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On the Laplace transforms of the first exit times in one-dimensional non-affine jump-diffusion models

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We compute the Laplace transforms of the first exit times for certain one-dimensional jump-diffusion processes from two-sided intervals. The method of proof is based on the solutions of the associated integro-differential boundary value problems for the corresponding value functions. We consider jump-diffusion processes solving stochastic differential equations driven by Brownian motions and several independent compound Poisson processes with multi-exponential jumps. The results are illustrated on the non-affine pure jump analogues of certain mean-reverting or diverting diffusion processes which represent closed-form solutions of the appropriate stochastic differential equations.

1 Introduction

The aim of this paper is to derive closed-form expressions for the Laplace transform in (2.5) of the first times at which the (non-affine) jump-diffusion process X defined by (2.1) exits a two-sided interval. It is assumed that the stochastic differential equation in (2.1) for X is driven by a standard Brownian motion and several independent compound Poisson processes with exponentially distributed jumps. We consider the case in which the equation in (2.1) can either be solved explicitly or reduced to the associated ordinary differential equation, by means of the appropriate integrating factor process. Such *solvable* stochastic differential equations were considered in Gard [13, Chapter IV] and Øksendal [22, Chapter V] for continuous diffusion processes, and then in [9] and [12] for their jump-diffusion analogues. The tractability of the resulting analytic solutions of this type of stochastic differential equations was shown in İyigünler, Çağlar, and Ünal [14], by analysing the accuracy of the numerical approximations obtained from the appropriate discretisation schemes. We obtain closed-form solutions to the integro-differential boundary value problems associated with the values of Laplace transforms of the first exit times as stopping problems for continuous-time Markov processes, including the (non-affine) pure-jump analogues of certain mean-reverting and diverting diffusions.

Optimal stopping problems for some mean-reverting and diverting jump-diffusion processes were initiated by Davis [4], Peskir and Shiryaev [25]-[26], Dayanik and Sezer [5]-[6], and [10]-[11] among others, with the aim to detect the change points in the associated discontinuous observable processes. Discounted optimal stopping problems for certain payoff functions depending on the current values of geometric compound Poisson processes with multi-exponential jumps and their various extensions were considered by Mordecki [20]-[21], Kou [16], and Kou and Wang [18] among others, with the aim of computing rational values for the perpetual American options. The analytical tractability of these jump-diffusion models, which are widely applied for the description of the

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dynamics of risky asset prices in financial markets, enabled to reproduce the leptokurtic property of the returns distributions. The main feature of the resulting optimal stopping problems and their equivalent free boundary problems was the breakdown of the smooth-fit conditions for the value functions at the optimal boundaries and their replacement by the continuous-fit conditions. In the context of optimal stopping problems, Asmussen et al. [2] obtained explicit expressions for the Laplace transforms of the first hitting times of more general phase-type Lévy processes over constant boundaries by means of the Wiener-Hopf factorization techniques.

Analytic expressions for the Laplace transforms of the first hitting times of compound Poisson processes over linear boundaries were computed in Zacks et al. [28] in the case of positive jumps and in Perry et al. [23]-[24] in certain cases of positive and negative jumps. Kou and Wang [17] and Sepp [27] derived closed-form expressions for the Laplace transforms of the first hitting times over constant boundaries for double-exponential jump-diffusion processes. Other related stopping problems arising from the computation of the Laplace transforms of the first-passage times of more complicated spectrally positive and negative Lévy processes over constant levels were recently considered by Mijatović and Pistorius [19]. Monte Carlo schemes for the computation of the distribution of the first exit time of jump-diffusion processes from a two-sided interval in the general size distribution case were developed in Fernandez et al. [8]. In contrast to the results of the most of the papers mentioned above, in the present paper, we consider the problem of computation of the Laplace transforms of the first exit times from intervals for jump-diffusion processes with drift coefficients being of general structure which may lead to the mean-reverting or diverting behaviour of the processes.

The paper is structured as follows. In Section 2, we introduce the setting and notation of the model with a jump-diffusion process satisfying a solvable stochastic differential equation. We define the Laplace transform of the first exit time of the process and formulate the associated boundary value problem for an integro-differential operator. In Section 3, we obtain a closed-form solution to the equivalent ordinary differential boundary value problem using the assumption that the jump sizes of the driving compound Poisson processes have exponential distribution. We derive explicit expressions in the cases of several mean-reverting and diverting pure-jump analogues of certain continuous diffusion processes. In Section 4, we show that the solution to the boundary value problem provides the original Laplace transform.

2 Preliminaries

In this section, we give a precise probabilistic formulation of the model and the stopping problem as well as its equivalent boundary-value problem.

2.1 Formulation of the problem. Suppose that on a probability space $(\Omega, \mathcal{F}, P)$ there exists a standard Brownian motion $W = (W_t)_{t \ge 0}$, Poisson processes $N^i = (N^i_t)_{t \ge 0}$ and $N^{m+j} = (N^{m+j}_t)_{t \ge 0}$ of intensities λ_i and λ_{m+j}, and $(\Xi^i_k)_{k \in \mathbb{N}}$ and $(\Xi^{m+j}_l)_{l \in \mathbb{N}}$ are sequences of independent exponentially distributed random variables with parameters $\alpha_i > 1$, $i = 1, \ldots, m$, and $\alpha_{m+j} > 0$, $j = 1, \ldots, n$, for some $m, n \in \mathbb{N}$, respectively. Assume that W, N^i, N^{m+j}, $(\Xi^i_k)_{k \in \mathbb{N}}$, and $(\Xi^{m+j}_l)_{l \in \mathbb{N}}$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, are independent. Let us consider a process $X = (X_t)_{t \ge 0}$ solving the stochastic differential equation

$$dX_t = \beta(X_t)\, dt + \sigma\, X_t\, dW_t + X_{t-} \int \Big(e^v - 1 \Big)\, (\mu - \nu)(dt, dv), \tag{2.1}$$

where $\beta(x)$ is a continuously differentiable function of at most linear growth, and $\sigma \ge 0$ is a given constant. Here $\mu(dt, dv)$ is a measure of jumps of the process $J = (J_t)_{t \ge 0}$ defined by

$$\mu((0,t] \times B) = \sum_{0 < s \le t} I_{\{\Delta J_s \in B\}} \quad \text{with} \quad J_t = \sum_{i=1}^{m} \sum_{k=1}^{N^i_t} \Xi^i_k - \sum_{j=1}^{n} \sum_{l=1}^{N^{m+j}_t} \Xi^{m+j}_l, \tag{2.2}$$

for any Borel subset B of $\mathbb{R} \setminus \{0\}$, and $\nu(dt, dv)$ is its compensator measure with

$$\nu(dt, dv) = dt\bigg(I_{\{v>0\}} \sum_{i=1}^{m} \lambda_i \, \alpha_i e^{-\alpha_i v} + I_{\{v<0\}} \sum_{j=1}^{n} \lambda_{m+j} \, \alpha_{m+j} e^{\alpha_{m+j} v} \bigg) \, dv, \tag{2.3}$$

where $I_{\{\cdot\}}$ denotes the indicator function (see, e.g. [15, Chapter II, Section 1] for the definitions of these notions). By virtue of the assumptions on the function $\beta(x)$ and the fact that the jump process J is of finite intensity, it thus follows from [15, Chapter III, Theorem 2.32] that the stochastic differential equation in (2.1) admits a (pathwise) unique solution.

Let us denote by $\mathcal{D}_X$ the state space of the process X and further assume that $\mathcal{D}_X = (d_0, d_1)$ for some $0 \le d_0 < d_1 \le \infty$. We also define the associated first passage (stopping) times

$$\tau_a = \inf\{t \ge 0 \,|\, X_t \le a\} \quad \text{and} \quad \zeta_b = \inf\{t \ge 0 \,|\, X_t \ge b\}, \tag{2.4}$$

for some $d_0 < a < b < d_1$ fixed. The main purpose of the present paper is to derive closed-form expressions for the Laplace transform of the random time $\tau_a \wedge \zeta_b$. We therefore need to compute the value function $V_*(x)$ of the following stopping problem given by

$$V_*(x) = E_x\big[e^{-\varkappa(\tau_a \wedge \zeta_b)} \, I_{\{\tau_a < \zeta_b\}}\big] \equiv E_x\big[e^{-\varkappa \tau_a} \, I_{\{\tau_a < \zeta_b\}}\big], \tag{2.5}$$

for any $x \in \mathcal{D}_X$ and some $\varkappa > 0$ fixed. Here E_x denotes the expectation with respect to the probability measure P_x under which the one-dimensional time-homogeneous (strong) Markov process X starts at $x \in \mathcal{D}_X \equiv (d_0, d_1)$.

2.2 The boundary value problem. By means of standard arguments based on the application of Itô's formula for semimartingales from [15, Chapter I, Theorem 4.57], it is shown that the infinitesimal generator $\mathbb{L}$ of the process X acts on a twice continuously differentiable bounded function $V(x)$ on $\mathcal{D}_X \equiv (d_0, d_1)$ according to the rule

$$(\mathbb{L}V)(x) = \frac{\sigma^2 x^2}{2} \, V''(x) + \bigg(\beta(x) - \bigg(\sum_{i=1}^{m} \frac{\lambda_i}{\alpha_i - 1} - \sum_{j=1}^{n} \frac{\lambda_{m+j}}{\alpha_{m+j} + 1} \bigg) \, x \bigg) \, V'(x) \tag{2.6}$$
$$+ \sum_{i=1}^{m} \lambda_i \int_0^\infty \Big(V(xe^y) - V(x) \Big) \, \alpha_i e^{-\alpha_i y} \, dy + \sum_{j=1}^{n} \lambda_{m+j} \int_{-\infty}^0 \Big(V(xe^y) - V(x) \Big) \, \alpha_{m+j} e^{\alpha_{m+j} y} \, dy,$$

for all $d_0 < x < d_1$. In order to find analytic expressions for the unknown value function $V_*(x)$ in (2.5), let us build on the results of the general theory of Markov processes (see, e.g. [7, Chapter V]). For this purpose, we formulate the boundary value problem

$$(\mathbb{L}V)(x) = \varkappa V(x), \quad \text{for} \quad a < x < b, \tag{2.7}$$
$$V(x) = 1, \quad \text{for} \quad x \le a, \quad \text{and} \quad V(x) = 0, \quad \text{for} \quad x \ge b, \tag{2.8}$$
$$V(a+) = 1 \quad \text{and} \quad V(b-) = 0, \tag{2.9}$$

for $d_0 < a < b < d_1$ fixed, where the continuity conditions of (2.9) hold in the cases in which the process X can pass continuously through the boundaries a and b, respectively. On the other hand, when $\sigma = 0$ holds, the stochastic differential equation in (2.1) for X does not contain a diffusion part, so that the function $V_*(x)$ may be discontinuous at the points a or b, depending on the sign of the local drift rate $\beta(x) - \gamma x$ with

$$\gamma = \sum_{i=1}^{m} \frac{\lambda_i}{\alpha_i - 1} - \sum_{j=1}^{n} \frac{\lambda_{m+j}}{\alpha_{m+j} + 1} \tag{2.10}$$

in the stochastic differential equation of (2.1). This property follows from the fact that X may pass through either of them only by jumping. Therefore, in order to determine which of the continuity conditions in (2.9) should hold for $V(x)$, let us assume that one of the following four cases is realised. If either the inequality $\beta(x) - \gamma x < 0$ or $\beta(x) - \gamma x > 0$ holds for all $a \le x \le b$, then the process X can pass through either a or b continuously, and thus, we assume that $V(x)$ satisfies either the left-hand or right-hand condition of (2.9), respectively. Moreover, if there exists some constant $a < c < b$ such that

$$\beta(x) - \gamma\, x < 0 \quad \text{for} \quad x < c, \quad \beta(x) - \gamma\, x > 0 \quad \text{for} \quad x > c, \quad \text{and} \quad \beta(c) - \gamma\, c = 0 \tag{2.11}$$

holds, so that the process X diverts from the level c in a continuous way, and thus, we assume that $V(x)$ satisfies both conditions of (2.9), since the process X can pass through a and b continuously. Finally, if there exists some constant $a < c < b$ such that

$$\beta(x) - \gamma\, x > 0 \quad \text{for} \quad x < c, \quad \beta(x) - \gamma\, x < 0 \quad \text{for} \quad x > c, \quad \text{and} \quad \beta(c) - \gamma\, c = 0 \tag{2.12}$$

holds, then the process X reverts to the level c in a continuous way, and thus, both conditions of (2.9) do not hold for $V(x)$ at either a or b, since the process X cannot pass through these points continuously.

3 Solutions of the boundary-value problem

In this section, we derive closed-form solutions of the integro-differential boundary value problem formulated above for various drift rate functions of the considered jump-diffusion process. For this purpose, we reduce the original integro-differential equation to the equivalent ordinary differential equation of order $m+n+2$ and solve the latter by means of the appropriate boundary conditions.

3.1 The equivalent ordinary differential problem. We now use the assumptions that the jump sizes $(\Xi_k^i)_{k\in\mathbb{N}}$ and $(\Xi_l^{m+j})_{l\in\mathbb{N}}$ are exponentially distributed and reduce the integro-differential boundary value problem of (2.6)+(2.7)-(2.9) to an ordinary differential one. For this purpose, by applying the conditions of (2.8), we obtain that the equation in (2.6)+(2.7) takes the form

$$\begin{aligned} & a_{2,0}(x)\, V''(x) + a_{1,0}(x)\, V'(x) + a_{0,0}(x)\, V(x) + b_0(x) \\ & + \sum_{i=1}^{m} \lambda_i\, x^{\alpha_i} \int_x^b V(z)\, \alpha_i z^{-\alpha_i - 1}\, dz + \sum_{j=1}^{n} \lambda_{m+j}\, x^{-\alpha_{m+j}} \int_a^x V(z)\, \alpha_{m+j} z^{\alpha_{m+j}-1}\, dz = 0, \end{aligned} \tag{3.1}$$

for $a < x < b$, where γ has the form of (2.10) and we set

$$a_{2,0}(x) = \sigma^2 x^2/2, \quad a_{1,0}(x) = \beta(x) - \gamma\, x, \tag{3.2}$$

$$a_{0,0}(x) = -\sum_{i=1}^{m} \lambda_i - \sum_{j=1}^{n} \lambda_{m+j} - \varkappa, \quad \text{and} \quad b_0(x) = \sum_{j=1}^{n} \lambda_{m+j} \left(\frac{a}{x}\right)^{\alpha_{m+j}}, \tag{3.3}$$

for all $x > 0$. Let us define the differential operators

$$\mathbb{L}_i = -x^{\alpha_i - \alpha_{i-1} + 1} \frac{d}{dx} \quad \text{and} \quad \mathbb{L}_{m+j} = x^{\alpha_{m+j-1} - \alpha_{m+j} + 1} \frac{d}{dx}, \tag{3.4}$$

and introduce the notation $\mathbb{L}_{k,k'} = \mathbb{L}_k \circ \mathbb{L}_{k+1} \circ \cdots \circ \mathbb{L}_{k'}$, with $\mathbb{L}_{k,k'}$ being equal to the identity operator when $k > k'$. Then, we can observe that the expressions

$$G_i(x) = (\mathbb{L}_{i+1,i'} G_{i'})(x), \quad \text{for} \quad i' = i, \ldots, m, \tag{3.5}$$

$$G_{m+j}(x) = (\mathbb{L}_{m+j+1,m+j'} G_{m+j'})(x), \quad \text{for} \quad j' = j, \ldots, n, \quad G_i(x) = (\mathbb{L}_{i+1,m+j} G_{m+j})(x) \tag{3.6}$$

hold, where we set

$$G_0(x) = V(x), \quad G_i(x) = \int_x^b G_{i-1}(z)\, z^{\alpha_{i-1}-\alpha_i-1}\, dz, \tag{3.7}$$

$$G_{m+1}(x) = \int_a^x G_m(z)\, z^{\alpha_{m+1}+\alpha_m-1}\, dz, \quad G_{m+j}(x) = \int_a^x G_{m+j-1}(z)\, z^{\alpha_{m+j}-\alpha_{m+j-1}-1}\, dz, \tag{3.8}$$

for all $a \le x \le b$ and every $i = 1, \dots, m$ and $j = 2, \dots, n$, with $\alpha_0 = 0$, so that $G_i(b) = 0$ and $G_{m+j}(a) = 0$ holds, for $i = 0, \dots, m$ and $j = 1, \dots, n$. Hence, representing the integro-differential equation from (3.1) in terms of the functions $G_k(x)$, $k = 0, \dots, m+n$, defined in (3.7)-(3.8) with the properties of (3.5)-(3.6) and integrating by parts the corresponding terms, we obtain that the equation in (3.1) is equivalent to each of the $m+n$ ordinary integro-differential equations for the functions $G_k(x)$, $k = 0, \dots, m+n$, with the boundary conditions given by

$$\sum_{k=0}^{i+2} a_{k,i}(x)\, G_i^{(k)}(x) + b_i(x) + (-1)^i \bigg(\sum_{k=1}^m \lambda_k\, \alpha_k\, x^{\alpha_k} \int_x^b G_i(z)\, z^{\alpha_i-\alpha_k-1}\, dz \prod_{k'=1}^i (\alpha_{k'} - \alpha_k) \tag{3.9}$$

$$+ \sum_{l=1}^n \lambda_{m+l}\, \alpha_{m+l}\, x^{-\alpha_{m+l}} \int_a^x G_i(z)\, z^{\alpha_i+\alpha_{m+l}-1}\, dz \prod_{k'=1}^i (\alpha_{k'} + \alpha_{m+l}) \bigg) = 0, \quad \text{for} \quad a < x < b,$$

$$(\mathbb{L}_{k+1,i} G_i)(b) = 0, \quad \text{for} \quad k = 1, \dots, i, \tag{3.10}$$

for $i = 1, \dots, m$, and

$$\sum_{l=0}^{m+j+2} a_{l,m+j}(x)\, G_{m+j}^{(l)}(x) + b_{m+j}(x) + (-1)^m \sum_{l=1}^n \lambda_{m+l} \alpha_{m+l}\, x^{-\alpha_{m+l}} \tag{3.11}$$

$$\times \int_a^x G_{m+j}(z)\, z^{\alpha_{m+l}-\alpha_{m+j}-1}\, dz \prod_{k=1}^m (\alpha_k + \alpha_{m+l}) \prod_{l'=1}^j (\alpha_{m+l'} - \alpha_{m+l}) = 0, \quad \text{for} \quad a < x < b,$$

$$(\mathbb{L}_{m+l+1,m+j} G_{m+j})(a) = 0, \quad l = 1, \dots, j, \quad (\mathbb{L}_{i+1,m+j} G_{m+j})(b) = 0, \quad i = 1, \dots, m, \tag{3.12}$$

for $j = 1, \dots, n$, where the coefficients are given by

$$a_{k,i}(x) = \sum_{k'=k}^{i+2} \frac{(k'-1)!}{(k'-k)!(k-1)!} \frac{a_{k'-1,i-1}(x)}{x^{\alpha_{i-1}-\alpha_i-k+k'-1}} (\alpha_i - \alpha_{i-1} + 2 - k' + k)_{k'-k}, \tag{3.13}$$

$$a_{0,i}(x) = (-1)^{i-1}\, x^{\alpha_i} \bigg(\sum_{l=1}^n \lambda_{m+l} \alpha_{m+l} \prod_{k=1}^{i-1} (\alpha_k + \alpha_{m+l}) - \sum_{k=1}^m \lambda_k \alpha_k \prod_{k'=1}^{i-1} (\alpha_{k'} - \alpha_k) \bigg), \tag{3.14}$$

$$b_i(x) = (-1)^i \sum_{l=1}^n \lambda_{m+l} \Big(\frac{a}{x} \Big)^{\alpha_{m+l}} \bigg(1 + \alpha_{m+l} \sum_{k=1}^i a^{\alpha_k} G_k(a) \prod_{k'=1}^{k-1} (\alpha_{k'} + \alpha_{m+l}) \bigg), \tag{3.15}$$

for $k = 1, \dots, i+2$ and $i = 1, \dots, m$, and

$$a_{l,m+j}(x) = \sum_{l'=l}^{m+j+2} \frac{(l'-1)!}{(l'-l)!(l-1)!} \, \frac{a_{l'-1,m+j-1}(x)}{x^{\alpha_{m+j}-\alpha_{m+j-1}-l+l'-1}} \, (\alpha_{m+j-1} - \alpha_{m+j} + 2 - l' + l)_{l'-l}, \tag{3.16}$$

$$a_{0,m+j}(x) = (-1)^m x^{-\alpha_{m+j}} \sum_{l=1}^{n} \lambda_{m+l} \alpha_{m+l} \prod_{k=1}^{m} (\alpha_k + \alpha_{m+l}) \prod_{l'=1}^{j-1} (\alpha_{m+l'} - \alpha_{m+l}), \tag{3.17}$$

$$b_{m+j}(x) = b_m(x) = (-1)^m \sum_{l=1}^{n} \lambda_{m+l} \Big(\frac{a}{x}\Big)^{\alpha_{m+l}} \bigg(1 + \alpha_{m+l} \sum_{k=1}^{m} a^{\alpha_k} G_k(a) \prod_{k'=1}^{k-1} (\alpha_{k'} + \alpha_{m+l})\bigg), \tag{3.18}$$

for $l = 1, \dots, m+j+2$ and $j = 1, \dots, n$. Here $(x)_k$ denotes the Pochhammer symbol $(x)_k = x(x+1)\cdots(x+k-1)$, and $(x)_0 = 1$, for any $x \in \mathbb{R}$ and $k > 0$ (see, e.g. [1, Chapter XIII]). We particularly observe that the equation in (3.11) for $j = n$ is an ordinary differential equation. Thus, taking into account the fact that $V(x) = G_0(x) = (\mathbb{L}_{1,m+n} G_{m+n})(x)$ holds for $a \le x \le b$, we see that the integro-differential boundary value problem of (2.6)+(2.7)-(2.9) for the unknown function $V(x)$ is equivalent to the ordinary differential equation for the function $G_{m+n}(x)$ with the boundary conditions

$$\sum_{k=0}^{m+n+2} a_{k,m+n}(x) \, G^{(k)}_{m+n}(x) + b_{m+n}(x) = 0, \quad \text{for} \quad a < x < b, \tag{3.19}$$

$$(\mathbb{L}_{m+j+1,m+n} G_{m+n})(a) = 0, \quad \text{for} \quad j = 1 \dots, n, \tag{3.20}$$

$$(\mathbb{L}_{i+1,m+n} G_{m+n})(b) = 0, \quad \text{for} \quad i = 1, \dots, m, \tag{3.21}$$

$$(\mathbb{L}_{1,m+n} G_{m+n})(a+) = 1, \quad \text{and} \quad (\mathbb{L}_{1,m+n} G_{m+n})(b-) = 0, \tag{3.22}$$

for the unknown function $G_{m+n}(x)$. It follows from the results of the general theory of linear ordinary differential equations that the general solution of the equation in (3.19) has the form

$$G_{m+n}(x) = \overline{G}_{m+n}(x) + \sum_{k=1}^{m+n+2} C_k \, U_k(x), \quad \text{for} \quad a < x < b, \tag{3.23}$$

where C_k, $k = 1, \dots, m+n+2$, are some arbitrary constants. Here $U_k(x)$, $k = 1, \dots, m+n+2$, constitute the fundamental system of $m+n+2$ solutions (i.e. nontrivial linearly independent particular solutions) of the homogeneous version of the $m+n+2$-th order linear ordinary differential equation in (3.19) and $\overline{G}_{m+n}(x)$ is a particular solution of (3.19). Then, applying the boundary conditions from (3.20)-(3.22) to the function in (3.23), we get that the equalities

$$(\mathbb{L}_{m+j+1,m+n} \overline{G}_{m+n})(a) + \sum_{k=1}^{m+n+2} C_k \, (\mathbb{L}_{m+j+1,m+n} U_k)(a) = 0, \quad \text{for} \quad j = 1, \dots, n, \tag{3.24}$$

$$(\mathbb{L}_{i+1,m+n} \overline{G}_{m+n})(b) + \sum_{k=1}^{m+n+2} C_k \, (\mathbb{L}_{i+1,m+n} U_k)(b) = 0, \quad \text{for} \quad i = 1, \dots, m, \tag{3.25}$$

$$(\mathbb{L}_{1,m+n} \overline{G}_{m+n})(a+) + \sum_{k=1}^{m+n+2} C_k \, (\mathbb{L}_{1,m+n} U_k)(a+) = 1, \tag{3.26}$$

$$(\mathbb{L}_{1,m+n} \overline{G}_{m+n})(b-) + \sum_{k=1}^{m+n+2} C_k \, (\mathbb{L}_{1,m+n} U_k)(b-) = 0 \tag{3.27}$$

hold, where the conditions in (3.26)-(3.27) are satisfied whenever $\sigma \neq 0$ in (2.1). Hence, the candidate solution for the system in (2.7)-(2.9) admits the representation

$$V(x;a,b) = (\mathbb{L}_{1,m+n}\overline{G}_{m+n})(x) + \sum_{k=1}^{m+n+2} C_k(a,b)\,(\mathbb{L}_{1,m+n}U_k)(x), \quad \text{for} \quad a<x<b, \tag{3.28}$$

where the constants $C_k(a,b)$, $k=1,\ldots,m+n+2$, are uniquely determined by the linear system of equations (3.24)-(3.27), due to the linear independence of the fundamental solutions $U_k(x)$, $k=1,\ldots,m+n+2$, of the equation in (3.19).

On the other hand, if $\sigma=0$ holds, the corresponding integro-differential equation in (2.6)+(3.1) admits the solution $V(x;a,b)$ of the form of (3.28), with $C_{m+n+2}(a,b)=0$. In order to specify the remaining constants $C_k(a,b)$, $k=1,\ldots,m+n+1$, let us study the cases depending on the sign of the drift rate $\beta(x)-\gamma x$ of the process X from (2.1). More precisely, if either $\beta(x)-\gamma x<0$ or $\beta(x)-\gamma x>0$ holds for all $a \le x \le b$, then the constants $C_k(a,b)$, $k=1,\ldots,m+n+1$, are uniquely determined by the linear system of equations in (3.24)-(3.25), with either (3.26) or (3.27), respectively. Moreover, if there exists some constant $a<c<b$ such that the properties in (2.11) hold, then c is a singularity point of the integro-differential equation (2.6)+(3.1) when $\sigma=0$. Hence, we obtain that the candidate solution for the system in (2.7)-(2.9) admits the representation

$$V^-(x;a,c) = (\mathbb{L}_{1,m+n}\overline{G}_{m+n})(x) + \sum_{k=1}^{m+n+1} C_k^-(a,c)\,(\mathbb{L}_{1,m+n}U_k)(x), \quad \text{for} \quad a<x<c, \tag{3.29}$$

$$V^+(x;c,b) = (\mathbb{L}_{1,m+n}\overline{G}_{m+n})(x) + \sum_{k=1}^{m+n+1} C_k^+(c,b)\,(\mathbb{L}_{1,m+n}U_k)(x), \quad \text{for} \quad c<x<b, \tag{3.30}$$

where the constants $C_k^-(a,c)$ and $C_k^+(c,b)$, $k=1,\ldots,m+n+1$, are uniquely determined by the linear systems in (3.24)-(3.25), with either (3.26) or (3.27), respectively.

Finally, if there exists some constant $a<c<b$ such that the properties in (2.12) hold, then the candidate solution for the system in (2.7)-(2.9) admits the representation of (3.28), where we have $|(\mathbb{L}_{1,m+n}U_{m+n+1})(c-)| = |(\mathbb{L}_{1,m+n}U_{m+n+1})(c+)| = \infty$ at the corresponding singularity point c of the equation in (2.6)+(2.7) when $\sigma=0$. In this case, we need to put $C_{m+n+1}(a,b)=0$ into (3.28), since otherwise $V(x;a,b)\to\pm\infty$ as $x\uparrow c$ or $x\downarrow c$, respectively, which must be excluded by virtue of the obvious fact that the value function $V_*(x)$ in (2.5) is bounded. Then, the remaining constants $C_k(a,b)$, $k=1,\ldots,m+n$, are uniquely determined by the linear system in (3.24)-(3.25) above with $C_{m+n+1}(a,b)=0$.

3.2 The case of a single driving compound Poisson process. Let us now find the solution of the boundary value problem of (2.6)-(2.9) in the setting with a single driving compound Poisson process with positive exponential jumps. Specifically, we put $m=1$, $n=0$, and $\sigma=0$, so that the compensator measure $\nu(dt,dv)$ from (2.3) has the form $\nu(dt,dv)=\lambda_1 dt I_{\{v>0\}}\alpha_1 e^{-\alpha_1 v}dv$, for some $\lambda_1>0$ and $\alpha_1>1$. Then, the system in (3.19)-(3.22) can be represented as

$$\begin{aligned}&\big(\beta(x)-\gamma\,x\big)\,x\,G_1''(x)-\lambda_1\alpha_1\,G_1(x) \\ &+\Big((\alpha_1+1)\,\big(\beta(x)-\gamma\,x\big)-(\lambda_1+\varkappa)\,x\Big)\,G_1'(x)=0, \quad \text{for} \quad a<x<b,\end{aligned} \tag{3.31}$$

$$G_1(b)=0, \quad G_1'(a+)=-a^{-\alpha_1-1}, \quad \text{and} \quad G_1'(b-)=0, \tag{3.32}$$

so that the general solution from (3.23) takes the form

$$G_1(x)=C_1\,U_1(x)+C_2\,U_2(x), \quad \text{for} \quad a<x<b, \tag{3.33}$$

and the system in (3.24)-(3.27) is given by

$$C_1\, U_1(b) + C_2\, U_2(b) = 0, \tag{3.34}$$

$$C_1\, U_1'(a+) + C_2\, U_2'(a+) = -a^{-\alpha_1-1}, \quad \text{and} \quad C_1\, U_1'(b-) + C_2\, U_2'(b-) = 0, \tag{3.35}$$

where C_k, $k=1,2$, are some arbitrary constants, and $U_k(x)$, $k=1,2$, constitute the fundamental system of solutions of the ordinary differential equation in (3.31) for $a<x<b$. Hence, we obtain that the candidate solution for the system in (2.7)-(2.9) admits the representation

$$V(x;a,b) = -x^{\alpha_1+1}\big(C_1(a,b)\, U_1'(x) + C_2(a,b)\, U_2'(x)\big), \quad \text{for} \quad a<x<b, \tag{3.36}$$

where the constants $C_k(a,b)$, $k=1,2$, are uniquely determined by (3.34) and either the left-hand or right-hand equation in (3.35), when either the inequality $\beta(x)-\gamma x<0$ or $\beta(x)-\gamma x>0$ holds for all $a\le x\le b$, respectively. On the other hand, when the condition of (2.11) holds, the candidate solution is of the form

$$V^-(x;a,c) = -x^{\alpha_1+1}\big(C_1^-(a,c)\, U_1'(x) + C_2^-(a,c)\, U_2'(x)\big), \quad \text{for} \quad a<x<c, \tag{3.37}$$

$$V^+(x;c,b) = -x^{\alpha_1+1}\big(C_1^+(c,b)\, U_1'(x) + C_2^+(c,b)\, U_2'(x)\big), \quad \text{for} \quad c<x<b, \tag{3.38}$$

where the constants $C_k^-(a,c)$ and $C_k^+(c,b)$, $k=1,2$, are uniquely determined by the equation in (3.34) and either of the equations in (3.35), respectively. Finally, when the condition of (2.12) holds, the candidate solution is of the form

$$V(x;a,b) = -x^{\alpha_1+1}\, C_1(a,b)\, U_1'(x), \quad \text{for} \quad a<x<b, \tag{3.39}$$

where the constant $C_1(a,b)$ is uniquely determined by (3.34).

Let us finally derive explicit expressions for the fundamental system of solutions $U_k(x)$, $k=1,2$, and thus, for the candidate value functions $V(x;a,b)$ from (3.36)+(3.39), or $V^-(x;a,c)$ and $V^+(x;c,b)$ from (3.37)-(3.38), for several drift rates $\beta(x)$ in the stochastic differential equation of (2.1), under the assumptions of this subsection.

Example 3.1. *(A pure jump analogue of the Ornstein-Uhlenbeck model.)* Let the drift coefficient $\beta(x)$ of the process X from (2.1) be given as $\beta(x)=\beta_0(1+x)$ for some constant β_0 and all $x\in\mathcal{D}_X=(0,\infty)$, and set $\beta_1=\beta_0-\gamma$. When $\beta_1\neq 0$ holds, we see from (3.31) that $G_1(x)$ satisfies the ordinary differential equation

$$\Big(\frac{\beta_0}{\beta_1}+x\Big)\, x\, G_1''(x) + \bigg(\Big(\alpha_1+1-\frac{\lambda_1+\varkappa}{\beta_1}\Big)\, x + \frac{\beta_0(\alpha_1+1)}{\beta_1}\bigg)\, G_1'(x) - \frac{\lambda_1\alpha_1}{\beta_1}\, G_1(x) = 0, \tag{3.40}$$

for $a<x<b$. When neither of the conditions (2.11)-(2.12) is satisfied, the equation in (3.40) does not have singular points for $a<x<b$. Hence, it follows from [29, Formulas 2.1.2.172 and 2.1.2.171] that the candidate solution for the system in (2.7)-(2.9) is of the form (3.36), where

$$U_1(x) = F(\eta_1,\eta_2;\zeta;H(x)), \tag{3.41}$$

$$U_2(x) = H(x)^{1-\zeta}\, F(\eta_1-\zeta+1,\eta_2-\zeta+1;2-\zeta;H(x)), \tag{3.42}$$

in the case when ζ, $\zeta-\eta_1-\eta_2$, and $\eta_1-\eta_2$ are not integers (for the other cases see [1, Chapter XV, Section 5] and [3, Chapter II, Sections 2 and 3]). Here $F(\eta,\zeta;\rho;z)$ denotes the Gauss' hypergeometric function, which admits the integral representation

$$F(\eta,\zeta;\rho;z) \equiv {}_2F_1(\eta,\zeta;\rho;z) = \frac{\Gamma(\rho)}{\Gamma(\zeta)\Gamma(\rho-\zeta)}\int_0^1 v^{\zeta-1}(1-v)^{\rho-\zeta-1}(1-vz)^{-\eta}\, dv \tag{3.43}$$

for $0 < \zeta < \rho$, and the series expansion

$$F(\eta, \zeta; \rho; z) \equiv {}_2F_1(\eta, \zeta; \rho; z) = 1 + \sum_{k=1}^{\infty} \frac{(\eta)_k (\zeta)_k}{(\rho)_k} \frac{z^k}{k!} \tag{3.44}$$

for $\rho \neq 0, -1, -2, \ldots$, where the series converges under all $|z| < 1$ (see [1, Chapter XV]), and we set

$$\eta_{1,2} = \frac{\alpha_1\beta_1 - (\lambda_1 + \varkappa) \pm \sqrt{(\lambda_1 + \varkappa - \alpha_1\beta_1)^2 + 4\lambda_1\alpha_1\beta_1}}{2\beta_1}, \quad \zeta = \alpha_1 + 1, \quad \text{and} \quad H(x) = -\frac{\beta_1}{\beta_0}\, x. \tag{3.45}$$

When the condition in (2.11) is satisfied, the equation in (3.40) has a singular point at $x = -\beta_0/\beta_1$ for $a < x < b$. However, it does not have a singular point for $a < x < -\beta_0/\beta_1$ and $-\beta_0/\beta_1 < x < b$, and the candidate solution for the system in (2.7)-(2.9) is of the form (3.37)-(3.38) where the functions $U_k(x)$, $k = 1, 2$, are given by (3.41)-(3.42). Finally, notice that when the condition in (2.12) is satisfied, we have that $a < -\beta_0/\beta_1 < b$, $\beta_1 < 0 < \beta_0 < \gamma$, and the equation in (3.40) has a singular point at $x = -\beta_0/\beta_1$, for $a < x < b$. It follows from [29, Formula 2.1.2.172] and [1, Chapter XV, Section 5] that the general solution of the second-order ordinary differential equation in (3.40) is of the form (3.33), where

$$U_1(x) = F(\eta_1, \eta_2; \eta_1 + \eta_2 + 1 - \zeta; 1 - H(x)), \tag{3.46}$$

$$U_2(x) = (1 - H(x))^{\zeta - \eta_1 - \eta_2}\, F(\zeta - \eta_2, \zeta - \eta_1; \zeta - \eta_1 - \eta_2 + 1; 1 - H(x)). \tag{3.47}$$

Hence, the properties that the equalities $F(\eta, \zeta; \rho; 0) = 1$ and $\partial_z F(\eta, \zeta; \rho; z) = (\eta\zeta/\rho) F(\eta + 1, \zeta + 1; \rho + 1; z)$ are satisfied yield the fact that $|U_2'(-\beta_0/\beta_1 \pm)| = \infty$ holds and the candidate solution for the system in (2.7)-(2.9) is of the form (3.39).

On the other hand, when we assume that $\beta_1 = 0$ holds, it follows from (3.31) that $G_1(x)$ satisfies the ordinary differential equation

$$\beta_0\, x\, G_1''(x) + \big(\beta_0(\alpha_1 + 1) - (\lambda_1 + \varkappa)\, x\big)\, G_1'(x) - \lambda_1\alpha_1\, G_1(x) = 0, \tag{3.48}$$

for $a < x < b$, and none of the conditions in (2.11)-(2.12) are satisfied. It follows from [29, Formulas 2.1.2.108 and 2.1.2.70] that the candidate solution for the system in (2.7)-(2.9) is of the form (3.36), where

$$U_1(x) = e^{\rho x}\, \Phi(\eta, \zeta; H(x)), \quad U_2(x) = e^{\rho x}\, \Psi(\eta, \zeta; H(x)), \tag{3.49}$$

and we set $\eta = (\alpha_1\varkappa/(\lambda_1 + \varkappa)) + 1$, $\zeta = \alpha_1 + 1$, $\rho = (\lambda_1 + \varkappa)/\beta_0$, and $H(x) = -(\lambda_1 + \varkappa)x/\beta_0$. Here, the functions $\Phi(\eta, \zeta; z)$ and $\Psi(\eta, \zeta; z)$ are the Kummer's and Tricomi's confluent hypergeometric functions (see, e.g. [1, Chapter XIII]), respectively, which admit the integral representations

$$\Phi(\eta, \zeta; z) = \frac{\Gamma(\zeta)}{\Gamma(\eta)\Gamma(\zeta - \eta)} \int_0^1 e^{zv}\, v^{\eta - 1}(1 - v)^{\zeta - \eta - 1}\, dv, \tag{3.50}$$

for $0 < \eta < \zeta$, and all $z \in \mathbb{R}$, and

$$\Psi(\eta, \zeta; z) = \frac{1}{\Gamma(\zeta)} \int_0^{\infty} e^{-zv}\, v^{\eta - 1}(1 + v)^{\zeta - \eta - 1}\, dv, \tag{3.51}$$

for $\zeta > 0$ and all $z > 0$. Here $\Phi(\eta, \zeta; z)$ also has the series expansion

$$\Phi(\eta, \zeta; z) = 1 + \sum_{k=1}^{\infty} \frac{(\eta)_k}{(\zeta)_k} \frac{z^k}{k!} \tag{3.52}$$

for $\zeta \neq 0, -1, -2, \ldots$, where the series converges under all $z > 0$ (see [1, Chapter XIII]), and $\Gamma(z)$ denotes the

Euler's gamma function.

Example 3.2. *(A pure jump analogue of the Cox-Ingersoll-Ross model I.)* Let the drift coefficient $\beta(x)$ of the process X from (2.1) be given as $\beta(x) = \beta_1 x \ln x$ for some constant β_1 and all $x \in \mathcal{D}_X = (1, \infty)$. Then, we conclude from (3.31) that $G_1(x)$ satisfies the ordinary differential equation

$$(\beta_1 \ln x - \gamma)\, x^2\, G_1''(x) + \big((\alpha_1 + 1)\,(\beta_1 \ln x - \gamma) - \lambda_1 - \varkappa\big)\, x\, G_1'(x) - \lambda_1 \alpha_1\, G_1(x) = 0, \tag{3.53}$$

for $a < x < b$. When we assume that $\beta_1 \neq 0$ holds and the condition in (2.12) is not satisfied, by performing the change of variable $y = \ln x$, it follows from [29, Formulas 2.1.2.108 and 2.1.2.70] that the candidate solution for the system in (2.7)-(2.9) is of the form (3.36) or (3.37)-(3.38), where

$$U_1(x) = x^\rho\, \Phi(\eta, \zeta; H(x)), \quad U_2(x) = x^\rho\, \Psi(\eta, \zeta; H(x)), \tag{3.54}$$

for $\zeta \neq 0, -1, -2, \ldots$, and

$$U_1(x) = x^\rho\, H(x)^{1-\zeta}\, \Phi(\eta - \zeta + 1, 2 - \zeta; H(x)), \tag{3.55}$$

$$U_2(x) = x^\rho\, H(x)^{1-\zeta}\, \Psi(\eta - \zeta + 1, 2 - \zeta; H(x)), \tag{3.56}$$

for $\zeta = 0, -1, -2, \ldots$. Here, we set $\eta = (-\varkappa I_{\{\beta_1 < 0\}} - \lambda_1 I_{\{\beta_1 > 0\}})/\beta_1$, $\zeta = -(\lambda_1 + \varkappa)/\beta_1$, $\rho = -\alpha_1 I_{\{\beta_1 < 0\}}$, and $H(x) = \mathrm{sgn}(\beta_1)\alpha_1(\gamma/\beta_1 - \ln x)$, and the functions $\Phi(\eta, \zeta; z)$ and $\Psi(\eta, \zeta; z)$ are defined in (3.50)-(3.51). When the condition in (2.12) is satisfied, we have that $a < e^{\gamma/\beta_1} < b$, $\beta_1 < 0$, and the equation in (3.53) has a singular point at $x = e^{\gamma/\beta_1}$, for $a < x < b$. Hence, the properties that the equalities $\Phi(\eta, \zeta; 0) = 1$, $\partial_z \Phi(\eta, \zeta; z) = (\eta/\zeta)\Phi(\eta + 1, \zeta + 1; z)$, $\partial_z \Psi(\eta, \zeta; z) = -\eta \Psi(\eta + 1, \zeta + 1; z)$, and $\Psi(\eta, \zeta; 0+) = \infty$, for $\zeta > 1$, are satisfied yield the fact that $|U_2'(e^{\gamma/\beta_1}\pm)| = \infty$ holds and the candidate solution for the system in (2.7)-(2.9) is of the form (3.39).

On the other hand, when we assume that $\beta_1 = 0$ holds, it follows from (3.53) that $G_1(x)$ satisfies the ordinary differential equation

$$\gamma\, x^2\, G_1''(x) + \big(\gamma(\alpha_1 + 1) + \lambda_1 + \varkappa\big)\, x\, G_1'(x) + \lambda_1 \alpha_1\, G_1(x) = 0, \tag{3.57}$$

for $a < x < b$, and none of the conditions (2.11)-(2.12) are satisfied. We can conclude from [29, Formula 2.1.123] that the candidate solution for the system in (2.7)-(2.9) is of the form (3.36), where

$$U_1(x) = x^{(1-\pi_1+2\pi_3)/2}, \;\; U_2(x) = x^{(1-\pi_1-2\pi_3)/2}, \quad \text{if } (1 - \pi_1)^2 > 4\pi_2, \tag{3.58}$$

$$U_1(x) = x^{(1-\pi_1)/2}, \;\; U_2(x) = x^{(1-\pi_1)/2} \ln x, \quad \text{if } (1 - \pi_1)^2 = 4\pi_2, \tag{3.59}$$

$$U_1(x) = x^{(1-\pi_1)/2} \sin(\pi_3 \ln x), \;\; U_2(x) = x^{(1-\pi_1)/2} \cos(\pi_3 \ln x), \quad \text{if } (1 - \pi_1)^2 < 4\pi_2, \tag{3.60}$$

and we set $\pi_1 = \alpha_1 + 1 + (\lambda_1 + \varkappa)/\gamma$, $\pi_2 = \lambda_1 \alpha_1$, and $\pi_3 = \sqrt{|(1 - \pi_1)^2 - 4\pi_2|}/2$.

Example 3.3. *(A pure jump analogue of the Cox-Ingersoll-Ross model II.)* Let the drift coefficient $\beta(x)$ of the process X from (2.1) be given as $\beta(x) = \beta_1 x / \ln x$ for some constant β_1 and $x \in \mathcal{D}_X = (0, 1)$. Then, we conclude from (3.31) that $G_1(x)$ satisfies the ordinary differential equation

$$\Big(\frac{\beta_1}{\ln x} - \gamma\Big)\, x^2\, G_1''(x) + \bigg((\alpha_1 + 1)\,\Big(\frac{\beta_1}{\ln x} - \gamma\Big) - \lambda_1 - \varkappa\bigg)\, x\, G_1'(x) - \lambda_1 \alpha_1\, G_1(x) = 0, \tag{3.61}$$

for $a < x < b$. By analogy with Example 3.2, when we assume that $\beta_1 \neq 0$ holds and the condition in (2.12) is not satisfied, it follows from [29, Formulas 2.1.2.108 and 2.1.2.70] that the candidate solution for the system in (2.7)-(2.9) is of the form (3.36) or (3.37)-(3.38), where the functions $U_k(x)$, $k = 1, 2$, are given by (3.54) for ζ being not a non-positive integer, and (3.55)-(3.56) otherwise, where we set $\Delta = (\varkappa - \gamma)^2 + 4\varkappa\gamma\alpha_1$, $\rho = -(\lambda_1 + \varkappa + \sqrt{\Delta})/2\gamma - \alpha_1/2$, $\eta = \beta_1(\alpha_1 + \rho)\rho/\sqrt{\Delta}$, $\zeta = \beta_1(\lambda_1 + \varkappa)/\gamma^2$, and $H(x) = \sqrt{\Delta}(\ln x - \beta_1/\gamma)/\gamma$.

When the condition in (2.12) is satisfied, we have that $a < e^{\beta_1/\gamma} < b$, $\beta_1 > 0$, and the equation in (3.53) has a singular point at $x = e^{\beta_1/\gamma}$ for $a < x < b$. Hence, by analogy with Example 3.2, we can conclude that $|U_2'(e^{\gamma/\beta_1}\pm)| = \infty$ and the candidate solution for the system in (2.7)-(2.9) is of the form (3.39). Finally, when we assume that $\beta_1 = 0$ holds, it follows from (3.61) that $G_1(x)$ satisfies the ordinary differential equation in (3.57), for $a < x < b$, and none of the conditions (2.11)-(2.12) are satisfied. We can conclude from [29, Formula 2.1.123] that the candidate solution for the system in (2.7)-(2.9) is of the form (3.36), where the functions $U_k(x)$, $k = 1, 2$, are given by (3.58)-(3.60).

4 Equivalence of the two problems

We now state and prove the corresponding assertion relating the solution of the boundary value problem to the original Laplace transform value function.

Theorem 4.1. *Suppose that the process X provides a (unique strong) solution of the stochastic differential equation in (2.1). Then, the Laplace transform $V_*(x)$ from (2.5) of the associated with X random variable $\tau_a \wedge \zeta_b$ over the event $\{\tau_a < \zeta_b\}$ from (2.4) admits the representation*

$$V_*(x) = V(x; a, b), \quad \textit{for} \quad a < x < b, \tag{4.1}$$

for any fixed $a, b \in \mathcal{D}_X$ such that $a < b$, where the function $V(x; a, b)$ is specified as follows:

(i) If $\sigma \neq 0$ then the function $V(x; a, b)$ admits the representation of (3.28) with the constants $C_k(a, b)$, $k = 1, \ldots, m + n + 2$, providing a unique solution to the system in (3.24)-(3.27).

(ii) If $\sigma = 0$ and either $\beta(x) - \gamma x < 0$ or $\beta(x) - \gamma x > 0$ holds for all $a \le x \le b$, then $V(x; a, b)$ admits the representation of (3.28) with $C_{m+n+2}(a, b) = 0$ and $C_k(a, b)$, $k = 1, \ldots, m + n + 1$, being a unique solution of the system in (3.24)-(3.25) with either (3.26) or (3.27), respectively. If $\sigma = 0$ and the condition of (2.11) holds, then $V(x; a, b)$ is given by $V^-(x; a, c)$ and $V^+(x; c, b)$ from (3.29)-(3.30) with the constants $C_k^-(a, c)$ and $C_k^+(c, b)$, $k = 1, \ldots, m + n + 1$, being a unique solution of the system (3.24)-(3.25) with either the conditions of (3.26) or (3.27), respectively. Finally, if $\sigma = 0$ and the condition of (2.12) holds, then $V(x; a, b)$ admits the representation of (3.28) with $C_{m+n+1}(a, b) = C_{m+n+2}(a, b) = 0$ and $C_k(a, b)$, $k = 1, \ldots, m + n$, being a unique solution of the system in (3.24)-(3.25).

Proof. In order to verify the assertion formulated above, we need to show that the function on the right-hand side of the expression in (4.1) coincides with the value function in (2.5). For this purpose, let us denote by $V(x)$ the right-hand side of the expression in (4.1).

(i) Let us first consider the case $\sigma \neq 0$. Following the idea of the proof in [17, Theorem 3.1], by using the property that $V(x)$ is bounded, we can introduce a sequence of twice continuously differentiable bounded functions $(V_k(x))_{k\in\mathbb{N}}$ on $\mathcal{D}_X$ such that $|V_k(x) - V(x)| \le 1$ holds for all $x \in \mathcal{D}_X$, and $V_k(x) = V(x)$ holds for $x \in \mathcal{D}_X \setminus ((a - 1/k, a) \cup (b, b + 1/k))$. Note that, by construction of the functions above, we clearly have $V_k(x) \to V(x)$ for all $x \in \mathcal{D}_X$ as $k \to \infty$. By applying the Itô's formula to the process $e^{-\varkappa t} V_k(X_t)$, we get that

$$e^{-\varkappa(t\wedge\tau_a\wedge\zeta_b)} \, V_k(X_{t\wedge\tau_a\wedge\zeta_b}) = V_k(x) + \int_0^{t\wedge\tau_a\wedge\zeta_b} e^{-\varkappa s} \, (\mathbb{L}V_k - \varkappa V_k)(X_s) \, ds + M^k_{t\wedge\tau_a\wedge\zeta_b} \tag{4.2}$$

holds for all $t \ge 0$ and $x \in \mathcal{D}_X$, where the process $M^k = (M^k_t)_{t\ge0}$ defined by

$$M^k_t = \int_0^t e^{-\varkappa s} \, V'(X_s) \, \sigma X_s \, dW_s + \int_0^t \int e^{-\varkappa s} \left(V_k(X_{s-}e^y) - V_k(X_{s-}) \right) (\mu - \nu)(ds, dy), \tag{4.3}$$

for any $k \in \mathbb{N}$, is a local martingale. It follows from the inequality $|V_k(x) - V(x)| \le 1$ for all $x \in \mathcal{D}_X$ that we have

$$\begin{aligned}\big|(\mathbb{L}V_k - \varkappa V_k)(x)\big| &\le \lambda \bigg(\sum_{i=1}^{m} \alpha_i \int_{\ln b - \ln x}^{\ln(b+1/k) - \ln x} \big|V_k(xe^y) - V(xe^y)\big|\, dy \\ &\quad + \sum_{j=1}^{n} \beta_j \int_{\ln(a-1/k) - \ln x}^{\ln a - \ln x} \big|V_k(xe^y) - V(xe^y)\big|\, dy \bigg) \\ &\le \lambda \bigg(\ln\Big(\frac{b+1/k}{b}\Big) \sum_{i=1}^{m} \alpha_i + \ln\Big(\frac{a}{a-1/k}\Big) \sum_{i=1}^{n} \beta_i \bigg) \to 0,\end{aligned} \tag{4.4}$$

for $x \in \mathcal{D}_X$ uniformly in x as $k \to \infty$. Hence, we obtain from the expression in (4.2) and the fact that $V_k(x)$ is bounded that the inequality

$$|M_t^k| \le C + \lambda \bigg(\ln\Big(\frac{b+1/k}{b}\Big) \sum_{i=1}^{m} \alpha_i + \ln\Big(\frac{a}{a-1/k}\Big) \sum_{i=1}^{n} \beta_i \bigg)\, t \tag{4.5}$$

holds for some constant $C > 0$ and all $t \ge 0$, so that the process $(M_{t \wedge \tau_a \wedge \zeta_b}^k)_{t \ge 0}$ is a martingale. Thus, taking the expectation with respect to P_x in (4.2), we get

$$E_x \bigg[e^{-\varkappa(t \wedge \tau_a \wedge \zeta_b)}\, V_k(X_{t \wedge \tau_a \wedge \zeta_b}) - \int_0^{t \wedge \tau_a \wedge \zeta_b} e^{-\varkappa s}\, (\mathbb{L}V_k - \varkappa V_k)(X_s)\, ds \bigg] = V_k(x), \tag{4.6}$$

for $t \ge 0$ and $x \in \mathcal{D}_X$. Note that, by virtue of the fact that $V_k(x) \to V(x)$ holds for all $x \in \mathcal{D}_X$, we get that $V_k(X_{t \wedge \tau_a \wedge \zeta_b}) \to V(X_{t \wedge \tau_a \wedge \zeta_b})$ (P_x-a.s.). Therefore, we have by the dominated convergence that

$$\lim_{k \to \infty} E_x \big[e^{-\varkappa(t \wedge \tau_a \wedge \zeta_b)}\, V_k(X_{t \wedge \tau_a \wedge \zeta_b}) \big] = E_x \big[e^{-\varkappa(t \wedge \tau_a \wedge \zeta_b)}\, V(X_{t \wedge \tau_a \wedge \zeta_b}) \big], \tag{4.7}$$

and by the uniform convergence in (4.4), we obtain

$$\lim_{k \to \infty} E_x \bigg[\int_0^{t \wedge \tau_a \wedge \zeta_b} e^{-\varkappa s}\, (\mathbb{L}V_k - \varkappa V_k)(X_s)\, ds \bigg] = 0, \tag{4.8}$$

for $t \ge 0$ and $x \in \mathcal{D}_X$. Hence, we conclude that

$$E_x \big[e^{-\varkappa(t \wedge \tau_a \wedge \zeta_b)}\, V(X_{t \wedge \tau_a \wedge \zeta_b}) \big] = \lim_{k \to \infty} V_k(x) = V(x) \tag{4.9}$$

holds for all $t \ge 0$ and $x \in \mathcal{D}_X$. Therefore, letting t go to infinity and using the conditions in (2.8)-(2.9) as well as the fact that $V(X_{\tau_a \wedge \zeta_b}) = I_{\{\tau_a < \zeta_b\}}$ on the set $\{\tau_a \wedge \zeta_b < \infty\}$, we can apply the Lebesgue dominated convergence theorem for (4.9) to obtain that the equalities

$$E_x \big[e^{-\varkappa(\tau_a \wedge \zeta_b)}\, I_{\{\tau_a < \zeta_b\}} \big] = E_x \big[e^{-\varkappa(\tau_a \wedge \zeta_b)}\, V(X_{\tau_a \wedge \zeta_b})\, I_{\{\tau_a \wedge \zeta_b < \infty\}} \big] = V(x) \tag{4.10}$$

hold for all $x \in \mathcal{D}_X$, which completes the proof in the case $\sigma \neq 0$.

(ii) Assume now that $\sigma = 0$ and $V(x)$ satisfies the right-hand condition in (2.9), so that $V(b-) = 0$ holds, but does not satisfy the left-hand condition there, so that $V(a+) \neq 1$ holds (the other cases can be dealt with similarly). This feature corresponds to the case in which the process X can pass through the boundary a only by jumping, and we particularly have that $P_x(X_{\tau_a} = a) = 0$ holds for $x \in \mathcal{D}_X \setminus \{a\}$. By analogy to case (i), we introduce a sequence of continuously differentiable bounded functions $(V_k(x))_{k \in \mathbb{N}}$ on $\mathcal{D}_X$ such that $V_k(a) = V(a+)$ and $|V_k(x) - V(x)| \le |V_k(a) - V(a)|$ holds for all $x \in \mathcal{D}_X$, and $V_k(x) = V(x)$ holds for $x \in \mathcal{D}_X \setminus ((a - 1/k, a] \cup (b, b + 1/k))$. Note that, by construction of the functions above, we clearly have $V_k(x) \to V(x)$ for all $x \in \mathcal{D}_X \setminus \{a\}$ as $k \to \infty$ and, since $P_x(X_{\tau_a} = a) = 0$ holds, we have that

$V_k(X_{t\wedge\tau_a\wedge\zeta_b}) \to V(X_{t\wedge\tau_a\wedge\zeta_b})$ (P_x-a.s.). The rest of the proof follows from the arguments in case (i). □

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