Yuri Kabanov, Constantinos Kardaras and Shiqi Song

No arbitrage of the first kind and local martingale numéraires

Article (Accepted version) (Refereed)

Original citation: Kabanov, Yuri, Kardaras, Constantinos and Song, Shiqi (2016) *No arbitrage of the first kind and local martingale numéraires.* Finance and Stochastics, 20 (4). pp. 1097-1108. ISSN 0949-2984

DOI: 10.1007/s00780-016-0310-6

Noname manuscript No.
(will be inserted by the editor)

No arbitrage of the first kind and local martingale numéraires

Yuri Kabanov · Constantinos Kardaras · Shiqi Song

Received: date / Accepted: date

Abstract A supermartingale deflator (resp., local martingale deflator) multiplicatively transforms nonnegative wealth processes into supermartingales (resp., local martingales). The supermartingale numéraire (resp., local martingale numéraire) is the wealth process whose reciprocal is a supermartingale deflator (resp., local martingale deflator). It has been established in previous literature that absence of arbitrage of the first kind (NA_1) is equivalent to existence of the supermartingale numéraire, and further equivalent to existence of a strictly positive local martingale deflator; however, under NA_1, the local martingale numéraire may fail to exist. In this work, we establish that, under NA_1, the supermartingale numéraire under the original probability P becomes the local martingale numéraire for equivalent probabilities, arbitrarily close to P in total-variation distance.

Keywords Arbitrage · Viability · Fundamental Theory of Asset Pricing · Numéraire · Local martingale deflator · σ-Martingale

Mathematics Subject Classification (2000) 91G10 · 60G44

JEL Classification C60 · G13

Laboratoire de Mathématiques, Université de Franche-Comté, 16 Route de Gray, 25030 Besançon, cedex, France, and International Laboratory of Quantitative Finance, Higher School of Economics, Moscow, Russia
E-mail: Youri.Kabanov@univ-fcomte.fr

London School of Economics, 10 Houghton Street, London, WC2A 2AE, UK
E-mail: k.kardaras@lse.ac.uk

Université d'Evry Val d'Essonne, Boulevard de France, 91037 Evry, cedex, France
E-mail: bachsuitepremier@gmail.com

1 Introduction

A central structural assumption in the mathematical theory of financial markets is the existence of so-called *local martingale deflators*, i.e., processes that act multiplicatively and transform nonnegative wealth processes into local martingales. Under the *No Free Lunch with Vanishing Risk* (NFLVR) condition of [5], [6], the density process of a local martingale (or, more generally, a σ-martingale) measure is a strictly positive local martingale deflator. However, strictly positive local martingale deflators may exist even if the market allows for free lunch with vanishing risk. Both from a financial and mathematical point of view, especially important is the case where a deflator is the reciprocal of a wealth process called *local martingale numéraire*; in this case, the prices of all assets (and, in fact, all wealth processes resulting from trading), when denominated in units of the latter local martingale numéraire, are (positive) local martingales.

The relevant, weaker than NFLVR, market viability property which turns out to be equivalent to the existence of supermartingale (or local martingale) numéraires was isolated by various authors under different names: *No Asymptotic Arbitrage of the 1st Kind* (NAA_1), *No Arbitrage of the 1st Kind* (NA_1), *No Unbounded Profit with Bounded Risk* (NUPBR), etc., see [10], [5], [9], [11], [13]. It is not difficult to show that all these properties are, in fact, equivalent, even in a wider framework than that of the standard semimartingale setting—for more information, see Appendix. In the present paper we opt to utilize the economically meaningful formulation NA_1, defined as the property of the market to assign a strictly positive superhedging value to any non-trivial positive contingent claim.

In the standard financial model studied here, the market is described by a d-dimensional semimartingale process S giving the discounted prices of basic securities. In [11], it was shown (even in a more general case of convex portfolio constraints) that the following statements are equivalent:

(i) *Condition* NA_1 *holds.*
(ii) *There exists a strictly positive supermartingale deflator.*
(ii') *The supermartingale numéraire exists.*

In [16], the previous list of equivalent properties was complemented by:

(iii) *There exists a strictly positive local martingale deflator.*

There are counterexamples (see, for example, [16]) showing that the local martingale numéraire may fail to exist even when there is an equivalent martingale measure (and, in particular, when condition NA_1 holds). Such examples are possible only in the case of discontinuous asset-price process: it was already shown in [2] that, for continuous semimartingales, among strictly positive local martingale deflators there exists one whose reciprocal is a strictly positive wealth process.

In the present note, we add to the above list of equivalences a further property:

(*iv*) *In any total-variation neighbourhood of the original probability, there exists an equivalent probability under which the local martingale numéraire exists.*

Establishing the chain $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$ is rather straightforward and well-known. The contribution of the note is proving the "closing" implication $(i) \Rightarrow (iv)$. It is an obvious corollary of the already known implication $(i) \Rightarrow (ii')$ and the following principal result of our note, a version of which was established previously only for the case $d = 1$ in [13]:

Proposition 1.1 *The supermartingale numéraire under P becomes the local martingale numéraire under probabilities $\tilde{P} \sim P$ which are arbitrarily close in total variation distance to P.*

Proposition 1.1 bears a striking similarity with the density result of σ-martingale measures in the set of all separating measures—see [6] and Theorem A.5 in the Appendix. In fact, coupled with certain rather elementary properties of stochastic exponentials, the aforementioned density result is the main ingredient of our proof of Proposition 1.1.

Importantly, we obtain in particular the main result of [16], utilising completely different arguments. The proof in [16] combines a change-of-numéraire technique and a reduction to the Delbaen–Schachermayer Fundamental Theorem of Asset Pricing (FTAP) in [5]. The latter is considered as one of the most difficult and fundamental results of Arbitrage Theory, and the search for simplified proofs is continued—see, e.g., [3]. In fact, it may be obtained as a by-product of our result and the version of the Optional Decomposition Theorem in [15], as has been explained in [12, Section 3].

2 Framework and main result

2.1 The set-up

In all that follows, we fix $T \in (0, \infty)$ and work on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \in [0,T]}, P)$ satisfying the usual conditions. Unless otherwise explicitly specified, all relationships between random variables are understood in the P-a.s. sense, and all relationships between stochastic processes are understood modulo P-evanescence.

Let $S = (S_t)_{t \in [0,T]}$ be a d-dimensional semimartingale. We denote by $L(S)$ the set of S-integrable processes, i.e., the set of all d-dimensional predictable processes for which the stochastic integral $H \cdot S$ is defined. We stress that we consider general vector stochastic integration—see [8].

An integrand $H \in L(S)$ such that $x + H \cdot S \geq 0$ holds for some $x \in \mathbb{R}_+$ will be called *x-admissible*. We introduce the set of semimartingales

$$\mathcal{X}^x := \{H \cdot S\colon\ H \text{ is } x\text{-admissible integrand}\},$$

and denote $\mathcal{X}^x_>$ its subset formed by processes X such that $x + X > 0$ and $x+X_- > 0$. These sets are invariant under equivalent changes of the underlying probability. Define also the sets of random variables $\mathcal{X}^x_T := \{X_T :\ X \in \mathcal{X}^x\}$.

For $\xi \in L^0_+$, we define

$$x(\xi) := \inf\{x \in \mathbb{R}_+ :\ \text{there exists } X \in \mathcal{X}^x \text{ with } x + X_T \geq \xi\},$$

with the standard convention $\inf \emptyset = \infty$.

In the special context of financial modeling:

- The process S represents the price evolution of d liquid assets, discounted by a certain baseline security.
- With H being x-admissible integrand, $x + H \cdot S$ is the value process of a self-financing portfolio with the initial capital $x \geq 0$, constrained to stay nonnegative at all times.
- A random variable $\xi \in L^0_+$ represents a contingent claim, and $x(\xi)$ is its *superhedging value* in the class of nonnegative wealth processes.

2.2 Main result

We define $|P - \tilde{P}|_{TV} = \sup_{A\in\mathcal{F}} |P(A) - \tilde{P}(A)|$ as the total variation distance between the probabilities P and $\tilde{P}$ on $(\Omega, \mathcal{F})$.

Theorem 2.1 *The following conditions are equivalent:*

(i) $x(\xi) > 0$ *for every* $\xi \in L^0_+ \setminus \{0\}$.

(ii) *There exists a strictly positive process* Y *such that the process* $Y(1+X)$ *is a supermartingale for every* $X \in \mathcal{X}^1$.

(iii) *There exists a strictly positive process* Y *such that the process* $Y(1+X)$ *is a local martingale for every* $X \in \mathcal{X}^1$.

(iv) *For any* $\varepsilon > 0$ *there exists* $\tilde{P} \sim P$ *with* $|\tilde{P} - P|_{TV} < \varepsilon$ *and* $\tilde{X} \in \mathcal{X}^1_>$ *such that* $(1 + X)/(1 + \tilde{X})$ *is a local* $\tilde{P}$*-martingale for every* $X \in \mathcal{X}^1$.

Remark 2.2 It is straightforward to check that statements (ii), (iii), and (iv) of Theorem 2.1 are equivalent to the same conditions where "for every $X \in \mathcal{X}^1$" is replaced by "for every $X \in \mathcal{X}^1_>$".

Theorem 2.1 is formulated in "pure" language of stochastic analysis. In the context of Mathematical Finance, the following interpretations regarding its statement should be kept in mind:

- Condition (i) states that any non-trivial contingent claim $\xi \geq 0$ has a strictly positive superhedging value. This is referred to as condition NA_1 (No Arbitrage of the 1st Kind); it is equivalent to the boundedness in probability of the set $\mathcal{X}^1_T$, or, alternatively, to condition NAA_1 (No Asymptotic Arbitrage of the 1st Kind)—see Appendix.
- The process Y in statement (ii) (resp., in statement (iii)) is called a strictly positive *supermartingale deflator* (resp., *local martingale deflator*).

– The process $\tilde{X}$ with the property in statement (iv) is called the *local martingale numéraire under the probability* $\tilde{P}$.

With the above terminology in mind, we may reformulate the properties (i) – (iv) as was done in Introduction.

3 Proof of Theorem 2.1

3.1 Proof of easy implications

The arguments establishing the implications $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$ in Theorem 2.1 are elementary and well known; however, for completeness of presentation, we shall briefly reproduce them here.

Assume statement (iv), and in its notation fix some $\varepsilon > 0$, let Z be the density process of $\tilde{P}$ with respect to P, and set $\tilde{Z} := 1/(1+\tilde{X})$. For any $X \in \mathcal{X}^1$, the process $\tilde{Z}(1+X)$ is a local $\tilde{P}$-martingale. Hence, with $Y := Z\tilde{Z}$, the process $Y(1+X)$ is a local P-martingale, i.e., (iii) holds.

Since a positive local martingale is a supermartingale, the implication $(iii) \Rightarrow (ii)$ is obvious.

To establish implication $(ii) \Rightarrow (i)$, suppose that Y is a strictly positive supermartingale deflator. It follows that $EY_T(1+X_T) \le 1$ holds for all $X \in \mathcal{X}^1$. Hence, the set $Y_T(1+\mathcal{X}^1_T)$ is bounded in L^1, and, *a fortiori*, bounded in probability. Since $Y_T > 0$, the set $\mathcal{X}^1_T$ is also bounded in probability. The latter property is equivalent to condition NA_1—see Lemma A.1 in the Appendix.

By [11, Theorem 4.12] and Lemma A.1 in the Appendix, condition (i) in the statement of Theorem 2.1 implies the existence of the supermartingale numéraire. Therefore, in order to establish implication $(i) \Rightarrow (iv)$ of Theorem 2.1 and complete its proof, it remains to prove Proposition 1.1. For this, we need some auxiliary facts presented in the next subsection.

3.2 Ratio of stochastic exponentials

We introduce the notation

$$B(S) := \{f \in L(S)\colon\ f\Delta S > -1\};$$

that is, $B(S)$ is the subset of integrands for which the trajectories of the stochastic exponentials $\mathcal{E}(f \cdot S)$ are bounded away from zero.

Note that the set $1 + \mathcal{X}^1_>$ coincides with the set of stochastic exponentials of integrals with respect to S:

$$1 + \mathcal{X}^1_> = \{\mathcal{E}(f \cdot S) :\ f \in B(S)\}.$$

Indeed, a stochastic exponential corresponding to the integrand $f \in B(S)$ is strictly positive, as is also its left limit, and satisfies the linear integral equation

$$\mathcal{E}(f \cdot S) = 1 + \mathcal{E}_-(f \cdot S) \cdot (f \cdot S) = 1 + (\mathcal{E}_-(f \cdot S)f) \cdot S.$$

Thus, $\mathcal{E}(f \cdot S) \in \mathcal{X}^1_>$. Conversely, if the process $V = 1 + H \cdot S$ is such that $V > 0$ and $V_- > 0$, then

$$V = 1 + (V_- V_-^{-1}) \cdot V = 1 + V_- \cdot (V_-^{-1} \cdot (H \cdot S)) = 1 + V_- \cdot ((V_-^{-1} H) \cdot S);$$

that is, $V = \mathcal{E}(f \cdot S)$, where $f = V_-^{-1} H \in B(S)$.

The above observations, coupled with Remark 2.2, imply that condition (iv) may be alternatively reformulated as follows:

(iv) *For any $\varepsilon > 0$, there exist $g \in B(S)$ and $\tilde{P} \sim P$ with $|\tilde{P} - P|_{TV} < \varepsilon$ such that $\mathcal{E}(f \cdot S)/\mathcal{E}(g \cdot S)$ is a local $\tilde{P}$-martingale for every $f \in B(S)$.*

Let S^c denote the continuous local martingale part of the semimartingale S. Recall that $\langle S^c \rangle = c \cdot A$, where A is a predictable increasing process and c is a predictable process with values in the set of positive semidefinite matrices; then, $g \in L(S^c)$ if and only if $|c^{1/2} g|^2 \cdot A_T < \infty$.

In the sequel, fix an arbitrary $g \in B(S)$, and set

$$S^g = S - cg \cdot A - \sum_{s \leq .} \frac{g_s \Delta S_s}{1 + g_s \Delta S_s} \Delta S_s. \tag{3.1}$$

As

$$\sum_{s \leq .} \left| \frac{g_s \Delta S_s}{1 + g_s \Delta S_s} \right| |\Delta S_s| \leq \frac{1}{2} \sum_{s \leq T} \left| \frac{g_s \Delta S_s}{1 + g_s \Delta S_s} \right|^2 + \frac{1}{2} \sum_{s \leq T} |\Delta S_s|^2 < \infty,$$

the last term in the right-hand side of (3.1) is a processes of bounded variation, implying that S^g is a semimartingale.

Noting that $\Delta S^g = \Delta S/(1 + g\Delta S)$, we obtain from (3.1) that

$$S = S^g + cg \cdot A + \sum_{s \leq .} (g_s \Delta S_s) \Delta S^g_s.$$

Lemma 3.1 *It holds that $L(S) = L(S^g)$.*

Proof Let $f \in L(S)$. Then

$$|(f, cg)| \cdot A_T \leq \frac{1}{2} |c^{1/2} f|^2 \cdot A_T + \frac{1}{2} |c^{1/2} g|^2 \cdot A_T < \infty,$$

$$\sum_{s \leq T} \frac{|g_s \Delta S_s f_s \Delta S_s|}{1 + g_s \Delta S_s} \leq \frac{1}{2} \sum_{s \leq T} |f_s \Delta S_s|^2 + \frac{1}{2} \sum_{s \leq T} \left| \frac{g_s \Delta S_s}{1 + g_s \Delta S_s} \right|^2 < \infty.$$

Thus, $L(S) \subseteq L(S^g)$. To show the opposite inclusion, take $f \in L(S^g)$. The conditions $g \in L(S)$ and $f \in L(S^g)$ imply that f and g are integrable with respect to $S^c = (S^g)^c$, i.e., that $|c^{1/2} g|^2 \cdot A_T < \infty$ and $|c^{1/2} f|^2 \cdot A_T < \infty$. As previously, it then follows that $|(f, cg)| \cdot A_T < \infty$. Since also

$$\sum_{s \leq t} |(g_s \Delta S_s)(f_s \Delta S^g_s)| \leq \frac{1}{2} \sum_{s \leq T} |g_s \Delta S_s|^2 + \frac{1}{2} \sum_{s \leq T} |f_s \Delta S^g_s|^2 < \infty,$$

we obtain that $f \in L(S)$, i.e., the inclusion $L(S^g) \subseteq L(S)$. $\square$

Lemma 3.2 *It holds that $B(S^g) = B(S) - g$.*

Proof Let $h = f - g$, where $f \in B(S)$. Then, $h \in L(S) = L(S^g)$ by Lemma 3.1, and

$$h\Delta S^g = (f-g)\Delta S^g = \frac{(f-g)\Delta S}{1+g\Delta S} = \frac{1+f\Delta S}{1+g\Delta S} - 1 > -1.$$

Conversely, let us start with $h \in B(S^g)$. Then, using again Lemma 3.1, we obtain that $f := h + g$ belongs to $L(S)$. Furthermore, recalling the relation $\Delta S = \Delta S^g/(1 - g\Delta S^g)$, we obtain that

$$f\Delta S = (h+g)\Delta S = \frac{(h+g)\Delta S^g}{1-g\Delta S^g} = \frac{1+h\Delta S^g}{1-g\Delta S^g} - 1 > -1,$$

which completes the proof. □

For $f \in B(S)$, Lemma 3.2 gives $(f - g) \in B(S^g)$; then, straightforward calculations using Yor's product formula

$$\mathcal{E}(U)\mathcal{E}(V) = \mathcal{E}(U + V + [U, V]),$$

valid for arbitrary semimartingales U and V, lead to the identity

$$\frac{\mathcal{E}(f \cdot S)}{\mathcal{E}(g \cdot S)} = \mathcal{E}((f-g) \cdot S^g). \tag{3.2}$$

(In this respect, see also [11, Lemma 3.4].) Then, invoking Lemma 3.2, we obtain the set-equality

$$1 + \mathcal{X}^1_>(S^g) = \mathcal{E}^{-1}(g \cdot S)(1 + \mathcal{X}^1_>(S)). \tag{3.3}$$

3.3 Proof of Proposition 1.1

Let the process $\mathcal{E}(g\cdot S)$, where $g \in B(S)$, be the supermartingale numéraire; in other words, the ratio $\mathcal{E}(f \cdot S)/\mathcal{E}(g\cdot S)$ is a supermartingale for each $f \in B(S)$. Passing to S^g and using Lemma 3.2, we obtain that $E\mathcal{E}_T(h \cdot S^g) \le 1$ holds for all $h \in B(S^g)$. Therefore, $EH \cdot S^g_T \le 0$ holds for every $H \in L(S^g)$ such that $H \cdot S^g > -1$ and $H \cdot S^g_- > -1$, thus, for every $H \in L(S^g)$ for which the process $H \cdot S^g$ is bounded from below. This means, in the terminology of [9], that the probability P is a separating measure for S^g. An application of [6, Proposition 4.7] (also, Theorem A.5) implies, for any $\varepsilon > 0$, the existence of probability $\tilde{P} \sim P$, depending on ε, with $|P - \tilde{P}|_{TV} < \varepsilon$, such that S^g is a σ-martingale with respect to $\tilde{P}$, that is, $S^g = G\cdot M$ where G is a $]0, 1]$-valued one-dimensional predictable process and M is a d-dimensional local $\tilde{P}$-martingale. Recall that a bounded from below stochastic integral with respect to a local martingale is a local martingale [1, Prop. 3.3]. The ratio $\mathcal{E}(f \cdot S)/\mathcal{E}(g \cdot S)$, being an integral with respect to S^g, hence with respect to M, is a $\tilde{P}$-local martingale for each $f \in B(S)$, which is exactly what we need. □

Remark 3.1 An inspection of the arguments in [11] used to establish the implication $(i) \Rightarrow (ii')$ reveals that in the case where the Lévy measures of S are concentrated on finite sets depending on (ω, t), the supermartingale numéraire is, in fact, the local martingale numéraire.

A No-arbitrage conditions, revisited

A.1 Condition NA_1: equivalent formulations

We discuss equivalent forms of condition NA_1 in the context of a general abstract setting, where the model is given by specifying the wealth processes set. The advantage of this generalization is that one may use only elementary properties without any reference to stochastic calculus and integration theory.

Let $\mathcal{X}^1$ be a convex set of càdlàg processes X with $X \geq -1$ and $X_0 = 0$, containing the zero process. For $x \geq 0$ we define the set $\mathcal{X}^x = x\mathcal{X}^1$, and note that $\mathcal{X}^x \subseteq \mathcal{X}^1$ when $x \in [0,1]$. Put $\mathcal{X} := \operatorname{cone} \mathcal{X}^1 = \mathbb{R}_+ \mathcal{X}^1$ and define the sets of terminal random variables $\mathcal{X}^1_T := \{X_T \colon X \in \mathcal{X}^1\}$ and $\mathcal{X}_T := \{X_T \colon X \in \mathcal{X}\}$. In this setting, the elements of $\mathcal{X}$ are interpreted as admissible wealth processes starting from zero initial capital; the elements of $\mathcal{X}^x$ are called x-admissible.

Remark A.1 ("Standard" model) In the typical example, a d-dimensional semimartingale S is given and $\mathcal{X}^1$ is the set of stochastic integrals $H \cdot S$ where H is S-integrable and $H \cdot S \geq -1$. Though our main result deals with the standard model, basic definitions and their relations with concepts of the arbitrage theory is natural to discuss in more general framework.

Define the set of strictly 1-admissible processes $\mathcal{X}^1_> \subseteq \mathcal{X}^1$ composed of $X \in \mathcal{X}^1$ such that $X > -1$ and $X_- > -1$. The sets $x + \mathcal{X}^x$, $x + \mathcal{X}^x_>$ etc., $x \in \mathbb{R}_+$, have obvious interpretation. We are particularly interested in the set $1 + \mathcal{X}^1_>$. Its elements are strictly positive wealth processes starting with unit initial capital, and may be thought as *tradeable numéraires*.

For $\xi \in L^0_+$, define the *superhedging value* $x(\xi) := \inf\{x : \ \xi \in x + \mathcal{X}^x_T - L^0_+\}$. We say that the wealth-process family $\mathcal{X}$ satisfies condition NA_1 (*No Arbitrage of the 1st Kind*) if $x(\xi) > 0$ holds for every $\xi \in L^0_+ \setminus \{0\}$. Alternatively, condition NA_1 can be defined via

$$\Big(\bigcap_{x>0} \{x + \mathcal{X}^x_T - L^0_+\} \Big) \cap L^0_+ = \{0\}.$$

The family $\mathcal{X}$ is said to satisfy condition NAA_1 (*No Asymptotic Arbitrage of the 1st Kind*) if for any sequence $(x^n)_n$ of positive numbers with $x^n \downarrow 0$ and any sequence of value processes $X^n \in \mathcal{X}$ such that $x^n + X^n \geq 0$, it holds that $\limsup_n P(x^n + X^n_T \geq 1) = 0$.

Finally, the family $\mathcal{X}$ satisfies condition NUPBR (*No Unbounded Profit with Bounded Risk*) if the set $\{X_T : \ X \in \mathcal{X}^1_>\}$ is P-bounded. Since we have $(1/2)\mathcal{X}^1_T = \mathcal{X}^{1/2}_T \subseteq \{X_T : \ X \in \mathcal{X}^1_>\}$, the set $\{X_T : \ X \in \mathcal{X}^1_>\}$ is P-bounded if and only if the set $\mathcal{X}^1_T$ is P-bounded.

The next result shows that all three market viability notions introduced above coincide.

Lemma A.1 *$NAA_1 \;\Leftrightarrow\; NUPBR \;\Leftrightarrow\; NA_1$.*

Proof $\mathrm{NAA}_1 \;\Rightarrow$ NUPBR: If $\{X_T:\; X\in\mathcal{X}^1_>\}$ fails to be P-bounded, $P(1+\tilde X^n_T\ge n)\ge\varepsilon>0$ holds for a sequence of $\tilde X^n\in\mathcal{X}^1_>$, and we obtain a violation of NAA_1 with $n^{-1}+n^{-1}\tilde X^n_T$.

NUPBR $\Rightarrow$ NA_1: If NA_1 fails, there exist $\xi\in L^0_+\setminus\{0\}$ and a sequence $X^n\in\mathcal{X}^{1/n}$ such that $1/n+X^n\ge\xi$. Then, the sequence $nX^n_T\in\mathcal{X}^1$ fails to be P-bounded, in violation of NUPBR.

$\mathrm{NA}_1 \;\Rightarrow\; \mathrm{NAA}_1$: If the implication fails, then there are sequences $x^n\downarrow 0$ and $X^n\ge -x^n$ such that $P(x^n+X^n_T\ge 1)\ge 2\varepsilon>0$. By the von Weizsäcker theorem (see [17]), any sequence of random variables bounded from below contains a subsequence converging in Cesaro sense a.s. as well as its all further subsequences. We may assume without loss of generality that for $\xi^n:=x^n+X^n_T$ the sequence $\bar\xi^n:=(1/n)\sum_{i=1}^n\xi_i$ converges to $\xi\in L^0_+$. Note that $\xi\ne 0$. Indeed,

$$
\begin{aligned}
\varepsilon(1-P(\bar\xi^n\ge\varepsilon)) &\ge \frac{1}{n}\sum_{i=1}^n E\xi^i I_{\{\bar\xi^n<\varepsilon\}} \ge \frac{1}{n}\sum_{i=1}^n E\xi^i I_{\{\xi^i\ge 1,\;\bar\xi^n<\varepsilon\}}\\
&\ge \frac{1}{n}\sum_{i=1}^n P(\xi^i\ge 1,\;\bar\xi^n<\varepsilon) \ge \frac{1}{n}\sum_{i=1}^n (P(\xi^i\ge 1)-P(\bar\xi^n\ge\varepsilon))\\
&\ge 2\varepsilon - P(\bar\xi^n\ge\varepsilon).
\end{aligned}
$$

It follows that $P(\bar\xi^n\ge\varepsilon)\ge\varepsilon/(1-\varepsilon)$. Thus,

$$E(\xi\wedge 1)=\lim_n E(\bar\xi^n\wedge 1)\ge\varepsilon^2/(1-\varepsilon)>0.$$

It follows that there exists $a>0$ such that $P(\xi\ge 2a)>0$. In view of Egorov's theorem, one can find a measurable set $\Gamma\subseteq\{\xi\ge a\}$ with $P(\Gamma)>0$ on which $x^n+X^n\ge a$ holds for all sufficiently large n. But this means that the random variable $aI_\Gamma\ne 0$ can be super-replicated starting with arbitrary small initial capital, in contradiction with the assumed condition NA_1.

Remark A.2 (On terminology and bibliography) Conditions NAA_1 and NA_1 have clear financial meanings, while P-boundedness of the set $\mathcal{X}^1_T$, at first glance, looks as a technical condition—see [5]. The concept of NAA_1 first appeared in [10] in a much more general context of large financial markets, along with another fundamental notion NAA_2 (No Asymptotic Arbitrage of the 2nd Kind). The P-boundedness of $\mathcal{X}^1_T$ was discussed in [9] (as the BK-property), in the framework of a model given by value processes; however, it was overlooked that it coincides with NAA_1 for the "stationary" model. This condition appeared under the acronym NUPBR in [11], and was shown to be equivalent to NA_1 in [12].

A.2 NA_1 and NFLVR

Remaining in the framework of the abstract model of the previous subsection, we provide here results on the relation of condition NA_1 with other fundamental notions of the arbitrage theory, cf. with [9].

Define the convex sets $C := (\mathcal{X}_T - L^0_+) \cap L^\infty$ and denote by $\bar{C}$ and $\bar{C}^*$, the norm-closure and weak*-closure of C in L^∞, respectively. Conditions NA, NFLVR, and NFL are defined correspondingly via

$$C \cap L^\infty_+ = \{0\}, \qquad \bar{C} \cap L^\infty_+ = \{0\}, \qquad \bar{C}^* \cap L^\infty_+ = \{0\}.$$

Consecutive inclusions induce the hierarchy of these properties:

$$\begin{array}{ccccc} C & \subseteq & \bar{C} & \subseteq & \bar{C}^* \\ NA & \Leftarrow & NFLVR & \Leftarrow & NFL. \end{array}$$

Lemma A.2 *NFLVR* $\Rightarrow$ *NA* & NA_1.

Proof Assume that NFLVR holds. Condition NA follows trivially. If NA_1 fails, then there exists $[0,1]$-valued $\xi \in L^0_+ \setminus \{0\}$ such that for each $n \geq 1$ one can find $X^n \in \mathcal{X}^{1/n}$ with $1/n + X^n_T \geq \xi$. Then the random variables $X^n_T \wedge \xi$ belong to C and converge uniformly to ξ, contradicting NFLVR. □

To obtain the converse implication in Lemma A.2, we need an extra property. We call a model *natural* if the elements of $\mathcal{X}$ are adapted processes and for any $X \in \mathcal{X}$, $s \in [0, T[$, and $\Gamma \in \mathcal{F}_s$ the process $\tilde{X} := I_{\Gamma \cap \{X_s \leq 0\}} I_{[s,T]}(X - X_s)$ is an element of $\mathcal{X}$. In words, a model is natural if an investor deciding to start trading at time s when the event Γ happened, can use from this time, if $X_s \leq 0$, the investment strategy that leads to the value process with the same increments as X.

Lemma A.3 *Suppose that the model is natural. If NA holds, then any* $X \in \mathcal{X}$ *admits the bound* $X \geq -\lambda$ *where* $\lambda = \|X^-_T\|_\infty$.

Proof If $P(X_s < -\lambda) > 0$, then $\tilde{X} := I_{\{X_s < -\lambda\}} I_{[s,T]}(X - X_s)$ belongs to $\mathcal{X}$, the random variable $\tilde{X}_T \geq 0$ and $P(\tilde{X}_T > 0) > 0$ in violation of NA. □

Proposition A.4 *Suppose that the model is natural, and, additionally, for every* $n \geq 1$ *and* $X \in \mathcal{X}$ *with* $X \geq -1/n$ *the process* $nX \in \mathcal{X}^1$. *Then, NFLVR* $\Leftrightarrow$ *NA* & NA_1.

Proof By Lemma A.2, we only have to show the implication $\Leftarrow$. If NFLVR fails, there are $\xi_n \in C$ and $\xi \in L^\infty_+ \setminus \{0\}$ such that $\|\xi_n - \xi\|_\infty \leq n^{-1}$. By definition, $\xi_n \leq \eta_n = X^n_T$ where $X^n \in \mathcal{X}$. Obviously, $\|\eta^-_n\|_\infty \leq n^{-1}$ and, since NA holds, $nX^n \in \mathcal{X}^1$ in virtue of Lemma A.3 and our hypothesis. By the von Weizsäcker theorem, we may assume that $\eta_n \to \eta$ a.s. Since $P(\eta > 0) > 0$, the sequence $nX^n_T \in \mathcal{X}^1_T$ tends to infinity with strictly positive probability, violating condition NUPBR, or, equivalently, NA_1. □

Examples showing that the conditions NFLVR, NA, and NA_1 are all different, even for the standard model (satisfying, of course, the hypotheses of the above proposition) can be found in [7].

Assume now that $\mathcal{X}^1$ is a subset of the space of semimartingales $\mathcal{S}$, equipped with the Emery topology given by the quasinorm

$$\mathbf{D}(X) := \sup\{E1 \wedge |H \cdot X_T| : \ H \text{ is predictable}, \ \ |H| \le 1\}.$$

Define the condition ESM as the existence of probability $\tilde{P} \sim P$ such that $\tilde{E}X_T \le 0$ for all processes $X \in \mathcal{X}$. A probability $\tilde{P}$ with such property is referred to as *equivalent separating measure*. According to the Kreps–Yan separation theorem, conditions NFL and ESM are equivalent. The next result is proven in [9] on the basis of paper [5] where this theorem was established for the "standard" model; see also [3].

Theorem A.3 *Suppose that $\mathcal{X}^1$ is closed in $\mathcal{S}$, and that the following concatenation property holds: for any $X, X' \in \mathcal{X}^1$ and any bounded predictable processes $H, G \ge 0$ such that $HG = 0$ the process $\tilde{X} := H \cdot X + G \cdot X'$ belongs to $\mathcal{X}^1$ if it satisfies the inequality $\tilde{X} \ge -1$.*

Then, under condition NFLVR it holds that $C = \bar{C}^*$ and, as a corollary, we have

$$NFLVR \Leftrightarrow NFL \Leftrightarrow ESM.$$

Remark A.4 It is shown in [14, Theorem 1.7] that the condition NA_1 is equivalent to the existence of a supermartingale numéraire in a setting where wealth-process sets are abstractly defined via a requirement of predictable convexity (also called fork-convexity).

In the case of the "standard" model with a finite-dimensional semimartingale S describing the price of the basic securities we have the following: if S is bounded (resp., locally bounded), a separating measure is a martingale measure (resp., local martingale measure). Without any local boundedness assumption on S, we have the following result from [6], a short proof of which is given in [9] and which we use here:

Theorem A.5 In any neighborhood in total variation of a separating measure there exists an equivalent probability under which S is a σ-martingale.

It follows that, if NFLVR holds, the process S is a σ-martingale with respect to some probability measure $P' \sim P$ with density process Z'. Therefore, for any process $X \equiv H \cdot S$ from $\mathcal{X}^1$, the process $1 + X$ is a local martingale with respect to P', or equivalently, $Z'(1 + X)$ is a local martingale with respect to P; therefore, Z' is a local martingale deflator.

Remark A.6 A counterexample in [4, Section 6] involving a simple one-step model shows that Theorem A.5 is not valid in markets with countably many assets. As a corollary, condition NFLVR (equivalent in this case to NA_1) is not sufficient to ensure the existence of a local martingale measure, or a local martingale deflator.

Acknowledgements The research of Yuri Kabanov is funded by the grant of the Government of Russian Federation n°14.12.31.0007. The research of Constantinos Kardaras is partially funded by the MC grant FP7-PEOPLE-2012-CIG, 334540. The authors would like to thank three anonymous referees for their helpful remarks and suggestions.

References

1. Ansel J.-P., Stricker C. Couverture des actifs contingents et prix maximum. *Ann. Inst. H. Poincaré Probab. Statist.*, volume 30, no 2 (1994), 303–315.
2. Choulli T., Stricker C. Deux applications de Galtchouk–Kunita–Watanabe decomposition. *Séminaire de Probabilités, XXX*, Lect. Notes Math., **1626** (1996), 12–23.
3. Cuchiero C., Teichmann J. A convergence result for the Emery topology and a variant of the proof of the fundamental theorem of asset pricing. *Finance and Stochastics*, **19** (2015), 4, 743–761.
4. Cuchiero, C., Klein, I., Teichmann, J. A new perspective on the fundamental theorem of asset pricing for large financial markets. *Theory of Probability and Its Applications*, **60** (2015), 4, 660-685.
5. Delbaen F., Schachermayer W. A general version of the fundamental theorem of asset pricing. *Math. Annalen*, **300** (1994), 463–520.
6. Delbaen F., Schachermayer W. The Fundamental Theorem of Asset Pricing for unbounded stochastic processes. *Math. Annalen*, **312** (1998), 215–250.
7. Herdegen M., Herrmann S. A class of strict local martingales. *Swiss Finance Institute Research Paper,* No. 14-18 (2014).
8. Jacod J., Shiryaev A.N. *Limit Theorems for Stochastic Processes.* Second edition, Springer, 2010.
9. Kabanov Yu.M. On the FTAP of Kreps–Delbaen–Schachermayer. *Statistics and Control of Random Processes. The Liptser Festschrift. Proceedings of Steklov Mathematical Institute Seminar*, World Scientific, Singapore, 1997, 191–203.
10. Kabanov Yu. M., Kramkov D. O. Large financial markets: asymptotic arbitrage and contiguity. *Probab. Theory and Its Applications*, **39** (1994) 1, 222–229.
11. Karatzas I., Kardaras C. The numéraire portfolio in semimartingale financial models. *Finance and Stochastics*, **9** (2007), 4, 447–493.
12. Kardaras C. Finitely additive probabilities and the fundamental theorem of asset pricing. In: *Contemporary Quantitative Finance. Essays in Honour of Eckhard Platen*, 1934, Springer, 2010.
13. Kardaras C. Market viability via absence of arbitrages of the first kind. *Finance and Stochastics*, **16** (2012), 4, 651–667.
14. Kardaras C. On the closure in the Emery topology of semimartingale wealth-process sets. *Ann. Appl. Probab.*, **23** (2013), 4, 1355–1376.
15. Stricker C., Yan J.A. Some remarks on the optional decomposition theorem. *Séminaire de Probabilités, XXXII*, Lecture Notes Math., **1686** (1998), 56–66.
16. Takaoka K., Schweizer M. A note on the condition of no unbounded profit with bounded risk. *Finance and Stochastics*. **18** (2014), 2, 393–405.
17. von Weizsäcker H. Can one drop L^1-boundedness in Komlós subsequence theorem? *Amer. Math. Monthly*, **111** (2004), 10, 900–903.