)\right) \end{aligned} \tag{3.5}$$

[3] We can assume $\hat{M} = \hat{M}'$ without loss of generality since $x \in \mathbb{S}^d_{++}$ implies $(\hat{M}x) = (\hat{M}'x) = (1/2)((\hat{M} + \hat{M}')x)$.

$$+ z\left(2(M_2^2+M_3^2)+2M_3+pr_1\right)$$
$$+\frac{y^2}{x}\left(-2q\rho^2(M_2+M_3)^2\right)$$
$$+pr_0+\ell^2(M_1+M_3).$$

As can be seen from (7) in Lemma C.1 below, the problem term y^2/x arises when evaluating $\bar{A}$ from (2.19), since for $d > n$,

$$\sqrt{X}\Theta(X)\sqrt{X} = X\sigma'(\sigma X\sigma')^{-1}\sigma X = \frac{1}{x}X\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}X = \begin{pmatrix}x & y\\ y & \frac{y^2}{x}\end{pmatrix}, \tag{3.6}$$

whereas, for arbitrary model coefficients, if $d \le n$ then $\sqrt{X}\Theta(X)\sqrt{X} = X$.

Thus, if $\mathfrak{F}[v] = \lambda$ for some constant λ it must be that each coefficient of $x, y, z, y^2/x$ in (3.5) is equal to zero. By considering y^2/x it follows that $M_2 + M_3 = 0$. Plugging this into the coefficient of y gives $M_2 = 0$ and hence $M_3 = 0$. Then the coefficient of z being zero yields $0 = pr_1$ a contradiction since $r_1 > 0$. Thus, the function $\hat{v}$ cannot be affine.

Appendix A. Proof of Proposition 3.6.

We first define a class of supermartingale deflators on $[0, T]$ for any $T > 0$. Given an $\mathbb{M}^d$-valued process η with $\int_0^T \|\eta_u\|^2 du < \infty$ a.s., define M^η via (note that for a function g of $\mathbb{S}^d_{++}$ we will write g_u for $g(X_u)$)

$$\begin{aligned} M_t^\eta &:= e^{-\int_0^t r_u du}\mathcal{E}\left(\int\left(-\nu_u'\sigma_u C_u dB_u\rho_u + \mathrm{Tr}\left(\eta_u dB_u'\right) - \rho_u'\eta_u'C_u'\Theta_u C_u dB_u\rho_u\right)\right)_t \\ &\quad\times\mathcal{E}\left(-\int\left(\nu_u'\sigma_u D_u + \rho_u'\eta_u'C_u'\Theta_u D_u\right)dW_u\right)_t \\ &= e^{-\int_0^t r_u du}\mathcal{E}\left(\int\sum_{k,l=1}^d dB_u^{kl}\left(-(C'\sigma'\nu)_k\rho_l + \eta_{kl} - (C'\Theta C\eta\rho)_k\rho_l\right)_u\right)_t \\ &\quad\times\mathcal{E}\left(-\int\sum_{k=1}^d dW_u^k\left((D'\sigma'\nu)_k + (D'\Theta C\eta\rho)_k\right)_u\right)_t, \qquad t\le T. \end{aligned} \tag{A.1}$$

When $\eta = 0$, $e^{\int_0^\cdot r_u du}M^\eta$ defines the *minimal martingale measure*, provided the stochastic exponentials are indeed martingales; cf [21]. Hence we call η a *risk premia*. For any admissible strategy π, $M^\eta\mathcal{W}^\pi$ is a positive supermartingale. Indeed, using (2.9), (2.10), and (A.1), the stochastic integration by parts formula shows that the drift of $M^\eta\mathcal{W}^\pi$ has the following integrand (omitting function arguments and time subscripts):

$$\begin{aligned} &M^\eta\mathcal{W}^\pi\pi'\left[\Sigma\nu + \sigma C\left(-C'\sigma'\nu\rho' + \eta - C'\Theta C\eta\rho\rho'\right)\rho - \sigma D\left(D'\sigma'\nu + D'\Theta C\eta\rho\right)\right] \\ &\quad = M^\eta\mathcal{W}^\pi\pi'\left[\Sigma\nu - \sigma\left(CC'\rho'\rho + DD'\right)\sigma'\nu + \sigma C\eta\rho - \sigma\left(CC'\rho'\rho + DD'\right)\Theta C\eta\rho\right] \\ &\quad = M^\eta\mathcal{W}^\pi\pi'\left[\sigma C\eta\rho - \sigma\Theta C\eta\rho\right] \\ &\quad = 0, \end{aligned}$$

where the second identity follows from $(CC'\rho'\rho + DD')(x) = 1_m$ and the third identity holds due to $\sigma\Theta = \sigma$. Therefore $M^\eta\mathcal{W}^\pi$ is a positive local martingale, hence a supermartingale.

The convergence of finite horizon problem (2.17) to the ergodic version (2.22) has been studied in [48] for a general matrix setting. Let us check that our current assumptions (Assumptions 2.1, 2.3, 2.4, and 3.1) verify [48, Assumptions 3.4–3.6]. Therefore, [48, Propositions 2.5, 2.7, and Theorem 3.9] can be used to study the long horizon problem.

Lemma A.1. *Let Assumptions 2.1, 2.3, 2.4, and 3.1 hold. Then [48, Assumptions 3.4–3.6] hold.*

Proof. In the current specification, the functions f and g introduced in [48, equation (3.11)] are $f(x) = x$ and $g(x) = \Lambda\Lambda'$, respectively, and the function B in [48, equation (3.3)] is $\bar{b}$.

Assumption 3.4 *in* [48]. Item (i) is clearly satisfied by the current specification of f and g. Item (ii) follows from Lemma A.3 below.

Assumption 3.5 *in* [48]. Item (i) follows from the specification of f and g and item (1) of Assumption 3.1. Item (ii) is exactly Assumption 3.1(2) here. Moreover β_1 in [48, Assumption 3.5] is exactly β here. Recall the form of V from (2.19). Since both r and $\nu'\Sigma\nu$ are assumed to be of at most linear growth in Assumption 3.1(1), V is of at most linear growth as well. To check V is bounded from above on $\|x\| \leq n_0$, we split into $p < 0$ and $0 < p < 1$ cases. When $p < 0$, since $q\nu'\Sigma\nu \geq 0$, we have $V(x) \leq pr(x)$, which is bounded from above due to Assumption 3.1(4)(a). When $0 < p < 1$, since $q < 0$ and both r and $\nu'\Sigma\nu$ are bounded from above (cf. Assumption 3.1(5)(a)), V is also bounded from above on $\|x\| \leq n_0$. Therefore [48, Assumption 3.5(iii)] is verified.

To check [48, Assumption 3.5(iv)], we discuss $p < 0$ and $0 < p < 1$ separately. When $p < 0$, since V has at most linear growth and we have seen that it is also bounded from above, therefore V is bounded from below or decays linearly. When either $\liminf_{\|x\|\to\infty} r(x)/\|x\| > 0$ or $\liminf_{\|x\|\to\infty} \nu'\Sigma\nu(x)/\|x\| > 0$, V decays linearly. Hence [48, Assumption 3.5(iv)(a)] is satisfied. When $\beta > 0$, [48, Assumption 3.5(iv)] is satisfied as well.

When $0 < p < 1$, observe that $\mathrm{Tr}\,(f(x))\,\mathrm{Tr}\,(g(x)) = \mathrm{Tr}\,(x)\,\mathrm{Tr}\,(\Lambda'\Lambda) \leq \sqrt{d}\,\mathrm{Tr}\,(\Lambda\Lambda')\,\|x\|$. Therefore the constant α_1 in [48, Assumption 3.5(i)] can be chosen as $\sqrt{d}\,\mathrm{Tr}\,(\Lambda\Lambda')$. Lemma A.3 shows that $\bar{\kappa}$ can be chosen as $1 - q\bar{\rho}^2$. Moreover, γ_1 in [48, Assumption 3.5(iii)] can be chosen as $-\chi$ here. Finally the constant 16 in [48, Assumption 3.5(iv)(b)] can be improved to 8 due to the currently specification of f and g. This is because $\bar{\kappa}$ in the second inequality of [48, equation (5.7)] can be improved to $\bar{\kappa}/2$. Indeed, let $\phi_0 = \phi_0^{(1)} + \phi_0^{(2)} + C = -\underline{c}\log(\det(x)) + \bar{c}\|x\|\eta(\|x\|) + C$ for some positive constants $\underline{c}, \bar{c}, C$ and a cutoff function η such that $\eta(r) = 1$ for $r > n_0 + 2$. Calculation shows

$$
\begin{aligned}
&\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi_0 \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D_{(kl)}\phi_0 \\
&= 4\mathrm{Tr}\left(x\left(-\underline{c}x^{-1} + \bar{c}\tfrac{x}{\|x\|}\right)\Lambda\Lambda'\left(-\underline{c}x^{-1} + \bar{c}\tfrac{x}{\|x\|}\right)\right) \\
&\leq 4\mathrm{Tr}\left(x\left(-\underline{c}x^{-1}\right)\Lambda\Lambda'\left(-\underline{c}x^{-1}\right)\right) + 4\mathrm{Tr}\left(x\bar{c}\tfrac{x}{\|x\|}\Lambda\Lambda'\bar{c}\tfrac{x}{\|x\|}\right) \\
&= \sum_{i,j,k,l=1}^{d} D_{(ij)}\phi_0^{(1)}\mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D_{(kl)}\phi_0^{(1)} + \sum_{i,j,k,l=1}^{d} D_{(ij)}\phi_0^{(2)}\mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D_{(kl)}\phi_0^{(2)}.
\end{aligned}
$$

Therefore the statement of [48, Lemma 5.2] still holds with $4\bar{\kappa}\alpha_1\bar{c}^2$ therein replaced by $2\bar{\kappa}\alpha_1\bar{c}^2$. In conclusion, Assumption 3.1(5)(b) verifies [48, Assumption 3.5(iv)(b)].

Assumption 3.6 *in* [48]. Calculation shows that

$$H_\epsilon(x;\bar{b}) = \mathrm{Tr}\left((\bar{b}(x) - (1+\epsilon+d)\Lambda\Lambda')x^{-1}\right).$$

Hence [48, Assumption 3.6(i)] is ensured by Assumption 3.1(3) here. For [48, Assumption 3.6(ii) and (iii)], the proof in [48] only requires these conditions to hold when $\|x\| \le n_0$; see [48, Lemma 5.5]. Observe that $\lim_{\det(x)\downarrow 0} \mathrm{Tr}\left(x^{-1}\right) + C\log(\det(x)) = \infty$ for any C. Therefore [48, Assumption 3.6(ii)] is satisfied when it is restricted to $\|x\| \le n_0$. Finally [48, Assumption 3.6(iii)] follows from either Assumption 3.1(4)(a) or (5)(a). ∎

After verifying assumptions in [48], existence and uniqueness of solution to (2.17) is established in Proposition 3.5. In order to prove the verification result Proposition 3.6, we need to introduce some notation. For a fixed $\phi \in C^{(1,2),\gamma}((0,\infty)\times\mathbb{S}^d_{++},\mathbb{R})$, the regularity assumptions on the coefficients and ellipticity assumption in (2.4) ensure that the *generalized* martingale problem on $\mathbb{S}^d_{++}$ for

(A.2)
$$\mathcal{L}^{\phi,T-t} := \frac{1}{2}\sum_{i,j,k,l=1}^d A_{(ij),(kl)}D_{(ij),(kl)} + \sum_{i,j=1}^d\left(\bar{b}_{ij} + \sum_{k,l=1}^d \bar{A}_{(ij),(kl)}D_{(kl)}\phi(T-t,\cdot)\right)D_{(ij)}, \quad t\le T,$$

has a unique solution $\left(\mathbb{P}^{\phi,T,x}\right)_{x\in\mathbb{S}^d_{++}}$; cf. [46]. When ϕ does not depend upon t we will write $\mathcal{L}^\phi$ and denote the solution as $\left(\mathbb{P}^{\phi,x}\right)_{x\in\mathbb{S}^d_{++}}$. The martingale problem for $\mathcal{L}^{\phi,T-\cdot}$ is *well-posed* if the coordinate process X does not hit the boundary $\mathbb{S}^d_{++}$, $\mathbb{P}^{\phi,T,x}$-a.s., before T for any $x\in\mathbb{S}^d_{++}$. Similarly, if ϕ does not depend upon time, then well-posedness follows if the coordinate process does not hit the boundary in finite time $\mathbb{P}^{\phi,x}$-a.s. for any $x\in\mathbb{S}^d_{++}$.

For the given ϕ, define the stochastic exponential

(A.3)
$$\begin{aligned} Z^{\phi,T}_t &:= \mathcal{E}\left(\int_0^\cdot \sum_{k,l=1}^d dB^{kl}_u\left(-q(C'\sigma'\nu)_k\rho_l + \sum_{i,j=1}^d\left(a^{ij}_{kl} - q(C'\Theta Ca^{ij}\rho)_k\rho_l\right)D_{(ij)}\phi\right)(T-u,X_u)\right)_t \\ &\quad\times\mathcal{E}\left(\int_0^\cdot \sum_{k=1}^m dW^k_u\left(-q(D'\sigma'\nu)_k - q\sum_{i,j=1}^d (D'\Theta Ca^{ij}\rho)_k D_{(ij)}\phi\right)(T-u,X_u)\right)_t, \quad t\le T. \end{aligned}$$

For ϕ not depending upon time, write Z^ϕ for $Z^{\phi,T}$ and note that Z^ϕ is defined for all $t\ge 0$. Recall from section 2.1 that Assumption 2.1 ensures the well-posedness of (2.1). Hence the martingale problem for L in (2.13) is well-posed. Now if the martingale problem for $\mathcal{L}^{\phi,T-\cdot}$ is also well-posed, it follows from [10, Remark 2.6] that the first stochastic exponential on the right-hand side of (A.3) is a $\mathbb{P}^x$-martingale on $[0,T]$. On the other hand, since X and W are $\mathbb{P}^x$-independent, it follows from [36, Lemma 4.8] that $Z^{\phi,T}$ is also a $\mathbb{P}^x$-martingale on $[0,T]$. Therefore, we may define a new measure $\mathbb{P}^{\phi,T,x}$ on $\mathcal{F}_T$ via $d\mathbb{P}^{\phi,T,x}/d\mathbb{P}^x|_{\mathcal{F}_T} = Z^{\phi,T}_T$. Moreover, Girsanov's theorem yields that X has generator $\mathcal{L}^{\phi,T-\cdot}$ under $\mathbb{P}^{\phi,T,x}$. When ϕ does not have a time argument and the martingale problem for $\mathcal{L}^{\phi}$ is well-posed, the same argument as above yields that Z^{ϕ} is a $\mathbb{P}^x$-martingale on $[0,\infty)$. Hence a new measure $\mathbb{P}^{\phi,x}$ is defined via $d\mathbb{P}^{\phi,x}/d\mathbb{P}^x|_{\mathcal{F}_T} = Z^{\phi}_T$, $T \geq 0$. Note that $\mathbb{P}^{\phi,x}$ is consistently defined on $\vee_{T\geq 0}\mathcal{F}_T$. Last we recall that $\mathbb{P}^{\phi}$ is *ergodic* if X is recurrent under $\mathbb{P}^{\phi}$ and there exists an invariant probability measure.

Remark A.2. Set $\phi = \hat{v}$ from Proposition 3.7; if $\mathbb{P}^{\hat{v},x}$ is well defined, then Girsanov's theorem together with (2.8) and (A.3) yield the following dynamics of S under $\mathbb{P}^{\hat{v},x}$:

$$\frac{dS^i_t}{S^i_t} = \left(r(X_t) + \frac{1}{1-p}\left(\Sigma\nu + \sum_{k,l=1}^{d} \sigma C a^{kl} \rho D_{(kl)}\hat{v}\right)(T-t,X_t)\right) dt + \sum_{j=1}^{m} \sigma_{ij}(X_t) d\hat{Z}^j_t, \quad i = 1,\dots,n,$$

where $\hat{Z}$ is a $\mathbb{P}^{\hat{v},x}$ Brownian motion. Comparing the previous dynamics with $\hat{\pi}$ in (2.23), it follows that $\hat{\pi}$ is the optimal strategy for a logarithmic investor under $\mathbb{P}^{\hat{v},x}$. Hence its associated wealth process $\hat{\mathcal{W}}$ has the *numéraire* property, i.e., $\mathcal{W}/\hat{\mathcal{W}}$ is a $\mathbb{P}^{\hat{v},x}$-supermartingale for any admissible wealth process $\mathcal{W}$.

For the proof of Proposition 3.6, we prepare the following two lemmas, whose proofs are postponed until after the proof of Proposition 3.6.

Lemma A.3. *Let Assumptions 2.3 and 2.4 hold. Let A and $\bar{A}$ be as in (2.19). Set*

$$\underline{\kappa} = \begin{cases} 1, & 0<p<1, \\ 1 - q\lambda_C^2\lambda_\rho^2, & p<0, \end{cases} \quad \text{and} \quad \overline{\kappa} = \begin{cases} 1 - q\lambda_C^2\lambda_\rho^2, & 0<p<1, \\ 1, & p<0, \end{cases} \tag{A.4}$$

where λ_C^2 is the supremum of eigenvalues of $C'C$ on $\mathbb{S}^d_{++}$ and $\lambda_\rho^2 = \sup_{x\in\mathbb{S}^d_{++}} \rho\rho'(x)$. Note that Assumption 2.4 implies $\lambda_C^2\lambda_\rho^2 \leq 1$, hence both $\underline{\kappa}$ and $\bar{\kappa}$ are strictly positive. For all $x \in \mathbb{S}^d_{++}$ and $\theta \in \mathbb{S}^d$, we have

$$\underline{\kappa} \sum_{i,j,k,l=1}^{d} \theta_{ij} A_{(ij),(kl)}(x)\theta_{kl} \leq \sum_{i,j,k,l=1}^{d} \theta_{ij} \overline{A}_{(ij),(kl)}(x)\theta_{kl} \leq \overline{\kappa} \sum_{i,j,k,l=1}^{d} \theta_{ij} A_{(ij),(kl)}(x)\theta_{kl}. \tag{A.5}$$

For $\eta \in C^{(1,2),\gamma}((0,\infty)\times\mathbb{S}^d_{++},\mathbb{R})$, define the function $\eta : \mathbb{S}^d_{++} \to \mathbb{M}^d$ via

$$\eta_{kl}(t,x;\phi) := \left(\sum_{i,j=1}^{d} a^{ij}_{kl} D_{(ij)}\phi\right)(t,x), \qquad k,l = 1,\dots,d,\ t\geq 0, x \in \mathbb{S}^d_{++}. \tag{A.6}$$

Define $\eta^T_t := \eta(T-t, X_t;\phi)$, $t\in[0,T]$. When ϕ is v from Proposition 3.5 (resp., $\hat{v}$ from Proposition 3.7), then $\eta(T-\cdot, X_\cdot; v)$ (resp., $\eta(X_\cdot;\hat{v})$) is expected to be the optimal risk premium for the dual problem of (2.12) (resp., its long-run analogue). The following result is the key to proving Proposition 3.6.

Lemma A.4. *Let $\phi \in C^{(1,2),\gamma}((0,\infty)\times\mathbb{S}^d_{++},\mathbb{R})$ satisfy $\phi_t = \mathfrak{F}[\phi]$ on $(0,\infty)\times\mathbb{S}^d_{++}$, where $\mathfrak{F}$ is defined in (2.18). For any $T\geq 0$, let $\pi_t = \pi(T-t, X_t;\phi)$, $\eta_t = \eta(T-t,X_t;\phi)$, for $t\in[0,T]$, and let $\mathcal{W}^\pi$ and M^η be the associated wealth process and supermartingale deflator, respectively. Then, the following identities hold:*

$$p\log\left(\mathcal{W}^\pi_T\right) - p\log\left(\mathcal{W}^\pi_t\right) + \phi(0,X_T) - \phi(T-t,X_t) = \log\left(Z^{\phi,T}_T\right) - \log\left(Z^{\phi,T}_t\right), \tag{A.7}$$

$$q\log\left(M^\eta_T\right) - q\log\left(M^\eta_t\right) + (1-q)(\phi(0,X_T)-\phi(T-t,X_t)) = \log\left(Z^{\phi,T}_T\right) - \log\left(Z^{\phi,T}_t\right),$$

where $Z^{\phi,T}$ is given in (A.3).

Using Lemmas A.3 and A.4, the proof of Proposition 3.6 is now given.

Proof of Proposition 3.6. From Lemma A.1, [48, Assumptions 3.4–3.6] are satisfied, and since these imply the assumptions of [48, Lemma 4.1] to the matrix case, the well-posedness of the martingale problem for $\mathcal{L}^{v,T-\cdot}$ follows from [48, Lemma 4.1]. Since the martingale problem for L is also well-posed, it then follows from the discussion after (A.3) that $Z^{v,T}$ is a $\mathbb{P}^x$-martingale. Applying Lemma A.4 to v, it then follows from (A.7) and $v(0,x)=0$ that

$$\mathbb{E}\left[\left(\frac{\mathcal{W}^\pi_T}{\mathcal{W}^\pi_t}\right)^p\middle|\mathcal{F}_t\right] = e^{v(T-t,X_t)} = \left(\mathbb{E}\left[\left(\frac{M^\eta_T}{M^\eta_t}\right)^q\middle|\mathcal{F}_t\right]\right)^{1/(1-q)} \qquad \text{for all } t\leq T. \tag{A.8}$$

Therefore the optimality of π follows from [26, Lemma 5] and (2.12) is verified using (A.8), (A.7) with $\phi = v$, and the martingale property of $Z^{v,T}$. ∎

Proof of Lemma A.3. From (2.18),

$$\sum_{i,j,k,l=1}^d \theta_{ij}\overline{A}_{(ij),(kl)}(x)\theta_{kl} = \sum_{i,j,k,l=1}^d \theta_{ij}\mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right)(x)\theta_{kl} - q\sum_{i,j,k,l=1}^d \theta_{ij}\rho'\left(a^{ij}\right)'C'\Theta C a^{kl}\rho\theta_{kl}.$$

Define the matrix Y via $Y_{kl} := \sum_{i,j=1}^d a^{ij}_{kl}\theta_{ij}$ for $k,l=1,\dots,d$. It then follows that

$$\sum_{i,j,k,l=1}^d \theta_{ij}\rho'\left(a^{ij}\right)'C'\Theta C a^{kl}\rho\theta_{kl} = \rho' Y' C'\Theta C Y\rho.$$

We claim that

$$0\leq \rho'Y'C'\Theta CY\rho \leq \lambda_C^2\lambda_\rho^2\mathrm{Tr}\left(YY'\right). \tag{A.9}$$

Admitting this fact and plugging back in for Y yields

$$0\leq \sum_{i,j,k,l=1}^d \theta_{ij}\rho'\left(a^{ij}\right)'C'\Theta C a^{kl}\rho\theta_{kl} \leq \lambda_C^2\lambda_\rho^2\sum_{i,j,k,l=1}^d \theta_{ij}\mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right)(x)\theta_{kl}. \tag{A.10}$$

If $p<0$, then $q>0$ and (A.5) holds for $\underline{\kappa} = 1-q\lambda_C^2\lambda_\rho^2$ and $\overline{\kappa}=1$. If $0<p<1$, then $q<0$ and hence (A.5) holds for $\underline{\kappa}=1$ and $\overline{\kappa} = 1-q\lambda_C^2\lambda_\rho^2$.

It remains to show (A.9). First, $\rho' Y' C' \Theta C Y \rho \geq 0$ follows from $\Theta \geq 0$. On the other hand, since by construction $\Theta \leq 1$ (see (2.16)), we have

$$\rho' Y' C' \Theta C Y \rho \leq \rho' Y' C' C Y \rho \leq \lambda_C^2 \rho' Y' Y \rho = \lambda_C^2 \mathrm{Tr}\left(Y \rho \rho' Y'\right),$$

where the second inequality holds since $C'C$ and CC' have the same eigenvalues. On the other hand, note $\mathrm{Tr}\,(NMN') \leq \lambda_M \mathrm{Tr}\,(NN')$ for any $N \in \mathbb{M}^d$ and $M \in \mathbb{S}^d$, where λ_M is the maximal eigenvalue of M. Therefore, using the fact that the largest eigenvalue of $\rho\rho'$ on $\mathbb{S}^d_{++}$ is λ_ρ^2, we obtain $\mathrm{Tr}\,(Y\rho\rho' Y) \leq \lambda_\rho^2 \mathrm{Tr}\,(YY')$ and confirm (A.9). ■

Proof of Lemma A.4. The following proof follows exactly the same steps as [27, Lemma B.3]. However, here we work with a semilinear equation and a matrix valued state variable. In particular, matrix valued coefficients complicate notation and calculations considerably. Therefore, for clarity, we present a detailed proof here.

First of all, set

$$\begin{aligned} \mathbf{A} &:= p \log\left(\mathcal{W}_T^\pi\right) - p\log\left(\mathcal{W}_t^\pi\right) + \phi(0, X_T) - \phi(T-t, X_t), \\ \mathbf{B} &:= q\log\left(M_T^\eta\right) - q\log\left(M_t^\eta\right) + (1-q)(\phi(0,X_T) - \phi(T-t, X_t)). \end{aligned} \tag{A.11}$$

The identities in (A.7) are verified in the following four steps:

(1) Use the dynamics for $\mathcal{W}^\pi$ in (2.10), the definition of M^η in (A.1), and the definitions of π, η in (2.15) and (A.6) to write

$$\begin{aligned} \mathbf{A} &= \int_t^T \mathbf{A1}_u du + \sum_{k,l=1}^d \int_t^T \mathbf{A2}_u^{kl} dB_u^{kl} + \sum_{k=1}^m \int_t^T \mathbf{A3}_u^k dW_u^k, \\ \mathbf{B} &= \int_t^T \mathbf{B1}_u du + \sum_{k,l=1}^d \int_t^T \mathbf{B2}_u^{kl} dB_u^{kl} + \sum_{k=1}^m \int_t^T \mathbf{B3}_u^k dW_u^k, \end{aligned} \tag{A.12}$$

where $\mathbf{A1}, \mathbf{B1} : [0,T] \times \mathbb{S}^d_{++} \to \mathbb{R}$, $\mathbf{A2}, \mathbf{B2} : [0,T]\times\mathbb{S}^d_{++} \to \mathbb{M}^d$, and $\mathbf{A3}, \mathbf{B3} : [0,T]\times \mathbb{S}^d_{++} \to \mathbb{R}^m$. These functions with time subscripts represent, for example, $\mathbf{A1}_u = \mathbf{A1}(T-u, X_u)$.

(2) Add and subtract

$$\begin{aligned} &\frac{1}{2}\sum_{k,l=1}^d \int_t^T \left(\mathbf{A2}_u^{kl}\right)^2 du + \frac{1}{2}\sum_{k=1}^m \int_t^T \left(\mathbf{A3}_u^k\right)^2 du, \\ &\frac{1}{2}\sum_{k,l=1}^d \int_t^T \left(\mathbf{B2}_u^{kl}\right)^2 du + \frac{1}{2}\sum_{k=1}^m \int_t^T \left(\mathbf{B3}_u^k\right)^2 du, \end{aligned} \tag{A.13}$$

to the right-hand side of $\mathbf{A}$ and $\mathbf{B}$, respectively, to obtain

$$\mathbf{A} = \int_t^T \left(\mathbf{A1}_u + \frac{1}{2}\sum_{k,l=1}^d \left(\mathbf{A2}_u^{kl}\right)^2 + \frac{1}{2}\sum_{k=1}^m \left(\mathbf{A3}_u^k\right)^2 \right) du + \log(\mathcal{Z}_T) - \log(\mathcal{Z}_t),$$

$$\mathbf{B} = \int_t^T \left(\mathbf{B1}_u + \frac{1}{2}\sum_{k,l=1}^d \left(\mathbf{B2}_u^{kl}\right)^2 + \frac{1}{2}\sum_{k=1}^m \left(\mathbf{B3}_u^k\right)^2 \right) du + \log(\tilde{\mathcal{Z}}_T) - \log(\tilde{\mathcal{Z}}_t),$$

where

(A.14)
$$\mathcal{Z}=\mathcal{E}\left(\int\sum_{k,l=1}^{d}\mathbf{A2}_u^{kl}dB_u^{kl}+\int\sum_{k=1}^{m}\mathbf{A3}_u^{k}dW_u^{l}\right),\quad \tilde{\mathcal{Z}}=\mathcal{E}\left(\int\sum_{k,l=1}^{d}\mathbf{B2}_u^{kl}dB_u^{kl}+\int\sum_{k=1}^{m}\mathbf{B3}_u^{k}dW_u^{k}\right).$$

(3) Show that for $u\le T$ and $x\in\mathbb{S}^d_{++}$,

$$\left(\mathbf{A1}+\frac{1}{2}\sum_{k,l=1}^{d}\left(\mathbf{A2}^{kl}\right)^2+\frac{1}{2}\sum_{k=1}^{m}\left(\mathbf{A3}^{k}\right)^2\right)(T-u,x)=\left(-\phi_t+\mathfrak{F}[\phi]\right)(T-u,x)=0,$$
$$\left(\mathbf{B1}+\frac{1}{2}\sum_{k,l=1}^{d}\left(\mathbf{B2}^{kl}\right)^2+\frac{1}{2}\sum_{k=1}^{m}\left(\mathbf{B3}^{k}\right)^2\right)(T-u,x)=\left(-\phi_t+\mathfrak{F}[\phi]\right)(T-u,x)=0.$$

(4) Show that $\mathcal{Z}=\tilde{\mathcal{Z}}=Z^{\phi,T}$.

Combining the above four steps, (A.7) is then verified.

Remark A.5. For notational ease the following conventions are used: (1) we will omit $\int_t^T$ and the integrator du from all integrals; (2) we will suppress the argument $(T-u,X_u)$ from all functions; (3) we will also drop all time subscripts. Thus, for example, we will write

$$f+g'dB\rho+h'dW=\int_t^T f(T-u,X_u)du+\int_t^T g(T-u,X_u)'dB_u\rho(X_u)+\int_t^T h(T-u,X_u)'dW_u.$$

The first identity in (A.7) is now shown. Using $\rho'\rho CC'+DD'=\mathbb{1}_m$ and the dynamics of $\mathcal{W}^\pi$ in (2.10), Itô's formula gives (A.12), where

(A.15)
$$\begin{aligned}\mathbf{A1}&=pr+p\pi'\Sigma\nu-\frac{1}{2}p\pi'\Sigma\pi-\phi_t+L\phi,\\ \mathbf{A2}^{kl}&=p(C'\sigma'\pi)_k\rho_l+\sum_{i,j=1}^{d}a^{ij}_{kl}D_{(ij)}\phi,\\ \mathbf{A3}^k&=p(D'\sigma'\pi)_k.\end{aligned}$$

While the second step follows from definitions of Z and $\tilde{Z}$, we move on to the third step. For $u\le T$ and $x\in\mathbb{S}^d_{++}$, it follows that

$$\begin{aligned}&\mathbf{A1}+\frac{1}{2}\sum_{k,l=1}^{d}\left(\mathbf{A2}^{kl}\right)^2+\sum_{k=1}^{m}\left(\mathbf{A3}^{k}\right)^2\\ &\quad=pr+p\pi'\Sigma\nu-\frac{1}{2}p\pi'\Sigma\pi-\phi_t+L\phi+\frac{1}{2}p^2\pi'\sigma CC'\sigma'\pi\rho'\rho+p\pi'\left(\sum_{i,j1}^{d}\sigma Ca^{ij}\rho D_{(ij)}\phi\right)\\ &\qquad+\frac{1}{2}\sum_{i,j,k,l=1}^{d}D_{(ij)}\phi\mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right)D_{(kl)}\phi+\frac{1}{2}p^2\pi'\sigma DD'\sigma'\pi\end{aligned}$$

$$= \frac{1}{2}p(p-1)\pi'\Sigma\pi + p\pi'\Sigma\nu + p\pi'\left(\sum_{ij=1}^{d}\sigma C a^{ij}\rho D_{(ij)}\phi\right)$$

$$+ pr - \phi_t + L\phi + \frac{1}{2}\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi \ \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D_{(kl)}\phi. \tag{A.16}$$

The terms above containing π are

$$\frac{1}{2}p(p-1)\pi'\Sigma\pi + p\pi'\left(\Sigma\nu + \sum_{i,j=1}^{d}\sigma C a^{ij}\rho D_{(ij)}\phi\right).$$

Using (2.15), we obtain the following expression for the quadratic function in the previous line:

$$-\frac{1}{2}q\nu'\Sigma\nu - q\sum_{i,j=1}^{d}\nu'\sigma C a^{ij}\rho D_{(ij)}\phi - \frac{1}{2}q\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi\ \rho'\left(a^{ij}\right)' C'\Theta C a^{kl}\rho D_{(kl)}\phi$$

for both cases $m \geq n$ or $m < n$. Thus, substituting the previous expression into (A.16), using the expressions for $\bar{A}, V$ in (2.19) and $\mathfrak{F}$ in (2.20) gives

$$\begin{aligned}
&\mathbf{A1} + \frac{1}{2}\sum_{k,l=1}^{d}\left(\mathbf{A2}^{kl}\right)^2 + \sum_{k=1}^{m}\left(\mathbf{A3}^{k}\right)^2 \\
&= pr - \frac{1}{2}q\nu'\Sigma\nu - q\sum_{i,j=1}^{d}\nu'\sigma C a^{ij}\rho D_{(ij)}\phi - \frac{1}{2}\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi\rho'\left(a^{ij}\right)' C'\Theta C a^{kl}\rho D_{(kl)}\phi \\
&\quad - \phi_t + L\phi + \frac{1}{2}\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D_{(kl)}\phi \\
&= -\phi_t + L\phi - q\sum_{i,j=1}^{d}\nu'\sigma C a^{ij}\rho D_{(ij)}\phi + \frac{1}{2}\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi\bar{A}_{(ij),(kl)}D_{(kl)}\phi + V \\
&= -\phi_t + \mathfrak{F}[\phi] \\
&= 0,
\end{aligned} \tag{A.17}$$

finishing the third step. For the last step, recall the definition of $Z^{\phi,T}$ from (A.3). Comparing with the definition of $\mathcal{Z}$ in (A.14), it suffices to show that

$$\begin{aligned}
\mathbf{A2}^{kl} &= -q(C'\sigma'\nu)_k\rho_l + \sum_{i,j=1}^{d}\left(a^{ij}_{kl} - q(C'\Theta C a^{ij}\rho)_k\rho_l\right) D_{(ij)}\phi, \\
\mathbf{A3}^{k} &= -q(D'\sigma'\nu)_k - q\sum_{i,j=1}^{d}\left(D'\Theta C a^{ij}\rho\right)_k D_{(ij)}\phi.
\end{aligned} \tag{A.18}$$

Using (2.15) for $m \geq n$ it follows that (recall $\Theta = \sigma'\Sigma^{-1}\sigma$ when $m \geq n$)

$$p(\sigma'\pi) = -q\sigma'\Sigma^{-1}\left(\Sigma\nu + \sum_{i,j=1}^{d} \sigma C a^{ij}\rho D_{(ij)}\phi\right) = -q\sigma'\nu - q\sum_{i,j=1}^{d} \Theta C a^{ij}\rho D_{(ij)}\phi.$$

Similarly, using (2.15) for $m < n$ gives (recall $\Theta = 1_m$ for $m < n$),

$$p(\sigma'\pi) = -q\sigma'\sigma(\sigma'\sigma)^{-1}\left(\sigma'\nu + \sum_{i,j=1}^{d} C a^{ij}\rho D_{(ij)}\phi\right) = -q\sigma'\nu - q\sum_{i,j=1}^{d} \Theta C a^{ij}\rho D_{(ij)}\phi.$$

Therefore, in both cases $m \geq n$, $m < n$ we have, using the definition of $\mathbf{A2}, \mathbf{A3}$ in (A.15), that

$$\mathbf{A2}^{kl} = p(C'\sigma'\pi)_k\rho_l + \sum_{i,j=1}^{d} a_{kl}^{ij} D_{(ij)}\phi = -q(C'\sigma'\nu)_k\rho_l + \sum_{i,j=1}^{d}\left(a_{kl}^{ij} - q(C'\Theta C a^{ij}\rho)_k\rho_l\right) D_{(ij)}\phi,$$

$$\mathbf{A3}^{k} = p(D'\sigma'\pi)_k = -q(D'\sigma'\nu)_k - q\sum_{i,j=1}^{d} (D'\Theta C a^{ij}\rho)_k D_{(ij)}\phi,$$

which verifies (A.18).

The proof for the second identity in (A.7) is similar. First, using the definition of M^η in (A.1), Itô's formula yields the second identity in (A.12), where

$$\begin{aligned}
\text{(A.19)} \qquad \mathbf{B1} &= -qr + (1-q)(-\phi_t + L\phi) \\
&\quad - \frac{1}{2}q\left(\sum_{k,l=1}^{d}\left(-(C'\sigma'\nu)_k\rho_l + \eta_{kl} - (C'\Theta C\eta\rho)_k\rho_l\right)^2 \right.\\
&\qquad \left. + \sum_{k=1}^{m}\left((D'\sigma'\nu)_k + (D'\Theta C\eta\rho)_k\right)^2\right), \\
\mathbf{B2}^{kl} &= q\left(-(C'\sigma'\nu)_k\rho_l + \eta_{kl} - (C'\Theta C\eta\rho)_k\rho_l\right) + (1-q)\sum_{i,j=1}^{d} a_{kl}^{ij} D_{(ij)}\phi, \\
\mathbf{B3}^{k} &= -q\left((D'\sigma'\nu)_k + (D'\Theta C\eta\rho)_k\right).
\end{aligned}$$

Using $(1-q)p = -q$ we obtain

$$\begin{aligned}
\text{(A.20)} \qquad \mathbf{B1} + \frac{1}{2}\sum_{k,l=1}^{d}\left(\mathbf{B2}^{kl}\right)^2 + \frac{1}{2}\sum_{k=1}^{m}(\mathbf{B3}^k)^2 &= (1-q)pr + (1-q)(-\phi_t + L\phi) \\
&\quad - \frac{1}{2}q(1-q)\left(\sum_{k,l=1}^{d}\left(-(C'\sigma'\nu)_k\rho_l + \eta_{kl} - (C'\Theta C\eta\rho)_k\rho_l\right)^2\right. \\
&\qquad \left. + \sum_{k=1}^{m}\left((D'\sigma'\nu)_k + (D'\Theta C\eta\rho)_k\right)^2\right)
\end{aligned}$$

$$+ q(1-q) \sum_{i,j,k,l=1}^{d} \left(-(C'\sigma'\nu)_k\rho_l + \eta_{kl} - (C'\Theta C\eta\rho)_k\rho_l\right) a^{ij}_{kl} D_{(ij)}\phi$$
$$+ \frac{1}{2}(1-q)^2 \sum_{k,l=1}^{d} \left(\sum_{i,j=1}^{d} a^{ij}_{kl} D_{(ij)}\phi\right)^2 .$$

Now, using $\rho'\rho CC' + DD' = \mathbb{1}_m$ gives

$$\begin{aligned}
&\sum_{k,l=1}^{d} \left(-(C'\sigma'\nu)_k\rho_l + \eta_{kl} - (C'\Theta C\eta\rho)_k\rho_l\right)^2 + \sum_{k=1}^{m} \left((D'\sigma'\nu)_k + (D'\Theta C\eta\rho)_k\right)^2 \\
&\quad = \nu'\sigma CC'\sigma'\nu\rho'\rho + \mathrm{Tr}\left(\eta'\eta\right) + \rho'\eta'C'\Theta CC'\Theta C\eta\rho\rho'\rho - 2\nu'\sigma C\eta\rho \\
&\qquad + 2\nu'\sigma CC'\Theta C\eta\rho\rho'\rho - 2\rho'\eta'C'\Theta C\eta\rho + \nu'\sigma DD'\sigma'\nu \\
&\qquad + \rho'\eta'C'\Theta DD'\Theta C\eta\rho + 2\nu'\sigma DD'\Theta C\eta\rho \\
&\quad = \nu'\sigma(CC'\rho'\rho + DD')\sigma'\nu + \rho'\eta'C'\Theta(CC'\rho'\rho + DD')\Theta C\eta\rho \\
&\qquad + 2\nu'\sigma(CC'\rho'\rho + DD')\Theta C\eta\rho + \mathrm{Tr}\left(\eta'\eta\right) - 2\nu'\sigma C\eta\rho - 2\rho'\eta'C'\Theta C\eta\rho \\
&\quad = \nu'\Sigma\nu + \rho'\eta'C'\Theta\Theta C\eta\rho + 2\nu'\sigma\Theta C\eta\rho + \mathrm{Tr}\left(\eta'\eta\right) - 2\nu'\sigma C\eta\rho - 2\rho'\eta'C'\Theta C\eta\rho \\
&\quad = \nu'\Sigma\nu + \mathrm{Tr}\left(\eta'\eta\right) - \rho'\eta'C'\Theta C\eta\rho,
\end{aligned}$$

where the last equality follows since the definition of Θ in (2.16) implies both $\Theta\Theta = \Theta$ and $\sigma\Theta = \sigma$. We also have

$$\begin{aligned}
&\sum_{i,j,k,l=1}^{d} \left(-(C'\sigma'\nu)_k\rho_l + \eta_{kl} - (C'\Theta C\eta\rho)_k\rho_l\right) a^{ij}_{kl} D_{(ij)}\phi \\
&\qquad = \sum_{i,j=1}^{d} \left(-\nu'\sigma C a^{ij}\rho + \mathrm{Tr}\left(\eta' a^{ij}\right) - \rho'\eta'C'\Theta C a^{ij}\rho\right) D_{(ij)}\phi,
\end{aligned}$$

$$\sum_{k,l=1}^{d} \left(\sum_{i,j=1}^{d} a^{ij}_{kl} D_{(ij)}\phi\right)^2 = \sum_{i,j,k,l=1}^{d} D_{(ij)}\phi \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D_{(kl)}\phi.$$

Plugging all of this into (A.20) yields

$$\frac{1}{1-q}\left(\mathbf{B1} + \frac{1}{2}\sum_{k,l=1}^{d}\left(\mathbf{B2}^{kl}\right)^2 + \frac{1}{2}\sum_{k=1}^{m}(\mathbf{B3}^k)^2\right) = pr - \phi_t + L\phi - \frac{1}{2}q\left(\nu'\Sigma\nu + \mathrm{Tr}\left(\eta'\eta\right) - \rho'\eta'C'\Theta C\eta\rho\right) \tag{A.21}$$

$$+ q \sum_{i,j=1}^{d} \left(-\nu'\sigma C a^{ij}\rho + \mathrm{Tr}\left(\eta' a^{ij}\right) - \rho'\eta' C\Theta C a^{ij}\rho\right) D_{(ij)}\phi$$
$$+ \frac{1}{2}(1-q) \sum_{i,j,k,l=1}^{d} D_{(ij)}\phi \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D_{(kl)}\phi.$$

On the right-hand side, terms involving η are

$$\text{(A.22)}\quad -\frac{1}{2}q\mathrm{Tr}\left(\eta'\eta\right) + \frac{1}{2}q\rho'\eta' C'\Theta C\eta\rho + q\sum_{i,j=1}^{d} \mathrm{Tr}\left(\eta' a^{ij}\right) D_{(ij)}\phi - q\sum_{i,j=1}^{d} \rho'\eta' C'\Theta C a^{ij}\rho D_{(ij)}\phi.$$

For η in (A.6), the following identities hold:

$$\mathrm{Tr}\left(\eta'\eta\right) = \sum_{i,j,k,l=1}^{d} D_{(ij)}\phi\ \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D_{(kl)}\phi,$$
$$\rho'\eta' C'\Theta C\eta\rho = \sum_{i,j,k,l=1}^{d} D_{(ij)}\phi\ \rho'(a^{ij})' C'\Theta C a^{kl}\rho D_{(kl)}\phi,$$
$$\sum_{i,j=1}^{d} \mathrm{Tr}\left(\eta' a^{ij}\right) D_{(ij)}\phi = \sum_{i,j,k,l=1}^{d} D_{(ij)}\phi\ \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D_{(kl)}\phi,$$
$$\sum_{i,j=1}^{d} \rho'\eta' C'\Theta C a^{ij}\rho D_{(ij)}\phi = \sum_{i,j,k,l=1}^{d} D_{(ij)}\phi\ \rho'(a^{ij})' C'\Theta C a^{kl}\rho D_{(kl)}\phi.$$

Using above identities in (A.22), we obtain the following expression for (A.22):

$$\frac{1}{2}q \sum_{i,j,k,l=1}^{d} D_{(ij)}\phi \left(\mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) - \rho'(a^{ij})' C'\Theta C a^{kl}\rho\right) D_{(kl)}\phi.$$

Inserting this into (A.21) gives

$$\frac{1}{1-q}\left(\mathbf{B1} + \frac{1}{2}\sum_{i,j=1}^{d}(\mathbf{B2}^{ij})^2 + \frac{1}{2}\sum_{l=1}^{m}(\mathbf{B3}^{l})^2\right)$$
$$= pr - \phi_t + L\phi - \frac{1}{2}q\nu'\Sigma\nu - q\sum_{i,j=1}^{d}\nu'\sigma C a^{ij}\rho D_{(ij)}\phi$$
$$+ \frac{1}{2}(1-q)\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi \mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) D_{(kl)}\phi$$
$$+ \frac{1}{2}q\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi\left(\mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) - \rho'(a^{ij})' C'\Theta C a^{kl}\rho\right) D_{(kl)}\phi$$

$$
\begin{aligned}
&= -\phi_t + L\phi - q\sum_{i,j=1}^{d} \nu'\sigma C a^{ij}\rho D_{(ij)}\phi \\
&\quad + \frac{1}{2}\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi\left(\operatorname{Tr}\left(a^{ij}\left(a^{kl}\right)'\right) - q\rho'(a^{ij})'C'\Theta C a^{kl}\rho\right) D_{(kl)}\phi + pr - \frac{1}{2}q\nu'\Sigma\nu \\
&= -\phi_t + \mathfrak{F}[\phi] \\
&= 0,
\end{aligned}
$$

where the second to last equality uses (2.19) and (2.20). Thus, the third step is complete.

Turning to the last step, comparing $Z^{\phi,T}$ in (A.3) with $\tilde{\mathcal{Z}}$ in (A.14), it suffices to show

$$
\begin{aligned}
\mathbf{B2}^{kl} &= -q(C'\sigma'\nu)_k\rho_l + \sum_{i,j=1}^{d}\left(a^{ij}_{kl} - q(C'\Theta C a^{ij}\rho)_k\rho_l\right) D_{(ij)}\phi, \\
\mathbf{B3}^{k} &= -q(D'\sigma'\nu)_k - q\sum_{i,j=1}^{d}\left(D'\Theta C a^{ij}\rho\right)_k D_{(ij)}\phi.
\end{aligned}
$$

Using the definitions of **B2** and **B3** in (A.19) it suffices to show that

$$
\begin{aligned}
q\eta_{kl} - q(C'\Theta C\eta\rho)_k\rho_l + (1-q)\sum_{i,j=1}^{d} a^{ij}_{kl}D_{(ij)}\phi &= \sum_{i,j=1}^{d}\left(a^{ij}_{kl} - q(C'\Theta C a^{ij}\rho)_k\rho_l\right) D_{(ij)}\phi, \\
(D'\Theta C\eta\rho)_k &= \sum_{i,j=1}^{d}(D'\Theta C a^{ij}\rho)_k D_{(ij)}\phi.
\end{aligned}
$$

Since $\eta_{kl} = \sum_{i,j=1}^{d} a^{ij}_{kl}D_{(ij)}\phi$ from (A.6) the last two identities readily follow, finishing the proof. ■

Appendix B. Proofs for section 3.2.

Proof of Theorem 3.8. Having verified all assumptions of [48, Theorems 2.11 and 3.9] in Lemma A.1, Statement 2.5(i) readily follows from the previous reference. Note that $\nabla h = \nabla v - \nabla\hat{v}$, Statement 2.5(ii) follows from $\nabla h(T,\cdot) \to 0$ in part (i) and the form of π in (2.15).

To prove part (iii), let us collect two facts from [48]. First [48, Proposition 2.3(i)] implies that $\mathbb{P}^{\hat{v},x}$, as the solution to the martingale problem for $\mathcal{L}^{\hat{v}}$, is a well-defined probability measure. Therefore the discussion after (A.3) proves that $\mathbb{P}^{\hat{v},x}$ is equivalent to $\mathbb{P}^x$ on $\mathcal{F}_t$ for any $t \geq 0$. Second,

$$
\text{(B.1)} \qquad \lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^{\hat{v},x}}\left[\int_0^t \sum_{i,j,k,l=1}^{d} D_{(ij)}h\bar{A}_{(ij),(kl)}D_{(kl)}h(T-u,X_u)\,du\right] = 0.
$$

Indeed, since the integrand in (B.1) is independent of the Brownian motion W, (B.1) is proved in [48, Theorems 2.9 and 3.9].

Let us use the previous two facts to prove (2.25) first. To this end, using (2.15), we obtain in either case $m \geq n$ or $m < n$,

$$
\begin{aligned}
&(\pi(T-t,x;v) - \pi(x;\hat{v}))' \, \Sigma(x) \, (\pi(T-t,x;v) - \pi(x;\hat{v})) \\
&= \frac{1}{(1-p)^2} \left(\sum_{i,j,k,l=1}^{d} D_{(ij)} h \rho' (a^{ij})' C' \Theta C a^{kl} \rho D_{(kl)} h \right) (T-t,x) \\
&\leq \frac{\lambda_C^2 \lambda_\rho^2}{(1-p)^2} \left(\sum_{i,j,k,l=1}^{d} D_{(ij)} h \mathrm{Tr} \left(a^{ij} \left(a^{kl} \right)' \right) D_{(kl)} h \right) (T-t,x) \\
&\leq \frac{\lambda_C^2 \lambda_\rho^2}{\underline{\kappa}(1-p)^2} \left(\sum_{i,j,k,l=1}^{d} D_{(ij)} h \bar{A}_{(ij),(kl)} D_{(kl)} h \right) (T-t,x),
\end{aligned}
$$

where the first inequality follows from (A.10) and the second inequality follows from the first inequality in (A.5). Then (B.1) yields

$$
\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^{\hat{v},x}} \left[\int_0^t \left(\pi_u^T - \hat{\pi}_u \right)' \Sigma(X_u) \left(\pi_u^T - \hat{\pi}_u \right) \, du \right] = 0.
$$

This implies the convergence in probability $\mathbb{P}^{\hat{v},x}$, hence in $\mathbb{P}^x$, since $\mathbb{P}^{\hat{v},x}$ is equivalent to $\mathbb{P}^x$ on $\mathcal{F}_t$.

To prove (2.24), apply the first identity of (A.7), where we choose $\phi = v$ from Proposition 3.5 and $\pi = \pi^T$ from (2.21). Taking the difference of this identity when $t = t$ and $t = 0$, respectively, yields

$$
\left(\frac{\mathcal{W}_t^T}{w} \right)^p = Z_t^{v,T} e^{v(T,x) - v(T-t,X_t)}.
$$

On the other hand, apply the first identity of (A.7) again, but choose $\pi = \hat{\pi}$ from (2.23) and $\phi(t,x) = \hat{\lambda} t + \hat{v}(x)$, where $(\hat{\lambda}, \hat{v})$ comes from Proposition 3.7 and the current choice of ϕ satisfies $\phi_t = \mathfrak{F}[\phi]$ due to (2.22). Taking the difference of this identity when $t = t$ and $t = 0$, respectively, we obtain

$$
\left(\frac{\hat{\mathcal{W}}_t}{w} \right)^p = Z_t^{\hat{v}} e^{\hat{\lambda} T + \hat{v}(x) - \hat{\lambda}(T-t) - \hat{v}(X_t)}.
$$

Therefore, the ratio between the previous two identities reads

$$
\frac{\mathcal{W}_t^T}{\hat{\mathcal{W}}_t} = \left(\frac{Z_t^{v,T}}{Z_t^{\hat{v}}} e^{h(T,x) - h(T-t,X_t)} \right)^{\frac{1}{p}}, \tag{B.2}
$$

where h is defined in Statement 2.5(i). It has been proved in part (i) that $h(T,\cdot) \to C$ for some constant C. Therefore $e^{h(T,x) - h(T-t,X_t)} \to 1$ a.s. as $T \to \infty$. In the next paragraph, we will show

(B.3) $$\mathbb{P}^{\hat{v},x} - \lim_{T\to\infty} \frac{Z_t^{v,T}}{Z_t^{\hat{v}}} = 1.$$

Plugging the previous two convergence back into (B.2), it follows that

$$\mathbb{P}^{\hat{v},x} - \lim_{T\to\infty} \frac{\mathcal{W}_t^T}{\hat{\mathcal{W}}_t} = 1.$$

Recall from Remark A.2 that $\mathcal{W}^T/\hat{\mathcal{W}}$ is a $\mathbb{P}^{\hat{v},x}$-supermartingale. Combining the previous convergence with Scheffé's lemma, we obtain

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^{\hat{v},x}}\left[\left|\frac{\mathcal{W}_t^T}{\hat{\mathcal{W}}_t} - 1\right|\right] = 0.$$

Applying [24, Lemma 3.9] under $\mathbb{P}^{\hat{v},x}$, the previous convergence then yields

$$\mathbb{P}^{\hat{v},x} - \lim_{T\to\infty} \sup_{0\le u\le t}\left|\frac{\mathcal{W}_u^T}{\hat{\mathcal{W}}_u} - 1\right| = 0.$$

Hence (2.24) is confirmed after utilizing the equivalence between $\mathbb{P}^{\hat{v},x}$ and $\mathbb{P}^x$.

It remains to prove (B.3). To this end, using (A.3) for v and $\hat{v}$, and the definition of h, it follows that $Z_t^{v,T}/Z_t^{\hat{v}} = \mathcal{E}(L^T)_t$, where the $\mathbb{P}^{\hat{v},x}$-local martingale L^T takes the form

$$\begin{aligned} L_t^T = &\int_0^t \sum_{k,l=1}^d d\hat{B}_u^{kl}\left(\sum_{i,j=1}^d \left(a_{kl}^{ij} - q(C'\Theta C a^{ij}\rho)_k\rho_l\right) D_{(ij)}h\right)(T-u,X_u)\\ &+\int_0^t \sum_{k=1}^m d\hat{W}_u^k\left(-q\sum_{i,j=1}^d (D'\Theta C a^{ij}\rho)_k D_{(ij)}h\right)(T-u,X_u), \qquad t\le T, \end{aligned}$$

where $\hat{B}$ and $\hat{W}$ are $\mathbb{P}^{\hat{v},x}$ independent $\mathbb{M}^d$ and $\mathbb{R}^m$ dimensional Brownian motions. Calculation using $\rho'\rho CC' + DD' = 1_m$ and $\Theta\Theta = \Theta$ shows that

$$\left[L^T, L^T\right]_t = \int_0^t \left(\sum_{i,j,k,l=1}^d D_{(ij)}h\left(\bar{A}_{(ij),(kl)} - q(1-q)\rho'\left(a^{ij}\right)' C'\Theta C a^{kl}\rho\right) D_{(kl)}h\right)(T-u,X_u)du.$$

Using (A.10) at $\theta = Dh \in \mathbb{S}^d$ it follows for $p<0$ $(0<q<1)$ that

$$[L^T, L^T]_t \le \int_0^t \left(\sum_{i,j,k,l=1}^d D_{(ij)}h\bar{A}_{(ij),(kl)}D_{(kl)}h\right)(T-u,X_u)du$$

and for $0<p<1$ $(q<0)$ that

$$\begin{aligned}[L^T, L^T]_t &\le \int_0^t \left(\sum_{i,j,k,l=1}^d D_{(ij)}h\left(\bar{A}_{(ij),(kl)} - q(1-q)\lambda_C^2\lambda_\rho^2\,\mathrm{Tr}\left(a^{ij}\left(a^{kl}\right)'\right)\right) D_{(kl)}h\right)(T-u,X_u)du\\ &\le \left(1 - \frac{1}{\underline{\kappa}}q(1-q)\lambda_C^2\lambda_\rho^2\right)\int_0^t \left(\sum_{i,j,k,l=1}^d D_{(ij)}h\bar{A}_{(ij),(kl)}D_{(kl)}h\right)(T-u,X_u)du,\end{aligned}$$

where the last inequality uses Lemma A.3. From (B.1) it thus follows that

$$\lim_{T\uparrow\infty} \mathbb{E}^{\mathbb{P}^{\hat{v},x}} \left[[L^T, L^T]_t\right] = 0,$$

which implies $\mathbb{P}^{\hat{v},x} - \lim_{T\to\infty}[L^T, L^T]_t = 0$. Combining the previous convergence and the fact that L^T is continuous local martingales, it follows that $\mathbb{P}^{\hat{v},x} - \lim_{T\to\infty} \mathcal{E}(L^T)_t = 1$, hence (B.3) holds. ■

Proof of Theorem 3.10. Given results in [48, Theorems 2.9 and 3.9], the statement follows from the same argument in [24, Theorem 2.9]. We now check that the assumptions in [24] are satisfied in the current setting. First, for each $T > 0$, there exists a probability measure $\mathbb{Q}^{T,x}$ such that $\mathbb{Q}^{T,x}$ is equivalent to $\mathbb{P}^x$ on $\mathcal{F}_T$ and such that $e^{-\int_0^\cdot r(X_u)du}S$ is a $\mathbb{Q}^{T,x}$-local martingale on $[0,T]$. Indeed, let $\theta : \mathbb{S}^d_{++} \mapsto \mathbb{R}^k$ be a continuous function and set

$$Z_t = \mathcal{E}\left(-\int_0^\cdot \sum_{k=1}^d \theta_k(X_u)dW_u^k\right)_t .$$

The continuity of θ and the $\mathbb{P}$ independence of X and W ensure that Z is also a $\mathbb{P}^x$-martingale; cf. [36, Lemma 4.8]. Under Assumption 3.9 we may choose $\theta = D'(DD')^{-1}\sigma'\nu$, and it follows that θ is continuous. Since Z is a $\mathbb{P}^x$-martingale, for each T we may define a probability $\mathbb{Q}^{T,x}$, which is equivalent to $\mathbb{P}^x$ on $\mathcal{F}_T$, via $d\mathbb{Q}^{T,x}/d\mathbb{P}^x|_{\mathcal{F}_T} = Z_T$. Using Girsanov's theorem, a direct calculation shows that $e^{-\int_0^\cdot r(X_u)du}S$ is a $\mathbb{Q}^{T,x}$-local martingale. Therefore [24, Assumption 2.3] is satisfied. On the other hand, Propositions 3.5 and 3.6 combined imply that the value of the optimization problem in (2.12) is finite for all $T \geq 0$. Therefore [24, Assumption 2.4] is satisfied as well. On the other hand, Assumptions 2.6 and 2.7 are exactly [24, Assumptions 2.1 and 2.2], respectively.

Therefore [24, Proposition 2.5] proves that, for all $\varepsilon > 0$,

(B.4)
$$\lim_{T\uparrow\infty} \mathbb{P}^{v,T,x}\left[\sup_{u\leq t}\left|\frac{\mathcal{W}_u^{1,T}}{\mathcal{W}_u^T} - 1\right| \geq \varepsilon\right] = 0,$$
$$\lim_{T\uparrow\infty} \mathbb{P}^{v,T,x}\left[\int_0^t \left(\pi_u^{1,T} - \pi_u^T\right)' \Sigma(X_u)\left(\pi_u^{1,T} - \pi_u^T\right) du \geq \varepsilon\right] = 0.$$

Here since the martingale problem for $\mathcal{L}^{v,T-\cdot}$ is well-posed (cf. [48, Lemma 4.1]), $\mathbb{P}^{T,v,x}$ is defined via (A.3) with $\phi = v$. From the definitions of $\mathbb{P}^{v,T,x}$ and $\mathbb{P}^{\hat{v},x}$, it follows that

$$\left.\frac{d\mathbb{P}^{v,T,x}}{d\mathbb{P}^{\hat{v},x}}\right|_{\mathcal{F}_t} = \frac{Z_t^{v,T}}{Z_t^{\hat{v}}}.$$

Note that both events on the left-hand side of (B.4) are $\mathcal{F}_t$-measurable. Therefore, (B.3) implies (B.4) holds when $\mathbb{P}^{v,T,x}$ is replaced by $\mathbb{P}^{\hat{v},x}$, hence also by $\mathbb{P}^x$, since $\mathbb{P}^{\hat{v},x}$ and $\mathbb{P}^x$ are equivalent on $\mathcal{F}_t$. Last, the extension to Statement 2.8 is immediate after utilizing Statement 2.5(iii). ■

Appendix C. Proofs for section 3.3.

Lemma C.1. *For $v = Tr(Mx)$ as in (3.1) it follows for $d \leq n$ that*

$$\mathfrak{F}[v](x) = Tr\left(x\left(2M\Lambda(1-q\rho\rho')\Lambda'M + K'M + MK - q\sigma'\nu\rho'\Lambda'M - qM\Lambda\rho\nu'\sigma + \frac{1}{2}\left(p(r_1+r_1') - q\sigma'\nu\nu'\sigma\right)\right)\right) + Tr\left(LL'M\right) + pr_0. \tag{C.1}$$

For $d > n$

$$\mathfrak{F}[v](x) = Tr\Bigg(x\Bigg(2M\Lambda\Lambda'M + K'M + MK - q\sigma'\nu\rho'\Lambda'M - qM\Lambda\rho\nu'\sigma + \frac{1}{2}\left(p(r_1+r_1') - q\sigma'\nu\nu'\sigma\right)\Bigg)\Bigg) - 2q\,Tr\left(x\sigma'\left(\sigma x\sigma'\right)^{-1}\sigma xM\Lambda\rho\rho'\Lambda'M\right) + Tr\left(LL'M\right) + pr_0. \tag{C.2}$$

Proof. Plugging in the model coefficients gives

$$\begin{aligned} b(x) &= LL' + Kx + xK', \qquad a^{ij}_{kl}(x) = \sqrt{x}_{ik}\Lambda_{jl} + \sqrt{x}_{jk}\Lambda_{il},\\ r(x) &= r_0 + \mathrm{Tr}\left(r_1x\right), \qquad \sigma(x) = \sigma\sqrt{x}, \qquad \nu(x) = \nu,\\ C(x) &= \mathbb{1}_d, \qquad \rho(x) = \rho. \end{aligned}$$

Therefore, using the definitions in (2.19), calculation shows that

$$\begin{aligned} \bar{b}_{ij}(x) &= (LL' + Kx + xK')_{ij} - q(x\sigma'\nu\rho'\Lambda')_{ij} - q(x\sigma'\nu\rho'\Lambda')_{ji},\\ A_{(ij),(kl)}(x) &= x_{ik}(\Lambda\Lambda')_{jl} + x_{il}(\Lambda\Lambda')_{jk} + x_{jk}(\Lambda\Lambda')_{il} + x_{jl}(\Lambda\Lambda')_{ik},\\ V(x) &= pr_0 + \frac{1}{2}p\mathrm{Tr}\left(x(r_1+r_1')\right) - \frac{1}{2}q\mathrm{Tr}\left(x\sigma'\nu\nu'\sigma\right) \end{aligned} \tag{C.3}$$

and

$$\begin{aligned} \bar{A}_{(ij),(kl)}(x) = {} & x_{ik}(\Lambda\Lambda')_{jl} - q(\sqrt{x}\Theta(x)\sqrt{x})_{ik}(\Lambda\rho\rho'\Lambda')_{jl}\\ & + x_{il}(\Lambda\Lambda')_{jk} - q(\sqrt{x}\Theta(x)\sqrt{x})_{il}(\Lambda\rho\rho'\Lambda')_{jk}\\ & + x_{jk}(\Lambda\Lambda')_{il} - q(\sqrt{x}\Theta(x)\sqrt{x})_{jk}(\Lambda\rho\rho'\Lambda')_{il}\\ & + x_{jl}(\Lambda\Lambda')_{ik} - q(\sqrt{x}\Theta(x)\sqrt{x})_{jl}(\Lambda\rho\rho'\Lambda')_{ik}. \end{aligned}$$

For the given v, $D_{(ij)}v = D_{(ji)}v = M_{ij}$ and $D_{(ij),(kl)}v = 0$. Therefore

$$\begin{aligned} &\sum_{i,j,k,l=1}^{d} A_{(ij),(kl)}D_{(ij),(kl)}v = 0,\\ &\sum_{i,j=1}^{d} \bar{b}_{ij}D_{(ij)}v = \mathrm{Tr}\left(x\left(K'M + MK - q\sigma'\nu\rho'\Lambda'M - qM\Lambda\rho\nu'\sigma\right)\right) + \mathrm{Tr}\left(LL'M\right), \end{aligned} \tag{C.4}$$

where we have used repeatedly that M, X are symmetric and that $\mathrm{Tr}\,(ABC) = \mathrm{Tr}\,(BCA) = \mathrm{Tr}\,(CAB)$ for matrices A, B, C. When $d \le n$, it follows that $\Theta(x) = \mathbb{1}_d$ and $\bar{A}$ from (7) simplifies to

$$\bar{A}_{(ij),(kl)}(x) = x_{ik}\left(\Lambda\Lambda' - q\Lambda\rho\rho'\Lambda'\right)_{jl} + x_{il}\left(\Lambda\Lambda' - q\Lambda\rho\rho'\Lambda'\right)_{jk} + x_{jk}\left(\Lambda\Lambda' - q\Lambda\rho\rho'\Lambda'\right)_{il} + x_{jl}\left(\Lambda\Lambda' - q\Lambda\rho\rho'\Lambda'\right)_{ik},$$

and hence using the symmetry for $\Lambda\Lambda' - q\Lambda\rho\rho'\Lambda'$,

$$\text{(C.5)} \qquad \frac{1}{2}\sum_{i,j,k,l=1}^{d} \bar{A}_{(ij),(kl)} D_{(ij)} v D_{(kl)} v = 2\mathrm{Tr}\left(x\left(M\Lambda(1 - q\rho\rho'\Lambda' M)\right)\right).$$

Therefore, (C.1) follows using (C.3), (C.4), (C.5) and the definition of $\mathfrak{F}$ in (2.18). When $d > n$,

$$\sqrt{x}\Theta(x)\sqrt{x} = \sqrt{x}\left(\sigma'\Sigma^{-1}\sigma\right)(x)\sqrt{x} = x\sigma'\left(\sigma x\sigma'\right)^{-1}\sigma x;$$

thus, using (7) it follows that

$$\text{(C.6)} \qquad \frac{1}{2}\sum_{i,j,k,l=1}^{d} \bar{A}_{(ij),(kl)} D_{(ij)} v D_{(kl)} v = 2\mathrm{Tr}\left(xM\Lambda\Lambda' M\right) - 2q\mathrm{Tr}\left(x\sigma'\left(\sigma x\sigma'\right)^{-1}\sigma x M\Lambda\rho\rho'\Lambda' M\right).$$

Equation (C.2) now follows from (C.3), (C.4), and (C.6). ■

Proof of Proposition 3.11. Using Lemma C.1 it follows for $d \le n$ that if M solves (3.2), then $\mathfrak{F}[v] = \lambda$ with $\lambda = \mathrm{Tr}\,(LL'M) + pr_0$. Now, with $D = -M$, (3.2) takes the form

$$D\left(2\Lambda(1 - q\rho\rho')\Lambda'\right)D - D(K - q\Lambda\rho\nu'\sigma) - (K - q\Lambda\rho\nu'\sigma)'D - \frac{1}{2}\left(-p(r_1 + r_1') + q\sigma'\nu\nu'\sigma\right) = 0.$$

Since the eigenvalues of $\rho\rho'$ are $\rho'\rho$ and 0, then

$$2\Lambda(1 - q\rho\rho')\Lambda' \ge 2(1 - q\rho'\rho)\Lambda\Lambda' > 0.$$

Furthermore, by assumption $-p(r_1 + r_1') + q\sigma'\nu\nu'\sigma > 0$. Thus, the Riccati equation takes the form

$$\text{(C.7)} \qquad D\mathbf{B}\mathbf{B}'D - D\mathbf{A} - \mathbf{A}'D - \mathbf{C}\mathbf{C}' = 0,$$

where $\mathbf{B} = \sqrt{2\Lambda(1 - q\rho\rho')\Lambda'}$, $\mathbf{A} = K - q\Lambda\rho\nu'\sigma$, and $\mathbf{C} = (1/\sqrt{2})\sqrt{-p(r_1 + r_1') + q\sigma'\nu\nu'\sigma}$. By [1, Lemma 2.4.1], if there exist matrices F_1 and F_2 such that $\mathbf{A} - \mathbf{B}F_1 < 0$[4] and $\mathbf{A}' - \mathbf{C}F_2 < 0$, then there is a unique solution $\hat{M} = -\hat{D}$ to the above such that

$$\text{(C.8)} \qquad \mathbf{A} - \mathbf{B}\mathbf{B}'\hat{D} = \mathbf{A} + \mathbf{B}\mathbf{B}'\hat{M} = (K - q\Lambda\rho\nu'\sigma) + 2\Lambda(1 - q\rho\rho')\Lambda'\hat{M} < 0.$$

[4]Here and in what follows, we write $M < 0$ for a given matrix $M \in \mathbb{M}^d$ with $M + M' < 0$.

Note that $F_1 = \mathbf{B}^{-1}(\mathbb{1}_d - \mathbf{A})$ and $F_2 = \mathbf{C}^{-1}(\mathbb{1}_d - \mathbf{A}')$ are two such matrices. Hence (C.7) admits a unique solution $\hat{M}$ such that (C.8) holds.

For $\phi = \hat{v} = \mathrm{Tr}(\hat{M}x)$, consider the generator $\mathcal{L}^{\hat{v}}$ from (A.2), which takes the form

$$\mathcal{L}^{\hat{v}} = \frac{1}{2}\sum_{i,j,k,l=1}^{d} A_{(ij),(kl)} D_{(ij),(kl)} + \sum_{i,j=1}^{d}\left(\bar{b}_{ij} + \sum_{k,l=1}^{d}\bar{A}_{(ij),(kl)}\hat{M}_{kl}\right) D_{(ij)}.$$

The drift (i.e., the first order term) above takes the form

$$\begin{aligned}
&\bar{b}^{ij} + \sum_{k,l=1}^{d}\bar{A}_{(ij),(kl)}\hat{M}_{kl}\\
&\quad= \left(LL' + \left(K - q\Lambda\rho\nu'\sigma + 2\Lambda(1-q\rho\rho')\Lambda'\hat{M}\right)x + x\left(K - q\Lambda\rho\nu'\sigma + 2\Lambda(1-q\rho\rho')\Lambda'\hat{M}\right)'\right)_{ij}\\
&\quad= \left(LL' + (\mathbf{A} + \mathbf{B}\mathbf{B}'\hat{M})x + x(\mathbf{A} + \mathbf{B}\mathbf{B}\hat{M})'\right)_{ij}.
\end{aligned}$$

Thus, we see that the process X with generator given by $\mathcal{L}^{\hat{v}}$ is a Wishart process of the form in (2.5). Moreover, (C.8) implies that $K := \mathbf{A} + \mathbf{B}\mathbf{B}'\hat{M} < 0$, hence X is ergodic. Indeed, $LL' > (d+1)\Lambda\Lambda' > 0$ ensures X does not explode to the boundary of $\mathbb{S}^d_{++}$. Furthermore, consider

$$u(x) = -\underline{c}\log(\det x) + \bar{c}\,\|x\|\,\eta(\|x\|),$$

where $\underline{c}, \bar{c}$ are two constants to be determined later, and $\eta(y)$ is a smooth function satisfying $0 \le \eta(y) \le 1$, $\eta(y) = 1$ for $y > 1$ and 0 for $y < 1/2$. Observe that $\lim_{\|x\|\to\infty} u(x) = \infty$ and $\lim_{\det(x)\to 0} u(x) = \infty$, where both limits are uniform as x approaches the boundaries. On the other hand, a calculation similar to that in [48, Lemmas 5.2 and 5.3] (with $\bar{\kappa}$ therein equal to 0) shows the existence of $\underline{c}, \bar{c}, \epsilon > 0$ and a sufficiently large subdomain $E \subset \mathbb{S}^d_{++}$ such that $\mathcal{L}^{\hat{v}}u(x) \le -\epsilon$ for all $x \in \mathbb{S}^d_{++}\setminus E$. Therefore [46, Theorem 6.1.3] shows that $\mathbb{P}^{\hat{v}}$ is ergodic. Hence $\hat{v}$ is equal to $\mathrm{Tr}(\hat{M}x)$ and $\hat{\lambda} = \mathrm{Tr}(LL'\hat{M}) + pr_0$. This fact follows from [48, Proposition 2.3] and [33, Theorems 2.1, 2.2] which shows the equivalency between $\mathcal{L}^{\hat{v}}$ being ergodic and $\hat{\lambda}$ being the smallest λ with accompanying solution v to $\mathfrak{F}[v] = \lambda$. ■

Lemma C.2. *In the setting of Example 3.12, for v as in (3.1), $\mathfrak{F}[v]$ takes the form in (3.5).*

Proof. $\mathfrak{F}[v]$ is given in (C.2) of Lemma C.1. Specifying to the example coefficients and using the representation for X,M from (3.4),

$$\begin{aligned}
&2M\Lambda\Lambda'M + K'M + MK - q\sigma'\nu\rho'\Lambda'M - qM\Lambda\rho\nu'\sigma + \frac{1}{2}\left(p(r_1 + r_1') - q\sigma'\nu\nu'\sigma\right)\\
&\quad= 2M^2 + 2M - q\rho\nu\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}M - q\rho\nu M\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} + pr_1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \frac{1}{2}q\nu^2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\\
&\quad= 2\begin{pmatrix}M_1^2 + M_2^2 & M_2(M_1+M_3)\\ M_2(M_1+M_3) & M_2^2 + M_3^2\end{pmatrix} + 2\begin{pmatrix}M_1 & M_2\\ M_2 & M_3\end{pmatrix} - q\rho\nu\begin{pmatrix}M_1 + M_2 & M_2 + M_3\\ 0 & 0\end{pmatrix},\\
&\qquad - q\rho\nu\begin{pmatrix}M_1 + M_2 & 0\\ M_2 + M_3 & 0\end{pmatrix} + pr_1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \frac{1}{2}q\nu^2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\\
&\quad= \begin{pmatrix}2(M_1^2 + M_2^2) + 2M_1 - 2q\rho\nu(M_1+M_2) + pr_1 - \frac{1}{2}q\nu^2 & 2M_2(M_1+M_3) + 2M_2 - q\rho\nu(M_2+M_3)\\ 2M_2(M_1+M_3) + 2M_2 - q\rho\nu(M_2+M_3) & 2(M_2^2 + M_3^2) + 2M_3 + pr_1\end{pmatrix}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\mathrm{Tr}\left(X\left(2M\Lambda'\Lambda' M + K'M + MK - q\sigma'\nu\rho'\Lambda' M - qM\Lambda\rho\nu'\sigma + \frac{1}{2}\left(p(r_1+r_1') - q\sigma'\nu\nu'\sigma\right)\right)\right)\\
&\quad = x\left(2(M_1^2+M_2^2) + 2M_1 - 2q\rho\nu(M_1+M_2) + pr_1 - (1/2)q\nu^2\right)\\
&\qquad + y\left(4M_2(M_1+M_3) + 4M_2 - 2q\rho\nu(M_2+M_3)\right)\\
&\qquad + z\left(2(M_2^2+M_3^2) + 2M_3 + pr_1\right).
\end{aligned} \tag{C.9}
$$

Now, as for the nonconstant term on the second line of (C.2), from (3.6) we have

$$
\begin{aligned}
&-2q\mathrm{Tr}\left(X\sigma'(\sigma X\sigma')^{-1}\sigma XM\Lambda\rho\rho'\Lambda' M\right)\\
&\quad = -2q\rho^2\mathrm{Tr}\left(\begin{pmatrix} x & y \\ y & y^2/x\end{pmatrix} M \begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix} M\right)\\
&\quad = -2q\rho^2\mathrm{Tr}\left(\begin{pmatrix} x & y \\ y & y^2/x\end{pmatrix}\begin{pmatrix} (M_1+M_2)^2 & (M_1+M_2)(M_2+M_3) \\ (M_1+M_2)(M_2+M_3) & (M_2+M_3)^2\end{pmatrix}\right)\\
&\quad = x\left(-2q\rho^2(M_1+M_2)^2\right) + y\left(-4q\rho^2(M_1+M_2)(M_2+M_3)\right) + \frac{y^2}{x}\left(-2q\rho^2(M_2+M_3)^2\right).
\end{aligned} \tag{C.10}
$$

Since $\mathrm{Tr}\,(LL'M) + pr_0 = \ell^2(M_1+M_3) + pr_0$, (3.5) follows from (C.9) and (C.10). ■

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