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Chromatic polynomials and toroidal graphs

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Dedicated to the memory of Dan Archdeacon

Abstract

The chromatic polynomials of some families of quadrangulations of the torus can be found explicitly. The method, known as 'bracelet theory' is based on a decomposition in terms of representations of the symmetric group. The results are particularly appropriate for studying the limit curves of the chromatic roots of these families. In this paper these techniques are applied to a family of quadrangulations with chromatic number 3, and a simple parametric equation for the limit curve is obtained. The results are in complete agreement with experimental evidence.

1 Introduction

A great deal of Dan Archdeacon's work was related to the study of toroidal graphs. In particular, he was a co-author of the paper [1] in which the quadrangulations of the torus with chromatic number 3 are classified. In this paper I shall discuss some properties of graphs of this kind, specifically regarding their chromatic polynomials. The fact that these polynomials can be calculated explicitly not only tells us the number of proper k-colourings for each value of k, but also explains some fascinating properties of the zeros of the polynomials.

2 Bracelets and toroidal graphs

A *bracelet* is a graph $G_n = G_n(K, L)$ formed by taking n copies of a graph K, ordered cyclically, and joining each copy to the next one by a fixed set of edges L. When K is a cycle graph the copies of K can be embedded as parallel non-bounding cycles on the surface of a torus and, in some cases, the edges L can be drawn without crossings on the annuli cut out by successive copies. In this way we obtain a family of toroidal embeddings of the graphs $\{G_n\}$.

In order to obtain a quadrangulation with chromatic number 3, we can take K as a 3-cycle with vertices $1, 2, 3$. Simple considerations lead to the result that, for a quadrangulation, we must have $|L| = 3$, and the number of quadrangles in each annulus is also 3. In fact there are essentially only three cases, L_X, L_Y, L_Z, as shown in Figure 1.

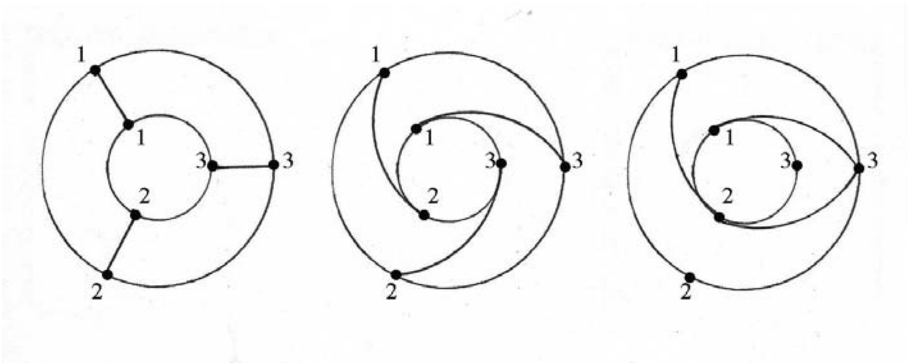


Fig. 1: $L_X = \{11, 22, 33\}$, $L_Y = \{12, 23, 31\}$, $L_Z = \{12, 31, 32\}$.

We shall refer to the bracelets defined by these linking sets as X_n, Y_n, Z_n respectively. They all yield quadrangulations of the torus with $3n$ vertices, $6n$ edges, and $3n$ faces.

The graph X_n can be regarded as the quotient of the plane square lattice obtained by identifying the sides of the rectangle with vertices $(0, 0)$, $(0, 3)$, $(n, 3)$, $(n, 0)$ in the usual way. The basic 3-cycles are defined by the vertices $(i.0)$, $(i, 1)$, $(i, 2)$, $(i, 3)$, where the first and last are identified. An explicit formula for the chromatic polynomial of X_n $(n \geq 2)$ is [3]

$$\begin{aligned}
&(z^3 - 6z^2 + 14z - 13)^n \\
&\quad +(z-1)\Big((-z^2 + 7z - 13)^n + 2(-z^2 + 4z - 4)^n\Big) \\
&\quad\quad +\frac{1}{2}z(z-3)\Big((z-5)^n + 2(z-2)^n\Big) \\
&\quad\quad\quad +\frac{1}{2}(z-1)(z-2)\Big(2(z-4)^n + (z-1)^n\Big) \\
&\quad\quad\quad\quad +(z^3 - 6z^2 + 8z - 1)(-1)^n. \qquad (**)
\end{aligned}$$

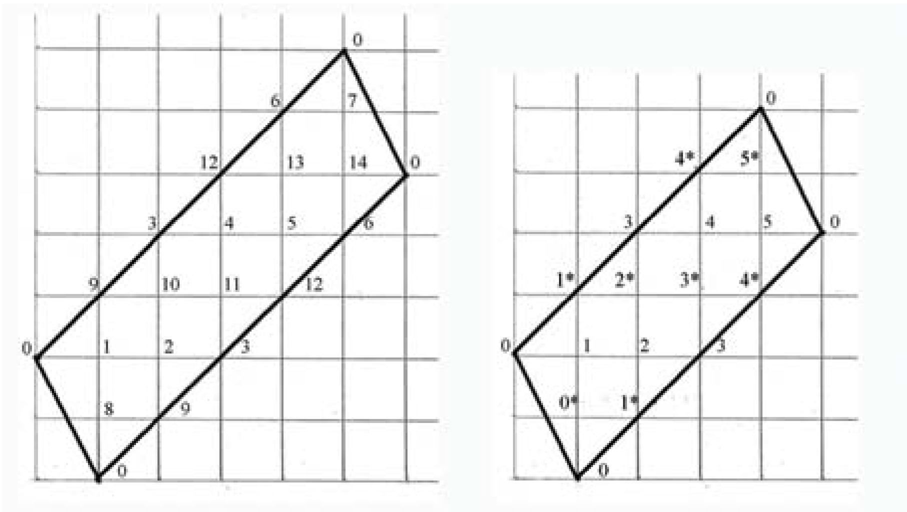
This formula tells us, for example, that the numbers of proper colourings of X_n with $0, 1, 2, 3$ colours are, respectively, 0, 0, 0, $2^{n+1} + 4$. The graphs Y_n can treated in the same way, and the results are very similar. Indeed it is intuitively clear that X_{3m} and Y_{3m} are isomorphic. Here we shall focus on the graphs Z_n, where there are some remarkable features.

Rather unexpectedly the graphs Z_n can also be represented as quotients of the plane square lattice. The fundamental region is the parallelogram with vertices $(0, 0), (-1, 2), (n-1, n+2), (n, n)$, and the basic 3-cycles are formed by the vertices

$$(x, x),\ (x, x+1),\ (x-1, x+1),\ (x-1, x+2),$$

where the first and last vertices are the same under identification. (Examples are shown in Figures 2a and 2b.)

When $3n$ is an odd number $2r + 1$, we can label the vertices of the fundamental region $0, 1, 2, \ldots, 2r$ in the following way: give the vertex $(-1, 2)$ the label 0 and proceed to the right along the 'horizontal' edges, taking account of the identifications on the boundaries of the fundamental region. In this way we obtain a Hamiltonian cycle, in which the vertex labelled i is joined to $i - 1$ and $i + 1$ modulo $2r + 1$. The 'vertical' edges join i to $i + r$ and $i - r$. Thus we obtain the graph known as the *quartic möbius ladder*. The case $3n = 15$ is shown in Figure 2a.



Figs. 2a and 2b: The graphs Z_5 and Z_4 as quotients of the plane square lattice

Suppose $3n$ is an even number $2r$. If we start labelling as before, we find an r-cycle $0, 1, 2, \ldots, r - 1$, in which i is joined to $i - 1$ and $i + 1$ modulo r. Starting again at $(0, 1)$ we obtain another r-cycle $0^*, 1^*, 2^*, \ldots, (r-1)^*$, and the vertical edges join i^* to i and $i + 1$ (all symbols modulo r here). The case $3n = 2r = 12$ is shown in Figure 2b. These graphs are known as the *quartic plane ladders*. Since they can be presented as bracelets in a much simpler way, using the base graph K_2, we shall not discuss them in detail here.

3 The theory of bracelets

The theory of bracelets [5] was developed in order to explain the distinctive form taken by the chromatic polynomials of graphs like the ones described in the previous section. It turns out that the chromatic polynomial of a bracelet $G_n(K, L)$, with K a complete graph K_b, can be expressed in a 'standard form' as a sum of terms, one for each partition π of an integer ℓ with $0 \leq \ell \leq b$:

$$P(G_n; z) \;=\; \sum_{\pi} m_\pi(z)\mathrm{tr}(T_{L,\pi}(z)^n).$$

Here $m_\pi(z)$ is a polynomial in z with rational coefficients (independent of L), and $T_{L,\pi}(z)$ is a matrix whose entries are polynomials in z with integer coefficients. In the explicit expression for $P(X_n; z)$ displayed $(**)$ in Section 2, we have $b = 3$ and the first term corresponds to the trivial partition [0] of 0, the second term to to partition [1] of 1, the third and fourth terms to the partitions [2] and [11] of 2, and the last term is the sum of contributions corresponding to the partitions]3], [21] and [111] of 3. In general the calculation of the terms is non-trivial, but a systematic procedure does exist, and in the case $b = 3$ it is relatively easy.

The standard formula is well-adapted to the study of families of bracelets $\{G_n\}$ because the dependence on n is clear. The implications of this remark will be discussed in the next Section, after we have obtained an explicit formula for the chromatic polynomials of the graphs Z_n. Since $b = 3$ in this case, we can apply the recipe given in section 6 of [5] for calculating the polynomials $m_\pi(z)$ and the matrices $T_{L,\pi}(z)$, where $L = \{12, 31, 32\}$. We shall use the notation of that paper.

The first step is to list the set $\mathcal{M}(L)$ of 'matchings' that form a subset of L: $\mathcal{M}(L) = \{\emptyset, \{12\}, \{31\}, \{32\}, \{12, 31\}\}$. Each $T_{L,\pi}$ matrix can be expressed in terms of U_M matrices, where $M \in \mathcal{M}(L)$,

$$T_{L,\pi} = U_\emptyset - (U_{12} + U_{31} + U_{32}) + U_{12,31}.$$

Matchings with $|M| \leq |\pi|$ make zero contribution and, since $\mathcal{M}(L)$ contains no matchings of size three, we need only consider the partitions π of $0, 1,$ and 2. According to the general theory we have $m_{[0]}(z) = 1$, and

$$m_{[1]}(z) = z - 1, \quad m_{[2]}(z) = \frac{1}{2}z(z-3), \quad m_{[11]}(z) = \frac{1}{2}(z-1)(z-2).$$

To calculate the matrices $T_{L,[0]}$, $T_{L,[1]}$, $T_{L,[2]}$, $T_{L,[11]}$, we proceed as follows. For $\pi = [0]$ the U_M matrices are 1×1 matrices:

$$U_\emptyset = [z(z-1)(z-2)], \quad U_{12} = U_{31} = U_{32} = [(z-1)(z-2)], \quad U_{12,31} = [z-2].$$

Hence $T_{L,[0]} = [z^3 - 6z^2 + 12z - 8] = [(z-2)^3]$.

For $\pi = [1]$ the U_M matrices are 3×3 matrices: $U_\emptyset$ is the zero matrix and

$$U_{12} = \begin{pmatrix} 0 & (z-1)(z-2) & 0 \\ 0 & -(z-2) & 0 \\ 0 & -(z-2) & 0 \end{pmatrix}, \qquad U_{31} = \begin{pmatrix} -(z-2) & 0 & 0 \\ -(z-2) & 0 & 0 \\ (z-1)(z-2) & 0 & 0 \end{pmatrix},$$

$$U_{32} = \begin{pmatrix} 0 & -(z-2) & 0 \\ 0 & -(z-2) & 0 \\ 0 & (z-1)(z-2) & 0 \end{pmatrix}, \qquad U_{12,31} = \begin{pmatrix} 0 & z-2 & 0 \\ -1 & -1 & 0 \\ z-2 & 0 & 0 \end{pmatrix}.$$

Hence

$$T_{L,[1]} = -(U_{12} + U_{31} + U_{32}) + U_{12,31} = \begin{pmatrix} z-2 & -(z-2)(z-3) & 0 \\ z-3 & 2z-5 & 0 \\ -(z-2)^2 & -(z-2)^2 & 0 \end{pmatrix}.$$

Note that the trace of $T^n_{L,[1]}$ is the same as the trace of the nth power of the 2×2 submatrix formed by the first two rows and columns.

Finally, for $\pi = [2]$ and $\pi = [11]$ there is only one non-zero matrix, $U_{12,31}$. In both cases it takes the form

$$U_{12,31} = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix},$$

where $\alpha = -1$ when $\pi = [2]$ and $\alpha = 1$ when $\pi = [11]$. It follows that the trace of $T^n_{L,[2]}$ is $(-1)^n$ and the trace of $T^n_{L,[11]}$ is 1^n, so the corresponding terms in the standard formula are

$$\frac{1}{2}z(z-3)(-1)^n + \frac{1}{2}(z-1)(z-2)1^n = \begin{cases} z^2 - 3z + 1 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 1 *Let $A(z)$ be the matrix*

$$\begin{pmatrix} z-2 & -(z-2)(z-3) \\ z-3 & 2z-5 \end{pmatrix},$$

and let $e_n(z)$ be 1 if n is even and $z^2 - 3z + 1$ if n is odd. Then the chromatic polynomial of the bracelet $G_n(K_3, L)$ with $L = \{12, 31, 32\}$ is

$$(z-2)^n + (z-1)\mathrm{tr}(A(z)^n) + e_n(z).$$

□

Note that $A(3)$ is the identity matrix, so the number of proper 3-colourings of Z_n is 6 for all n, as can be seen directly.

4 Equimodular curves and limit curves

For convenience, let $w = z - 2$. Then when n is odd the chromatic polynomial of Z_n can be written in the form

$$w^{3n} + (w+1)(\lambda_1(w)^n + \lambda_2(w)^n) + 1^n,$$

where $\lambda_1(w)$ and $\lambda_2(w)$ are the eigenvalues of

$$\begin{pmatrix} w & -w(w-1) \\ w-1 & 2w-1 \end{pmatrix}.$$

The significance of this formula is that it allows us to study the behaviour of the zeros of the chromatic polynomials as $n \to \infty$. The basic result is a general theorem of Beraha, Kahane and Weiss [2], which implies that the zeros cluster around parts of certain curves. In the case of the graphs Z_n these curves comprise the points w such that two of the four values

$$w^3,\ \lambda_1(w),\ \lambda_2(w),\ 1,$$

are equal in modulus. A dominance condition must also be satisfied: the two values equal in modulus must be greater in modulus than the other two.

The situation can be analysed by the using the general algebraic methods developed in an earlier paper on equimodular curves [4]. But fortunately in this case a more elementary method is sufficient. The key observation is that $\lambda_1(w)$ and $\lambda_2(w)$ are the roots of the characteristic equation of the matrix displayed above, which is:

$$\lambda^2 - (3w-1)\lambda + w^3 = 0.$$

Thus $\lambda_1(w)\lambda_2(w) = w^3$. Using the dominance condition we can now dispose of most of the six pairings of the four values. If $|w^3| = 1$ then $|\lambda_1(w)|\,|\lambda_2(w)| = 1$, so one of these values dominates (unless all four have modulus 1). If $|\lambda_1(w)| = |\lambda_2(w)|$ then their common value is $|w|^{3/2}$, and this is dominated by $|w^3|$ if $|w| > 1$ and by 1 if $|w| < 1$. Finally, if $|\lambda_1(w)| = |w^3|$ then $|\lambda_2(w)| = 1$, and vice versa. Hence we need only consider the curves where one of $\lambda_1(w), \lambda_2(w)$ has modulus 1.

The discriminant of the quadratic equation for λ_1 and λ_2 is

$$(3w-1)^2 - 4w^3 = (1-w)^2(1-4w),$$

which suggests that we should put $1 - 4w = t^2$. We find

$$\lambda_1 = \left(\frac{1+t}{2}\right)^3, \quad \lambda_2 = \left(\frac{1-t}{2}\right)^3.$$

If $|\lambda_1| = 1$, we have $|1+t| = 2$, so there is a ϕ in the range $0 \le \phi < 2\pi$ such that $t = 2e^{i\phi} - 1$, and

$$w = \frac{1}{4}(1-t^2) = e^{i\phi} - e^{2i\phi}.$$

Note that if $|\lambda_2| = 1$ then t is replaced by $-t$, and we get the same form for w. Hence we have our main result.

Theorem 2 *As $n \to \infty$ the roots of the chromatic polynomials of the graphs Z_n (n odd) approach the curve with parametric equation*

$$w = e^{i\phi} - e^{2i\phi} \qquad (0 \le \phi < 2\pi).$$

□

The form of the curve can easily be deduced from the parametric equation. Let $w = u + iv$, so that $u = \cos\phi - \cos 2\phi$ and $v = \sin\phi - \sin 2\phi$. When $\phi = 0$, we have $w = 0$. For $0 < \phi < \pi/3$, u is positive and v is negative, and when $\phi = \pi/3$, $w = 1$. For $\pi/3 < \phi < 2\pi/3$ u and v are both positive, and when $\phi = 2\pi/3$, $w = i\sqrt{3}$. Similarly we find segments of the curve passing through the points $-2, -i\sqrt{3}, 1$, and returning to 0, as illustrated in Figure 3.

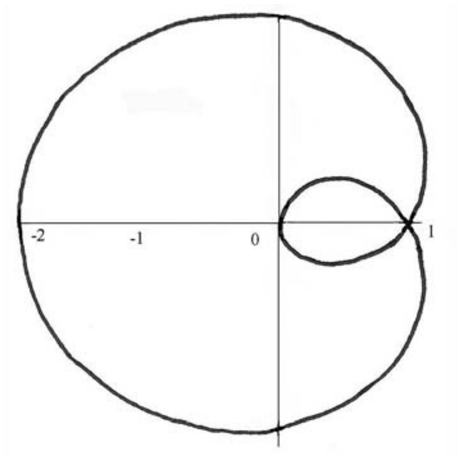


Fig. 3: The curve $w = e^{i\phi} - e^{2i\phi}$.

The curve is known as a *limaçon*. If $w = re^{i\theta}$ in polar coordinates, then $r^2 = u^2 + v^2 = 2 - 2\cos\phi$ and a simple calculation shows that

$$r^4 - 3r^2 = 2(\cos 2\phi - \cos\phi) = -2u = -2r\cos\theta.$$

From this we can write down the Cartesian equation of the curve in rational terms. This result is confirmed by a plot of the roots of the chromatic polynomials of all quartic möbius ladders with up to 59 vertices (Figure 4). This data was kindly supplied by Gordon Royle, whose interest stemmed from some work on matroids [7].

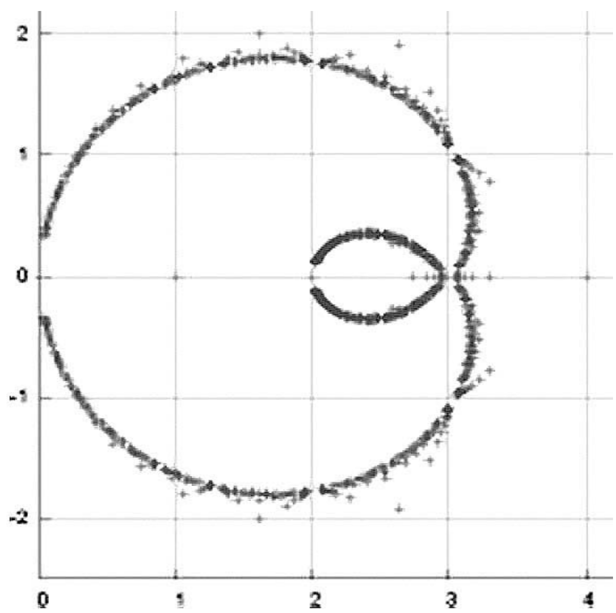


Fig. 4: Royle's plot of the roots of quartic möbius ladders ($z = w + 2$).

It is worth pointing out that the methods of bracelet theory can be applied to the flow polynomial, and to the two-variable Tutte polynomial [6]. Many features of the results are, as yet, unexplained. Even in the case of the chromatic polynomial, Figures 3 and 4 suggests some interesting questions. What is the significance of the singularity at $z = 3$ (recall that each polynomial takes the value 6 at that point)? Is there something strange happening in the vicinity of the points $z = 3 \pm i$ and $z = 2 \pm i\sqrt{3}$? And what is the significance (if any) of the points where $\cos\phi$ is $\frac{1}{8}$, at which $|z|$ attains its maximum value?

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