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On the sequential testing and quickest change-point detection problems for Gaussian processes

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We study the sequential hypothesis testing and quickest change-point (disorder) detection problems with linear delay penalty costs for certain observable time-inhomogeneous Gaussian diffusions and fractional Brownian motions. The method of proof consists of the reduction of the initial problems into the associated optimal stopping problems for one-dimensional time-inhomogeneous diffusion processes and the analysis of the associated free boundary problems for partial differential operators. We derive explicit estimates for the Bayesian risk functions and optimal stopping boundaries for the associated weighted likelihood ratios and obtain their exact asymptotic growth rates under large time values.

1 Introduction

The problems of statistical sequential analysis seek to determine the distributional properties of continuously observable stochastic processes with minimal costs. The problem of sequential testing for two simple hypotheses about the drift rate of an observable Gaussian process is to detect the form of its drift rate from one of the two given alternatives. In the Bayesian formulation of this problem, it is assumed that these alternatives have an a priori given distribution. The problem of quickest change-point (or disorder) detection for an observable Gaussian process is to find a stopping time of alarm τ which is as close as possible to the unknown time of change-point θ at which the local drift rate of the process changes from one form to another. In the classical Bayesian formulation, it is assumed that the random time θ takes the value 0 with probability π and is exponentially distributed given that $\theta > 0$. Such problems found

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applications in many real-world systems in which the amount of observation data is increasing over time (see, e.g. Carlstein, Müller, and Siegmund [5] for an overview).

The sequential testing and quickest change-point detection problems were originally formulated and solved for sequences of observable independent identically distributed random variables (see, e.g. Shiryaev [40, Chapter IV, Sections 1 and 3]). The first solutions of these problems in the continuous-time setting were obtained in the case of observable Wiener processes with constant drift rates (see Shiryaev [40, Chapter IV, Sections 2 and 4]). The standard disorder problem for observable Poisson processes with unknown intensities was introduced and solved in Davis [9], under certain restrictions on the model parameters. Peskir and Shiryaev [34, 35] solved both problems of sequential analysis for Poisson processes in full generality (see also [36, Chapter VI, Sections 23 and 24]). The method of solution of these problems was based on the reduction of the associated optimal stopping problems to the equivalent free boundary problems for ordinary (integro-)differential operators and a unique characterization of the Bayesian risks by means of the smooth and continuous fit conditions for the value functions at the optimal stopping boundaries. Further investigations of both problems for observable Wiener processes were studied in [18, 19] in the finite horizon setting, and for certain more general time-homogeneous diffusions in [20, 21] on infinite time intervals.

These two classical problems of sequential analysis for the case of observable compound Poisson processes, in which the unknown characteristics were the intensities and distributions of jumps, were investigated in Dayanik and Sezer [11, 12]. Some multidimensional extensions of the problems with several observable independent compound Poisson and Wiener processes were considered in Dayanik, Poor, and Sezer [10] and in Dayanik and Sezer [13]. Other formulations of the change-point detection problem for Poisson processes for various types of probabilities of false alarms and delay penalty costs were studied in Bayraktar, Dayanik, and Karatzas [1]. More general versions of the standard Poisson disorder problem were solved by Bayraktar, Dayanik, and Karatzas [2], where the intensities of the observable processes changed to certain unknown values. These problems for observable jump processes were solved by successive approximations of the value functions of the corresponding optimal stopping problems. The same method was also applied for the solution of the disorder problem for observable Wiener process in Sezer [38], in which disorder occurs at one of the arrival times of an observable Poisson process.

The aim of this paper is to address these problems of statistical sequential analysis in their Bayesian formulations for certain Gaussian processes with non-stationary increments. We formulate a unifying optimal stopping problem for the appropriate time-inhomogeneous diffusion likelihood ratio processes and show that the optimal stopping times are the first times at which these processes exit from certain regions restricted by time-dependent curved boundaries. It is verified that the value functions and the stopping boundaries provide a unique solution of the equivalent free boundary problem for a partial differential operator of parabolic type. Since the latter problem does not admit an explicit solution, we consider an associated auxiliary ordinary differential free boundary problem in which the time variable plays the role of a parameter. We derive analytic expressions for the value functions and optimal boundaries in the auxiliary problem and specify their exact asymptotic behavior under large time values. The resulting solutions determine the asymptotic growth rates of the value functions and optimal stopping boundaries in the original time-inhomogeneous problems.

The paper is organized as follows. In Section 2, we formulate a unifying optimal stopping problem for the time-inhomogeneous diffusion likelihood ratio processes and show how this problem arises from the Bayesian sequential testing and quickest change-point detection settings. We formulate an equivalent free boundary problem and derive explicit solutions of the auxiliary ordinary free boundary problems which have the time variable as a parameter. In Section 3, we study the asymptotic behavior of the resulting estimates for stopping boundaries under large time values, by means of deriving their Taylor expansions with respect to the local drift rate of the observable processes. These estimates determine the upper rates of decrease and the lower rates of increase for their true counterparts given certain assumptions on the parameters of the model. In Section 4, we apply these results to the models with observable fractional Brownian motions, by proving that the optimal stopping times have finite expectations. The latter fact is needed for the verification that the solution of the free boundary problem provides the solution of the original optimal stopping problem, which is done in Section 5.

2 Preliminaries

In this section, we give a formulation of the unifying optimal stopping problem for a one-dimensional time-inhomogeneous regular diffusion process and consider the associated partial and ordinary differential free boundary problems.

2.1 For a precise formulation of the problem, let us consider a probability space $(\Omega, \mathcal{G}, P)$ with a standard Brownian motion $\overline{B} = (\overline{B}_t)_{t \geq 0}$. Let $\Phi = (\Phi_t)_{t \geq 0}$ be a one-dimensional time-inhomogeneous diffusion process with the state space $[0, \infty)$, which is a pathwise (strong) solution of the stochastic differential equation

$$d\Phi_t = \eta(t, \Phi_t)\, dt + \zeta(t, \Phi_t)\, d\overline{B}_t \quad (\Phi_0 = \phi), \tag{2.1}$$

where $\eta(t, \phi)$ and $\zeta(t, \phi) > 0$ are some continuously differentiable functions of at most linear growth in ϕ on $[0, \infty)$. Let us consider an optimal stopping problem with the value function

$$V_*(t, \phi) = \inf_\tau E_{t,\phi} \left[G(\Phi_{t+\tau}) + \int_0^\tau F(\Phi_{t+s})\, ds \right], \tag{2.2}$$

where $E_{t,\phi}$ denotes the expectation under the assumption that $\Phi_t = \phi$, for some $\phi \in [0, \infty)$. Here, the gain function $G(\phi)$ and the cost function $F(\phi)$ are assumed to be non-negative, continuous and bounded, $G(\phi)$ is concave and continuously differentiable on $((0, c') \cup (c', \infty))$ for some $c' \in [0, \infty]$, and the infimum in (2.2) is taken over all stopping times τ such that the integral above has a finite expectation, so that $E_{t,\phi}\tau < \infty$ holds. Such time-inhomogeneous optimal stopping problems for diffusion processes within a finite horizon setting have been considered in McKean [27], van Moerbeke [42], Jacka [23], Broadie and Detemple [4], Myneni [28], Peskir [33, 32], and [18, 19] among others (see also Peskir and Shiryaev [36, Chapter VII] and Detemple [14] for an overview and further references). Other time-inhomogeneous optimal stopping problems with infinite time horizon were recently considered in [17].

Example 2.1 (Sequential testing problem.) Suppose that we observe a continuous process $X = (X_t)_{t \geq 0}$ of the form $X_t = \theta\mu(t) + B_t$, where $\mu(t) > 0$ is increasing and two times continuously differentiable function for $t > 0$, $\mu(0) = 0$, and $B = (B_t)_{t \geq 0}$ is a standard

Brownian motion which is independent of the random variable θ. We assume that $P(\theta = 1) = \pi$ and $P(\theta = 0) = 1 - \pi$ holds for some $\pi \in (0,1)$ fixed. The problem of sequential testing of two simple hypotheses about the values of the parameter θ can be embedded into the optimal stopping problem of (2.2) with $G(\phi) = ((a\phi) \wedge b)/(1+\phi)$ and $F(\phi) = 1$, where $a, b > 0$ are some given constants (see, e.g. [40, Chapter IV, Section 2] and [36, Chapter VI, Section 21]). In this case, the *likelihood ratio* process Φ takes the form

$$\Phi_t = \frac{\pi}{1-\pi} L_t \quad \text{with} \quad L_t = \exp\bigg(\int_0^t \mu'(s)\, dX_s - \int_0^t \frac{(\mu'(s))^2}{2}\, ds \bigg), \tag{2.3}$$

and thus solves the stochastic differential equation in (2.1) with the coefficients $\eta(t,\phi) = (\mu'(t)\phi)^2/(1+\phi)$ and $\zeta(t,\phi) = \mu'(t)\phi$, where the process $\overline{B} = (\overline{B}_t)_{t \geq 0}$ defined by

$$\overline{B}_t = X_t - \int_0^t \frac{\mu'(s)\Phi_s}{1+\Phi_s}\, ds \tag{2.4}$$

is the *innovation* standard Brownian motion generating the same filtration $(\mathcal{F}_t)_{t \geq 0}$ as the process X.

Example 2.2 (Quickest change-point detection problem.) Suppose that we observe a continuous process $X = (X_t)_{t \geq 0}$ of the form $X_t = (\mu(t) - \mu(\theta))^+ + B_t$, where $\mu(t) > 0$ is increasing and two times continuously differentiable function for $t > 0$, $\mu(0) = 0$, and $B = (B_t)_{t \geq 0}$ is a standard Brownian motion which is independent of the random variable θ. We assume that $P(\theta = 0) = \pi$ and $P(\theta > t \,|\, \theta > 0) = e^{-\lambda t}$ holds for all $t \geq 0$, and some $\pi \in (0,1)$ and $\lambda > 0$ fixed. The problem of quickest detection of the change-point parameter θ can be embedded into the optimal stopping problem of (2.2) with $G(\phi) = 1/(1+\phi)$ and $F(\phi) = c\phi/(1+\phi)$, where $c > 0$ is a given constant (see, e.g. [40, Chapter IV, Section 4] and [36, Chapter VI, Section 22]). In this case, the *likelihood ratio* process Φ takes the form

$$\Phi_t = \frac{L_t}{e^{-\lambda t}} \bigg(\frac{\pi}{1-\pi} + \int_0^t \frac{\lambda e^{-\lambda s}}{L_s}\, ds \bigg) \quad \text{with} \quad L_t = \exp\bigg(\int_0^t \mu'(s)\, dX_s - \int_0^t \frac{(\mu'(s))^2}{2}\, ds \bigg), \tag{2.5}$$

and thus solves the stochastic differential equation in (2.1) with the coefficients $\eta(t,\phi) = \lambda(1+\phi) + (\mu'(t)\phi)^2/(1+\phi)$ and $\zeta(t,\phi) = \mu'(t)\phi$, where the innovation standard Brownian motion $\overline{B} = (\overline{B}_t)_{t \geq 0}$ is given by (2.4).

Observe also that the problem setting in both Examples 2.1 and 2.2 above includes the canonical case of the observable Brownian motion with the linear drift function $\mu(t) = \rho t$, for some constant $\rho > 0$ (see, e.g. [40, Chapter IV] and [36, Chapter VI]). Note that the time-dependent function $(\mu(t) - \mu(\theta))^+$ can be considered in the models in which certain known seasonal effects may change the local drift rate of the observable process. Another possible natural extension $\mu((t-\theta)^+)$ of the original linear drift function $\mu(t) = \rho t$ is not studied in the paper. The problem setting in the case of observable diffusion processes in which the local drift rate depends on the running value of the observations was considered in [20] and [21].

2.2 It follows from the general theory of optimal stopping for Markov processes (see, e.g. [36, Chapter I, Section 2.2]) that the optimal stopping time in the problem of (2.2) is given by

$$\tau_* = \inf\{s \geq 0 \,|\, V_*(t+s, \Phi_{t+s}) = G(\Phi_{t+s})\} \tag{2.6}$$

whenever it exists. We further search for an optimal stopping time of the form

$$\tau_* = \inf\{s \geq 0 \,|\, \Phi_{t+s} \notin (g_*(t+s), h_*(t+s))\} \tag{2.7}$$

for some functions $0 \leq g_*(t) < h_*(t) \leq \infty$ to be determined (see, e.g. [36, Chapter IV, Section 14] for a time-inhomogeneous finite-horizon setting).

2.3 By means of standard arguments (see, e.g. [24, Chapter V, Section 5.1]), it can be shown that the infinitesimal generator $\mathbb{L}$ of the process $(t, \Phi) = (t, \Phi_t)_{t \geq 0}$ is given by the expression

$$\mathbb{L} = \partial_t + \eta(t, \phi)\, \partial_\phi + \frac{\zeta^2(t, \phi)}{2}\, \partial^2_{\phi\phi} \tag{2.8}$$

for all $(t, \phi) \in (0, \infty)^2$. In order to find analytic expressions for the unknown value function $V_*(t, \phi)$ from (2.2) and the unknown boundaries $g_*(t)$ and $h_*(t)$ from (2.7), we use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [40, Chapter III, Section 8] and [36, Chapter IV, Section 8]). We formulate the associated free boundary problem

$$(\mathbb{L}V)(t, \phi) = -F(\phi) \quad \text{for} \quad g(t) < \phi < h(t) \tag{2.9}$$

$$V(t, g(t)+) = G(g(t)) \quad \text{and} \quad V(t, h(t)-) = G(h(t)) \quad (\textit{instantaneous stopping}) \tag{2.10}$$

$$V(t, \phi) = G(\phi) \quad \text{for} \quad \phi < g(t) \quad \text{and} \quad \phi > h(t) \tag{2.11}$$

$$V(t, \phi) < G(\phi) \quad \text{for} \quad g(t) < \phi < h(t) \tag{2.12}$$

$$(\mathbb{L}G)(\phi) > -F(\phi) \quad \text{for} \quad \phi < g(t) \quad \text{and} \quad \phi > h(t) \tag{2.13}$$

for some $0 \leq g(t) < c' < h(t) \leq \infty$ and all $t \geq 0$. Note that the superharmonic characterization of the value function (see, e.g. [40, Chapter III, Section 8] and [36, Chapter IV, Section 9]) implies that $V_*(t, \phi)$ from (2.2) is the largest function satisfying (2.9)-(2.13) with the boundaries $g_*(t)$ and $h_*(t)$. Moreover, since the system in (2.9)-(2.13) may admit multiple solutions, we need to use some additional conditions which would uniquely determine the value function and the optimal stopping boundaries for the initial problem of (2.2). For this reason, we will need to assume that the *smooth-fit* conditions

$$\partial_\phi V(t, g(t)+) = \partial_\phi G(g(t)) \quad \text{and} \quad \partial_\phi V(t, h(t)-) = \partial_\phi G(h(t)) \quad (\textit{smooth fit}) \tag{2.14}$$

hold for all $t > 0$.

We further provide an analysis of the parabolic free boundary problem of (2.9)-(2.13), satisfying the conditions of (2.14), and such that the resulting boundaries are continuous and of bounded variation. Since such free boundary problems cannot normally be solved explicitly, the existence and uniqueness of classical as well as viscosity solutions of the variational inequalities, arising in the context of optimal stopping problems, have been extensively studied in the literature (see, e.g. Friedman [16], Bensoussan and Lions [3], Krylov [25], or Øksendal [30]). Although the necessary conditions for existence and uniqueness of such solutions in [16, Chapter XVI, Theorem 11.1], [25, Chapter V, Section 3, Theorem 14] with [25, Chapter VI, Section 4, Theorem 12], and [30, Chapter X, Theorem 10.4.1] can be verified by virtue of the regularity of the coefficients of the diffusion process in (2.1), the application of these classical results would still have rather inexplicit character. We therefore continue with the following verification assertion related to the free boundary problem formulated above, which is proved in the Appendix.

Theorem 2.3 *Let the process Φ be a pathwise unique solution of the stochastic differential equation in (2.1). Suppose that the functions $G(\phi)$ and $F(\phi)$ are bounded and continuous, and G is concave and continuously differentiable on $((0,c') \cup (c',\infty))$ for some $c' \in [0,\infty]$. Assume that the couple $g_*(t)$ and $h_*(t)$, such that $0 \le g_*(t) < c' < h_*(t) \le \infty$, together with $V(t,\phi; g_*(t), h_*(t))$ form a solution of the free boundary problem of (2.9)-(2.14), while the boundaries $g_*(t)$ and $h_*(t)$ are continuous and of bounded variation. Define the stopping time τ_* as the first exit time of the process Φ from the interval $(g_*(t), h_*(t))$ as in (2.7), and assume that $E_{t,\phi}\tau_* < \infty$ holds. Then, the value function $V_*(t,\phi)$ takes the form*

$$V_*(t,\phi) = \begin{cases} V(t,\phi; g_*(t), h_*(t)), & \text{if } g_*(t) < \phi < h_*(t) \\ G(\phi), & \text{if } \phi \le g_*(t) \text{ or } \phi \ge h_*(t) \end{cases} \tag{2.15}$$

with

$$V(t,\phi; g_*(t), h_*(t)) = E_{t,\phi}\left[G(\Phi_{t+\tau_*}) + \int_0^{\tau_*} F(\Phi_{t+s})\, ds \right], \tag{2.16}$$

and the boundaries $g_*(t)$ and $h_*(t)$ are uniquely determined by the smooth-fit conditions of (2.14).

2.4 Note that the solution of the free boundary problem in (2.9)-(2.14) cannot be found in an explicit form for the sequential testing and quickest change-point detection problems formulated in Examples 2.1 and 2.2 above, due to the presence of a time derivative in the infinitesimal generator $\mathbb{L}$. This is in contrast with the canonical time-homogeneous setting where the solution is explicit (see, e.g. Theorems 21.1 and 22.1 in [36, Chapter VI]). In this respect, let us introduce the function $\widehat{V}(t,\phi)$ and the boundaries $\widehat{g}(t)$ and $\widehat{h}(t)$ which satisfy the second-order ordinary differential equation

$$(\mathbb{L}V)(t,\phi) = -F(\phi) + \partial_t V(t,\phi) \quad \text{for } g(t) < \phi < h(t), \tag{2.17}$$

and the conditions of (2.10)-(2.14), where the variable t plays the role of a parameter. We further provide a connection of the original and the auxiliary free boundary problems associated with the differential equations in (2.9) and (2.17), respectively. In particular, we will show that, under certain conditions, the lower and upper optimal stopping boundaries $\widehat{g}(t)$ and $\widehat{h}(t)$ of the auxiliary problem provide lower and upper estimates of the optimal stopping boundaries $g_*(t)$ and $h_*(t)$ of the original problem. Moreover, we can use the same techniques for finding $\widehat{g}(t)$ and $\widehat{h}(t)$ as in the time-homogeneous setting.

Let us first state the corresponding verification assertion for the modified free boundary problem which directly follows from Theorem 2.3.

Corollary 2.4 *Let the process Φ be a pathwise unique solution of the stochastic differential equation in (2.1). Suppose that the functions $G(\phi)$ and $F(\phi)$ are bounded and continuous, and G is concave and continuously differentiable on $((0,c') \cup (c',\infty))$ for some $c' \in [0,\infty]$. Assume that the couple $\widehat{g}(t)$ and $\widehat{h}(t)$, such that $0 \le \widehat{g}(t) < c' < \widehat{h}(t) \le \infty$, together with $V(t,\phi;\widehat{g}(t),\widehat{h}(t))$ form a unique solution of the ordinary differential free boundary problem of (2.17)+(2.10)-(2.14), the derivative $\partial_t V(t,\phi;\widehat{g}(t),\widehat{h}(t))$ exists and is continuous, and the boundaries $\widehat{g}(t)$ and $\widehat{h}(t)$ are continuous and of bounded variation. Then, the function $\widehat{V}(t,\phi)$ defined by*

$$\widehat{V}(t,\phi) = \begin{cases} V(t,\phi;\widehat{g}(t),\widehat{h}(t)), & \text{if } \widehat{g}(t) < \phi < \widehat{h}(t) \\ G(\phi), & \text{if } \phi \le \widehat{g}(t) \text{ or } \phi \ge \widehat{h}(t) \end{cases} \tag{2.18}$$

is the value function for the optimal stopping problem

$$\widehat{V}(t,\phi) = \inf_\tau E_{t,\phi}\bigg[G(\Phi_{t+\tau}) + \int_0^\tau \Big(F(\Phi_{t+s}) - \partial_t \widehat{V}(t+s,\Phi_{t+s})\, I\big(\Phi_{t+s} \in (\widehat{g}(t+s),\widehat{h}(t+s))\big)\Big)\, ds\bigg] \tag{2.19}$$

where $I(\cdot)$ denotes the indicator function and the stopping time $\widehat{\tau}$ of the form

$$\widehat{\tau} = \inf\{s \ge 0 \,|\, \Phi_{t+s} \notin (\widehat{g}(t+s),\widehat{h}(t+s))\} \tag{2.20}$$

is optimal in (2.19), whenever the integral above is of finite expectation, and $\widehat{\tau} = 0$ otherwise.

Remark 2.5 Let us fix some $t \ge 0$ and assume that $\partial_t \widehat{V}(t+s,\phi) \ge 0$ holds for all $s \ge 0$ and $\phi \in (\widehat{g}(t+s),\widehat{h}(t+s))$. Then, the value function $\widehat{V}(t+s,\phi)$ of the auxiliary optimal stopping problem in (2.19) represents a lower estimate for the value function $V_*(t+s,\phi)$ of (2.2), i.e. $\widehat{V}(t+s,\phi) \le V_*(t+s,\phi)$ for all $s \ge 0$ and $\phi > 0$. Indeed, it follows from the fact that $\partial_t \widehat{V}(t+s,\phi) \ge 0$ for all $s \ge 0$ and $\phi \in (\widehat{g}(t+s),\widehat{h}(t+s))$ that the stopping times τ over which the infimum is taken in (2.19) include those for which $E_{t,\phi}\tau < \infty$ holds. Hence, comparing the right-hand sides of (2.2) and (2.19), and using again the property $\partial_t \widehat{V}(t+s,\phi) \ge 0$, we obtain $\widehat{V}(t+s,\phi) \le V_*(t+s,\phi)$ for all $s \ge 0$ and $\phi > 0$. It thus follows from the structure of the optimal stopping times τ_* and $\widehat{\tau}$ in (2.7) and (2.20) that the inequality $\tau_* \le \widehat{\tau}$ should hold ($P_{t,\phi}$-a.s.). In this case, the optimal stopping boundaries $\widehat{g}(t+s)$ and $\widehat{h}(t+s)$ from (2.20) are lower and upper estimates for the original optimal stopping boundaries $g_*(t+s)$ and $h_*(t+s)$ in (2.7), that is $\widehat{g}(t+s) \le g_*(t+s)$ and $h_*(t+s) \le \widehat{h}(t+s)$ for all $s \ge 0$. We also note that, by means of the arguments based on the comparison results for stochastic differential equations similar to the ones used in [20, Lemma 2.1] and [21, Lemma 3.1], it can be shown that that $g_*(t+s)$ is increasing (decreasing) and $h_*(t+s)$ is decreasing (increasing) whenever $\mu'(t+s)$ is increasing (decreasing).

Example 2.6 (Sequential testing problem.) Let us first solve the free-boundary problem in (2.17)+(2.10)–(2.14) with $G(\phi) = (a\phi \wedge b)/(1+\phi)$ and $F(\phi) = 1$ as in Example 2.1 above. For this, we follow the arguments of [40, Chapter IV, Section 2] and [36, Chapter VI, Section 21] and integrate the second-order ordinary differential equation in (2.17) twice with respect to the variable $\phi/(1+\phi)$ as well as use the conditions of (2.10) and (2.14) at the upper boundary $\widehat{h}(t)$ to obtain

$$V(t,\phi;\widehat{g}(t),\widehat{h}(t)) = \frac{b}{1+\phi} - \frac{2}{(\mu'(t))^2}\bigg(\bigg(\frac{\widehat{h}(t)}{1+\widehat{h}(t)} - \frac{\phi}{1+\phi}\bigg)\Upsilon(\widehat{h}(t)) - \Psi(\widehat{h}(t)) + \Psi(\phi)\bigg), \quad (2.21)$$

where we denote

$$\Psi(\phi) = -\frac{1-\phi}{1+\phi}\ln\phi \quad \text{and} \quad \Upsilon(\phi) = \phi - \frac{1}{\phi} + 2\ln\phi, \quad (2.22)$$

for all $\phi > 0$. Then, applying the conditions of (2.10) and (2.14) at the lower boundary $\widehat{g}(t)$, we obtain that the functions $\widehat{g}(t)$ and $\widehat{h}(t)$ solve the system of arithmetic equations

$$\frac{a(\mu'(t))^2 g(t)}{2(1+g(t))} = \frac{b(\mu'(t))^2}{2(1+g(t))} - \Upsilon(h(t))\left(\frac{h(t)}{1+h(t)} - \frac{g(t)}{1+g(t)}\right) + \Psi(h(t)) - \Psi(g(t)), \quad (2.23)$$

$$\frac{(b+a)(\mu'(t))^2}{2} = \Upsilon(h(t)) - \Upsilon(g(t)), \quad (2.24)$$

which is equivalent to the system

$$\frac{(b-a)(\mu'(t))^2}{2} = h(t) + \frac{1}{h(t)} - g(t) - \frac{1}{g(t)}, \quad (2.25)$$

$$\frac{b(\mu'(t))^2}{2} = h(t) + \ln h(t) - g(t) - \ln g(t), \quad (2.26)$$

for all $t > 0$. It is shown in [40, Chapter IV, Section 2] and [36, Chapter VI, Section 21] that the system in (2.25)-(2.26) admits the unique solution $0 < \widehat{g}(t) < b/a < \widehat{h}(t) < \infty$, for any $\mu'(t)$ and $t \geq 0$ fixed. Moreover, by using the implicit function theorem, we can differentiate (2.25)-(2.26) to get

$$(b-a)\,\mu'(t)\mu''(t) = h'(t) - \frac{h'(t)}{h^2(t)} - g'(t) + \frac{g'(t)}{g^2(t)}, \quad (2.27)$$

$$b\,\mu'(t)\mu''(t) = h'(t) + \frac{h'(t)}{h(t)} - g'(t) - \frac{g'(t)}{g(t)}, \quad (2.28)$$

from which we deduce that

$$g'(t) = \frac{\mu'(t)\mu''(t)(b-ah(t))g^2(t)}{(g(t)+1)(h(t)-g(t))} \quad \text{and} \quad h'(t) = \frac{\mu'(t)\mu''(t)(b-ag(t))h^2(t)}{(h(t)+1)(h(t)-g(t))} \quad (2.29)$$

holds for all $t > 0$. In particular, we also obtain that the partial derivative $\partial_t\widehat{V}(t,\phi)$ exists and is continuous. It can also be shown directly from the analysis of the system in (2.25)-(2.26) that $\widehat{g}(t)$ is increasing (decreasing) and $\widehat{h}(t)$ is decreasing (increasing) whenever $\mu'(t)$ is increasing (decreasing).

Example 2.7 (Quickest change-point detection problem.) Let us now solve the free-boundary problem in (2.17)+(2.10)–(2.14) with $G(\phi) = 1/(1+\phi)$ and $F(\phi) = c\phi/(1+\phi)$ as in Example 2.2 above, where we set $\widehat{g}(t) = 0$ for all $t \geq 0$. For this, we follow the arguments of [40, Chapter IV, Section 4] or [36, Chapter VI, Section 22] and integrate the second-order ordinary differential equation in (2.17) twice with respect to the variable $\phi/(1+\phi)$ as well as use the conditions of (2.10) and (2.14) at the upper boundary $\widehat{h}(t)$ to obtain

$$V(t, \phi; \widehat{h}(t)) = \frac{1}{1+\widehat{h}(t)} + \int_{\phi}^{\widehat{h}(t)} \frac{C(t)}{(1+y)^2} \int_0^y \exp\Big(- \Lambda(t) \left(H(y) - H(x) \right) \Big) \frac{1+x}{x} \, dx \, dy, \tag{2.30}$$

where we denote

$$C(t) = \frac{2c}{(\mu'(t))^2}, \quad \Lambda(t) = \frac{2\lambda}{(\mu'(t))^2}, \quad \text{and} \quad H(x) = \ln x - \frac{1+x}{x}, \tag{2.31}$$

for all $t \geq 0$ and $\phi > 0$. It thus follows from the condition of (2.14) that the boundary $\widehat{h}(t)$ solves the arithmetic equation

$$C(t) \int_0^{h(t)} \exp\Big(- \Lambda(t) \; \left(H(h(t)) - H(x) \right) \Big) \frac{1+x}{x} \, dx = 1, \tag{2.32}$$

for all $t \geq 0$. It is shown in [40, Chapter IV, Section 4] and [36, Chapter VI, Section 22] that the equation in (2.32) admits the unique solution $\lambda/c \leq \widehat{h}(t)$, for any $\mu'(t)$ and $t \geq 0$ fixed. Moreover, by using the implicit function theorem, we can also obtain that $\widehat{h}(t)$ is continuosly differentiable, as well as the partial derivative $\partial_t \widehat{V}(t, \phi)$ exists and is continuous. It can also be shown directly from the analysis of the equation in (2.32) with the notation of (2.31) that $\widehat{h}(t)$ is decreasing (increasing) whenever $\mu'(t)$ is increasing (decreasing).

3 Asymptotic behaviour of the stopping boundaries

In this section, we are interested in how the optimal stopping boundaries $\widehat{g}(t)$ and $\widehat{h}(t)$ in the modified problem behave asymptotically with respect to the derivative $\mu'(t)$ of the drift function $\mu(t)$ in Example 2.1 and Example 2.2, as $t \to \infty$. More precisely, we will obtain the limits and the asymptotic expansions of $\widehat{g}(t)$ and $\widehat{h}(t)$ with respect to $\mu'(t)$ in some particular cases, when either $\mu'(t) \to 0$ or $\mu'(t) \to \infty$ holds as $t \to \infty$. Observe that in the canonical case of the observable Brownian motion with linear drift $\mu(t) = \rho t$ we have $\mu'(t) = \rho$, for some constant $\rho > 0$, so that the results of this section automatically provide an asymptotic analysis of the time-constant optimal stopping boundaries $a_* = a_*(\rho)$ and $b_* = b_*(\rho)$ with respect to the signal-to-noise ratio coefficient ρ.

Example 3.1 (Sequential testing problem.) Let us introduce the function $W(x)$ which is the inverse of $e^x x$, and thus, solves the equation

$$e^{W(x)}W(x) = x \quad \text{for} \quad x \geq 0 \tag{3.1}$$

(see, e.g. [8, Formula (1.5)]). Note that $W(x)$ is strictly increasing and satisfy the properties $W(0) = 0$, and $W(x) \to \infty$ as $x \to \infty$, and it has the asymptotic series expansion

$$W(x) \sim \ln(x) - \ln(\ln(x)) \quad \text{as} \quad x \to \infty \tag{3.2}$$

(see, e.g. [8, Formula (4.19)]). Then, by solving the quadratic equation in (2.25) for $h(t)$, we obtain that $\widehat{g}(t)$ and $\widehat{h}(t)$ satisfy

$$h_{\pm}(t) = \frac{g(t)}{2} + \frac{1}{2g(t)} + \frac{(b-a)(\mu'(t))^2}{4} \pm \sqrt{\left(\frac{g(t)}{2} + \frac{1}{2g(t)} + \frac{(b-a)(\mu'(t))^2}{4}\right)^2 - 1}, \tag{3.3}$$

where $\widehat{h}(t) = \widehat{h}_-(t)$ or $\widehat{h}(t) = \widehat{h}_+(t)$, for all $t \geq 0$. Hence, by substituting the expression of (3.3) into the formula of (2.26) and taking exponentials on both sides, we have that $\widehat{g}(t)$ satisfies the following equation

$$\begin{aligned} &\frac{g(t)}{2} + \frac{1}{2g(t)} + \frac{(b-a)(\mu'(t))^2}{4} \pm \sqrt{\left(\frac{g(t)}{2} + \frac{1}{2g(t)} + \frac{(b-a)(\mu'(t))^2}{4}\right)^2 - 1} \\ &= W\big(e^{g(t)+b(\mu'(t))^2/2}g(t)\big), \end{aligned} \tag{3.4}$$

which contains both the positive and negative branch of the function on the left-hand side, depending on the root which we have chosen for $\widehat{h}(t)$ in (3.3). If we rearrange the terms and square both sides of the expression in (3.4), we get that $\widehat{g}(t)$ should satisfy

$$1 + W^2\big(e^{g(t)+b(\mu'(t))^2/2}g(t)\big) = \left(g(t) + \frac{1}{g(t)} + \frac{(b-a)(\mu'(t))^2}{2}\right) W\big(e^{g(t)+b(\mu'(t))^2/2}g(t)\big), \tag{3.5}$$

for all $t \geq 0$.

Let us first consider the case in which $b > a$ and $\mu'(t) \to \infty$ holds as $t \to \infty$. If we assume that $\widehat{h}(t) = \widehat{h}_-(t)$, by using the assumption that $b > a$ and $0 < \widehat{g}(t) < b/a$, we obtain that $\widehat{h}_-(t) \to 0$, which contradicts the fact that $b/a < \widehat{h}(t) < \infty$ holds for all $t \geq 0$. It follows that $\widehat{h}(t) = \widehat{h}_+(t)$ and $\widehat{g}(t)$ should solve the equation in (3.4) with the positive branch of the function taken on the left-hand side. Hence, the left-hand side of the expression in (3.4) converges to ∞ as $t \to \infty$, so that $e^{\widehat{g}(t)+b(\mu'(t))^2/2}\widehat{g}(t) \to \infty$ holds by virtue of the properties of the function $W(x)$ defined in (3.1). In particular, the functions on both sides of (3.5) converge to ∞ with the same speed, and thus, the following expression holds

$$W(e^{\widehat{g}(t)+b(\mu'(t))^2/2}\widehat{g}(t)) \sim \frac{(b-a)(\mu'(t))^2}{2} + \widehat{g}(t) + \frac{1}{\widehat{g}(t)} \quad \text{as} \quad t \to \infty. \tag{3.6}$$

Furthermore, taking into account the asymptotic series expansion of (3.2), we see that

$$W(e^{\widehat{g}(t)+b(\mu'(t))^2/2}\widehat{g}(t)) \sim \frac{b(\mu'(t))^2}{2} + \widehat{g}(t) + \ln(\widehat{g}(t)) \quad \text{as} \quad t \to \infty. \tag{3.7}$$

Since $\widehat{g}(t)$ is bounded from above by b/a for all $t \geq 0$ and using the equation of (3.3) for $\widehat{h}(t)$, we therefore conclude that

$$\widehat{g}(t) \sim \frac{2}{a(\mu'(t))^2} \quad \text{and} \quad \widehat{h}(t) \sim \frac{b(\mu'(t))^2}{2} \quad \text{as} \quad t \to \infty. \tag{3.8}$$

Let us now consider the case in which $b < a$ and $\mu'(t) \to \infty$ holds as $t \to \infty$. Since the function on the left-hand side of (2.25) converges to $-\infty$ as $t \to \infty$, taking into account the fact that $\widehat{g}(t) < b/a < \widehat{h}(t)$ holds for $t \geq 0$, we obtain that $\widehat{g}(t) \to 0$ as $t \to \infty$. Assuming that $W(e^{\widehat{g}(t)+b(\mu'(t))^2/2}\widehat{g}(t))$ does not converge to ∞ implies that there exists a sequence $(t_n)_{n\in\mathbb{N}}$, such that $t_n \to \infty$ and $\widehat{g}(t_n) = O(e^{-b(\mu'(t_n))^2/2})$ as $n \to \infty$. Now, when $\widehat{h}(t) = \widehat{h}_+(t)$, we obtain that $\widehat{h}(t_n) \to \infty$ as $n \to \infty$, while the assumption that the right-hand side of (3.4) does not converge to ∞ leads to contradiction. On the other hand, when $\widehat{h}(t) = \widehat{h}_-(t)$, we obtain that $\widehat{h}(t) \to 0$, which contradicts the assumption that $b/a < \widehat{h}(t) < \infty$ holds for all $t \geq 0$. We therefore obtain that $W(e^{\widehat{g}(t)+b(\mu'(t))^2/2}\widehat{g}(t)) \to \infty$, and by the same considerations as in the case $b > a$ above, regarding the asymptotic behaviour of both the left and right sides of (3.5), we obtain (3.8).

Let us finally consider the case in which $\mu'(t) \to 0$ holds as $t \to \infty$. Since the left-hand side of (2.26) converges to 0 in this case, by using the fact that the function $x + \ln x$ is strictly increasing for $x > 0$, and $0 < \widehat{g}(t) < b/a < \widehat{h}(t) < \infty$ holds for all $t \geq 0$, we may conclude that $\widehat{g}(t) \to b/a$ and $\widehat{h}(t) \to b/a$ holds as $t \to \infty$.

Example 3.2 (Quickest change-point detection problem.) Integrating by parts and using the notations of (2.31), we obtain

$$C(t) \int_0^y \frac{(1+x)}{x} \exp\Big(- \Lambda(t)\,(H(y) - H(x))\Big)\, dx = \frac{cy}{\lambda}\left(1 - \frac{Q(-\Lambda(t)-1, \Lambda(t)/y)}{\Lambda(t)+1}\right), \tag{3.9}$$

where we denote

$$Q(z,y) = -zy^{-z}e^{y}\Gamma(z,y) \quad \text{with} \quad \Gamma(z,y) = \int_y^\infty e^{-u}u^{z-1}\, du, \tag{3.10}$$

for all $z \leq 0$ and $y \geq 0$. In this case, the expression in (2.32) takes the form

$$h(t)\left(1 - \frac{Q(-\Lambda(t)-1, \Lambda(t)/h(t))}{\Lambda(t)+1}\right) = \frac{\lambda}{c}, \tag{3.11}$$

for all $t \geq 0$. We also recall the properties of the function $Q(z,y)$ in [41, Section 9] (see also [22, Section 2.5]) and note that $0 \leq Q(z,y) \leq 1$ as well as $Q(z,0) = 1$ holds for all $z \leq 0$.

Let us first consider the case in which $\mu'(t) \to \infty$, and thus $\Lambda(t) \to 0$ as $t \to \infty$. Since $\lambda/c \leq \widehat{h}(t)$ holds, we have $\Lambda(t)/\widehat{h}(t) \to 0$, so that $Q(-\Lambda(t)-1, \Lambda(t)/\widehat{h}(t)) \to 1$ as $t \to \infty$. Therefore, by using the fact that $\widehat{h}(t)$ satisfies the equation in (3.11), we get that $\widehat{h}(t) \to \infty$

holds as $t \to \infty$.

Suppose that $\mu'(t) \to 0$, so that $\Lambda(t) \to \infty$ holds as $t \to \infty$. Then, using the property $0 \le Q(z,y) \le 1$, it follows from (3.11) that

$$\widehat{h}(t) \sim \frac{\lambda}{c} \quad \text{as} \quad t \to \infty. \tag{3.12}$$

Let us now determine the exact rate of increase for $\widehat{h}(t)$ in the case in which $\mu'(t) \to \infty$ as $t \ge \infty$. In this case, the expression in (2.32) can be written as

$$\Lambda(t) \int_0^{h(t)} \exp\Big(\Lambda(t)\, H(x)\Big) \frac{1+x}{x}\, dx = \frac{\lambda}{c} \exp\Big(\Lambda(t)\, H(h(t))\Big), \tag{3.13}$$

for $t \ge 0$. Then, using the definition of the function $H(x)$ in (2.31), we obtain the expansion on the right-hand side of (3.13) in the form

$$\frac{\lambda}{c} \exp\Big(\Lambda(t)\, H(\widehat{h}(t))\Big) \sim \frac{\lambda\, \widehat{h}(t)^{\Lambda(t)}}{c}, \tag{3.14}$$

under $\mu'(t) \to \infty$. Note that the assumption of

$$\limsup_{t \to \infty} \widehat{h}(t)^{\Lambda(t)} = \infty \tag{3.15}$$

implies that there exists a sequence $(t_n)_{n \in \mathbb{N}}$, such that $t_n \to \infty$ and $\exp(\Lambda(t_n) H(\widehat{h}(t_n))) \to \infty$ as $n \to \infty$. Since we have $\widehat{h}(t) \to \infty$, there exists $t' \ge 0$ such that $2\lambda/c < \widehat{h}(t)$ holds for all $t \ge t'$. Moreover, since the function $H(x)$ is strictly increasing for $x > 0$, by evaluating the left-hand side of (3.13) at $\widehat{h}(t)$, we obtain that

$$\begin{aligned} &\int_0^{\widehat{h}(t)} \Lambda(t) \exp\Big(\Lambda(t)\, H(x)\Big) \frac{1+x}{x}\, dx = \int_0^{\widehat{h}(t)} x\, d \exp\Big(\Lambda(t) H(x)\Big) \\ &> \int_{\frac{2\lambda}{c}}^{\widehat{h}(t)} x\, d \exp\Big(\Lambda(t)\, H(x)\Big) > \frac{2\lambda}{c} \Bigg(\exp\Big(\Lambda(t)\, H(\widehat{h}(t))\Big) - \exp\Big(\Lambda(t) H\Big(\frac{2\lambda}{c}\Big)\Big)\Bigg) \end{aligned} \tag{3.16}$$

holds for all $t \ge t'$. This fact means that the leading term of the left-hand side of (3.13) is larger than the leading term on the right-hand side of (3.13) along the sequence t_n as $n \to \infty$, and thus, the assumption of (3.15) cannot be satisfied. Therefore, we conclude that $\widehat{h}(t)^{\Lambda(t)}$ is finite under $\widehat{h}(t) \to \infty$ and $\Lambda(t) \to 0$, so that $\ln \widehat{h}(t) = O((\mu'(t))^2)$ as $t \to \infty$.

4 The fractional Brownian motion setting

In this section, we apply the asymptotic results obtained above to demonstrate the existence of solutions in the problems of sequential analysis for an observable fractional Brownian motion with linear drift. In particular, we will prove that the optimal stopping time τ_* has a finite expectation. Observe that we consider the application of the results obtained above on an optimal stopping problem for the time-inhomogeneous diffusion process which is driven by a fundamental martingale associated with the observable fractional Brownian motion. Note that several optimal stopping and sequential analysis problems for fractional Brownian motions were considered by Elliot and Chan [15], Prakasa Rao [37], Çetin, Novikov and Shiryaev [6], and Chronopoulou and Fellouris [7].

Example 4.1 (Sequential testing problem.) Suppose that in the setting of Example 2.1 the observable continuous process $X \equiv Y^H = (Y^H_t)_{t \ge 0}$ is given by $Y^H_t = \theta \rho t + B^H_t$, where $B^H = (B^H_t)_{t \ge 0}$ is a fractional Brownian motion with parameter $H \in (1/2, 1)$ independent of θ, and $\rho > 0$ is a constant. Introduce the process $\overline{M}^H = (\overline{M}^H_t)_{t \ge 0}$ by

$$\overline{M}^H_t = Z^H_t - c_1 \int_0^t \rho \, \frac{s^{1-2H} \Phi_s}{1 + \Phi_s} \, ds \quad \text{with} \quad \langle \overline{M}^H \rangle_t = \langle Z^H \rangle_t = \frac{c_1 t^{2-2H}}{2 - 2H}, \tag{4.1}$$

where the process $Z^H = (Z^H_t)_{t \ge 0}$ is defined by

$$Z^H_t = \int_0^t \frac{s^{1/2-H} (t-s)^{1/2-H}}{2H \Gamma(3/2 - H) \Gamma(H + 1/2)} \, dY^H_s \quad \text{and} \quad c_1 = \frac{\Gamma(3/2 - H)}{2H \Gamma(H + 1/2) \Gamma(2 - 2H)}, \tag{4.2}$$

and the likelihood ratio process Φ is given by

$$\Phi_t = \frac{\pi}{1 - \pi} \, L_t \quad \text{with} \quad L_t = \exp\left(\rho Z^H_t - \frac{\rho^2 \langle Z^H \rangle_t}{2} \right), \tag{4.3}$$

for all $t \ge 0$. It follows from the result of [29, Theorem 3.1] that the process $\overline{M}^H$ is a fundamental martingale with respect to the filtration $(\mathcal{F}_t)_{t \ge 0}$ and thus admits the representation with respect to the innovation standard Brownian motion $\overline{B}$ of the form

$$\overline{M}^H_t = \sqrt{c_1} \int_0^t s^{1/2-H} \, d\overline{B}_s \quad \text{and} \quad \overline{B}_t = \frac{1}{\sqrt{c_1}} \int_0^t s^{H-1/2} \, d\overline{M}^H_s, \tag{4.4}$$

for all $t \ge 0$ (see, e.g. [29, Section 5.2]). In this case, the process Φ from (4.3) satisfies the stochastic differential equation in (2.1) with $\eta(t, \phi)$ and $\zeta(t, \phi)$ as in Example 2.1, where $\mu'(t) = \rho \sqrt{c_1} t^{1/2-H}$ holds for all $t \ge 0$. Hence, the analysis from the previous section can be applied for the drift rate $\mu'(t) \to 0$ when $1/2 < H < 1$ as $t \to \infty$.

Let us fix a starting time $t \ge 0$ and introduce the deterministic time change $\beta(t, s)$ with the rate $(\mu'(s))^2$ defined as

$$\beta(t, s) = \int_t^{t+s} (\mu'(u))^2 \, du \equiv \frac{c_1 \rho^2 ((t+s)^{2-2H} - t^{2-2H})}{2 - 2H}, \tag{4.5}$$

and its inverse $\gamma(t, s)$ shifted by t, so that $\beta(t, \gamma(t, s) - t) = s$ holds for all $s \ge 0$. Since the process Φ satisfies the stochastic differential equation of (2.1), by applying the time-change formula for Itô integrals from [30, Theorems 8.5.1 and 8.5.7], we obtain

$$\Phi_{\gamma(t,s)} = \Phi_t \, \exp\left(\widetilde{B}_s - \frac{s}{2} + \int_0^s \frac{\Phi_{\gamma(t,u)}}{1+\Phi_{\gamma(t,u)}} \, du\right) \quad \text{with} \quad \widetilde{B}_s = \int_t^{\gamma(t,s)} \mu'(u) \, d\overline{B}_u, \tag{4.6}$$

where $\widetilde{B} = (\widetilde{B}_s)_{s\geq 0}$ is a standard Brownian motion with respect to the filtration $(\mathcal{F}_{\gamma(t,s)})_{s\geq 0}$. Therefore, by using the definition of $\widehat{\tau}$ in (2.20) and taking into consideration the time change $\beta(t,s)$ from (4.5), we conclude that the stopping time $\beta(t,\widehat{\tau})$ with respect to the filtration $(\mathcal{F}_{\gamma(t,s)})_{s\geq 0}$ can be represented as

$$\beta(t,\widehat{\tau}) = \inf\left\{ s \geq 0 \,\middle|\, \widetilde{B}_s - \frac{s}{2} + \int_0^s \frac{\Phi_{\gamma(t,u)}}{1+\Phi_{\gamma(t,u)}} \, du + \ln \Phi_t \notin \big(\ln \widehat{g}(\gamma(t,s)), \ln \widehat{h}(\gamma(t,s))\big) \right\}, \tag{4.7}$$

for all $t \geq 0$.

Assume that $b \neq a$ in Example 2.1. In this case, noticing from (4.5) that $\gamma(t,s) \to \infty$ and using the fact that $\widehat{g}(t) \to b/a$ and $\widehat{h}(t) \to b/a$ as $t \to \infty$, it follows that for any $\varepsilon > 0$ there exists $t_* > 0$ large enough such that the inequalities

$$\frac{b}{a} - \varepsilon < \widehat{g}(\gamma(t,s)) < \frac{b}{a} < \widehat{h}(\gamma(t,s)) < \frac{b}{a} + \varepsilon \tag{4.8}$$

hold for all $t > t_*$ and $s \geq 0$. Let us now fix an arbitrary $\varepsilon > 0$ such that $\varepsilon < b/a$ and further assume that $t > t_*$. Then, introducing the sets of sample paths $A_0 = \{\omega \in \Omega \,|\, \widehat{g}(t) < \Phi_t < \widehat{h}(t)\}$,

$$A_s = \big\{\omega \in A_0 \,\big|\, \widehat{g}(\gamma(t,s)) < \Phi_{\gamma(t,s)} < \widehat{h}(\gamma(t,s))\big\}, \text{ and } C_s = \big\{\omega \in \Omega \,\big|\, |\Phi_{\gamma(t,s)} - b/a| < \varepsilon\big\}, \tag{4.9}$$

and using the inequalities in (4.8), we get the inclusion $A_s \subseteq C_s$ for any $s \geq 0$. Therefore, by the definition of the event C_s, for the upper bounds $c_1(\varepsilon)$ and $c_2(\varepsilon)$ defined below, we have

$$c_1(\varepsilon) \equiv \frac{b - a\varepsilon}{a + b - a\varepsilon} < \frac{\Phi_{\gamma(t,s)}}{1+\Phi_{\gamma(t,s)}} < \frac{b + a\varepsilon}{a + b + a\varepsilon} \equiv c_2(\varepsilon), \quad \text{for} \quad \omega \in A_s, \tag{4.10}$$

for any $\varepsilon > 0$. It follows from the notations in (4.6) and the structure of the event A_0 that $A_s \subseteq D_s$ holds, where we set

$$D_s = \left\{ \omega \in \Omega \,\middle|\, \widetilde{B}_s - \frac{s}{2} \in \left(\ln\left(\frac{\widehat{g}(\gamma(t,s))}{\widehat{h}(t)}\right) - c_2(\varepsilon)\, s, \ln\left(\frac{\widehat{h}(\gamma(t,s))}{\widehat{g}(t)}\right) - c_1(\varepsilon)\, s\right) \right\}, \tag{4.11}$$

for all $s \geq 0$. Define the stopping time $\overline{\tau}$ as

$$\overline{\tau} = \inf\left\{ s \geq 0 \,\middle|\, \widetilde{B}_s - \frac{s}{2} \notin \left(\ln\left(\frac{\widehat{g}(\gamma(t,s))}{\widehat{h}(t)}\right) - c_2(\varepsilon)\, s, \ln\left(\frac{\widehat{h}(\gamma(t,s))}{\widehat{g}(t)}\right) - c_1(\varepsilon)\, s\right) \right\}, \tag{4.12}$$

and notice that the stopping times $\beta(t, \widehat{\tau}) = \beta(t, \widehat{\tau}(\omega))$ and $\overline{\tau} = \overline{\tau}(\omega)$ admit the representations

$$\beta(t, \widehat{\tau}(\omega)) = \sup\left\{ s \ge 0 \,\middle|\, \omega \in \bigcap_{0 \le u \le s} A_u \right\} \quad \text{and} \quad \overline{\tau}(\omega) = \sup\left\{ s \ge 0 \,\middle|\, \omega \in \bigcap_{0 \le u \le s} D_u \right\}, \tag{4.13}$$

for any $\omega \in \Omega$. Then, it follows from the inclusion $A_s \subseteq D_s$ for $s \ge 0$ that $\beta(t, \widehat{\tau}) \le \overline{\tau}$ holds. Because of the assumption $b \ne a$, we can choose $\varepsilon < b/a$ such that either $1 - \varepsilon > b/a$ holds when $b < a$ or $1 + \varepsilon < b/a$ holds when $b > a$. Hence, assuming that $b < a$, we have $1/2 - c_2(\varepsilon) > 0$. Thus, it follows from the expressions in (4.8) and (4.12) that $\overline{\tau} \le \tau'$ holds, where we set

$$\tau' = \inf\big\{ s \ge 0 \,|\, \widetilde{B}_s \le \ln(b - a\varepsilon) - \ln(a\widehat{h}(t)) + (1/2 - c_2(\varepsilon))s \big\}, \tag{4.14}$$

which is a stopping time with polynomial moments of all orders (see, e.g. [39, Chapter IV]). Therefore, it follows from the fact that $\beta(t, \widehat{\tau}) \le \overline{\tau} \le \tau'$ holds and the structure of the time change in (4.5) that $E_{t,\phi}\widehat{\tau} \le E_{t,\phi}\gamma(t, \tau') - t < \infty$ is satisfied, and we get the same inequalities in the case of $b > a$, similarly.

Let us now prove that $\partial_t V(t, \phi; \widehat{g}(t), \widehat{h}(t)) > 0$ holds for all $\phi \in (\widehat{g}(t), \widehat{h}(t))$ and $t > 0$ large enough. For this purpose, by differentiating the expression in (2.21) and using the expressions in (2.22) and (2.29), we get

$$\begin{aligned} &\partial_t V(t, \phi; \widehat{g}(t), \widehat{h}(t)) = 2(2H-1)(\Psi(\widehat{h}(t)) - \Psi(\phi))/(t(\mu'(t))^2) \\ &- \frac{2}{(\mu'(t))^2}\left(\frac{\widehat{h}(t)}{1+\widehat{h}(t)} - \frac{\phi}{1+\phi}\right)\left(\frac{(2H-1)\xi(\widehat{h}(t))}{t} + \frac{\widehat{h}'(t)(\widehat{h}(t)+1)^2}{\widehat{h}(t)^2}\right) = \frac{2(2H-1)\Xi(t,\phi)}{t(\mu'(t))^2}, \end{aligned} \tag{4.15}$$

where we denote

$$\Xi(t, \phi) = \phi + \ln\phi - \widehat{h}(t) - \ln\widehat{h}(t) + \frac{\phi}{1+\phi}\big(\Upsilon(\widehat{h}(t)) - \Upsilon(\phi)\big) + \frac{(\widehat{h}(t) - \phi)(b - a\widehat{g}(t))}{2(\widehat{h}(t) - \widehat{g}(t))(1+\phi)}, \tag{4.16}$$

for all $t > 0$ and $\phi > 0$. It is clear that $\Xi(t, \widehat{h}(t)) = 0$ holds, and thus, we obtain from the expressions in (2.24) and (2.26) that

$$\Xi(t, \widehat{g}(t)) = \frac{(\mu'(t))^2}{2}\left(-b + \frac{\widehat{g}(t)(a+b)}{1+\widehat{g}(t)}\right) + \frac{(b - a\widehat{g}(t))}{2(1+\widehat{g}(t))} = \frac{(b - a\widehat{g}(t))}{2(1+\widehat{g}(t))}\left(1 - \frac{(\mu'(t))^2}{2}\right), \tag{4.17}$$

holds for $t > 0$. Since $b/a > \widehat{g}(t) > 0$ is satisfied, and there exists $t' > 0$ such that $\mu'(t) < \sqrt{2}$ holds for all $t \ge t'$, we have $\Xi(t, \widehat{g}(t)) > 0$ for $t \ge t'$. Then, by differentiating the expression in (4.16), we get

$$\partial_\phi \Xi(t, \phi) = \frac{1}{(1+\phi)^2}\left(\Upsilon(\widehat{h}(t)) - \Upsilon(\phi) - \frac{(b - a\widehat{g}(t))(1+\widehat{h}(t))}{2(\widehat{h}(t) - \widehat{g}(t))}\right), \tag{4.18}$$

for all $t > 0$ and $\phi > 0$. Observe that, since $\Upsilon(\phi)$ is an increasing function, it follows that $\partial_\phi \Xi(t, \phi)$ changes its sign at most once in the region $\phi \in (\widehat{g}(t), \widehat{h}(t))$ for all $t \ge t'$. It is easily seen that the inequality $\partial_\phi \Xi(t, \widehat{h}(t)) < 0$ holds, which means that either $\Xi(t, \phi)$ is decreasing for $\phi \in (\widehat{g}(t), \widehat{h}(t))$ or there exists some $\phi_* \in (\widehat{g}(t), \widehat{h}(t))$ such that $\Xi(t, \phi)$ is increasing for $\phi \in (\widehat{g}(t), \phi_*]$ and decreasing for $\phi \in (\phi_*, \widehat{h}(t))$. Hence, since $\Xi(t, \widehat{g}(t)) > 0$ and $\Xi(t, \widehat{h}(t)) = 0$ holds, we get that $\Xi(t, \phi) > 0$ is satisfied in both cases for $\phi \in (\widehat{g}(t), \widehat{h}(t))$ and $t \geq t'$. For $1/2 < H < 1$, it follows from the expressions in (4.15) that the inequality $\partial_t V(t, \phi; \widehat{g}(t), \widehat{h}(t)) > 0$ holds for all $\phi \in (\widehat{g}(t), \widehat{h}(t))$ and $t \geq t'$. We can therefore apply the assertions of Remark 2.5 and use the fact that $E_{t,\phi}\widehat{\tau} < \infty$ to obtain that $E_{t,\phi}\tau_* \leq E_{t,\phi}\widehat{\tau} < \infty$ holds when the starting time t satisfies $t > t' \vee t_*$.

Example 4.2 (Quickest disorder detection problem.) Suppose that in the setting of Example 2.2 the observable continuous process $X \equiv Y^H = (Y_t^H)_{t \geq 0}$ is given by $Y_t^H = \rho(t - \theta)^+ + B_t^H$, where $B^H = (B_t^H)_{t \geq 0}$ is a fractional Brownian motion with parameter $H \in (1/2, 1)$ independent of θ, and $\rho > 0$ is a constant. In this case, the likelihood ratio process Φ is given by

$$\Phi_t = \frac{L_t}{e^{-\lambda t}} \left(\frac{\pi}{1 - \pi} + \int_0^t \frac{\lambda e^{-\lambda s}}{L_s} \, ds \right) \quad \text{with} \quad L_t = \exp\left(\rho Z_t^H - \frac{\rho^2 \langle Z^H \rangle_t}{2} \right), \tag{4.19}$$

for all $t \geq 0$. Therefore, by using the same reasoning as in Example 4.1, we obtain that the process Φ from (4.19) satisfies the stochastic differential equation in (2.1) with $\eta(t, \phi)$ and $\zeta(t, \phi)$ as in Example 2.2, where $\mu'(t) = \rho\sqrt{c_1} t^{1/2 - H}$ holds for all $t \geq 0$. Hence, the analysis from the previous section can be applied for the drift rate $\mu'(t) \to 0$ when $1/2 < H < 1$ as $t \to \infty$.

Let us fix a starting time $t \geq 0$ and define the deterministic time change $\beta(t, s)$ and its inverse $\gamma(t, s)$ as in (4.5) for all $s \geq 0$. By using the expression in (4.19), we get that $\Phi_s \geq \Phi_0 e^{\lambda s} L_s$ holds for all $s \geq 0$. Therefore, if we define the stopping time $\widetilde{\tau}$ as

$$\widetilde{\tau} = \inf\{ s \geq 0 \,|\, \Phi_0 \, e^{\lambda(t+s)} L_{t+s} \geq \widehat{h}(t + s) \}, \tag{4.20}$$

we have that $\widehat{\tau} \leq \widetilde{\tau}$ holds, where $\widehat{\tau}$ is defined in (2.20). In order to simplify further notations, we define the process $\widetilde{\Phi} = (\widetilde{\Phi}_s)_{s \geq 0}$ by $\widetilde{\Phi}_s = \Phi_0 e^{\lambda \gamma(t,s)} L_{\gamma(t,s)}$ for $s \geq 0$. Since the process L has the form of (4.19), by applying the time-change formula for Itô integrals from [30, Theorems 8.5.1 and 8.5.7], we obtain

$$\widetilde{\Phi}_s = \widetilde{\Phi}_0 \exp\left(\widetilde{B}_s - \frac{s}{2} + \lambda(\gamma(t, s) - t) + \int_0^s \frac{\widetilde{\Phi}_u}{1 + \widetilde{\Phi}_u} \, du \right), \tag{4.21}$$

where the process $\widetilde{B} = (\widetilde{B}_s)_{s \geq 0}$ defined in (4.6) is a standard Brownian motion. Therefore, by using the definition of $\widetilde{\tau}$ in (4.20) and taking into consideration the time change from (4.5), the stopping time $\beta(t, \widetilde{\tau})$ can be represented as

$$\beta(t, \widetilde{\tau}) = \inf\left\{ s \geq 0 \,\middle|\, \widetilde{B}_s - \frac{s}{2} + \lambda(\gamma(t, s) - t) + \int_0^s \frac{\widetilde{\Phi}_u}{1 + \widetilde{\Phi}_u} \, du + \ln \widetilde{\Phi}_0 \geq \ln \widehat{h}(\gamma(t, s)) \right\}. \tag{4.22}$$

Since $\gamma(t, s) \to \infty$ as $t \to \infty$, it follows from (3.12) that for any $\varepsilon > 0$ there exists $t^* > 0$ large enough such that the inequalities

$$\frac{\lambda}{c} < \widehat{h}(\gamma(t,s)) < \frac{\lambda}{c} + \varepsilon \tag{4.23}$$

hold for all $t > t^*$ and $s \geq 0$.

Let us now fix an arbitrary $\varepsilon > 0$ and further assume that $t > t^*$. By using the fact that $\widetilde{\Phi}$ is a nonnegative process, we obtain from (4.5) that the inequalities

$$\lambda(\gamma(t,s) - t) + \int_0^s \frac{\widetilde{\Phi}_u}{1 + \widetilde{\Phi}_u}\, du \geq \lambda(\gamma(t,s) - t) \geq \lambda \Big(\frac{s(2-2H)}{c_1 \rho^2} \Big)^{1/(2-2H)} \tag{4.24}$$

hold for all $s \geq 0$. Define the random variable Δ_t as

$$\Delta_t = \sup_{s \geq 0} \bigg(s + \ln \Big(\frac{\lambda + c\varepsilon}{c\, \widetilde{\Phi}_0} \Big) - \lambda \Big(\frac{s(2-2H)}{c_1 \rho^2} \Big)^{1/(2-2H)} + \frac{s}{2} \bigg), \tag{4.25}$$

and notice that it follows from the inequalities in (4.23) and (4.24) that

$$\ln \Big(\frac{\widehat{h}(\gamma(t,s))}{\widetilde{\Phi}_0} \Big) - \lambda(\gamma(t,s) - t) - \int_0^s \frac{\widetilde{\Phi}_u}{1 + \widetilde{\Phi}_u} du + \frac{s}{2} < \Delta_t - s \tag{4.26}$$

holds for all $s \geq 0$ and any $t > t^*$ fixed. Subsequently, we obtain from (4.22) that $\beta(t, \widetilde{\tau}) \leq \tau''$, where we set

$$\tau'' = \inf\{ s \geq 0 \,|\, \widetilde{B}_s \geq \Delta_t - s \}, \tag{4.27}$$

for any $t > t^*$. Moreover, by introducing the event $A = \{\omega \in \Omega \,|\, \widetilde{\Phi}_0 < \widehat{h}(t)\}$, we also obtain that $\beta(t, \widetilde{\tau}) = 0$ on $\Omega \setminus A$, and hence, we conclude that $\beta(t, \widetilde{\tau}) \leq \tau'' I(A)$ holds. Since we have that $\Delta_t > 0$ on the event A and $\Delta_t < \infty$ ($P_{t,\phi}$-a.s.), for $1/2 < H < 1$, we get that $\tau'' I(A)$ has polynomial moments of all orders (see, e.g. [39, Chapter IV]). Therefore, it follows from the fact that $\beta(t, \widehat{\tau}) \leq \beta(t, \widetilde{\tau}) \leq \tau'' I(A)$ holds and the structure of the time change in (4.5) that $E_{t,\phi} \widehat{\tau} \leq E_{t,\phi} \gamma(t, \tau'') - t < \infty$ is satisfied.

Let us finally show that $\partial_t V(t, \phi; \widehat{h}(t)) > 0$ holds for all $\phi \in (0, \widehat{h}(t))$ and $t > 0$. For this purpose, differentiating the expression in (2.30) and using the expressions in (2.32) and (3.9), we get

$$\begin{aligned} \partial_t V(t, \phi; \widehat{h}(t)) &= \int_\phi^{\widehat{h}(t)} \frac{\partial}{\partial t} \bigg(\frac{C(t)}{(y+1)^2} \int_0^y \frac{(1+x)}{x} \exp\Big(- \Lambda(t)(H(y) - H(x)) \Big) dx \bigg) dy \\ &= \int_\phi^{\widehat{h}(t)} \frac{cy}{\lambda(y+1)^2} \frac{\partial}{\partial t} \bigg(1 - \frac{Q(-\Lambda(t) - 1, \Lambda(t)/y)}{\Lambda(t) + 1} \bigg) dy \end{aligned} \tag{4.28}$$

for all $\phi < \widehat{h}(t)$ and $t > 0$. Note that we also have

$$x^{-a}\, e^x\, \Gamma(a, x) = x^{-a}\, e^x \int_x^\infty e^{-u} u^{a-1}\, du = \int_0^\infty e^{-xu}\, (u+1)^{a-1}\, du, \tag{4.29}$$

for $a < 0$ and $x > 0$. It is shown by differentiation of the expressions in (4.29) that the function $x^{-a} e^x \Gamma(a, x)$ is decreasing in x and increasing in a (see, e.g. [22, Section 2.5] for similar results). Hence, the function

$$\frac{y}{x+1} \, Q\Big(- x - 1, \frac{x}{y} \Big) = y \, \Big(\frac{x}{y} \Big)^{x+1} e^{x/y} \, \Gamma\Big(- x - 1, \frac{x}{y} \Big) \tag{4.30}$$

is decreasing in x for $x, y > 0$, where the functions $Q(z, y)$ and $\Gamma(z, y)$ are defined in (3.10). Recall that, for $1/2 < H < 1$, the function $\mu'(t)$ is decreasing, so that $\Lambda(t)$ is increasing in t. Hence, by using the formulas from (4.30), we obtain from the expressions in (4.28) that $V(t, \phi; \widehat{h}(t))$ is increasing in t that leads to $\partial_t V(t, \phi; \widehat{h}(t)) > 0$ for all $\phi \in (0, \widehat{h}(t))$ and $t > 0$. We can therefore apply the assertions of Remark 2.5 and use the fact that $E_{t,\phi} \widehat{\tau} < \infty$ to conclude that $E_{t,\phi} \tau_* \le E_{t,\phi} \widehat{\tau} < \infty$, when the starting time t satisfies $t > t^*$.

5 Appendix

Let us now prove the verification assertion stated in Theorem 2.3 above.

Proof: In order to verify the assertions stated above, let us denote by $V(t, \phi)$ the right-hand side of the expression in (2.15). Then, using the fact that the function $V(t, \phi)$ satisfies the conditions of (2.11)-(2.13) by construction, we can apply the local time-space formula from Peskir [31] (see also [36, Chapter II, Section 3.5] for a summary of the related results and further references) to obtain

$$\begin{aligned} V(t+u, \Phi_{t+u}) + \int_0^u F(\Phi_{t+s}) \, ds &= V(t, \phi) + M_u + K_u \\ &+ \int_0^u (\mathbb{L} V + F)(t+s, \Phi_{t+s}) \, I\big(\Phi_{t+s} \neq g_*(t+s), \Phi_{t+s} \neq h_*(t+s)\big) \, ds \end{aligned} \tag{5.1}$$

for all $t \ge 0$, where the process $M = (M_u)_{u \ge 0}$ defined by

$$M_u = \int_0^u V_\phi(t+s, \Phi_{t+s}) \, \zeta(t+s, \Phi_{t+s}) \, I\big(\Phi_{t+s} \neq g_*(t+s), \Phi_{t+s} \neq h_*(t+s)\big) \, d\overline{B}_s \tag{5.2}$$

is a continuous local martingale with respect to the probability measure $P_{t,\phi}$. Here, the process $K = (K_u)_{u \ge 0}$ is given by

$$\begin{aligned} K_u = \frac{1}{2} \int_0^u \Delta_\phi V(t+s, g_*(t+s)) \, I\big(\Phi_{t+s} = g_*(t+s)\big) \, d\ell_s^{g_*} \\ + \frac{1}{2} \int_0^u \Delta_\phi V(t+s, h_*(t+s)) \, I\big(\Phi_{t+s} = h_*(t+s)\big) \, d\ell_s^{h_*} \end{aligned} \tag{5.3}$$

where $\Delta_\phi V(t+s, g_*(t+s)) = V_\phi(t+s, g_*(t+s)+) - V_\phi(t+s, g_*(t+s)-)$, $\Delta_\phi V(t+s, h_*(t+s)) = V_\phi(t+s, h_*(t+s)+) - V_\phi(t+s, h_*(t+s)-)$, and the processes $\ell^{g_*} = (\ell_u^{g_*})_{u \ge 0}$ and $\ell^{h_*} = (\ell_u^{h_*})_{u \ge 0}$

defined by

$$\ell_u^{g_*} = P_{t,\phi} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^u I\big(g_*(t+s) - \varepsilon < \Phi_{t+s} < g_*(t+s) + \varepsilon\big)\, \zeta^2(t+s, \Phi_{t+s})\, ds \tag{5.4}$$

and

$$\ell_u^{h_*} = P_{t,\phi} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^u I\big(h_*(t+s) - \varepsilon < \Phi_{t+v} < h_*(t+s) + \varepsilon\big)\, \zeta^2(t+s, \Phi_{t+s})\, ds \tag{5.5}$$

are the local times of Φ at the curves $g_*(t)$ and $h_*(t)$, at which $V_\phi(t, \phi)$ may not exist. It follows from the concavity and continuous differentiability of the gain function $G(\phi)$ in (2.2), and the stopping time τ_* in (2.7), that the inequalities $\Delta_\phi V(t, g_*(t)) \le 0$ and $\Delta_\phi V(t, h_*(t)) \le 0$ should hold for all $t \ge 0$, so that the continuous process K defined in (5.3) is non-increasing. We may therefore conclude that $K_u = 0$ can hold for all $u \ge 0$ if and only if the smooth-fit conditions of (2.14) are satisfied.

Using the assumption that the inequality in (2.13) holds for the function $G(\phi)$ with the boundaries $g_*(t)$ and $h_*(t)$, we conclude that $(\mathbb{L}V + F)(t, \phi) \ge 0$ holds for any $\phi \ne g_*(t)$ and $\phi \ne h_*(t)$. Moreover, it follows from the conditions in (2.10)-(2.12) that the inequality $V(t, \phi) \le G(\phi)$ holds for all $(t, \phi) \in [0, \infty)^2$. Thus, for any stopping time τ such that $E_{t,\phi}\tau < \infty$, the expression in (5.1) yields the inequalities

$$G(\Phi_{t+\tau}) + \int_0^\tau F(\Phi_{t+s}) ds - K_\tau \ge V(t+\tau, \Phi_{t+\tau}) + \int_0^\tau F(\Phi_{t+s}) ds - K_\tau \ge V(t, \phi) + M_\tau. \tag{5.6}$$

Let $(\tau_n)_{n \in \mathbb{N}}$ be the localizing sequence of stopping times for the process M such that $\tau_n = \inf\{u \ge 0 \mid |M_u| \ge n\}$. Then, taking the expectations with respect to the probability measure $P_{t,\phi}$ in (5.6), by means of the optional sampling theorem (see, e.g. [26, Chapter III, Theorem 3.6] or [24, Chapter I, Theorem 3.22]), we get the inequalities

$$\begin{aligned} & E_{t,\phi}\bigg[G(\Phi_{t+\tau\wedge\tau_n}) + \int_0^{\tau\wedge\tau_n} F(\Phi_{t+s})\, ds - K_{\tau\wedge\tau_n}\bigg] \\ & \ge E_{t,\phi}\bigg[V(t+\tau\wedge\tau_n, \Phi_{t+\tau\wedge\tau_n}) + \int_0^{\tau\wedge\tau_n} F(\Phi_{t+s})\, ds - K_{\tau\wedge\tau_n}\bigg] \ge V(t, \phi) + E_{t,\phi}\, M_{\tau\wedge\tau_n} = V(t, \phi). \end{aligned} \tag{5.7}$$

Hence, letting n go to infinity and using Fatou's lemma, we obtain

$$\begin{aligned} & E_{t,\phi}\bigg[G(\Phi_{t+\tau}) + \int_0^\tau F(\Phi_{t+s})\, ds - K_\tau\bigg] \\ & \ge E_{t,\phi}\bigg[V(t+\tau, \Phi_{t+\tau}) + \int_0^\tau F(\Phi_{t+s})\, ds - K_\tau\bigg] \ge V(t, \phi) \end{aligned} \tag{5.8}$$

for any stopping time τ such that $E_{t,\phi}\tau < \infty$ and $E_{t,\phi}K_\tau > -\infty$, and all $(t, \phi) \in [0, \infty)^2$, where $K_\tau = 0$ holds whenever the conditions of (2.14) are satisfied. By virtue of the structure of the stopping time in (2.7) and the conditions of (2.11), it is readily seen that the equalities in (5.6) hold with τ_* instead of τ when either $\phi \le g_*(t)$ or $\phi \ge h_*(t)$, respectively.

Let us now show that the equalities are attained in (5.8) when τ_* replaces τ and the smooth-fit conditions of (2.14) hold for $g_*(t) < \phi < h_*(t)$. By virtue of the fact that the function $V(t, \phi)$ and the boundaries $g_*(t)$ and $h_*(t)$ solve the partial differential equation in (2.9) and satisfy the conditions in (2.10) and (2.14), it follows from the expression in (5.1) and the structure of the stopping time in (2.7) that

$$
\begin{aligned}
&G(\Phi_{t+\tau_*\wedge\tau_n}) + \int_0^{\tau_*\wedge\tau_n} F(\Phi_{t+s})\, ds \\
&\geq V(t+\tau_*\wedge\tau_n, \Phi_{t+\tau_*\wedge\tau_n}) + \int_0^{\tau_*\wedge\tau_n} F(\Phi_{t+s})\, ds = V(t,\phi) + M_{\tau_*\wedge\tau_n}
\end{aligned}
\tag{5.9}
$$

holds for $g_*(t) < \phi < h_*(t)$. Hence, taking expectations and letting n go to infinity in (5.9), using the assumptions that $G(\phi)$ is bounded and the integral in (2.16) is of finite expectation, we apply the Lebesgue dominated convergence theorem to obtain the equality

$$
E_{t,\phi}\Big[G(\Phi_{t+\tau_*}) + \int_0^{\tau_*} F(\Phi_{t+s})\, ds\Big] = V(t,\phi) \tag{5.10}
$$

for all $(t,\phi) \in [0,\infty)^2$. We may therefore conclude that the function $V(t,\phi)$ coincides with the value function $V_*(t,\phi)$ of the optimal stopping problem in (2.2) whenever the smooth-fit conditions of (2.14) hold.

In order to prove the uniqueness of the value function $V_*(t,\phi)$ and the boundaries $g_*(t)$ and $h_*(t)$ as solutions to the free-boundary problem in (2.9)-(2.13) with the smooth-fit conditions of (2.14), let us assume that there exist other continuous boundaries of bounded variation $\widetilde{g}(t)$ and $\widetilde{h}(t)$ such that $0 \leq \widetilde{g}(t) < c' < \widetilde{h}(t) \leq \infty$ holds. Then, define the function $\widetilde{V}(t,\phi)$ as in (2.15) with $\widetilde{V}(t,\phi;\widetilde{g}(t),\widetilde{h}(t))$ satisfying (2.9)-(2.14) and the stopping time $\widetilde{\tau}$ as in (2.7) with $\widetilde{g}(t)$ and $\widetilde{h}(t)$ instead of $g_*(t)$ and $h_*(t)$, respectively, such that $E_{t,\phi}\widetilde{\tau} < \infty$. Following the arguments from the previous part of the proof and using the fact that the function $\widetilde{V}(t,\phi)$ solves the partial differential equation in (2.9) and satisfies the conditions of (2.10) and (2.14) with $\widetilde{g}(t)$ and $\widetilde{h}(t)$ instead of $g(t)$ and $h(t)$ by construction, we apply the change-of-variable formula from [31] to get

$$
\begin{aligned}
&\widetilde{V}(t+u, \Phi_{t+u}) + \int_0^u F(\Phi_{t+s})\, ds = \widetilde{V}(t,\phi) + \widetilde{M}_u \\
&+ \int_0^u (\mathbb{L}\widetilde{V} + F)(t+s, \Phi_{t+s})\, I\big(\Phi_{t+s} \notin (\widetilde{g}(t+s), \widetilde{h}(t+s))\big)\, ds
\end{aligned}
\tag{5.11}
$$

where the process $\widetilde{M} = (\widetilde{M}_u)_{u\geq 0}$ defined as in (5.2) with $\widetilde{V}_\phi(t,\phi)$ instead of $V_\phi(t,\phi)$ is a continuous local martingale with respect to the probability measure $P_{t,\phi}$. Thus, taking into account the structure of the stopping time $\widetilde{\tau}$, we obtain from (5.11) that

$$
\begin{aligned}
&G(\Phi_{t+\widetilde{\tau}\wedge\widetilde{\tau}_n}) + \int_0^{\widetilde{\tau}\wedge\widetilde{\tau}_n} F(\Phi_{t+s})\, ds \\
&\geq \widetilde{V}(t+\widetilde{\tau}\wedge\widetilde{\tau}_n, \Phi_{t+\widetilde{\tau}\wedge\widetilde{\tau}_n}) + \int_0^{\widetilde{\tau}\wedge\widetilde{\tau}_n} F(\Phi_{t+s})\, ds = \widetilde{V}(t,\phi) + \widetilde{M}_{\widetilde{\tau}\wedge\widetilde{\tau}_n}
\end{aligned}
\tag{5.12}
$$

holds for $\widetilde{g}(t) < \phi < \widetilde{h}(t)$ and the localizing sequence $(\widetilde{\tau}_n)_{n \in \mathbb{N}}$ of $\widetilde{M}$ such that $\widetilde{\tau}_n = \inf\{u \geq 0 \mid |\widetilde{M}_u| \geq n\}$. Hence, taking expectations and letting n go to infinity in (5.12), using the assumptions that $G(\phi)$ and $F(\phi)$ are bounded and the integral in (2.16) taken up to $\widetilde{\tau}$ is of finite expectation, by means of the Lebesgue dominated convergence theorem, we have that the equality

$$E_{t,\phi}\bigg[G(\Phi_{t+\widetilde{\tau}}) + \int_0^{\widetilde{\tau}} F(\Phi_{t+s})\, ds\bigg] = \widetilde{V}(t,\phi) \tag{5.13}$$

is satisfied. Therefore, recalling the fact that τ_* is the optimal stopping time in (2.2) and comparing the expressions in (5.10) and (5.13), we see that the inequality $\widetilde{V}(t,\phi) \geq V(t,\phi)$ should hold for all $(t,\phi) \in [0,\infty)^2$.

We finally show that $\widetilde{g}(t)$ and $\widetilde{h}(t)$ should coincide with $g_*(t)$ and $h_*(t)$. By using the fact that $\widetilde{V}(t,\phi)$ and $V(t,\phi)$ satisfy (2.10)-(2.12), and $\widetilde{V}(t,\phi) \geq V(t,\phi)$ holds for all $(t,\phi) \in [0,\infty)^2$, we get that $g_*(t) \leq \widetilde{g}(t)$ and $\widetilde{h}(t) \leq h_*(t)$. Then, inserting $\tau_* \wedge \widetilde{\tau}_n$ into (5.11) in place of u and using the assumptions that $G(\phi)$ is bounded and the appropriate integrals are of finite expectation, by means of the arguments similar to the ones above, we obtain

$$\begin{aligned} E_{t,\phi}\bigg[\widetilde{V}(t+\tau_*, \Phi_{t+\tau_*}) + \int_0^{\tau_*} F(\Phi_{t+s})\, ds\bigg] &= \widetilde{V}(t,\phi) \\ + E_{t,\phi} \int_0^{\tau_*} (\mathbb{L}\widetilde{V} + F)(t+s, \Phi_{t+s})\, I\big(\Phi_{t+s} \notin (\widetilde{g}(t+s), \widetilde{h}(t+s))\big)\, ds & \end{aligned} \tag{5.14}$$

for all $(t,\phi) \in [0,\infty)^2$. Thus, since we have $\widetilde{V}(t,\phi) = V(t,\phi) = G(\phi)$ for $\phi = g_*(t)$ and $\phi = h_*(t)$, and $\widetilde{V}(t,\phi) \geq V(t,\phi)$, we see from the expressions in (5.10) and (5.14) that the inequality

$$E_{t,\phi} \int_0^{\tau_*} (\mathbb{L}\widetilde{V} + F)(t+s, \Phi_{t+s})\, I\big(\Phi_{t+s} \notin (\widetilde{g}(t+s), \widetilde{h}(t+s))\big)\, ds \leq 0, \tag{5.15}$$

should hold. Due to the assumption of continuity of $\widetilde{g}(t)$ and $\widetilde{h}(t)$, we may therefore conclude that $g_*(t) = \widetilde{g}(t)$ and $h_*(t) = \widetilde{h}(t)$, so that $\widetilde{V}(t,\phi)$ coincides with $V(t,\phi)$ for all $(t,\phi) \in [0,\infty)^2$. □

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