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Recursive Inspection Games

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Abstract

We consider a sequential inspection game where an inspector uses a limited number of inspections over a larger number of time periods to detect a violation (an illegal act) of an inspectee. Compared with earlier models, we allow varying rewards to the inspectee for successful violations. As one possible example, the most valuable reward may be the completion of a sequence of thefts of nuclear material needed to build a nuclear bomb. The inspectee can observe the inspector, but the inspector can only determine if a violation happens during a stage where he inspects, which terminates the game; otherwise the game continues.

Under reasonable assumptions for the payoffs, the inspector's strategy is independent of the number of successful violations. This allows to apply a recursive description of the game, even though this normally assumes fully informed players after each stage. The resulting recursive equation in three variables for the equilibrium payoff of the game, which generalizes several other known equations of this kind, is solved explicitly in terms of sums of binomial coefficients.

We also extend this approach to non-zero-sum games and, similar to Maschler (1966), "inspector leadership" where the inspector commits to (the same) randomized inspection schedule, but the inspectee acts legally (rather than mixes as in the simultaneous game) as long as inspections remain.

Keywords: inspection game, multistage game, recursive game, Stackelberg leadership, binomial coefficients

MSC2010 subject classification: 91A05, 91A20

OR/MS subject classification: Games/group decisions: Noncooperative; Military: Search/surveillance

JEL subject classification: C72

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1 Introduction

Inspection games model situations where an *inspector* with limited resources verifies by means of *inspections* that an *inspectee* adheres to legal obligations, which the inspectee has an incentive to *violate*. Inspection games have been applied to arms control and disarmament, tax auditing, fare evasion, environmental pollution, and homeland security; for a survey see Avenhaus, von Stengel, and Zamir [5], Avenhaus and Canty [2], and other recent works such as [9] or [8].

This paper presents a generalization of a classical sequential inspection game by Dresher [10]. In Dresher's game, the inspector has to distribute a given number of inspections over a larger number of inspection periods to detect a violation that the inspectee, who can count the inspector's visits, performs in at most one of these periods. In our extension of this game, the inspectee may violate more than once, and collect a possibly different reward for each successful violation; for example, a violation may be the diversion of a certain amount of nuclear material in a time period, with the highest reward to the inspectee once he has diverted enough material to build a nuclear bomb. As in Dresher's game, the game ends if a violation is discovered by the inspector who inspects at the same time. This is in line with an application to arms control, and may also apply in other contexts where an identified violator of legal rules becomes subject to much tighter surveillance.

A central aspect of our model and its analysis, and the reason for its choice of parameters, is that the inspector's mixed strategy in equilibrium does not depend on whether a successful violation took place during a time period without an inspection, about which the inspector is normally not informed. As we will explain in §3, the game can therefore, despite this lack of information, be described recursively by a sequence of 2×2 games for each stage. As long as there are remaining inspections (but fewer than the number of remaining time periods) and intended violations, the inspector and inspectee randomize at each stage whether to inspect and to violate. For the payoffs and mixed strategy probabilities in equilibrium we give explicit solutions in terms of the game parameters.

Our analysis starts with a zero-sum game, which is then extended to non-zero-sum payoffs. Furthermore, if, as in Maschler [19], the inspector can *commit* to his mixed equilibrium strategy, the inspectee will act legally as long as inspections remain. This commitment power, known as "inspector leadership", increases the inspector's payoff.

The main precursor to this work is [26], which, however, only considers the extra parameter of a varying number of intended violations, not different rewards for them. Inspection games with two parameters (time periods and inspections) were considered by Dresher [10], Thomas and Nisgav [24], and Baston and Bostock [6]. Maschler [19] introduced non-zero-sum games and inspector leadership. Höpfinger [16] and Avenhaus and von Stengel [4] extended Dresher's model to non-zero-sum payoffs. Rinderle [21] studied the case that inspections may have probabilities of false alarms and non-detection of a violation. Avenhaus and Canty [1] considered sequential inspections where timeliness of detection matters. Games with a third parameter of intended violations were considered by Kuhn [18], Sakaguchi [22, 23], and Ferguson and Melolidakis [13]. In these models, the game continues even after a detected violation, unlike in our model. In addition, the inspector is fully informed after each stage whether a violation took place or not even when he did not inspect. As already noted by Kuhn [18, p. 174], this full information is implicit in a recursive description.

In §2, we describe the inspection game and its parameters. The recursive equation for the value of this zero-sum game is solved explicitly. The equilibrium strategy of the inspector depends only on the number of remaining time periods and inspections. Section 3 discusses the key property that the strategy of the inspector does not depend on the inspectee's intended violations and their rewards, which allows to apply the solution also when the inspector has no information about undiscovered violations. We also show that our model is as general as possible to achieve this property. In §4, we show how to extend the solution relatively easily to the non-zero-sum game where a detected violation incurs a negative cost to both players compared to the case of legal action and no inspection. The "inspector leadership" game is studied in §5. In §6, we discuss possible extensions of our model, and general aspects of the recursive games we consider, in particular computational advantages compared to games in extensive form.

2 Zero-sum inspection game with multiple violations

We consider a two-player game $\Gamma(n,m,k)$, were n,m,k are three nonnegative integer parameters. The game is played over n discrete *time periods*. The number m is the number of *inspections* available to the inspector (the first player). The number k is the maximum number of *intended violations* of the inspectee (the second player). In each time period, the inspector can use one of his inspections (if $m > 0$) or not, and simultaneously (if $k > 0$) the inspectee chooses between legal action and violation. The game has also a real-valued "penalty" parameter b and nonnegative "reward" parameters $r_k, r_{k-1}, \ldots, r_1$ that determine the payoffs, as follows.

In this section, we assume that the payoffs are zero-sum. Let $v(n,m,k)$ be the value of the game $\Gamma(n,m,k)$, as the equilibrium payoff to the inspector. If $n = 0$, then the game is over and $v(n,m,k) = 0$. More generally, if $m \geq n$, then the inspector can inspect in every remaining time period, where we assume that the inspectee acts legally throughout, with

$$v(n,m,k) = 0 \qquad \text{if } m \geq n. \tag{1}$$

If $n > 0$, the game $\Gamma(n,m,k)$ is described recursively. Suppose first that $n > m > 0$, so that the inspector decides whether to use one of his inspections or not, and $k > 0$, so the inspectee decides whether to act legally or to violate. The recursive description of $\Gamma(n,m,k)$, with value $v(n,m,k)$, is given by the following payoffs to the inspector, which are the costs to the inspectee, at the first time period.

inspector \ inspectee	legal action	violation
inspection	$v(n-1,m-1,k)$	$b \cdot r_k$
no inspection	$v(n-1,m,k)$	$v(n-1,m,k-1) - r_k$

(2)

Of the four possible combinations of the player's actions in the first period, one of them *terminates* the game, namely when the inspector inspects and the inspectee performs a violation, which we assume is caught with certainty. In all other cases, the game continues. If the inspectee acts legally and the inspector inspects, then the game continues as the game $\Gamma(n-1,m-1,k)$. If the inspectee acts legally and the inspector does not inspect, then the game continues as the game $\Gamma(n-1,m,k)$. If the inspectee violates and the inspector did not inspect, then the game continues as the game $\Gamma(n-1,m,k-1)$, where in addition the inspectee collects the *reward* r_k which he can keep even if he is caught in a later time period. The corresponding bottom-right cell in (2) has therefore payoff entry $v(n-1,m,k-1)-r_k$ to the inspector.

If the game terminates because the inspectee is caught, we assume that inspectee has to pay the *penalty* $b \cdot r_k$, which is proportional to his reward r_k if the violation had been successful, multiplied by the penalty factor b. We assume only that $b > -1$, to allow for the possibility (in particular when payoffs are no longer zero-sum, discussed in §4) that even a caught violation is less preferred by the inspector than legal action (with reference payoff zero). A single successful (uncaught) violation with payoff $-r_k$ should however still be worse for the inspector than a caught violation with payoff $b \cdot r_k$, hence the requirement that $b > -1$. This condition holds obviously when $b > 0$ where a caught violation creates an actual penalty to the inspectee that is worse than legal action.

The nonnegative rewards to the inspectee $r_k, r_{k-1}, \ldots, r_1$ are numbered in that order to identify them from the game parameter k in $\Gamma(n,m,k)$ as the game progresses. That is, r_k is the reward for the first successful violation, r_{k-1} for the second, and so on until the reward r_1 for the kth and last violation if the game has not ended earlier. The inspectee can perform at most one violation per time period. Hence, if there are no inspections left ($m = 0$), then the inspectee can violate in each of the remaining n time periods up to k times in total, that is,

$$v(n,0,k) = -\sum_{i=1}^{\min\{k,n\}} r_{k+1-i} \,. \tag{3}$$

We allow some rewards to be zero. If all remaining rewards are zero, then this gives the same payoffs as when the inspectee only acts legally from now on, so this may instead be represented by a smaller k. However, the term $v(n,0,k)$ in (3) may be zero even if some remaining rewards are nonzero. This case may arise in the course of the game after some time periods without violations so that $n < k$, for example when $n = 1$, $k = 2$, $r_2 = 0$, $r_1 = 1$, so we allow for this possibility.

The game $\Gamma(n,m,k)$ is completely described by the "base cases" (1) and (3) (both of which imply $v(0,m,k) = 0$) and the recursive description (2). This description of the game assumes that both players are fully informed about the other player's action after each time period, and thus know in which of the four cells in (2) the game continues. We call this the game with *full information* and will weaken this assumption in §3.

The following main theorem gives an explicit formula for the game value $v(n,m,k)$ and the optimal inspection strategy. A large part of the proof is to show that (2) has a circular preference structure and hence a mixed equilibrium. The most important, but very direct part of the proof (from (28) onwards) is that the explicit representation (6) holds. We discuss a possible derivation of the term $t(n,m,k)$ in §5 after Theorem 5.

Theorem 1 *Let n,m,k be nonnegative integers, $b>-1$, and $r_k, r_{k-1}, \cdots, r_1 \geq 0$. Define*

$$s(n,m) = \sum_{i=0}^{m} \binom{n}{i} b^{m-i} \tag{4}$$

and

$$t(n,m,k) = \sum_{i=1}^{k} r_{k+1-i} \binom{n-i}{m}. \tag{5}$$

Then the zero-sum game $\Gamma(n,m,k)$ defined by (2) for $n>m>0$ and $k>0$ and by (1) and (3) otherwise has value

$$v(n,m,k) = \frac{-t(n,m,k)}{s(n,m)}\ . \tag{6}$$

For $n>m>0$ and $k>0$, the game (2) has a completely mixed equilibrium where the inspector inspects with probability p and the inspectee violates with probability q, where

$$p = \frac{s(n-1,m-1)}{s(n,m)}\ , \qquad 1-p = \frac{s(n-1,m)}{s(n,m)}\ , \tag{7}$$

and

$$q = \frac{v(n-1,m,k) - v(n-1,m-1,k)}{v(n-1,m,k) - v(n-1,m-1,k) + b\cdot r_k - v(n-1,m,k-1) + r_k}\ . \tag{8}$$

This equilibrium is unique, unless $r_{k+1-i} = 0$ for $1 \leq i \leq \min\{k, n-m\}$, in which case all entries in (2) are zero and the players can play arbitrarily.

Proof. Proof. We first consider some properties of $s(n,m)$ as defined in (4). Clearly,

$$s(n,0) = 1, \qquad s(n,n) = (1+b)^n \tag{9}$$

and

$$b\cdot s(n-1,m-1) = \sum_{i=0}^{m-1} \binom{n-1}{i} b^{m-i} = s(n-1,m) - \binom{n-1}{m}. \tag{10}$$

Furthermore,

$$s(n,m) = s(n-1,m-1) + s(n-1,m) \qquad (0<m<n) \tag{11}$$

which holds because

$$\begin{aligned} s(n,m) &= \sum_{i=0}^{m} \binom{n}{i} b^{m-i} = \binom{n}{0} b^m + \sum_{i=1}^{m} \left(\binom{n-1}{i} + \binom{n-1}{i-1}\right) b^{m-i} \\ &= \binom{n-1}{0} b^m + \sum_{i=1}^{m} \binom{n-1}{i} b^{m-i} + \sum_{i=0}^{m-1} \binom{n-1}{i} b^{m-1-i} \\ &= s(n-1,m) + s(n-1,m-1)\,. \end{aligned} \tag{12}$$

This means $s(n,m)$ is uniquely defined inductively by (9) and (11). Recall that $\binom{x}{0} = 1$ for any x [12, p. 50]. The following alternative representation

$$s(n,m) = \sum_{i=0}^{m} \binom{n-1-i}{m-i} (1+b)^i \tag{13}$$

holds because it also fulfills (9) and (11), which is shown similarly to (12). Because $b > -1$, we have $s(n,m) > 0$ for $0 \le m \le n$ by (13) (or by (9) and (11)).

The main assertion to prove is the explicit representation (6) for $v(n,m,k)$. Clearly, $v(n,m,0) = 0$, and (3) and (1) hold because in (5), $\binom{n-i}{m} = 0$ if $i > n-m$. Hence, we can assume that $n > m > 0$ and $k > 0$ where the recursive description (2) applies. By induction on n, we can assume as inductive hypothesis that

$$A = v(n-1,m-1,k), \qquad C = v(n-1,m,k), \qquad D = v(n-1,m,k-1) - r_k \tag{14}$$

are given using (6). These numbers and $B = b \cdot r_k$ define the game (2) as

	$1-q$	q
p	A	B
$1-p$	C	D

(15)

which also shows the probabilities p, $1-p$ and $1-q$, q of playing the rows and columns. To complete the induction, we will show that this game has value $v(n,m,k)$ as in (6).

If $r_{k+1-i} = 0$ for $1 \le i \le \min\{k, n-m\}$, then by (5) and (6) $A = B = C = D = 0$, so this is the all-zero game with value zero in agreement with (6), and arbitrary equilibrium strategies of the players. So assume that this is not the case, so that

$$t(n-1,m-1,k) > 0 \tag{16}$$

and hence $A < 0$.

Intuitively, (2) has a mixed equilibrium because the inspector prefers not to inspect if the inspectee acts legally and to inspect if he violates, and the inspectee prefers to act legally if inspected and to violate otherwise. In (15), this holds if

$$A < C, \quad B > D, \quad A < B, \quad C > D\,. \tag{17}$$

It is easy to see (and well known) that then with

$$p = \frac{C-D}{B-A+C-D}\,, \qquad 1-p = \frac{B-A}{B-A+C-D}\,, \tag{18}$$

$$1-q = \frac{B-D}{C-A+B-D}\,, \qquad q = \frac{C-A}{C-A+B-D}\,, \tag{19}$$

the game has value $v(n,m,k)$, where

$$v(n,m,k) \;=\; p \cdot A + (1-p) \cdot C \;=\; p \cdot B + (1-p) \cdot D\,, \tag{20}$$

$$v(n,m,k) \;=\; (1-q) \cdot A + q \cdot B \;=\; (1-q) \cdot C + q \cdot D\,, \tag{21}$$

where p in (18) is uniquely determined by (20) and q in (19) is uniquely determined by (21), and p, $1-p$, q, and $1-q$ are all positive by (17).

For (17), we first show $A < C$. By (6), this is equivalent to

$$\frac{-t(n-1,m-1,k)}{s(n-1,m-1)} < \frac{-t(n-1,m,k)}{s(n-1,m)}$$

or, by (16), to

$$\frac{s(n-1,m)}{s(n-1,m-1)} > \frac{t(n-1,m,k)}{t(n-1,m-1,k)} = \frac{r_k\binom{n-2}{m} + r_{k-1}\binom{n-3}{m} + \cdots + r_1\binom{n-1-k}{m}}{r_k\binom{n-2}{m-1} + r_{k-1}\binom{n-3}{m-1} + \cdots + r_1\binom{n-1-k}{m-1}}\,. \quad (22)$$

Assume that $k \le n-m$, otherwise replace k by $n-m$ because $\binom{n-1-i}{m-1} = \binom{n-1-i}{m} = 0$ for $i > n-m$. We show that the right expression in (22) is largest when $r_k > 0$ and $r_{k-1} = \cdots = r_1 = 0$. Namely, for general nonnegative $\rho_1, \ldots, \rho_k$, not all zero, positive $h_1, \ldots, h_k$, and any $g_1, \ldots, g_k$ so that

$$\frac{g_1}{h_1} \ge \frac{g_2}{h_2} \ge \cdots \ge \frac{g_k}{h_k}\,, \quad (23)$$

we have

$$\frac{g_1}{h_1} \ge \frac{\rho_1 g_1 + \cdots + \rho_k g_k}{\rho_1 h_1 + \cdots + \rho_k h_k} \quad (24)$$

which is seen by induction as follows. By omitting the terms where $\rho_i = 0$, we can assume $\rho_i > 0$ for all i. For $k=1$, (24) is true. For $k>1$, let $G = \rho_1 g_1 + \rho_2 g_2$ and $H = \rho_1 h_1 + \rho_2 h_2$. Then

$$\frac{g_1}{h_1} \ge \frac{G}{H} = \frac{\rho_1 g_1 + \rho_2 g_2}{\rho_1 h_1 + \rho_2 h_2} \ge \frac{g_2}{h_2} \quad (25)$$

because the left inequality in (25) is equivalent to $g_1(\rho_1 h_1 + \rho_2 h_2) \ge h_1(\rho_1 g_1 + \rho_2 g_2)$ and thus to $g_1 \rho_2 h_2 \ge h_1 \rho_2 g_2$ which holds by (23); the right inequality in (25) is shown similarly. This shows (24) for $k=2$, and for $k>2$ using the inductive hypothesis

$$\frac{G}{H} \ge \frac{g_3}{h_3} \ge \cdots \ge \frac{g_k}{h_k} \qquad \Rightarrow \qquad \frac{G}{H} \ge \frac{G + \rho_3 g_3 + \cdots + \rho_k g_k}{H + \rho_3 h_3 + \cdots + \rho_k h_k}\,.$$

With $\rho_i = r_{k+1-i}$, $g_i = \binom{n-1-i}{m}$, $h_i = \binom{n-1-i}{m-1}$ for $1 \le i \le k$ we have $\frac{g_i}{h_i} = \frac{n-i-m}{m}$ and thus (23) and (24), so (22) holds if

$$\frac{s(n-1,m)}{s(n-1,m-1)} > \frac{\binom{n-2}{m}}{\binom{n-2}{m-1}} = \frac{n-1-m}{m} \quad (26)$$

which we now show. By (13), the following are equivalent:

$$\begin{aligned}
s(n-1,m) &> \frac{n-1-m}{m} \cdot s(n-1,m-1)\,, \\
\sum_{i=0}^{m} \binom{n-1-i}{m-i}(1+b)^i &> \frac{n-1-m}{m} \cdot \sum_{i=0}^{m-1} \binom{n-1-i}{m-1-i}(1+b)^i\,, \\
(1+b)^m + \sum_{i=0}^{m-1} \binom{n-1-i}{m-1-i} \frac{n-m}{m-i}(1+b)^i &> \sum_{i=0}^{m-1} \binom{n-1-i}{m-1-i} \frac{n-1-m}{m}(1+b)^i\,,
\end{aligned}$$

which is true because $0 < m < n$ and thus $\frac{n-m}{m-i} > \frac{n-1-m}{m}$ for $0 \le i < m$. This shows (26) and thus $A < C$.

The remaining inequalities in (17) are seen as follows. Because $b > -1$ and $r_k \ge 0$, we have $B = b \cdot r_k \ge 0 - r_k \ge v(n-1,m,k-1) - r_k = D$, with inequality possible only if $r_k = 0$ and $t(n-1,m,k-1) = 0$, which because

$$t(n-1,m,k-1) \;=\; r_{k-1}\binom{n-2}{m} + r_{k-2}\binom{n-3}{m} + \cdots + r_1\binom{n-k}{m} \tag{27}$$

means $r_{k+1-i} = 0$ for $1 \le i \le \min\{k, n-m\}$ which we have excluded. So $B > D$. Suppose p given by (7) fulfills (20), which implies (18). By (11), the real number p defined in (7) is indeed a probability. Also, $p > 0$ and $1 - p > 0$, so that by (18) either $C < D$ and $B < A$ or $C > D$ and $B > A$. The former can be excluded because it would imply $B < A < C < D < B$. This proves (17).

So it remains to show (20), that is,

$$v(n,m,k) = p \cdot v(n-1,m-1,k) + (1-p) \cdot v(n-1,m,k)\,, \tag{28}$$

$$v(n,m,k) = p \cdot b \cdot r_k \qquad\qquad + (1-p) \cdot (v(n-1,m,k-1) - r_k)\,. \tag{29}$$

After multiplication with $s(n,m)$, (28) and (29) are by (6) and (7) equivalent to

$$-t(n,m,k) = -t(n-1,m-1,k) \;-\; t(n-1,m,k)\,, \tag{30}$$

$$-t(n,m,k) = s(n-1,m-1) \cdot b \cdot r_k \;+\; (-t(n-1,m,k-1) - s(n-1,m) \cdot r_k)\,. \tag{31}$$

Equation (30) holds because, by (5),

$$\begin{aligned} t(n-1,m-1,k) + t(n-1,m,k) &= \sum_{i=1}^{k} r_{k+1-i}\binom{n-1-i}{m-1} + \sum_{i=1}^{k} r_{k+1-i}\binom{n-1-i}{m} \\ &= \sum_{i=1}^{k} r_{k+1-i}\binom{n-i}{m} \qquad = t(n,m,k)\,. \end{aligned} \tag{32}$$

Equation (31) holds because, by (10) and (27),

$$\begin{aligned} & s(n-1,m-1) \cdot b \cdot r_k \qquad + \; (-t(n-1,m,k-1) - s(n-1,m) \cdot r_k) \\ =\; & (s(n-1,m) - \tbinom{n-1}{m}) \cdot r_k \;+\; (-t(n-1,m,k-1) - s(n-1,m) \cdot r_k) \\ =\; & -\tbinom{n-1}{m} \cdot r_k \qquad\qquad - \; t(n-1,m,k-1) \\ =\; & -t(n,m,k)\,. \end{aligned} \tag{33}$$

This shows (28) and (29), which completes the induction on n.

The inspectee's violation probability q in (8) is just given by (19). □

By Theorem 1, the game in (2) has a unique mixed equilibrium (unless all payoffs are zero). This uniqueness applies recursively to all stages of $\Gamma(n,m,k)$ if the players use behavior strategies. The same probabilities for their actions could result from mixed strategies that correlate these actions, which we do not consider because behavior strategies suffice [17].

A special case of Theorem 1 has been shown in [26], namely when $r_i = 1$ for $k \geq i \geq 1$. In that case, $t(n,m,k)$ in (5) can be written as

$$t(n,m,k) = \sum_{i=1}^{k} \binom{n-i}{m} = \binom{n}{m+1} - \binom{n-k}{m+1} \tag{34}$$

(see Feller [12, p. 63, equation (12.6)]).

Dresher [10] actually considered two special cases of this game for $b = 1$, namely $k = 1$ and $k = n - m$, where (34) simplifies to

$$t(n,m,1) = \binom{n-1}{m} \qquad \text{and} \qquad t(n,m,n-m) = \binom{n}{m+1}. \tag{35}$$

The corresponding expressions (6) were stated and proved by Dresher [10], and, apparently independently, by Sakaguchi [23].

3 Discussion of the model and interpretation of the main theorem

In this section, we discuss the main Theorem 1, in particular the fact that the inspector's equilibrium strategy depends only on the number of time periods and inspections. Consequently, the same strategy also applies to a new game $\Gamma'(n,m,k)$ where the inspector is not informed about violations at previous time periods when he did not inspect, which we call the game *without full information*. In a basic form, this assumption is implicit in the models by Dresher [10].

The recursive definition of $\Gamma(n,m,k)$ as in (2) allows to compute the game value even without an explicit formula as stated in (6). If a game as in (15) fulfills the inequalities (17) so that the game has a mixed equilibrium, then the equilibrium probabilities (18) and (19) give the value of the game as $\frac{BC-AD}{B-A+C-D}$. Sakaguchi [23] recursively computes the game value $v(n,m,k)$ in this way for different entries in (2).

As mentioned, the recursive description (2) assumes that, in particular, the inspector knows whether the inspectee chose legal action or violation even after a time period where the inspector did not inspect. In practice, it may be rather questionable how the inspector would obtain this knowledge.

In the games studied by Dresher [10], it actually does not matter whether the inspector has this knowledge or not. In Dresher's first game, the inspectee has only a single intended violation, corresponding to $k = 1$ in our model (and, throughout, $r_i = 1$ for all i). Then the lower-right entry in (2) given by $v(n-1,m,0) - r_1$ is equal to -1. In that case, because the inspectee has successfully violated once and will not violate further, the game is effectively over because the inspectee acts legally from then on and will not be caught. Then any action of the inspector is optimal, and so the inspector can act as if the violation is still to take place. That is, if the inspector does know whether he is in the game $\Gamma(n-1,m,1)$ or $\Gamma(n-1,m,0)$ (the latter with added payoff -1 due to the uncaught violation), then he can always act as if he is in the game $\Gamma(n-1,m,1)$ because that is the only situation where his strategy matters. Therefore, the recursive description is justified.

In the second game described by Dresher [10], the inspectee tries to violate as often as possible. This corresponds to our game $\Gamma(n,m,n-m)$ because the inspectee can only violate once per time period and will therefore not violate more than $n-m$ times because otherwise he would be caught with certainty. Then the bottom-left and bottom-right entries in (2) are $v(n-1,m,n-m)$ and $v(n-1,m,n-m-1)-1$, respectively. However, the lower-left game $\Gamma(n-1,m,n-m)$ (where the inspectee has "missed out" to violate during an uninspected time period) is equivalent to the game $\Gamma(n-1,m,n-m-1)$, that is, again a game with a maximal number of intended violations. The bottom-right game is the same, except for the added -1 to the inspector's payoff, so again it does not matter whether the inspector knows if the inspectee violated or not.

Dresher [10] gave explicit values for these two games as in (35). However, Dresher did not compute the optimal inspection probabilities, because he would otherwise most likely have noted that they are the same in the two games $\Gamma(n,m,1)$ and $\Gamma(n,m,n-m)$. These inspection probabilities are given by (7). A key aspect of our model is that they hold, independently of k, in the game $\Gamma(n,m,k)$ with the number k of intended violations as a new parameter.

Because of this independence of k, the equilibrium strategy of the inspector, and the game value $v(n,m,k)$, apply also to the game $\Gamma'(n,m,k)$ without full information where the inspector does *not* know if a violation occurred or not in an uninspected time period. Namely, by induction the inspection strategy is the same in the two games $\Gamma'(n-1,m,k)$ and $\Gamma'(n-1,m,k-1)$ which correspond to the two bottom cells in (2), the latter with an additional loss of $-r_k$ to the inspector, as long as the inspectee has still an incentive to violate; if that is not the case, as in the game $\Gamma'(n-1,m,0)$ which has value zero, then any inspection strategy is optimal and so the inspector should act as if there are still violations to take place because only then his action matters, as in Dresher's first game.

Formally, the game $\Gamma'(n,m,k)$ without full information is not described recursively. However, it can be modelled as an extensive form game with information sets [17] that represent the inspector's lack of information. If we then change the game to the game with full information, then these information sets are "cut", which transforms $\Gamma'(n,m,k)$ into the recursively described game $\Gamma(n,m,k)$ in (2). Because the inspector's behavior strategy in $\Gamma(n,m,k)$ is the same at all information sets obtained from "cutting" an information set h, say, in the original game $\Gamma'(n,m,k)$, it can also be defined uniquely as the behavior at h and thus defines a behavior strategy for $\Gamma'(n,m,k)$. In particular, the value of $\Gamma'(n,m,k)$ stays the same at $v(n,m,k)$. This (straightforward) manipulation of information sets is described in detail in [26]. In summary:

Corollary 1 *The equilibrium payoff and the equilibrium strategies for the inspection game with full information described in Theorem 1 also apply in the game* $\Gamma'(n,m,k)$ *without full information where the inspector is not informed about the action of the inspectee after a time period without inspection.*

In the game $\Gamma'(n,m,k)$ without full information, the inspectee has typically additional equilibrium behavior strategies compared to $\Gamma(n,m,k)$. As an example, let $n,m,k=3,1,2$

and $b = r_2 = r_1 = 1$. Then the bottom cells of (2) both correspond to the game $\Gamma(2,1,1)$, with added payoff -1 in the bottom-right cell. In $\Gamma(2,1,1)$, which is (15) with $A,B,C,D = -1,1,0,1$, the optimal strategies are $p = q = 1/3$, with $v(2,1,1) = -1/3$. At the first stage in $\Gamma(3,1,2)$, they are $p = 1/4$ and $q = 5/12$, which also applies to $\Gamma'(3,1,2)$. However, in the game $\Gamma'(3,1,2)$, the inspector does not know if the inspectee violated in the first time period or not, which gives the inspectee additional optimal behavior strategies for the second time period. For example, the inspectee can violate with probability $4/7$ if he acted legally in the first period and violate with probability zero if he violated in the first period. Another such coordinated different behavior in the second time period would be to violate with probability zero following legal action in the first period and to violate with probability $4/5$ following a violation in the first period.

We next discuss the rewards to the inspectee $r_k,\ldots,r_1$ for successful violations, and the corresponding scaled penalty $-b \cdot r_k$ to the inspectee in (2). These parameters are new compared to von Stengel [26], who proved Theorem 1 with $r_i = 1$ for $k \geq i \geq 1$. With general nonnegative rewards r_i, it seems that one can dispense with the parameter k and simply assume that only the first k rewards r_i are nonzero if the inspectee intends only k violations. In one respect this is a different game than when the inspectee will not carry out more than k violations, because when all rewards are zero, the inspectee can violate and be caught without penalty, which just terminates the game; one may argue that this is an acceptable game outcome that just has to be interpreted appropriately. The main reason for the parameter k in the recursive description of the game is to identify the next reward to the inspectee after a successful violation when the game continues in the bottom-right cell in (2). The number of intended violations serves as a "counter" for the rewards, which we have therefore numbered in the order $r_k, r_{k-1}, \ldots, r_1$. Such a counter is needed for the recursive description in one way or another.

The payoff $b \cdot r_k$ to the inspector for a caught violation may seem strange in the game $\Gamma'(n,m,k)$ where the inspector is not informed about k. However, we think it is justifiable to make the "stakes" of a violation proportional to r_k even if the inspector does not know r_k, because the inspectee knows what is at stake. We have chosen this payoff as $b \cdot r_k$ because otherwise the optimal inspection strategy would not be independent of k as it is according to the solution (7). In fact, the next theorem states that the payoffs in (2) are as general as possible so that this independence holds. For simplicity, we assume that the marginal gain r_j to the inspectee for the next of j remaining violations is always positive, and that the game has a circular preference structure.

Theorem 2 *Suppose that n,m,k are the number of time periods, inspections, and intended violations in a zero-sum inspection game where the inspectee can violate at most once per time period, where his overall payoff depends only on (and is strictly increasing in) the total number of successful violations, and whether he is ever caught (in which case the game terminates) or not. Consider this game with full information and value $v(n,m,k)$. Then the most general form of this game fulfills* (1) *and* (3), *and for $n > m > 0$ and $k > 0$ is the game*

p	$v(n-1,m-1,k)$	$f(k)$
$1-p$	$v(n-1,m,k)$	$v(n-1,m,k-1) - r_k$

(36)

similar to (2)*, where we assume that* (36) *has a unique completely mixed equilibrium. Here* $f(k)$ *is the marginal penalty and* r_k *is the marginal gain to the inspectee for the first of* k *remaining violations,* $r_k > 0$. *Then the probability* p *of inspection is independent of* k *(so that it can be applied to the game without full information) if and only if there is some* $b > -1$ *so that* $f(k) = b \cdot r_k$ *for all* k*, as in* (2).

Proof. Proof. Consider the game with full information. At the beginning of the game, we can assume $k \le n-m$ because the inspectee will not perform more than $n-m$ violations because he would otherwise be caught with certainty. For $1 \le i \le n$, let r_{k-i+1} be the marginal gain to the inspectee for the ith successful violation, which by assumption is strictly positive. Suppose that over the n time periods, the inspectee performs i successful violations, $0 \le i \le k$, and thus gains $r_k + r_{k-1} + \cdots + r_{k-i+1}$. This is his payoff (and loss to the inspector), in completely general form, if he is not caught. If the inspectee is caught when attempting the $(i+1)$st violation, then he pays the penalty $f(k-i)$, which is subtracted from this sum (this penalty may include, for example, repaying all previous gains); the inspector's payoff is then $-r_k - r_{k-1} - \cdots - r_{k-i+1} + f(k-i)$. Then $v(n,m,k) = 0$ when $k = 0$ or $m \ge n$ as in (1) (for legal action throughout), and $v(n,0,k)$ given by (3). For $n > m > 0$ and $k > 0$, the game with value $v(n,m,k)$ is given by (36), which is therefore the general form of an inspection game under the stated assumptions.

If $k = 1$, there is only one parameter $f(1)$ so that we can set $b = f(1)/r_1$ and the inspector's strategy can be applied to the game without full information; this is essentially the first game by Dresher [10]. Hence, we can assume $k \ge 2$.

Let $j \ge 2$ and suppose that the inspectee has performed $k-j$ successful violations (and therefore, so far, gained $r_k + r_{k-1} + \cdots + r_{j+1}$), that the inspector has performed $m-1$ inspections, and that $n-3$ time periods have passed. The successful violations and the inspections have to take place in different time periods, which is possible because $k - j + m - 1 \le n - m - 2 + m - 1 = n - 3$, and this occurs with positive probability because of the mixed equilibrium at every stage of the game. Then at this stage there are three time periods, one inspection, and j intended violations remaining, and the remaining game has value $v(3,1,j)$ and is of the form

$v(2,0,j)$	$f(j)$
$v(2,1,j)$	$v(2,1,j-1) - r_j$

. (37)

In the bottom left cell of (37), $v(2,1,j)$ is the value of

$v(1,0,j)$	$f(j)$
$v(1,1,j)$	$v(1,1,j-1) - r_j$

, that is, of

p	$-r_j$	$f(j)$
$1-p$	0	$-r_j$

. (38)

In the bottom right cell of (37), the inspectee collects a reward of r_j, and $v(2,1,j-1)$ is the value of the game

$v(1,0,j-1)$	$f(j-1)$
$v(1,1,j-1)$	$v(1,1,j-2) - r_{j-1}$

, that is, of

p	$-r_{j-1}$	$f(j-1)$
$1-p$	0	$-r_{j-1}$

. (39)

The two games in (38) and (39) correspond to the two cells in the bottom row of (37) and both must have the same probability p of inspection if this is to be applied to the game without full information. That is, according to (18),

$$\frac{1-p}{p} = \frac{1}{p}-1 = \frac{f(j)+r_j}{r_j} = \frac{f(j)}{r_j}+1 = \frac{f(j-1)+r_{j-1}}{r_{j-1}} = \frac{f(j-1)}{r_{j-1}}+1\,. \quad (40)$$

For $j=2$, this shows $f(2)/r_2 = f(1)/r_1 =: b$, where $b > -1$ because $1/p - 1 > 0$. For $j=3$ it shows $f(3)/r_3 = f(2)/r_2$, and so on, so that $f(j)/r_j = b$ for all $1 \le j \le k$, as claimed. □

Another question is if there is an intuitive reason that the inspector's optimal strategy in $\Gamma(n,m,k)$ does not depend on k (for any m, not just for $m=1$ as in the proof of Theorem 2). For example, Ferguson and Melolidakis [14] have applied a "game with finite resources" due to Gale [15] to a different inspection game where the solution also applies when one of the players lacks information. However, we have not been able to apply the highly symmetrical strategy in this game to our setting. At present, the very canonical proof (see equation (28) and onwards) of the explicit representation (7) and (6) seems to be the best explanation.

To conclude this section, we discuss the solution of the game $\Gamma(n,m,k)$ for some simple special cases. If $m=1$, then it is easy to see that the inspector uses his single inspection in the first $n-1$ time periods with equal probability, which for the last time period is multiplied with $1+b$, so if $b>0$ then higher probability is given to the last period.

The case $b=0$, where a caught violation terminates the game but no further penalty applies, has also some easily described properties. Then $s(n,m) = \binom{n}{m}$ and thus $p = m/n$ in (7), which means that all m-sets of the n time periods are equally likely to be inspected. Moreover, if $r_i = 1$ for $k \ge i \ge 1$, then $t(n,m,k) = \sum_{i=1}^{k} \binom{n-k}{m}$ by (5), and $-v(n,m,k) = t(n,m,k)/s(n,m)$ can be interpreted as the expected number of successful violations, as follows: The inspectee is indifferent between all possible time periods for choosing his k violations, and thus gets payoff $-v(n,m,k)$ if he violates in the first k time periods. Then, if the m-set of inspections does not include period 1 (with $\binom{n-1}{m}$ out of $\binom{n}{m}$ choices), the first violation succeeds. If this set also does not include period 2 (with further $\binom{n-2}{m}$ choices), then the first and second violation succeed, and so on.

The probability q of violation in the first period depends on k, and for $b=0$ and $r_i = 1$ for $k \ge i \ge 1$ has the following form. If $k=1$, then $q = 1/n$, independently of m. If $k = n-m$ (where the inspectee violates as often as possible), then $q = 1/(m+1)$, independently of n. Unfortunately, there is no straightforward simple extension of these values for intermediate values of k. In general, we have only found complicated expressions for the inspectee's strategy, which is why we have left it in the form (8) derived from the well-known representation (19) in terms of the game payoffs.

4 Non-zero-sum payoffs

In this section, we extend the zero-sum-game $\Gamma(n,m,k)$ to a non-zero-sum game $\hat{\Gamma}(n,m,k)$. The reason to consider non-zero-payoffs is that a caught violation as the outcome of the game is typically less preferred by both inspector and inspectee than legal action, because for the inspector it means the failure of the inspection regime. This is a standard assumption in inspection games, first proposed by Maschler [19].

We denote the equilibrium payoffs in $\hat{\Gamma}(n,m,k)$ by $v(n,m,k)$ for the inspector and by $w(n,m,k)$ for the inspectee (which are unique as shown in Theorem 4 below). The reference payoff for legal action throughout is zero for both players. As before, we assume that the inspectee acts legally if the inspector can inspect in every remaining period, that is,

$$v(n,m,k) = w(n,m,k) = 0 \qquad \text{if } m \geq n. \tag{41}$$

Also as before, if the inspector has run out of inspections, then the inspectee collects a nonnegative reward $r_k, r_{k-1}, \ldots$ which is a cost to the inspector for each remaining period up to the maximum number k of intended violations, that is,

$$-v(n,0,k) = w(n,0,k) = \sum_{i=1}^{\min\{k,n\}} r_{k+1-i}\ . \tag{42}$$

For the case that a violation is caught we introduce two parameters a and b as costs to inspector and inspectee (scaled by the reward r_k for a successful violation), where

$$0 < a < 1, \qquad b \geq 0, \tag{43}$$

so that for $n > m > 0$ and $k > 0$ the game $\hat{\Gamma}(n,m,k)$ (with full information) has the following recursive description:

inspector \ inspectee	legal action	violation
inspection	$w(n-1,m-1,k)$ $v(n-1,m-1,k)$	$-b \cdot r_k$ $-a \cdot r_k$
no inspection	$w(n-1,m,k)$ $v(n-1,m,k)$	$w(n-1,m,k-1) + r_k$ $v(n-1,m,k-1) - r_k$

(44)

In (44), the arrows represent the circular preferences of the players, which have been proved for $\Gamma(n,m,k)$ as (17). In particular, if $k = 1$, then the bottom-right cell in (44) for an uncaught violation has payoff $-r_k$ to the inspector, whereas the top-right cell has payoff $-a \cdot r_k$. Because $a < 1$, the inspector therefore prefers a caught violation to an uncaught one, as it should be the case.

Due to (43), the game $\hat{\Gamma}(n,m,k)$ does not include the zero-sum game $\Gamma(n,m,k)$ as a special case. However, the more general conditions $a < 1$ and $b > -1$ do include it when $a = -b$. The following theorem is essentially a corollary to Theorem 1.

Theorem 3 *Let n,m,k be nonnegative integers, let the reals a and b be as in* (43)*, and let $r_k, r_{k-1}, \cdots, r_1 \geq 0$. Define $s(n,m,k)$ as in* (4)*, $t(n,m,k)$ as in* (5)*, and $\hat{s}(n,m)$ by*

$$\hat{s}(n,m) = \sum_{i=0}^{m} \binom{n}{i} (-a)^{m-i} \ . \tag{45}$$

Then the non-zero-sum game $\hat{\Gamma}(n,m,k)$ defined by (44) *for $n > m > 0$ and $k > 0$ and by* (41) *and* (42) *otherwise has equilibrium payoffs to inspector and inspectee*

$$v(n,m,k) = \frac{-t(n,m,k)}{\hat{s}(n,m)} \ , \qquad w(n,m,k) = \frac{t(n,m,k)}{s(n,m)} \ . \tag{46}$$

For $n > m > 0$ and $k > 0$, the game (44) *has a completely mixed equilibrium where the inspector inspects with probability p according to* (7)*, and the inspectee violates with probability q according to* (8)*. This is the unique subgame perfect Nash equilibrium (SPNE) of the game, unless $r_{k+1-i} = 0$ for $1 \leq i \leq \min\{k, n-m\}$, in which case all entries in* (44) *are zero and the players can play arbitrarily. Each player's strategy is the min-max strategy for the payoffs of his opponent.*

Proof. Proof. If we modify the game $\hat{\Gamma}(n,m,k)$ to a zero-sum game with the payoffs $v(n,m,k)$ to the inspector (and thus $-v(n,m,k)$ to the inspectee), then it fulfills the assumptions of Theorem 1 with $b = -a > -1$. In this game, the inspector prefers not to inspect when the inspectee acts legally and to inspect when the inspectee violates, as shown with the vertical arrows in (44). The inspectee's strategy is as in (8) and is a min-max strategy for the inspector's payoff. It makes the inspector indifferent between his two actions, with the inspector's payoff $v(n,m,k)$ as in (46). Note that in (45), $\hat{s}(n,m) > 0$ for $0 \leq m \leq n$ due to the alternative representation (13) where $1 + b = 1 - a > 0$.

Similarly, if we modify the game $\hat{\Gamma}(n,m,k)$ to a zero-sum game based on the payoffs $w(n,m,k)$ to the inspectee (and thus $-w(n,m,k)$ to the inspector), then it fulfills also the assumptions of Theorem 1 with $b \geq 0 > -1$. Then the inspectee prefer to act legally when he is inspected and to violate otherwise, as shown with the horizontal arrows in (44). In this game, the inspector's strategy is given by (7), and is a min-max strategy for the inspectee's payoff. It makes the inspectee indifferent between his two actions in (44), with the inspectee's payoff $w(u,m,k)$ as in (46).

So the game in (44) has a circular preference structure and a unique mixed equilibrium as described (except when $r_{k+1-i} = 0$ for $1 \leq i \leq \min\{k, n-m\}$), which by induction defines the unique SPNE of $\hat{\Gamma}(n,m,k)$. □

In Theorem 3, the inspector's strategy in $\hat{\Gamma}(n,m,k)$ does not depend on k. As argued in §3, this strategy can therefore also be applied to the game $\hat{\Gamma}'(n,m,k)$ without full information. That is, we obtain the analogous statement to Corollary 1.

Corollary 2 *The equilibrium payoff and the equilibrium strategies for the non-zero-sum inspection game described in Theorem 3 with full information also apply in the game $\hat{\Gamma}'(n,m,k)$ without full information.*

As mentioned after Corollary 1, the game $\Gamma'(n,m,k)$ without full information may have additional equilibrium strategies for the inspectee, which applies in the same way to the game $\hat{\Gamma}'(n,m,k)$.

Because the games $\hat{\Gamma}(n,m,k)$ and $\hat{\Gamma}'(n,m,k)$ are not zero-sum, the question arises if they have other Nash equilibrium payoffs. The following theorem asserts that this is not the case.

Theorem 4 *All Nash equilibria of the non-zero-sum inspection game $\hat{\Gamma}(n,m,k)$ and the game $\hat{\Gamma}'(n,m,k)$ without full information have the payoffs described in Theorem 3.*

Proof. Proof. Consider first the game $\hat{\Gamma}(n,m,k)$ with full information. Let the game be represented as an extensive game. Call a *stage* of the game a particular time period together with the history of past actions. At each stage, we let, as in [26], the inspector move first and the inspectee second, where the decision nodes of the inspectee belong to a two-node information set so that the inspectee is not informed about the action of the inspector at the current stage, but knows everything else. The information set of the inspector is a singleton (this is different in the game $\hat{\Gamma}'(n,m,k)$ that we consider later).

Consider a Nash equilibrium of this game. Suppose that there is a stage of the game that is reached with positive probability where the players do not behave according to the SPNE described in Theorem 3, and let there be no later such stage. That is, each of four cells in (44) at this stage either has the SPNE payoffs as entries or is reached with probability zero. We claim that the equilibrium property is violated at this stage. If all cells have positive probability, then at least one player gains because they do not play the unique equilibrium at this stage. If some cells have probability zero, then one player plays deterministically. For example, suppose the inspectee acts legally. Then if the inspector inspects with positive probability, he gets the SPNE payoff $v(n-1,m-1,k)$. However, this is not his best response, because when he does not inspect at this stage, he gets at least $v(n-1,m,k)$ because that is also his min-max payoff which he can guarantee by playing a max-min strategy after no inspection at the current stage. Because we are in equilibrium, the inspector therefore responds with no inspection to the inspectee's certain legal action at this stage. However, then the inspectee can improve on his SPNE payoff $w(n-1,m,k)$ by violating and subsequently playing a max-min strategy, which contradicts the assumed equilibrium. This reasoning follows from the strictly circular payoff structure in (44) and holds for any assumed unplayed strategy. Hence, players always mix and the SPNE of the game $\hat{\Gamma}(n,m,k)$ is its unique Nash equilibrium (in behavior strategies, as always).

The crucial condition used is that SPNE payoffs are min-max payoffs, and the argument is similar to an analogous known result on finitely repeated games where all stage equilibrium payoffs are min-max payoffs (see Osborne and Rubinstein, [20, Proposition 155.1]).

Before we consider the game $\hat{\Gamma}'(n,m,k)$, we discuss a potential "threat" of the inspectee to use a violation even in the case $m \geq n$ when the inspector can inspect in every remaining period. If we assume that in this case the inspector has the choice not to inspect, this defines a game where legal action gives payoff zero to both players but violation gives a negative payoff to both players. The min-max payoff to the inspector is then $-a \cdot r_k$ rather than zero, when the inspectee irrationally violates later (which is his "threat") and the inspector inspects. If we use this payoff $-a \cdot r_k$ in the bottom-left cell of (44) rather than the assumed zero SPNE payoff $v(n-1,n-1,k)$, we show that nevertheless $-a \cdot r_k > v(n-1,n-2,k)$ so that the preceding reasoning still applies, that is, the inspector's best response to legal action at the current stage (and violation later) is still no inspection. Namely, by (46), and (9) and (10) with $b=-a$ and $m=n-1$,

$$v(n-1,n-2,k) = \frac{-t(n-1,n-2,k)}{\hat{s}(n-1,n-2)} = \frac{-r_k\binom{n-2}{n-2}}{((1-a)^{n-1}-\binom{n-1}{n-1}))/(-a)} = \frac{-a \cdot r_k}{1-(1-a)^{n-1}}$$

and thus $-a \cdot r_k > v(n-1,n-2,k)$ as claimed because $0 < a < 1$ by (43). That is, the inspector still prefers not to inspect in response to legal action and a later "threatened" violation that will be caught. Hence there is no "threat" of the inspectee that could induce a Nash equilibrium other than the SPNE.

Consider now a Nash equilibrium in behavior strategies of the game $\hat{\Gamma}'(n,m,k)$ without full information. The inspector's lack of information is represented by information sets of the inspector that comprise multiple decision nodes with the same history of the inspector's own past actions, but different past actions of the inspectee at the stages where the inspector did not inspect. Consider such an information set h of the inspector that is reached with positive probability where the inspector does *not* use the min-max strategy against the inspectee in Theorem 3, and assume that there is no later information set of this kind. Then at this stage, that is, for *all* information sets of the inspectee that immediately follow this move of the inspector at h, the inspectee will have the same action (legal action or violation) as a best response, which he therefore chooses with certainty because we are in equilibrium. However, in response the inspector would have to make a move at h against which the moves of the inspectee are not optimal. This contradicts the equilibrium property. Hence, the inspector has to choose the min-max strategy throughout, so that the inspectee's payoff is as in Theorem 3.

Now suppose that the inspector's payoff is different from his min-max payoff. This has to be a larger payoff because the inspector can guarantee his min-max payoff by playing a max-min strategy. Then at some information set h of the inspector that is reached with positive probability, again looking at the latest such set, the inspectee does at this stage not play a min-max strategy against the inspector, that is, a strategy that does not equalize the inspector's payoffs. To this the inspector plays a unique pure best response at h. This response is different from the inspector's strategy in Theorem 3, but we have just shown that this cannot be the case.

Hence, the players' payoffs are uniquely given according to Theorem 3, as claimed. □

5 Inspector leadership

The game by Dresher [10] with a single intended violation has been studied by Maschler [19] in a *leadership* variant where the inspector can announce and commit to his mixed strategy. We extend these considerations to our game with k intended violations, and simplify some of Maschler's arguments.

A two-player game in strategic form is changed to a leadership game by declaring one player as leader and the other as follower. The leader chooses and commits to a strategy about which the follower is fully informed and chooses, as in a subgame perfect equilibrium, a best response to every commitment of the leader. Both players receive the payoffs of the original game. A leadership game is often called a "Stackelberg game", following von Stackelberg [25] who modified in this manner the simultaneous model of quantity competition by Cournot [7] to a sequential game. We consider the leadership game for the mixed extension of a finite two-player where the leader can commit to a mixed strategy, as analyzed in full generality by von Stengel and Zamir [28].

Inspection games model situations where an inspectee is obliged to act legally and hence cannot openly declare that he will violate. However, the inspector can become a leader and commit to a mixed strategy, using a "roulette wheel" or other randomization device that decides with a verifiable probability in each time period (simultaneously to the choice of the inspectee) whether to inspect or not. Maschler [19] observed that in a non-zero-sum recursive game, similar to (44) for $k = 1$, the inspector can commit to essentially the same mixed strategy as before, but that the inspectee acts legally with certainty as long as inspections remain. We first consider a general 2×2 game as it arises in our context.

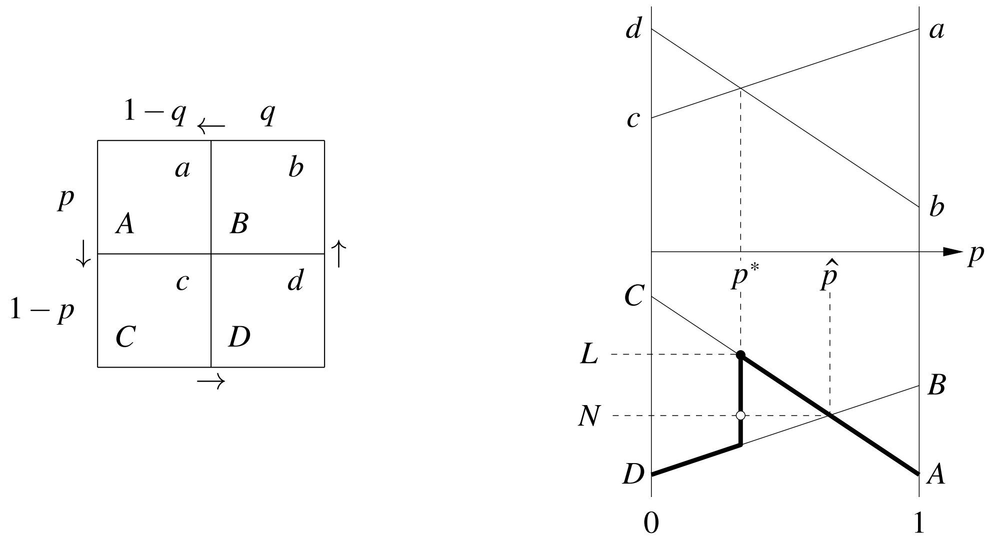

Figure 1 Left: 2×2 game with probability p for playing the top row and q for playing the right column. Right: Payoffs to column and row player if (47) and (49) hold in the leadership game where the row player commits to p.

Proposition 1 *Consider the 2×2 game on the left in Fig. 1 where the payoffs A, B, C, D to the row player and a, b, c, d to the column player fulfill*

$$A < C, \quad B > D, \quad a > b, \quad c < d\,. \tag{47}$$

Let

$$p^* = \frac{d - c}{a - b + d - c} \tag{48}$$

and assume that

$$p^*A + (1-p^*)C > p^*B + (1-p^*)D\ . \qquad (49)$$

Then the game has a unique mixed equilibrium where p^* is the equilibrium probability that the row player plays the top row, with Nash payoff $N = \frac{BC-AD}{B-D+C-A}$ to the row player. In the leadership game where the row player is the leader and can commit to a mixed strategy p, the unique subgame perfect equilibrium is that the row player commits to p^* and the column player responds with $q = 1$ if $p < p^*$ and $q = 0$ if $p \geq p^*$, in particular with $q = 0$ on the equilibrium path where $p = p^*$. In the leadership game, the payoff to the leader is $L = p^*A + (1-p^*)C$, and $L > N$. The payoff to the follower is $p^*a + (1-p^*)c$, the same as in the simultaneous game.

Proof. Proof. By (47), the game has a unique mixed equilibrium where the row player plays p^* and the column player plays $q^* = \frac{C-A}{B-D+C-A}$, and the row player gets payoff N and the column player gets $p^*a + (1-p^*)c$.

The claims about the leadership game can be seen from the right picture in Figure 1, which shows the players' payoffs as a function of p. For illustration, the column player's payoffs are assumed to be positive (as it typically holds in our inspection games, with the exception of the payoff b) and those of the column player as negative, here for the case that $B > A$ (so that (49) can only hold if $C > D$).

For $p < p^*$ the follower's best response is the right column ($q = 1$), with expected payoff $pb + (1-p)d$ to the follower and $pB + (1-p)D$ to the leader. For $p > p^*$ the follower's best response is the left column ($q = 0$), with expected payoff $pa + (1-p)c$ to the follower and $pA + (1-p)C$ to the leader. For the commitment to $p = p^*$ the follower is indifferent and in principle could reply with any q in $[0,1]$. The payoff to the leader as a function of p is shown as the bold line in the figure, including the vertical part for $p = p^*$ for $q \in [0,1]$. By (47), this leader payoff is increasing in p for $p < p^*$ and decreasing in p for $p > p^*$, so it has its maximum L if $p = p^*$ and, by (49), if the follower's response is $q = 0$, shown by a full dot in the picture. This reaction of the follower defines in fact the unique SPNE in the leadership game, because for $p = p^*$ the follower, even though indifferent, has to choose the response, here $q = 0$, that maximizes the leader's payoff, because otherwise the leader could *induce* this behavior by changing his commitment to $p^* + \varepsilon$ for some arbitrarily small positive ε, which contradicts the SPNE condition.

This has essentially been observed by Maschler [19], who postulated a "Pareto-optimal" response of the follower if he is indifferent, and noted that otherwise the leader can get a payoff arbitrarily close to L with a commitment to $p^* + \varepsilon$. The SPNE argument has been made by Avenhaus, Okada, and Zamir [3], and in generality by von Stengel and Zamir [28] who also give further references.

In addition to the leader payoff L, Fig. 1 shows the Nash payoff N (with a hollow dot) further below on the vertical line, which is less than L because it is given by $N = q^*L + (1-q^*)M$ where $M = p^*B + (1-p^*)D$ is the minimum payoff to the leader if the follower responds to p^* by choosing the right column $q = 1$, and $L > M$ by (49). If $B > A$ and thus $C > D$, then N is also the max-min payoff to the row player where his expected payoffs are the same for both columns, when he plays his max-min strategy $\hat{p} = \frac{C-D}{B-A+C-D}$, also shown in the picture. □

We want to apply Proposition 1 to the inspection game (44). Similar to Maschler [19, p. 18], the leadership game for n time periods, m inspections, and k intended violations is described as follows:

inspector \ inspectee	legal action	violation
roulette wheel calls for inspection	$w(n-1,m-1,k)$ $u(n-1,m-1,k)$	$-b\cdot r_k$ $-c(k)$
roulette wheel calls for no inspection	$w(n-1,m,k)$ $u(n-1,m,k)$	$w(n-1,m,k-1)+r_k$ $u(n-1,m,k-1)-r_k$

(50)

If the assumptions of Proposition 1 are met, then in this leadership game the inspector chooses the same strategy as in the simultaneous game so that the inspectee is indifferent between legal action and violation. However, the inspectee acts legally as long as $m > 0$, that is, there will never be a caught violation. For that reason, the result will hold for any negative cost $-c(k)$ to the inspector in that cell of the table.

In the game (50), the inspectee as follower should, by Proposition 1, get the same recursively defined payoff $w(n,m,k)$ as in Theorem 3, but the inspector gets a new payoff $u(n,m,k)$. The following consideration shows what this payoff should be. First, if $m = 0$, then the inspectee can and will safely use his k intended violations, as far as possible, in each of the remaining n time periods, so that as in (42),

$$-u(n,0,k) = w(n,0,k) = \sum_{i=1}^{\min\{k,n\}} r_{k+1-i}\ , \tag{51}$$

as well as

$$u(n,m,k) = w(n,m,k) = 0 \qquad \text{if } m \geq n. \tag{52}$$

For $n > m > 0$ and $k > 0$, the game (50) applies, where the inspectee gets the same payoff $w(n,m,k)$ for legal action and violation, given by (46). In particular, this is the inspectee's payoff if he always acts legally, as we assume he does in the leadership game. Once the inspector has run out of inspections, the inspectee gets the same payoff $w(n,0,k)$ as in (42) and (51), which is the negative of the inspector's payoff. By induction, the inspector's payoff should therefore in general simply be $u(n,m,k) = -w(n,m,k)$.

In the following theorem, subgame perfection refers to the leadership game that assumes best responses of the follower even off the equilibrium path, namely for all other inspection probabilities that the inspector could commit to. In terms of information about the history of the game, the probability of the "roulette wheel" at each stage is a function of n and m but not of k, as before.

Theorem 5 *Let n,m,k be nonnegative integers, $r_k, r_{k-1}, \cdots, r_1 \geq 0$, $b \geq 0$ and $c(k) > 0$. Then in the leadership game defined by* (52), (51), *and* (50) *for $n > m > 0$ and $k > 0$, the* *unique subgame perfect equilibrium payoff is $w(n,m,k)$ as in (46) to the inspectee, and $u(n,m,k) = -w(n,m,k)$ to the inspector. The inspector commits to the same inspection probability $p^* = p$ as in the game with simultaneous actions in each time period according to (7). For $m > 0$, the inspectee always acts legally, and for $m = 0$ he violates in each remaining time period, up to k times, as in (51). Compared to the simultaneous game in Theorem 3, the inspector's cost is smaller by the factor $\hat{s}(n,m)/s(n,m)$; the inspectee's payoff is the same.*

Proof. Proof. By induction on n. For $m = 0$, we have $s(n,m) = s(n,0) = 1$, so that $w(n,0,k) = t(n,0,k)$ which fulfills (51) by (5). Similarly, $t(n,m,k) = 0$ if $m \geq n$, which implies (52). In the same way, (51) holds for the inspector's payoff $u(n,0,k)$, and so does (52).

Let $n > m > 0$ and $k > 0$. In (50), the probability for inspection p should make the inspectee indifferent between legal action and violation, so that, as in Theorem 3, the inspectee gets the payoff $w(n,m,k)$ as claimed, and p is given by (7). If the inspectee always acts legally, then the inspector's payoff is recursively defined by

$$u(n,m,k) = p \cdot u(n-1,m-1,k) + (1-p) \cdot u(n-1,m,k) \tag{53}$$

as in (28), which has been shown to be true in (30) and (32).

It remains to show that (49) in Proposition 1 applies, that is, the inspectee indeed acts legally because the inspector's payoff for legal action is higher than for violation. By (50), using (53), this is equivalent to

$$u(n,m,k) > p \cdot (-c(k)) + (1-p) \cdot (u(n-1,m,k-1) - r_k). \tag{54}$$

Now, because $u(n,m,k) = -w(n,m,k)$, we know that, analogous to (29) which has been shown with (31) and (33),

$$u(n,m,k) = p \cdot b \cdot r_k + (1-p) \cdot (u(n-1,m,k-1) - r_k),$$

so that (54) is equivalent to $b \cdot r_k > -c(k)$, which is true. So the recursive equation for $u(n,m,k)$ in (53) is indeed justified.

To compare the payoff $u(n,m,k)$ to the inspector in the leadership game with his payoff $v(n,m,k)$ in the game with simultaneous action in each time period, (46) gives

$$u(n,m,k) = \frac{-t(n,m,k)}{s(n,m)} = \frac{\hat{s}(n,m)}{s(n,m)} \cdot \frac{-t(n,m,k)}{\hat{s}(n,m)} = \frac{\hat{s}(n,m)}{s(n,m)} \cdot v(n,m,k) \tag{55}$$

as claimed, where the factor $\hat{s}(n,m)/s(n,m)$ is smaller than 1 by (4) and (45), possibly significantly so, depending on the parameters a and b in (43). □

We conclude this section with two further observations. The first is that even if one does not see that $u(n,m,k)$ is just $-w(n,m,k)$ as described following (50), the recursive equation (53) shows what $u(n,m,k)$ should be, and also why $t(n,m,k)$ should be as in (5). Namely, suppose that we do not yet know $t(n,m,k)$ and assume that $u(n,m,k) = -t(n,m,k)/s(n,m)$. Then, by (7), equation (53) is equivalent to

$$\frac{-t(n,m,k)}{s(m,n)} = \frac{s(m-1,n-1)}{s(n,m)} \cdot \frac{-t(n-1,m-1,k)}{s(m-1,n-1)} + \frac{s(m,n-1)}{s(m,n)} \cdot \frac{-t(n-1,m,k)}{s(m,n-1)}$$

which is just (30). So $t(n,m,k)$ fulfills the equation of a "generalized Pascal triangle"

$$t(n,m,k) = t(n-1,m-1,k) + t(n-1,m,k) \tag{56}$$

in n and m (with k as a fixed parameter) and "base cases" (by (51) and because $s(n,0) = 1$)

$$t(n,0,k) = \sum_{i=1}^{k} r_{k+1-i}\ , \qquad t(n,n,k) = 0\ .$$

Writing down the numbers $t(n,m,k)$ in a triangle as functions of r_k, r_{k-1}, etc., one sees that the smallest n where r_{k+1-i} appears in $t(n,m,k)$ is for $n = i$ and $m = 0$. Due to (56), this becomes the root $\binom{0}{0}$ of an ordinary "Pascal triangle" for the coefficient of r_{k+1-i} in $t(n,m,k)$, which is therefore $\binom{n-i}{m}$, as in (5).

The second observation addresses the question if the inspector always prefers the inspectee to act legally in the game (44) where his payoffs are given by $v(n,m,k)$, and not recursively by $u(n,m,k)$ as in (50). As an application of Proposition 1, this would apply to a leadership game where the inspector can only commit to the probability of inspecting in the first time period, but acts without commitment in all subsequent periods. This is not a very natural game to look at, but the preference of the inspector in the simultaneous game is nevertheless of interest. As expected, the inspector indeed prefers that the inspectee acts legally. We found only a relatively long – but "canonical" – proof, which we present here for its possible interest concerning the manipulation of sums of binomial coefficients.

Theorem 6 *Consider the game* (44) *with entries as in Theorem 3 as a* 2×2 *game on the left in Fig. 1, and the equilibrium probability* $p^* = p$ *as in* (7) *for the inspector. Then* (49) *in Proposition 1 applies, that is, the inspector prefers that in response to* p^* *the inspectee acts legally.*

Proof. Proof. In the game (44), we have A, C, D as in (14) and $B = -a \cdot r_k$, where $-a > -1$ by (43). To show (49) directly we would have to compare terms involving $s(n,m)$, $t(n,m,k)$, and $\hat{s}(n,m)$ according to (7), (45) and (46). Instead, we apply Theorem 1 with $b = -a$ to the zero-sum game (15) with entries A, B, C, D, where the inspector has the max-min strategy of inspecting with probability

$$\hat{p} = \frac{\hat{s}(n-1,m-1)}{\hat{s}(n,m)}\ .$$

Because $A < B$ and $C > D$ as shown in (17), $\hat{p}$ is also the probability that equalizes the expected payoffs to the row player for the two columns of the game, $\hat{p} = \frac{C-D}{B-A+C-D}$, as shown on the right in Fig. 1. Then (49) holds if $\hat{p} > p^*$, because the expected payoff for the left column is $pA+(1-p)C = C+p(A-C)$ which is strictly decreasing in p, and for the right column it is $pB+(1-p)D = D+p(B-D)$ which is strictly increasing in p, so that $\hat{p} > p^*$ implies

$$p^*A+(1-p^*)C \;>\; \hat{p}A+(1-\hat{p})C \;=\; \hat{p}B+(1-\hat{p})D \;>\; p^*B+(1-p^*)D$$

that is, (49).

To show $\hat{p} > p^* = p$, we define

$$S(n,m,x) = \sum_{i=0}^{m} \binom{n}{i} (x-1)^i,$$

so that by (45) and (7)

$$\hat{s}(n,m) = S(n,m,1-a), \qquad s(n,m) = S(n,m,1+b).$$

As in (13) we have

$$S(n,m,x) = \sum_{i=0}^{m} \binom{n-1-i}{m-i} x^i. \tag{57}$$

Then we want to show, for $n > m > 0$, that

$$\hat{p} = \frac{\hat{s}(n-1,m-1)}{\hat{s}(n,m)} \;>\; p = \frac{s(n-1,m-1)}{s(n,m)}\ , \tag{58}$$

or equivalently, by (11),

$$\frac{1-\hat{p}}{\hat{p}} = \frac{\hat{s}(n-1,m)}{\hat{s}(n-1,m-1)} \;<\; \frac{1-p}{p} = \frac{s(n-1,m)}{s(n-1,m-1)}\ ,$$

that is,

$$\frac{S(n-1,m,1-a)}{S(n-1,m-1,1-a)} \;<\; \frac{S(n-1,m,1+b)}{S(n-1,m-1,1+b)}\ ,$$

which clearly holds if $S(n-1,m,x)/S(n-1,m-1,x)$ is strictly increasing in x for $x > 0$, which is what we will show. By (10) we have with $x = 1+b$

$$S(n-1,m,x) = (x-1)\cdot S(n-1,m-1,x) + \binom{n-1}{m}$$

so that

$$\frac{S(n-1,m,x)}{S(n-1,m-1,x)} = x-1 \;+\; \frac{\binom{n-1}{m}}{S(n-1,m-1,x)}$$

where we want to show that this term has positive derivative with respect to x, that is,

$$\frac{d}{dx}\left(\frac{S(n-1,m,x)}{S(n-1,m-1,x)}\right) \;=\; 1-\binom{n-1}{m}\frac{\frac{d}{dx}S(n-1,m-1,x)}{S(n-1,m-1,x)^2} > 0$$

or

$$S(n-1,m-1,x)^2 \;>\; \binom{n-1}{m}\frac{d}{dx}S(n-1,m-1,x)\ .$$

To simplify this expression, which we want to show for $n>m>0$, we equivalently show for $n>m\geq 0$ that

$$S(n,m,x)^2 \;>\; \binom{n}{m+1}\frac{d}{dx}S(n,m,x)$$

which by (57) says

$$\left(\sum_{i=0}^{m}\binom{n-1-i}{m-i}x^i\right)^2 > \binom{n}{m+1}\sum_{i=1}^{m} i\binom{n-1-i}{m-i}x^{i-1}\ . \tag{59}$$

In (59), we have $x>0$ and (because $n-1\geq m$) positive coefficients of x^m (and of higher powers of x) on the left hand side, whereas the highest power of x on the right hand side is x^{m-1}. Hence, it suffices to show that for $0\leq i\leq m-1$, each coefficient of x^i on the left hand side in (59) is at least as large as the coefficient of x^i on the right hand side, that is,

$$\sum_{k=0}^{i}\binom{n-1-k}{m-k}\binom{n-1-i+k}{m-i+k} \geq \binom{n}{m+1}(i+1)\binom{n-2-i}{m-1-i}. \tag{60}$$

There are $i+1$ summands on the left of (60), so it suffices to show that each of them, for $0\leq k\leq i$, fulfills

$$\binom{n-1-k}{m-k}\binom{n-1-i+k}{m-i+k} \geq \binom{n}{m+1}\binom{n-2-i}{m-1-i}. \tag{61}$$

Because

$$\binom{n}{m+1} = \binom{n-1-k}{m-k}\prod_{j=0}^{k}\frac{n-k+j}{m+1-k+j}$$

and

$$\binom{n-1-i+k}{m-i+k} = \binom{n-2-i}{m-1-i}\prod_{j=0}^{k}\frac{n-1-i+j}{m-i+j}\ ,$$

(61) is equivalent to

$$\prod_{j=0}^{k}\frac{n-1-i+j}{m-i+j} \;\geq\; \prod_{j=0}^{k}\frac{n-k+j}{m+1-k+j}$$

which holds if for $0\leq j\leq k$

$$\frac{n-1-i+j}{m-i+j} \;\geq\; \frac{n-k+j}{m+1-k+j}\ ,$$

that is,

$$1 \;+\; \frac{n-m-1}{m-i+j} \;\geq\; 1 \;+\; \frac{n-m-1}{m+1-k+j}\ . \tag{62}$$

If $n=m+1$, then (62) holds as equality. Otherwise, $n-m-1>0$, and (62) is equivalent to

$$m-i+j \;\leq\; m+1-k+j$$

or $k\leq i+1$ which is true. This proves the claim and thus (58), and the theorem. $\square$

6 Conclusions

We have presented an inspection game that extends existing models of n time periods and m inspections with a general model of k intended violations. Each violation may have a different marginal reward to the inspectee (which is therefore completely general) and a proportional penalty when caught. As shown in Theorem 2, this proportional penalty is necessary and sufficient for applying the recursively described game with full information to the game without full information where the inspector in uninformed about the inspectee's past actions in uninspected time periods, which is a realistic condition. We have studied three variants of this game: A zero-sum game, a non-zero-sum game where both inspector and inspectee get negative payoffs for a caught violation, and a leadership game. In the leadership game, the inspector commits to the same mixed strategy but receives a higher payoff because the inspectee acts legally as long as inspections remain (which evidently requires commitment power of the inspector who would otherwise not inspect in response). We were able to give explicit solutions for these games by proving nontrivial binomial identities and inequalities, as in the proofs of Theorems 1 and 6.

Further extensions of the model could involve inspections with statistical errors of false alarms and non-detected violations (as studied for a single intended violation in [21], and in a different model where time to detection matters in [1]). The fixed number n of time periods could be replaced by a random variable that counts "suspicious events". Rather than a fixed number m of inspections, an overall frequency (with a public randomization device) could part of the treaty for an inspection regime. Also, there could be multiple inspectees, with different targets for violations.

Such extensions pose without doubt new challenges for analysis. In practice, a new game of this sort is most likely not solved explicitly, as in this paper, but with the help of computer algorithms. In fact, variations of the model investigated in [26] led the author to the study of algorithms for solving extensive games, and the computationally efficient "sequence form" described in [27].

A recursive game is a much more compact description than an extensive game. The inspection game $\Gamma(n,m,k)$ in (2) is defined recursively in terms of the game values for simpler games. There are on the order of $n \cdot m \cdot k$ simpler games, a polynomial number in n,m,k. In contrast, the extensive game has an exponential number of nodes, because every set of m out of n time periods that the inspector inspects defines a different history of the game. However, from this history only the number of remaining inspections matters, which is captured by the recursive description.

Everett [11] defined recursive games as stochastic games where only absorbing states have nonzero payoffs. The recursive games we consider could be put in this form by awarding all payoffs (with the gains $r_k, r_{k-1}, \ldots$ to the inspectee for successful violations) at the end of the game, which is possible because the number k of remaining violations is part of the state description as (n,m,k). More importantly, in our games no state is ever revisited during play because n is decreased each time. The game graph of state transitions is acyclic, but not a tree.

Can such recursive games be equipped with nontrivial information structures? By Corollary 2, the inspection game $\hat{\Gamma}'(n,m,k)$ without full information has the same solution as the recursive game $\hat{\Gamma}(n,m,k)$ with full information. Here this is possible because the information that is implicit in the recursive description does not matter, which is due to the special payoff structure. Otherwise the equilibrium strategy of the inspector depends on k. In that case, can the equilibrium of the game with full information be used to solve the game without full information? (This question was posed to the author by the late Michael Maschler in 1991.) An interesting area of future work could be models of "small" games like recursive games that allow for lack of information, similar to information sets in extensive games, and corresponding solution methods.

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