Nikhil R. Devanur, Jugal Garg and László A. Végh

A rational convex program for linear Arrow-Debreu markets

A Rational Convex Program for Linear Arrow-Debreu Markets*

Nikhil R. Devanur† Jugal Garg‡ László A. Végh§

March 3, 2017

Abstract

We give a new, flow-type convex program describing equilibrium solutions to linear Arrow-Debreu markets. Whereas convex formulations were previously known ([17, 15, 6]), our program exhibits several new features. It gives a simple necessary and sufficient condition and a concise proof of the existence and rationality of equilibria, settling an open question raised by Vazirani [21]. As a consequence we also obtain a simple new proof of Mertens's [16] result that the equilibrium prices form a convex polyhedral set.

1 Introduction

The *exchange market model* is a classical model of a market along with a notion of equilibrium, introduced by Walras in [24]. In this model, agents arrive at the market with an initial endowment of divisible goods, and a utility function for consuming goods. A market equilibrium assigns prices to the goods such that when every agent uses the revenue from selling her initial endowment for purchasing a bundle of goods that maximizes her utility, the market *clears*, i.e, the total demand for every good is equal to its supply. The celebrated theorem by Arrow and Debreu [1] proves the existence of a market equilibrium under mild necessary conditions on the utility functions - therefore it is commonly known as the Arrow-Debreu market model. Since then, understanding equilibrium behavior and computing equilibrium prices has been extensively studied in mathematical economics and more recently in theoretical computer science.

In this paper we study the *linear Arrow-Debreu model*, where the utility functions of agents are *linear*. Let us first mention results pertaining to a well-studied further special case, the *linear Fisher model*, that was formulated by Fisher in 1891, who also studied the computability of equilibrium, via a hydraulic machine no less! (See Brainard and Scarf [4] for a fascinating account.) In this model, the agents are separated into two types, buyers and sellers; buyers arrive to the market with a certain amount of money they wish to spend on goods offered by the sellers. This model turned out to be substantially easier from a computational perspective than the linear Arrow-Debreu model. A convex programming formulation was given by Eisenberg and Gale [11]. The problem of equilibrium computation was introduced to the theoretical computer science community by Devanur et al. [8], who gave a polynomial time combinatorial primal-dual algorithm. This initiated an intensive line of research, most notable among which is a strongly polynomial time algorithm by Orlin [19]; for a survey, see [18, Chapter 5] or [21]. Also, Shmyrev [20] gave a new type of convex program (which was discovered independently by Birnbaum et al. [2]) capturing the equilibria.

Let us now turn to the linear Arrow-Debreu model. The first important algorithmic result was a finite Lemke-type path following algorithm for finding an equilibrium solution by Eaves [10]. A remarkable

*Part of this work was done while the third author visited the first one at Microsoft Research, Redmond in March 2013.
†Microsoft Research, Redmond, USA. (nikdev@microsoft.com)
‡College of Computing, Georgia Institute of Technology, Atlanta, USA. (jgarg@cc.gatech.edu)
§Department of Management, London School of Economics, London, UK. (l.vegh@lse.ac.uk)

consequence of this algorithm is that when the utilities are given by rational numbers, there also exists an equilibrium among rational numbers.

The history of convex programming formulations for the linear Arrow-Debreu model is somewhat convoluted. Jain [15] formulated a convex program whose solutions correspond to market equilibria; this can be used to obtain a polynomial time algorithm via the Ellipsoid algorithm. It turned out later that the same convex program was already formulated by Nenakov and Primak in [17]. Interestingly, the computer science community so far seems to have been unaware of the paper by Cornet [6] giving a similar, yet better convex program. This is not mentioned even in the survey paper by Codenotti et al. [5] exploring the background of the problem. (These convex programs will be exhibited in Section 3.)

An unsatisfactory aspect of the program in [15, 17] is that it fails to show the existence of an equilibrium; it only shows that *if* there exists an equilibrium, then any feasible solution to the convex program is one. In contrast, Cornet's program provides a proof of existence assuming that a stronger sufficient condition given by Gale in [13] holds. However it fails to show it for the weaker necessary and sufficient condition given by Gale in [14]. An efficient interior point algorithm to compute an equilibrium was given by Ye [25] based on the convex program in [15, 17]. An important recent result is a combinatorial primal-dual algorithm by Duan and Mehlhorn [9]; this does not rely on the convex programming formulation but adapts techniques from the algorithm by Devanur et al. [8] for linear Fisher markets.

The convex programs [17, 15] and [6] for the linear Arrow-Debreu model is of substantially different nature from those [11, 20, 2] for the linear Fisher model. The latter ones have linear constraints only, with separable convex objectives, in contrast to the nonlinear constraints in [17, 15]. Whereas [6] is formulated with only very simple linear constraints, the max-min type objective in fact hides similar nonlinear constraints.

The feasible region for both formulations for the linear Fisher model are indeed classical polyhedra, [11] a generalized flow polyhedron and [20, 2] a circulation polyhedron. This also explains why for the linear Fisher model, classical flow techniques are applicable (see [23, 22]) and strongly polynomial time algorithms exist. Also interestingly, the convex programs of [11] and [20, 2] fall into the class of *rational convex programs*, defined by Vazirani [21]: for a rational input, there exists a rational optimal solution with bitsize bounded polynomially in the input size. For the previous programs [6, 17, 15] the proof of the existence of a rational optimal solution requires further nontrivial arguments (e.g. [10, 9]). An open problem in [21] asks for the existence of a rational convex program for the linear Arrow-Debreu model with a simple, direct proof of rationality.

In this paper, *we exhibit a rational convex program for the linear Arrow-Debreu model, that also guarantees the existence of an equilibrium*, thus settling the open questions of [21]. Our convex program draws from the convex programs in [11, 20, 2]; more precisely, it is a combination of the convex program of [20, 2] and a dual program described in [7]. The objective function has terms from both the convex programs and there are two sets of constraints, one describing a circulation polyhedron as in [20, 2], and another from the dual program. In fact, in Section 4 we explain how we obtained the formulation from these two programs for linear Fisher markets. The main technical contribution is to show the existence of an equilibrium based on the Karush-Kuhn-Tucker (KKT) conditions for this convex program. Our program is feasible if and only if Gale's [14] necessary and sufficient conditions on the existence of equilibria hold. The existence of a rational optimal solution for rational input follows by showing that there exists an optimal solution that is an extreme point of the feasible region.

Now we give a formal description of the model and give our convex program. We are given set A of n agents, and assume that there is a one-to-one mapping between agents and goods, every agent $i \in A$ arrives with one divisible unit of good of type i. This is without loss of generality: the general case with an arbitrary set of goods and arbitrary initial endowments can be easily reduced to this setting; see Section 3. The utility of agent i for the good of agent j is $u_{ij} \geq 0$. The directed graph (A, E) contains an arc ij for every pair with $u_{ij} > 0$; it may also contain loops expressing that some agents are interested in their own goods. We make the standard assumption that for each agent $i \in A$, E contains at least one incoming and one outgoing arc incident to i.

By a *market equilibrium*, we mean a set of prices $p : A \to \mathbb{R}_+$ and allocations $x : E \to \mathbb{R}_+$ satisfying the following conditions.

- *Market clearing:* Demand equals supply.
  - $\sum_{i\in A} x_{ij} = 1$, for every $j \in A$, i.e., every good is fully sold.
  - $p_i = \sum_{j\in A} x_{ij} p_j$ for every $i \in A$, i.e., the money spent by agent i equals his income p_i.
- *Optimal bundle:* Every agent is allocated a utility maximizing bundle subject to its budget constraint. That reduces to
  - For every $i \in A$, if $x_{ij} > 0$ then u_{ij}/p_j is the maximal value over $j \in A$.
  - $p_i > 0$ for every $i \in A$;

It is easy to see that the following condition is necessary for the existence of an equilibrium:

$$\begin{gathered}\text{For every strongly connected component } S \subseteq E \text{ of the digraph } (A, E),\\ \text{if } |S| = 1 \text{ then there is a loop incident to the node in } S.\end{gathered} \tag{$\star$}$$

Indeed, assume $\{k\}$ is a singleton strongly connected component without a loop. Let T denote the set of nodes different from k that can be reached on a directed path in E from k. In an equilibrium allocation, the agents in $T \cup \{k\}$ spend all their money on the goods of the agents in T; this implies $p_k = 0$, contrary to our assumption.

We formulate the following convex program, with variables p_i representing the prices, the β_i's the inverse best bang-per-bucks, and y_{ij} the money paid by agent i to agent j.

$$\begin{aligned} \min \sum_{i\in A} p_i \log \frac{p_i}{\beta_i} - \sum_{ij\in E} y_{ij} \log u_{ij} & \\ \sum_{i:ij\in E} y_{ij} = p_j \quad & \forall j \in A \\ \sum_{j:ij\in E} y_{ij} = p_i \quad & \forall i \in A \\ u_{ij}\beta_i \le p_j \quad & \forall ij \in E \\ p_i \ge 1 \quad & \forall i \in A \\ y, \beta \ge 0 & \end{aligned} \tag{CP}$$

Theorem 1. *Consider an instance of the linear Arrow-Debreu market given by the graph (A, E) and the utilities $u : E \to \mathbb{R}_+$. The convex program (CP) is feasible if and only if ($\star$) holds, and in this case the optimum value is 0, and the prices p_i in an optimal solution give a market equilibrium with allocations $x_{ij} = y_{ij}/p_j$. Further, if all utilities are rational numbers, then there exists a market equilibrium with all prices and allocations also rational, of bitsize polynomially bounded in the input size.*

Here, the bitsize of the rational number p/q is defined as $\lceil \log_2 p\rceil + \lceil \log_2 q\rceil$. The rational optimum property follows by observing that there exists an optimal extremal point solution. The following results easily follow from the above theorem:

Corollary 2. *The following hold for linear Arrow-Debreu markets.*

(i) For every agent the utility is the same at every equilibrium.

(ii) The vectors (y, p) at equilibrium form a convex set. In particular, the set of price vectors at equilibrium is convex.

Property *(i)* was already proved by Gale [14] and also follows from Cornet [6]. Whereas the convexity of equilibrium prices was proved by Mertens [16] and by Florig [12], both these proofs are quite involved, whereas it is a straightforward consequence of Theorem 1. We are not aware of previous proofs on the convexity of y. In contrast, Cornet [6] proved that $(x, \log p)$ is convex at equilibria; here $x_{ij} = y_{ij}/p_j$ is the amount of good j allocated to agent i.

The Lagrangian dual of (CP) is similar to Cornet's program [6] (see (CP-C) in Section 3) but is different from it. Also, analyzing the optimal Lagrange multipliers for (CP) we can derive the feasibility convex program [17, 15]; these correspondences will be explained in Section 3. Our program exhibits some new and advantageous features as compared to Cornet's:

- The program (CP) provides necessary and sufficient condition of the existence of equilibria. In contrast, Cornet's program provides only a stronger sufficient condition given by Gale [13].
- The program (CP) is feasible if and only if there exists an equilibrium. In contrast, Cornet's program can be feasible also if there exists no equilibrium; in this case the objective is unbounded.
- The program (CP) demonstrates the existence of a rational equilibrium in a very simple way. Rationality can also be derived for the previous programs, but it requires further nontrivial arguments.
- All constraints in (CP) are linear.
- Our program establishes links to known convex programs for the Fisher model.

We think that the discovery of this convex program will pave the way for more efficient (and in particular, strongly polynomial time) algorithms for this model.

The rest of the paper is structured as follows. Section 2 is dedicated to the proof of Theorem 1. This is based on the KKT conditions, however, the argument is not straightforward, in contrast to similar arguments for the convex programs of [11, 20, 2]. Section 3 shows the equivalence of our existence condition ($\star$) to previous results by Gale [13, 14], exhibits the previous convex programs [6, 17, 15], and explains the correspondence between our formulation (CP) and these programs. The final Section 4 describes the intuition that lead us to the formulation (CP).

2 Proof of Theorem 1

Let us first verify that (CP) is actually a convex program. The feasible region is defined by linear constraints, so we only have to check that the objective is convex. The terms corresponding to the y_{ij}'s are linear. The term $\sum_{i\in A} p_i \log \frac{p_i}{\beta_i}$ is the relative entropy of p and β and is well-known to be convex in the nonnegative variables p_i and β_i.[1] Let us now verify the feasibility claim.

Claim 3. *The convex program (CP) is feasible if and only if ($\star$) holds.*

Proof. Assume that ($\star$) is violated, that is, there is a strongly connected component consisting of a single node i_0, and there is no loop in E incident to i_0 (that is, $u_{i_0 i_0} = 0$.) For a contradiction, assume (CP) admits a feasible solution (y, p, β). Then y gives a feasible circulation on the graph (A, E) such that there is a positive amount of flow entering (and leaving) every node. The circulation y can be decomposed to a weighted sum of directed cycles: $y = \sum_{k=1}^{t} w_k \chi_{C_k}$, where for each $1 \le k \le t$, χ_{C_k} is the 0-1 incidence

[1] Let us give a simple proof. We need to verify that for every $q, b, q', b' \ge 0$ and $0 < \lambda < 1$, we have

$$\lambda q \log \frac{q}{b} + (1-\lambda) q' \log \frac{q'}{b'} \ge (\lambda q + (1-\lambda)q') \log \frac{\lambda q + (1-\lambda) q'}{\lambda b + (1-\lambda) b'}.$$

This can be derived using the convexity of $x \log x$ for q/b, q'/b' with the linear combination $\lambda^* = \frac{\lambda b}{\lambda b + (1-\lambda) b'}$.

vector of a directed cycle C_k, and $w_k \geq 0$. Clearly every cycle C_k must be contained inside a strongly connected component. Hence no cycle may be incident to i_0, that is, the flow entering this node is 0, a contradiction.

Assume now that ($\star$) is satisfied. Consequently, there is a directed cycle C_i in (V, A) incident to every node i. Set $y = \sum_{i \in A} \chi_{C_i}$, and let p_i denote the amount of y entering the node i. This gives a feasible solution to (CP) with $\beta_i = \min_{j \in A} \frac{p_j}{u_{ij}}$. □

Claim 4. *The objective value in* (CP) *is non-negative, and it is 0 if and only if the prices p_i and the allocations $x_{ij} = y_{ij}/p_j$ form a market equilibrium. Conversely, for every market equilibrium p'_i, x'_{ij}, we get an optimal solution to* (CP) *by setting $p_i = \alpha p'_i$, $y_{ij} = \alpha p'_j x'_{ij}$, $\beta_i = \min_{j \in A} \alpha p'_j / u_{ij}$, where $\alpha = 1/\min\{1, \min_{i \in A} p_i\}$.*

Proof. By the third inequality, $-\log u_{ij} \geq \log \beta_i - \log p_j$. Hence the second term in the objective is at least

$$\sum_{ij \in E} (\log \beta_i - \log p_j) y_{ij} = \sum_{i \in A} \log \beta_i \left(\sum_{j: ij \in E} y_{ij} \right) - \sum_{j \in A} \log p_j \left(\sum_{i: ij \in E} y_{ij} \right) =$$
$$\sum_{i \in A} p_i \log \beta_i - \sum_{j \in A} p_j \log p_j = -\sum_{i \in A} p_i \log \frac{p_i}{\beta_i}.$$

This implies that the objective value is ≥ 0. Moreover, the lower bound is tight if and only if $u_{ij}\beta_i = p_j$ whenever $y_{ij} > 0$. This is equivalent to all transactions being best bang-per-buck purchases. It is easy to verify that the solution represents a market equilibrium. The second part also follows easily. □

The proof of the assertion in Theorem 1 that optimal solutions to (CP) correspond to market equilibria is complete by the following lemma.

Lemma 5. *Whenever (CP) is feasible, the optimum value is 0.*

Let us now formulate the Karush-Kuhn-Tucker conditions on optimality. Since all constraints in (CP) are linear, these are necessary and sufficient for optimality. Consider an optimal solution (p, y, β), and let us associate Lagrange multipliers δ_j, γ_i, w_{ij} and τ_i to the inequalities in the order as described in (CP). We obtain the following conditions.

$$-\delta_j + \gamma_i \leq -\log u_{ij} \quad \forall ij \in E \tag{1}$$
$$\delta_i - \gamma_i + \sum_{j: ji \in E} w_{ji} + \tau_i = \log \frac{p_i}{\beta_i} + 1 \quad \forall i \in A \tag{2}$$
$$-\sum_{j: ij \in E} u_{ij} w_{ij} \leq -\frac{p_i}{\beta_i} \quad \forall i \in A \tag{3}$$

Also, (1) must be tight for all $y_{ij} > 0$, and (3) must be tight for all $\beta_i > 0$. Further, $\tau_i > 0$ implies $p_i = 1$, and $w_{ij} > 0$ implies $u_{ij}\beta_i = p_j$. Note that in an optimal solution every $\beta_i > 0$, and hence (3) always holds with equality. We can therefore derive the following from (3):

$$p_i = \sum_{j: ij \in E} u_{ij} \beta_i w_{ij} = \sum_{j: ij \in E} p_j w_{ij}. \tag{4}$$

The following remark can be interpreted as a "self-duality" property: a market equilibrium does not only provide a primal optimal solution to (CP) but also optimal Lagrange multipliers.

Remark 6. *Assume there exists a market equilibrium (p, x); by re-scaling, we may assume $p_i \geq 1$ for all $i \in A$. As in Claim 4, p, $y_{ij} = p_j x_{ij}$ and $\beta_i = \min_{i \in A} p_j / u_{ij}$ give an optimal solution to (CP). It is straightforward to check that $\gamma_j = \log \beta_j$, $\delta_j = \log p_j$, $w_{ij} = x_{ij}$ and $\tau = 0$ give optimal Lagrange multipliers.*

The next claim expresses the optimum objective value of (CP) in terms of the Lagrange multipliers.

Claim 7. *Let (y,p,β) be a primal optimal solution, and let (γ,δ,w,τ) be optimal Lagrange multipliers. Then*

$$\sum_{i\in A} p_i \log\frac{p_i}{\beta_i} - \sum_{ij\in E} y_{ij}\log u_{ij} = \sum_{i\in A}\tau_i$$

Proof. By complementary slackness, (1) is tight whenever $y_{ij} > 0$. Taking the combination of these equalities multiplied by y_{ij}, we get

$$-\sum_{ij\in E} y_{ij}\log u_{ij} = \sum_{ij\in E} y_{ij}(\gamma_i - \delta_j) = \sum_{i\in A}(\gamma_i-\delta_i)p_i.$$

In the second equality, we used the flow conservation constraints in (CP). Next, let us add the equalities (2) multiplied by p_i. We obtain

$$\sum_{i\in A}\left(p_i\log\frac{p_i}{\beta_i}+p_i\right) = \sum_{i\in A}(\delta_i-\gamma_i)p_i + \sum_{i\in A}\sum_{j:ji\in E} w_{ji}p_i + \sum_{i\in A}\tau_i p_i =$$
$$\sum_{i\in A}(\delta_i-\gamma_i)p_i + \sum_{i\in A}p_i + \sum_{i\in A}\tau_i.$$

Here we used (4) for the second term, and that $p_i = 1$ whenever $\tau_i > 0$ for the third term. Adding this to the previous inequality proves the claim. □

Using the previous claim, Lemma 5 follows from the next lemma.

Lemma 8. *For the optimal Lagrange multipliers (γ,δ,w,τ), it follows that*

$$\tau_i = 0 \quad \forall i\in A.$$

Proof. The proof is by induction on the number of agents $|A|$. We assume that for all markets with $< |A|$ agents, the assertion holds. Let us introduce $q_i := e^{\delta_i}$ and $\theta_i := e^{\gamma_i}$. These are quantities playing a similar role to p_i and β_i: the conditions (1) can be rewritten as

$$u_{ij}\theta_i \le q_j \quad \forall ij\in E,$$

and furthermore by complementary slackness it follows that if $y_{ij} > 0$ then equality must hold. The θ_i's are therefore the inverse best bang-per-buck values for the prices q. Let $F\subseteq E$ denote the set of arcs with $u_{ij}\beta_i = p_j$ and $H\subseteq E$ the set of arcs with $u_{ij}\theta_i = q_j$. By complementary slackness, $\mathrm{supp}(y)\subseteq H$ and $\mathrm{supp}(w)\subseteq F$. Let us define

$$\alpha := \max_{i\in A}\frac{q_i}{p_i}, \quad S := \left\{ i\in A : \frac{q_i}{p_i} = \alpha\right\}.$$

Claim 9. *We have $\frac{p_i}{\beta_i} \le \frac{q_i}{\theta_i}$ for every $i\in S$. Further, if $ij\in F$, $i\in S$ and $\frac{p_i}{\beta_i} = \frac{q_i}{\theta_i}$, then $j\in S$ holds.*

Proof. The first claim is equivalent to $\frac{\theta_i}{\beta_i}\le\alpha$ if $i\in S$. This follows since

$$\theta_i = \min_{j\in A}\frac{q_j}{u_{ij}} \le \min_{j\in A}\frac{\alpha p_j}{u_{ij}} = \alpha\beta_i.$$

For the second part, assume for a contradiction that $q_j < \alpha p_j$ for some best bang-per-back arc $ij\in F$ with $i\in S$. This would imply that the inequality above is strict, giving a contradiction. □

Together with (2), this gives

$$\sum_{i \in A} w_{ij} \le 1 - \tau_j \quad \forall j \in S, \tag{5}$$

with equality only if $\frac{p_j}{\beta_j} = \frac{q_j}{\theta_j}$. Let

$$T := \{i \in A : j \in S \ \forall ij \in F\}$$

denote the sets of agents having all their best bang-per-buck goods in S with respect to prices p. Recall that $\text{supp}(w) \subseteq F$. By the definition of T, we get from (4) that

$$\sum_{j \in S} w_{ij} p_j = p_i \quad \forall i \in T. \tag{6}$$

Combining this with the straightforward $\sum_{j \in S} y_{ij} \le p_i$, for all $i \in T$, we obtain

$$\sum_{i \in T} \sum_{j \in S} w_{ij} p_j \ge \sum_{i \in T} \sum_{j \in S} y_{ij}.$$

Rearranging the sums gives

$$\sum_{j \in S} p_j \sum_{i \in T} w_{ij} \ge \sum_{j \in S} \sum_{i \in T} y_{ij} \tag{7}$$

The next step requires the following observation.

Claim 10. *For every arc $ij \in H$ with $j \in S$, it follows that $i \in T$.*

Proof. For a contradiction, assume $i \notin T$, that is, there exists a good $j' \notin S$ with $ij' \in F$. Then

$$\theta_i = \frac{q_j}{u_{ij}} = \alpha \frac{p_j}{u_{ij}} \ge \alpha \beta_i = \alpha \frac{p_{j'}}{u_{ij'}} > \frac{q_{j'}}{u_{ij'}} \ge \theta_i,$$

a contradiction. □

Recall that $\text{supp}(y) \subseteq H$, and therefore if $j \in S$ and $y_{ij} > 0$, then $i \in T$ must hold by the above Claim. Hence if $j \in S$, then $\sum_{i \in T} y_{ij} = p_j$. Combining this with (5) and (7), we get

$$\sum_{j \in S} (1 - \tau_j) p_j \ \ge \ \sum_{j \in S} p_j \sum_{i \in A} w_{ij} \ \ge \ \sum_{j \in S} p_j \sum_{i \in T} w_{ij} \ \ge \ \sum_{j \in S} \sum_{i \in T} y_{ij} \ = \sum_{j \in S} p_j. \tag{8}$$

We must have equality throughout, and therefore for all $j \in S$ it follows that $\tau_j = 0$ and $\frac{p_j}{\beta_j} = \frac{q_j}{\theta_j}$; the latter was a necessary condition for equality in (5). Now the second part of Claim 9 guarantees that $S \subseteq T$.

Using (6), we have $\sum_{i \in T} \sum_{j \in S} w_{ij} p_j = \sum_{i \in T} p_i$. On the other hand, the above equalities guarantee $\sum_{i \in T} \sum_{j \in S} w_{ij} p_j = \sum_{i \in S} p_i$. We can therefore conclude $S = T$. Moreover, the following holds.

Claim 11. *No arc in $\text{supp}(y) \cup \text{supp}(w)$ enters or leaves the set S.*

Proof. Recall that $\text{supp}(y) \subseteq H$ and $\text{supp}(w) \subseteq F$. Since $S = T$, the definition of T implies that no arc $ij \in F$ leaves S; recall that $\text{supp}(w) \subseteq F$. The second inequality in (8) must hold with equality, implying that $w_{ij} = 0$, whenever $i \in A \setminus S$, $j \in S$. Claim 10 implies that no arc $ij \in H$ enters S, and $\text{supp}(y) \subseteq H$. The first to equalities in (CP) imply that $\sum_{i \in S, j \in A \setminus S} y_{ij} = \sum_{i \in A \setminus S, j \in S} y_{ij}$. Hence no arc with $y_{ij} > 0$ may leave S. □

If $A = S$, then the proof of Lemma 8 is complete. If $S \subsetneq A$, then consider the restrictions of (p, y, β) and $(\gamma, \delta, w, \tau)$ to $A \setminus S$, and to the arcs inside $A \setminus S$. The first gives a feasible solution to (CP) on the restricted graph, whereas the second give optimal Lagrange multipliers, since the primal-dual slackness conditions are satisfied. According to our assumption on S being a minimal counterexample, it follows that $\tau_i = 0$ for all $i \in A \setminus S$, completing the proof. □

To complete the proof of Theorem 1, it is left to verify the claim on the existence of a rational optimal solution. This will follow from the next structural observation; note that the feasible region is a polyhedron.

Claim 12. *There exists an optimal solution to (CP) that is an extremal point of the feasible region.*

Proof. Consider an optimal solution $z = (p, y, \beta)$ to (CP); by the above, we know that it corresponds to a market equilibrium. As every point in the feasible region, z can be written as the sum of extremal rays and a convex combination of extremal points. Pick an arbitrary extremal point $z^* = (p^*, y^*, \beta^*)$ from the combination. We claim that this is also an optimal solution to (CP). By Claim 4, it suffices to show that it corresponds to a market equilibrium, which is equivalent to $u_{ij}\beta_i^* = p_j^*$ whenever $y_{ij}^* > 0$. For a contradiction, assume $u_{ij}\beta_i^* < p_j^*$ and $y_{ij}^* > 0$ holds for an $ij \in E$. Since z^* is included in the convex combination giving z, every strict inequality for z^* must also be strict for z; this would contradict the optimality of z. □

Since every extremal point of a rational polyhedron is rational with polynomially bounded size, the proof of Theorem 1 is complete. Next we derive the bound on the values of equilibrium prices and allocation. For this, we assume that all u_{ij}'s are integers, since scaling them by a constant does not change the equilibrium.

Lemma 13. *Assume all utilities are integers $\le U$ and we let $\Delta := 2^{n-1}(n+3)^{n+\frac{1}{2}}U^n$. Then there exists equilibrium prices p that are quotients of two integers $\le \Delta$, along with allocations x that are quotients of two integers $\le \Delta^2$.*

Proof. From Claim 12, an optimal solution to (CP) is achieved at an extremal point, say z^*, of the associated polyhedron. Let m denote the number of non-zero y_{ij}'s at z^*. We claim that $m \le 2n - 1$. Indeed, consider the bipartite graph (A, A, E'), where $E' = \{(i, j) \mid y_{ij} > 0\}$, $|E'| = m$. If this graph contains a cycle, then the y_{ij}'s can be modified such that every binding constraint remains binding and we get one more pair (i, j) with $y_{ij} = 0$, in a contradiction with v being a vertex.

Let $Cz = b$ denote a subset of binding constraints for z^* in the linear system defining the feasible region of (CP), after removing the columns corresponding to the $y_{ij} = 0$ variables. The number of columns is $m + 2n \le 4n - 1$. Note that the $2n$ equalities corresponding to the nodes are linearly dependent, and therefore the rank of the matrix C is at most $m + 2n - 1$.

By Cramer's rule, every y_{ij}, p_j and β_i is quotient of two integers bounded by the maximum subdeterminant of (C, b). Using Hadamard's bound, this is at most the product of the largest $(m + 2n - 1)$ column norms of (C, b). Note that $||b|| \le \sqrt{n} < \sqrt{n+3}$, as the only constraints containing nonzero constants are the $p_i \ge 1$ inequalities. The norm of each of the m the columns corresponding to the y_{ij} variables is $\sqrt{2}$ as each y_{ij} is contained in two constraints with coefficient 1. Similarly, the norm of each of the n columns corresponding to the p_i's is at most $\sqrt{n+3}$, and the norm of each of the n columns corresponding to the β_i's is at most $\sqrt{n}U$. We need the largest $m + 2n - 1$ columns and therefore may remove one of those of norm 2. From this, we can conclude that every p_j and y_{ij} is quotient of two integers bounded by Δ. Since the allocation $x_{ij} = y_{ij}/p_j$, we get that every x_{ij} is quotient of two integers bounded by Δ^2. □

Remark 14. *The above bound can be further strengthened to $\Delta = n!U^n$.*

3 Relation to previous work

3.1 Existence results

The Arrow-Debreu market is traditionally formulated in a more general setting. Besides the set of agents A, there is a set of goods G, and each agent arrives to the market with an initial endowment $w_{ig} \geq 0$ of good g. A market is given as $\mathcal{M} = (A, G, u, w)$. Our setting corresponds to the special case when $G = A$, and $w_{ij} = 1$ if $i = j$ and 0 otherwise. We shall refer to our special case as *bijective markets*.

Again, a market equilibrium consists of prices $p : G \to \mathbb{R}_{>0}$ and allocations of goods $x_{ijg} : A \times A \times G \to \mathbb{R}_+$, where x_{ijg} represents the amount of good g sold by agent j to agent i such that:

- $\sum_{i \in A} x_{ijg} = w_{jg}$, $\forall j \in A, g \in G$, i.e., every good of every agent is fully sold.
- For every $i \in A$, whenever $x_{ijg} > 0$ for some $g \in G$ and $j \in A$, then u_{ig}/p_g is the maximal value over $g \in G$.
- $\sum_{j \in A, g \in G} x_{ijg} p_g = \sum_{g \in G} w_{ig} p_g$, $\forall i \in A$, that is, the money spent by agent i equals his income.
- $p_i > 0$ for every $i \in A$.

The general case can be easily reduced to bijective markets (see e.g. Jain [15]). First if a good is included in the initial endowment of multiple agents, we give a different name for each such occurrence. If an agent has k goods in the endowment, we split the agent into k copies with the same utility function, each owning one of the goods.

Consider now a market in the general form $\mathcal{M} = (A, G, u, w)$. We say that a subset S of agents is *self-sufficient* whenever $u_{ig} > 0$, for some $i \in S$ implies that $w_{i'g} = 0, \forall i' \in A \setminus S$. That is, agents in S are not interested in the goods owned by agents not in S. We say that a market is *irreducible* if there exists no self-sufficient proper subset of the agents. The following sufficient condition was given by Gale in 1957:

Theorem 15 ([13])**.** *If the market $\mathcal{M} = (A, G, u, w)$ is irreducible then there exists an equilibrium.*

The above condition is sufficient but not necessary. Later, Gale [14] gave a strengthening of the above theorem. We say that a subset S of agents is *super self-sufficient* if in addition to above, $\exists i \in S$ such that $w_{ig} > 0$ and $u_{i'g} = 0, \forall i' \in S$. That is, an agent in S owns a good for which no agent in S is interested.

Theorem 16 ([14])**.** *There exists an equilibrium in the market $\mathcal{M} = (A, G, u, w)$ if and only if no subset of A is super self-sufficient.*

We show that in our special case of bijective markets (i.e. $G = A$, and $w_{ij} = 1$ if $i = j$ and 0 otherwise), the existence condition in Theorem 1 is equivalent to that in Theorem 16.

Lemma 17. *A bijective market is irreducible if and only if the directed graph (A, E) is strongly connected. Further, $(\star)$ holds if and only if no subset of A is super self-sufficient.*

Proof. The first part follows since in a bijective market a subset $S \subseteq A$ of agents is self-sufficient if and only if no arc enters S in the directed graph (A, E). For the second part, assume first that $(\star)$ is violated for node k, and let T denote the set of nodes different from k that can be reached on a directed path in E from k. Now let $S = T \cup \{k\}$. It is easy to check that S is super self-sufficient, since $w_{kk} > 0$ and $u_{ik} = 0, \forall i \in S$.

Conversely, assume there exists a super self-sufficient set S. According to the condition, there exist $k \in S$, such that $w_{kk} > 0$ and $u_{ik} = 0, \forall i \in S$. Clearly k is a singleton component with no self-loop in the strongly connected components of graph (A, E), verifying $(\star)$. □

3.2 Previous convex programs

Let us first exhibit Cornet's convex program [6]. It was originally given for the general case of arbitrary endowments, but we present it here for bijective markets. Also, it was originally formulated with a max-min objective over the feasible region $\sum_i x_{ij} \le 1$ for all $j \in A$, $x \ge 0$; we unfold the max-min objective here in the natural way. The variable x_{ij} corresponds to the amount of good j purchased by agent i, whereas q_i corresponds to the logarithm of the price of good i.

$$\begin{aligned} &\max t \\ &t \le \sum_{k:ik\in E} u_{ik}x_{ik} - u_{ij}e^{q_i-q_j} \quad \forall ij \in E \\ &\sum_{j:ji\in E} x_{ji} \le 1 \quad \forall i \in A \\ &x \ge 0 \end{aligned} \tag{CP-C}$$

Theorem 18 ([6])**.** *If (CP-C) is bounded then $t = 0$, and (t, x, q) is an optimal solution if and only if (x, p) corresponds to a market equilibrium where $p_i = e^{q_i}$ for all $i \in A$. Further, if the market is irreducible then (CP-C) is bounded.*

The proof uses a nontrivial argument on Lagrangian duality. Note that the existence of equilibrium follows on under Gale's sufficient condition from 1957 (Theorem 15), as opposed to (CP), where it follows under the necessary and sufficient condition in Theorem 16.

According to Theorem 18 and Lemma 17, if the market is irreducible then $t = 0$, and $\sum_{j:ji\in E} x_{ji} = 1$ must hold for every $i \in A$. By taking logarithms we get that the following convex program has a feasible solution:

$$\begin{aligned} &q_i - q_j \le \log\left(\sum_{k:ik\in E} u_{ik}x_{ik}\right) - \log u_{ij} \quad \forall ij \in E \\ &\sum_{j:ji\in E} x_{ji} = 1 \quad \forall i \in A \\ &x \ge 0 \end{aligned} \tag{CP-J}$$

This is precisely the convex program by Nenakov and Primak [17], and by Jain [15].

We can write the Lagrangian dual of our program (CP), see Boyd and Vandenberghe [3]. This gives

$$\begin{aligned} &\max \sum_{i\in A} \tau_i \\ &\delta_i - \delta_j + \tau_i \le 1 - \sum_{k:ki\in E} w_{ki} + \log\left(\sum_{k:ik\in E} u_{ik}w_{ik}\right) - \log u_{ij} \quad \forall ij \in E \\ &\tau, w \ge 0 \end{aligned} \tag{CP-D}$$

Note that the variables in an optimal solution correspond to optimal Lagrange multipliers satisfying the KKT-conditions (1)-(3). Theorem 1 implies that strong duality holds: if (CP) is feasible then there exists a market equilibrium, that easily provides a solution (CP-D).

Despite certain similarities, this formulation appears to be different from (CP-C), namely, it has a larger feasible region. Indeed, for every feasible solution of (CP-C), $\delta = q$, $w = x$, $\tau_i = t$ gives a feasible solution to (CP-D). Nevertheless, the converse is not true since $\sum_{i:ij\in E} w_{ij} \le 1$ may not hold for feasible solutions of (CP-D).

We further note that following the argument of Section 2, we can derive the feasibility of (CP-C). It follows that in an optimal solution we must have $\sum_{j:ji\in E} w_{ji} = 1$ and $\tau_i = 0$ for all $i \in A$. Using these, we can substitute $x = w$, $q = \delta$. This yields a feasible solution to (CP-C).

4 Intuition leading to the formulation

In this section we explain the intuition that lead us to the formulation (CP). The motivation was the standard reduction of optimization LP to feasibility LP. Consider the primal and dual pair of linear programs in the standard form:

$$\begin{aligned} \max\ & c^T x \\ Ax &\le b \\ x &\ge 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & y^T b \\ A^T y &\ge c \\ y &\ge 0 \end{aligned}$$

From weak duality, $c^T x \le b^T y$ for any feasible primal and dual solutions. Let us put all constraints together and add $b^T y \le c^T x$:

$$\begin{aligned} Ax &\le b \\ A^T y &\ge c^T \\ b^T y &\le cx \\ x, y &\ge 0 \end{aligned}$$

Optimal pairs of primal and dual solutions to the previous pair of programs are in one-to-one correspondence with the feasible solutions to this program.

Let us now consider two convex programs for the linear Fisher market: one of Shmyrev [20] and Birnbaum et al. [2], and the other, the dual of the Eisenberg-Gale convex program (see [7]). In Fisher's model there is a set of buyers A and another set of goods G. Buyer i arrives to the market with a budget m_i, and has linear utility $u_{ij} \ge 0$ on good j; let $E \subseteq A \times G$ denote the set of pairs with $u_{ij} > 0$. As we study bijective markets, we assume $A = G$, and good i is initially owned by buyer i.

$$\begin{aligned} \max\ & \sum_{j \in A} (p_j - p_j \log p_j) + \sum_{ij \in E} y_{ij} \log u_{ij} \\ & \sum_{i: ij \in E} y_{ij} = p_j \quad \forall j \in A \\ & \sum_{j: ij \in E} y_{ij} = m_i \quad \forall i \in A \\ & y, p \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & \sum_{i \in A} (p_i - m_i \log \beta_i) \\ & u_{ij}\beta_i \le p_j \quad \forall ij \in E \\ & \beta \ge 0 \end{aligned}$$

One can verify that the maximum is always at most the minimum. Let us now put the constraints of the two programs together, set the objective as minimizing the dual minus the primal objective, and formally substitute the constant m_i with the variable p_i. This leads to the convex program (CP).

References

[1] K. J. Arrow and G. Debreu. Existence of an equilibrium for a competitive economy. *Econometrica: Journal of the Econometric Society*, pages 265–290, 1954.

[2] B. E. Birnbaum, N. R. Devanur, and L. Xiao. Distributed algorithms via gradient descent for Fisher markets. In *ACM Conference on Electronic Commerce*, pages 127–136, 2011.

[3] S. P. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, 2004.

[4] W. C. Brainard and H. E. Scarf. How to compute equilibrium prices in 1891, 2000. Cowles Foundation Discussion Paper 1272.

[5] B. Codenotti, S. Pemmaraju, and K. Varadarajan. The computation of market equilibria. *ACM SIGACT News*, 35(4):23–37, 2004.

[6] B. Cornet. Linear exchange economies. Technical report, Cahier Eco-Math, Université de Paris, 1989.

[7] N. R. Devanur. Fisher markets and convex programs. Unpublished manuscript, 2009.

[8] N. R. Devanur, C. H. Papadimitriou, A. Saberi, and V. V. Vazirani. Market equilibrium via a primal–dual algorithm for a convex program. *Journal of the ACM (JACM)*, 55(5):22, 2008.

[9] R. Duan and K. Mehlhorn. A combinatorial polynomial algorithm for the linear arrow–debreu market. *Information and Computation*, 2014.

[10] C. B. Eaves. A finite algorithm for the linear exchange model. *Journal of Mathematical Economics*, 3(2):197–203, 1976.

[11] E. Eisenberg and D. Gale. Consensus of subjective probabilities: The pari-mutuel method. *The Annals of Mathematical Statistics*, 30(1):165–168, 1959.

[12] M. Florig. Equilibrium correspondence of linear exchange economies. *Journal of optimization theory and applications*, 120(1):97–109, 2004.

[13] D. Gale. Price equilibrium for linear models of exchange. *Rand report P-1156*, 1957.

[14] D. Gale. The linear exchange model. *Journal of Mathematical Economics*, 3(2):205–209, 1976.

[15] K. Jain. A polynomial time algorithm for computing an Arrow-Debreu market equilibrium for linear utilities. *SIAM Journal on Computing*, 37(1):303–318, 2007.

[16] J.-F. Mertens. The limit-price mechanism. *Journal of Mathematical Economics*, 39(5):433–528, 2003.

[17] E. I. Nenakov and M. E. Primak. One algorithm for finding solutions of the Arrow-Debreu model. *Kibernetica*, 3:127–128, 1983.

[18] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani. *Algorithmic game theory*. Cambridge University Press, 2007.

[19] J. B. Orlin. Improved algorithms for computing Fisher's market clearing prices. In *Proceedings of STOC*, pages 291–300. ACM, 2010.

[20] V. I. Shmyrev. An algorithm for finding equilibrium in the linear exchange model with fixed budgets. *Journal of Applied and Industrial Mathematics*, 3(4):505–518, 2009.

[21] V. V. Vazirani. The notion of a rational convex program, and an algorithm for the Arrow-Debreu Nash bargaining game. *Journal of the ACM (JACM)*, 59(2):7, 2012.

[22] L. A. Végh. Strongly polynomial algorithm for a class of minimum-cost flow problems with separable convex objectives. In *Proceedings of STOC*, pages 27–40. ACM, 2012.

[23] L. A. Végh. Concave generalized flows with applications to market equilibria. *Mathematics of Operations Research*, 39(2):573–596, 2013.

[24] L. Walras. Eléments d'économie politique pure, ou théorie de la richesse sociale (in French), 1874. English translation: Elements of pure economics; or, the theory of social wealth. American Economic Association and the Royal Economic Society, 1954.

[25] Y. Ye. A path to the Arrow–Debreu competitive market equilibrium. *Mathematical Programming*, 111(1-2):315–348, 2008.