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ON THE KRULL INTERSECTION THEOREM IN FUNCTION ALGEBRAS

RAYMOND MORTINI, RUDOLF RUPP, AND AMOL SASANE

Abstract. A version of the Krull Intersection Theorem states that for Noetherian domains the *Krull intersection* $ki(I)$ of every proper ideal I is trivial; that is

$$ki(I) := \bigcap_{n=1}^{\infty} I^n = \{0\}.$$

We investigate the validity of this result for various function algebras R, present ideals I of R for which $ki(I) \neq \{0\}$, and give conditions on I so that $ki(I) = \{0\}$.

1. Introduction

The aim of this note is to investigate the validity of the Krull Intersection Theorem in various function algebras. We begin by recalling the following version of the Krull Intersection Theorem [5, Corollary 5.4, p.152]. See also [19] for a simple proof. As usual, given an ideal I, I^n is the ideal of all elements of the form

$$\sum_{i=1}^{m} a_{1,i} \cdots a_{n,i}, \quad m \in \mathbb{N},\ a_{k,i} \in I.$$

Proposition 1.1 (Krull Intersection Theorem)**.** *If R is a Noetherian integral domain, and I a proper ideal of R, that is $I \subsetneq R$, then the* Krull intersection $ki(I)$ *of I, defined by*

$$ki(I) := \bigcap_{n=1}^{\infty} I^n,$$

is trivial, that is, $ki(I) = \{0\}$.

We note that neither of the assumptions on R can be dropped. Here are some examples.

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Example 1.2 (Not Noetherian, and not an integral domain)**.** This is based on [5, p.153]. Let $R = C^\infty(\mathbb{R})$, the ring of all infinitely differentiable real-valued functions on $\mathbb{R}$. Then R is not Noetherian (since

$$I_n := \{f \in C^\infty(\mathbb{R}) : f(x) = 0 \text{ for } x > n\}, \quad n \in \mathbb{N} := \{1, 2, 3, \cdots\},$$

form an ascending chain of ideals) and is not an integral domain. Let I be the ideal $\langle x \rangle$ generated by $x \mapsto x$. Let

$$f(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in C^\infty(\mathbb{R})$. For $n \in \mathbb{N}$, set

$$f_n(x) = \begin{cases} f(x)/x^n & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \in C^\infty(\mathbb{R})$ too, and so $f = f_n x^n \in I^n$. So we have $0 \neq f \in ki(I)$. $\diamond$

Example 1.3 (Not Noetherian, but an integral domain)**.** Let

$$R = H(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C} : f \text{ is entire}\}.$$

Denote the zero-set of a function $f \in H(\mathbb{C})$ by

$$Z(f) = \{z \in \mathbb{C} : f(z) = 0\}.$$

If $z_0 \in Z(f)$, let $\text{ord}(z_0, f)$ be the order of z_0 as a zero of f. Define

$$I := \{0\} \cup \left\{ f \in H(\mathbb{C}) \middle| \begin{array}{l} \exists N \in \mathbb{N}, \forall n \in \mathbb{N} : \text{if } n > N, \text{ then } f(n) = 0 \\ \text{and } \lim\limits_{n \to \infty} \text{ord}(n, f) = \infty \end{array} \right\}.$$

Then it can be seen that I is an ideal. We will show that $I^2 = I$. To this end, let $0 \neq f \in I$. Let $f_1 \in H(\mathbb{C})$ be an entire function with $Z(f_1) = Z(f) \cap \mathbb{N}$, but such that for each $n \in Z(f_1)$,

$$\text{ord}(n, f_1) := \max \left\{ 1, \left\lfloor \frac{\text{ord}(n, f)}{2} \right\rfloor \right\}.$$

Here for $x \in \mathbb{R}$, the notation $\lfloor x \rfloor$ stands for the largest integer $\leq x$. Then $f_1 \in I$. Set $f_2 = f/f_1$. Then $f_2 \in I$ as well. Finally, $f = f_1 \cdot f_2 \in I \cdot I = I^2$. $\diamond$

Example 1.4 (Noetherian, but not an integral domain)**.** The ring $\mathbb{C}[z]$ is Noetherian, and so it follows that $R := \mathbb{C}[z]/\langle z^2 - z \rangle$ is Noetherian too. But R is not an integral domain because $[z][z-1] = [0]$. With $I := \langle [z] \rangle$, the ideal generated by $[z]$ in R, we see that $I^2 = I$, because $[z]$ is an idempotent, and so $I = \langle [z] \rangle = \langle [z]^2 \rangle \subseteq I^2 \subseteq I$. $\diamond$

Example 1.5 (Sequence spaces)**.** Consider the sequence algebras

$$\begin{aligned} \ell^2 &:= \Big\{(a_n)_{n\in\mathbb{Z}} \;\Big|\; \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty\Big\}, \\ \ell^\infty &:= \Big\{(a_n)_{n\in\mathbb{Z}} \;\Big|\; \sup_{n\in\mathbb{Z}} |a_n| < \infty\Big\}, \\ s'(\mathbb{Z}) &:= \Big\{(a_n)_{n\in\mathbb{Z}} \;\Big|\; \exists M>0,\ \exists m>0,\ \forall n\in\mathbb{Z}:\ |a_n| \le M(1+|n|)^m\Big\}, \end{aligned}$$

endowed with termwise addition, termwise scalar multiplication, and termwise (Hadamard) multiplication. Then for any of the above algebras R, $I := c_{00}$, the set of all sequences with compact support, is a proper ideal in R. If $a := (a_1, \cdots, a_N, 0, \cdots) \in c_{00}$, then with b_n any complex number such that $b_n^2 = a_n$, $n = 1, \cdots, N$, and with $b := (b_1, \cdots, b_N, 0, \cdots) \in c_{00} = I$, we have that $I \ni a = b \cdot b \in I^2$. So $I^2 = I$, and hence $ki(I) = I \neq \{0\}$. We remark that $\ell^2(\mathbb{Z})$ with the termwise operations is isomorphic to $L^2(\mathbb{T})$ with convolution, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, and $s'(\mathbb{Z})$ with termwise operations is isomorphic to the algebra of periodic distributions $\mathcal{D}'(\mathbb{T})$ with convolution. $\diamond$

Here is an example of a non-Noetherian ring for which $\bigcap_{n=1}^\infty I^n = \{0\}$ for every proper ideal I. (This example is included in [1, Theorem 4], but we offer an elementary direct proof below.)

Example 1.6 (Non-Noetherian, but $\displaystyle\bigcap_{n=1}^{\infty} I^n = \{0\}$ for all ideals $I \subsetneq R$)**.** Let I be a proper ideal in $R := \mathbb{C}[z_1, z_2, z_3, \cdots]$. Then I is contained in some maximal ideal M of R. But then $ki(I) \subseteq ki(M)$. We will show that the maximal ideals M of R are just of the form $\langle z_n - \zeta_n : n \in \mathbb{N}\rangle$ for some sequence of complex numbers $\zeta_1, \zeta_2, \zeta_3, \cdots$. Then we will use this structure to show $ki(M) = \{0\}$, and hence we can conclude that also $ki(I) = \{0\}$ for every proper ideal I in R.

Claim: M is a maximal ideal in $\mathbb{C}[z_1, z_2, z_3, \cdots]$ if and only if there exists a sequence $\zeta_1, \zeta_2, \zeta_3, \cdots$ of complex numbers such that

$$M = \langle z_n - \zeta_n : n \in \mathbb{N}\rangle.$$

(This result is known; see [15]. Nevertheless, we include an elementary self-contained proof, fashioned along the same lines as the proof of Hilbert's Nullstellensatz; see [22].)

If $\boldsymbol{\zeta} = (\zeta_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers, then we first observe that the ideal $M_{\boldsymbol{\zeta}} := \langle z_n - \zeta_n : n \in \mathbb{N}\rangle$ is maximal as follows. We can look at the evaluation homomorphism $\varphi_{\boldsymbol{\zeta}}$ from $R := \mathbb{C}[z_1, z_2, z_3, \cdots]$ to $\mathbb{C}$ sending for every $n \in \mathbb{N}$ the indeterminate z_n to ζ_n. Then $\varphi_{\boldsymbol{\zeta}}$ is surjective, and $\ker \varphi_{\boldsymbol{\zeta}} = \{p \in R : p(\boldsymbol{\zeta}) = 0\}$. But as $p(\boldsymbol{\zeta}) = 0$, it follows from the Taylor series centered at $\boldsymbol{\zeta}$ for p that p belongs to $M_{\boldsymbol{\zeta}}$. Hence by the isomorphism theorem $R/M_{\boldsymbol{\zeta}}$ is isomorphic to $\mathbb{C}$, and thus $M_{\boldsymbol{\zeta}}$ is maximal.

Now suppose that M is maximal. Let $k \in \mathbb{N}$, and consider the ring homomorphism

$$\begin{array}{rcl} \varphi_k : \mathbb{C}[z_k] & \longrightarrow & \mathbb{C}[z_1, z_2, z_3, \cdots]/M =: \mathbb{F} \\ p & \longmapsto & [p] := p + M. \end{array}$$

If $\varphi_k(pq) = 0$, then since $\mathbb{F}$ is a field, either $\varphi_k(p) = 0$ or $\varphi_k(q) = 0$. Hence $\ker \varphi_k$ is a prime ideal of $\mathbb{C}[x_k]$. We will show first that $\ker \varphi_k \neq \{0\}$.

Suppose that $\ker \varphi_k = \{0\}$. Then $\varphi_k : \mathbb{C}[z_k] \to \mathbb{F}$ is an injective map. Thus there exists an extension of φ_k to $\Phi_k : \mathbb{C}(z_k) \to \mathbb{F}$, namely

$$\Phi_k\Big(\Big[\frac{p}{q}\Big]\Big) = \frac{\varphi_k(p)}{\varphi_k(q)}, \quad \Big[\frac{p}{q}\Big] \in \mathbb{C}(z_k),\ p \in \mathbb{C}[z_k],\ q \in \mathbb{C}[z_k] \setminus \{0\}.$$

It is straightforward to check that Φ_k is an injective homomorphism. Now $\mathbb{F}$ is a $\mathbb{C}$-vector space which is spanned by a countable number of elements: namely $m + M$, where m is any monomial in $\mathbb{C}[z_1, z_2, z_3, \cdots]$. Hence its subspace, namely $\Phi_k(\mathbb{C}(z_k))$ is also spanned by a countable sequence of vectors, say $\{v_n : n \in \mathbb{N}\}$. As these v_n are in $\Phi_k(\mathbb{C}(z_k))$, there exist $r_n \in \mathbb{C}(z_k)$ such that $\Phi_k(r_n) = v_n$. But then $\mathbb{C}(z_k)$ will be spanned by the r_n: indeed, if $r \in \mathbb{C}(z_k)$, then $\Phi_k(r) \in \mathbb{C}(z_k)$ and so there exist $\alpha_1, \cdots, \alpha_m \in \mathbb{C}$ such that $\Phi_k(r) = \alpha_1 v_{\ell_1} + \cdots + \alpha_m v_{\ell_m} = \Phi_k(\alpha_1 r_{\ell_1} + \cdots + \alpha_m r_{\ell_m})$, and so $r = \alpha_1 r_{\ell_1} + \cdots + \alpha_m r_{\ell_m}$, thanks to the injectivity of Φ_k. So the $\mathbb{C}$-vector space $\mathbb{C}(z_k)$ is also spanned by a countable number of vectors. However, it is easy to see that

$$\left\{ z_k \mapsto \frac{1}{z_k - \zeta} : \zeta \in \mathbb{C} \right\},$$

is an uncountable linear independent set in $\mathbb{C}(z_k)$, a contradiction. Hence $\ker \varphi_k \neq \{0\}$. The kernel of φ_k is proper, because $\varphi_k(1) = [1]$ (note that due to $\varphi_k(1) = \varphi_k(1^2)$, the only other possibility would be $\varphi_k(1) = [0]$, giving $1 + M = [0]$; a contradiction). Since $\ker \varphi_k$ is a nonzero proper prime ideal in $\mathbb{C}[x_k]$, it follows that there is a $\zeta_k \in \mathbb{C}$ such that $z_k - \zeta_k \in \ker \varphi_k$. Hence $z_k - \zeta_k \in M$.

As the choice of $k \in \mathbb{N}$ was arbitrary, we get a sequence $\zeta_1, \zeta_2, \zeta_3, \cdots$ of complex numbers such that $z_n - \zeta_n \in M$, and so $\langle z_n - \zeta_n : n \in \mathbb{N} \rangle \subseteq M$. But $\langle z_n - \zeta_n : n \in \mathbb{N} \rangle$ is maximal. Thus $M = \langle z_n - \zeta_n : n \in \mathbb{N} \rangle$. This completes the proof of the claim.

Let $M = \langle z_n - \zeta_n : n \in \mathbb{N} \rangle$. Suppose that $0 \neq f \in ki(M)$, and let k be such that $f \in \mathbb{C}[z_1, \cdots, z_k]$. Let $\Pi_k : \mathbb{C}[z_1, z_2, z_3, \cdots] \to \mathbb{C}[z_1, \cdots, z_k]$ be the evaluation homomorphism that sends z_n to ζ_n for $n > k$, and z_n to z_n itself if $n \leq k$. Then

$$f \in \bigcap_{n=1}^{\infty} \big(\Pi_k(M)\big)^n.$$

Since $\Pi_k(M)$ is just the maximal ideal $\langle z_n - \zeta_n : 0 \leq n \leq k \rangle$ in the ring $\mathbb{C}[z_1, \cdots, z_k]$, we conclude from the Krull Intersection Theorem (Proposition 1.1) applied to the Noetherian integral domain $\mathbb{C}[z_1, \cdots, z_k]$, that $f = 0$. Consequently, $ki(M) = \{0\}$. $\diamond$

In this rest of this article, we will investigate $ki(I)$ for (mainly maximal) ideals I in algebras of (mainly holomorphic) functions. The organization of the subsequent sections is as follows:

(1) In Section 2, we will determine $ki(I)$ for certain ideals in the algebra $H(D)$ of holomorphic functions in a domain $D \subseteq \mathbb{C}$.
(2) In Section 3, we will determine $ki(I)$ for certain ideals in uniform algebras.
(3) In Section 4, we will determine $ki(I)$ for certain ideals in the algebra $H^\infty(\mathbb{D})$ of bounded holomorphic functions in the unit disk $\mathbb{D}$.

2. $ki(I)$ for ideals I in $H(D)$

Example 1.3 above can be generalized to the following.

Example 2.1. Let $D \subseteq \mathbb{C}$ be a domain (that is, an open path-connected set). Let $R = H(D)$, the algebra of holomorphic functions in D with pointwise operations. Then there exists a proper ideal I of R such that $ki(I) \neq \{0\}$. We construct such an ideal I as follows. Let $(\zeta_n)_{n\in\mathbb{N}}$ be any sequence in D that converges to a point in the boundary ∂D of D (or more generally, without accumulation points in D). Let h be any Weierstrass product with simple zeros at ζ_n, $n \in \mathbb{N}$. Consider the proper ideal I of R generated by the functions f_n, $n \in \mathbb{N}$, given by

$$f_n(z) := \frac{h(z)}{(z-\zeta_1)\cdots(z-\zeta_n)}, \quad z \in D.$$

Let $g \in R \setminus \{0\}$ be a Weierstrass product which vanishes exactly at ζ_n of order $\mathrm{ord}(g, \zeta_n) = n$, for each $n \in \mathbb{N}$. We claim that $g \in I^n$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}$, set

$$q_n := \prod_{k=1}^{n} (z-\zeta_k)^k.$$

Then $G := g/q_n$ has the zero set $Z_D(G) = \{\zeta_{n+1}, \zeta_{n+2}, \cdots\}$, with orders of zeros ζ_k given by $\mathrm{ord}(G, \zeta_k) = k$, for $k > n$. Again by the Weierstrass' Factorization Theorem, we must have that f_n^n divides G in $H(D)$, and hence there exist $h_n \in H(D)$ such that $g = q_n G = q_n h_n f_n^n \in I^n$. Since $n \in \mathbb{N}$ was arbitrary, $g \in ki(I)$. ◇

On the other hand, we have the following result saying that for *non-free/fixed* ideals I of $H(D)$, $ki(I) = \{0\}$.

Definition 2.2 (Free ideals in $H(D)$)**.** Let $D \subseteq \mathbb{C}$ be a domain. For an element $f \in H(D)$, let $Z_D(f)$ denote the set of zeros of f. An ideal I in $H(D)$ is called *free* if the *zero set* of I,

$$Z_D(I) := \bigcap_{f\in I} Z_D(f)$$

is the empty set $\emptyset$, and *fixed/non-free* otherwise.

Proposition 2.3. *For a proper fixed ideal I of $H(D)$, $ki(I) = \{0\}$.*

Proof. Suppose that $\zeta \in Z_D(I)$. Let

$$m := \min\{\operatorname{ord}(f, \zeta) : f \in I\}.$$

Then $m \geq 1$ and each function $g \in I^n$ has ζ as a zero of order at least mn. But any holomorphic function belonging to I^n for all $n \in \mathbb{N}$ must therefore be identically zero since D is a domain. □

For maximal ideals of $H(D)$, one can say more, and we have the following results given in Theorem 2.5. But first we proof a helpful lemma, which will be used in the proof of Theorem 2.5 (and also in the subsequent result).

Lemma 2.4. *Let D be a domain, and M be a maximal ideal in the ring $H(D)$. If $f \in M$, and $h \in H(D)$ is such that $Z_D(f) \subseteq Z_D(h)$ (disregarding multiplicities), then $h \in M$ too.*

Proof. Suppose that h is not in M. Then the ideal $\langle h \rangle + M$, which strictly contains M, must be $H(D)$, thanks to the maximality of M. Thus there exists $m \in M$ and $g \in H(D)$ such that $1 = gh + m$. Hence we have that $Z_D(h) \cap Z_D(m) = \emptyset$. Thus $Z_D(f) \cap Z_D(m) = \emptyset$. By the Nullstellensatz for $H(D)$, it follows that there exist $u, v \in H(D)$ such that $1 = uf + vm \in M$. This is absurd, because $m, f \in M$ and M is proper. □

Theorem 2.5. *Let D be a domain in $\mathbb{C}$, and M be a free maximal ideal of $H(D)$. Then*

(1) $\{0\} \cup \left\{ f \in M : f \neq 0 \text{ and } \lim\limits_{\zeta \in Z_D(f)} \operatorname{ord}(f, \zeta) = \infty \right\} \subset ki(M)$.

(2) $ki(M) \subseteq \{0\} \cup \left\{ f \in M : f \neq 0 \text{ and } \sup\limits_{\zeta \in Z_D(f)} \operatorname{ord}(f, \zeta) = \infty \right\}$.

Hence $ki(M) = \{0\}$ if and only if there exists a $\zeta \in D$ such that $M = \langle z - \zeta \rangle$.

Here, by assumption that

$$\lim_{\zeta \in Z_D(f)} \operatorname{ord}(f, \zeta) = \infty,$$

we mean that given any $n > 0$, there exists a finite set $K \subset Z_D(f)$ such that $\operatorname{ord}(f, \zeta) > n$ for all $\zeta \in Z_D(f) \setminus K$.

Proof. (1) First we observe that M contains no polynomial. (Otherwise, if a polynomial $p \in M$, it follows, by using the fact that M is in particular prime, that M contains a linear factor $z - w$ of p. But then we have that $M \subseteq M_w := \{f \in H(D) : f(w) = 0\}$. Since the later ideal is proper, and M is maximal, $M = M_w$ would be a fixed ideal.)

Let $f \in M \setminus \{0\}$ with

$$\lim_{\zeta \in Z_D(f)} \operatorname{ord}(f, \zeta) = \infty.$$

Suppose that $n \in \mathbb{N}$. Then it is possible to factorize f as $f = f_n p$, where p is a polynomial and the orders of all zeros of f_n are at least n. By the primeness of (the maximal!) ideal M, and the fact that M contains no polynomials, it follows that $f_n \in M$ too. But now we can write $f_n = g_1 \cdots g_n$, where each of the functions g_k have the same zero set (disregarding multiplicities). Again the primeness of M, and Lemma 2.4, allow us to conclude that all the g_k belong to M. Hence $f \in M^n$. As the choice of $n \in \mathbb{N}$ was arbitrary, we obtain that $f \in ki(M)$.

(2) Assume that $f \in ki(M)$ and let $n \in \mathbb{N}$. Then f can be decomposed into a finite sum of the form

$$f = \sum_{k=1}^{N} f_{1,k} \cdots f_{n,k},$$

with each $f_{j,k} \in M$. All these functions $f_{j,k} \in M$ must have a common zero, since otherwise (by the Nullstellensatz for $H(D)$), we can generate 1 in M, a contradiction to the fact that M is proper. But then the order of this common zero of f must be at least n. As the choice of $n \in \mathbb{N}$ was arbitrary, it follows that $\sup\limits_{\zeta \in Z_D(f)} \operatorname{ord}(f, \zeta) = \infty$. □

We remark that a somewhat different characterization of $ki(M)$ was provided in [11, Theorem 3,p.714] for the algebra $H(\mathbb{C})$ of entire functions. We extend Henriksen's result to domains, and then compare our result above with his result below. Since Henriksen's proof was, in our viewpoint, very condensed, we provide all details in the more general case. Since prime ideals appear very naturally in the description of $ki(M)$, we include a nice property shared by this class of ideals. Also that result is known; [11, Theorem 1].

Given $f \in H(D)$, let

$$\mathbf{o}(f) := \sup_{\zeta \in Z_D(f)} \operatorname{ord}(f, \zeta).$$

If $q \equiv 0$, we set $\mathbf{o}(q) := \infty$.

Lemma 2.6. *Let P be a prime ideal in $H(D)$, where D is a domain in $\mathbb{C}$. Then P is non-maximal if and only if $\mathbf{o}(f) = \infty$ for every $f \in P$.*

Proof. The only if direction is a direct consequence of Lemma 2.4. So suppose that P is prime and contains an element f with $N := \mathbf{o}(f) < \infty$. Let M be a maximal ideal with $P \subseteq M$. We show that $P = M$. Write $f = f_1 \dots f_N$, where each zero of any f_j is simple. Since P is prime, at least one of the N factors belongs to P. Say it is f_1. Fix $g \in M$ and let $d = \gcd(f_1, g)$. Then, by Wedderburn's Theorem [20, p.119], $d \in \langle f_1, g \rangle \subseteq M$. Now $f_1 = dh$ for some $h \in H(D)$. Since the zeros of f_1 are simple, $Z(d) \cap Z(h) = \emptyset$. Hence, h cannot belong to $P \subseteq M$, because otherwise $H(D) = \langle d, h \rangle \subseteq M$, a contradiction. Thus $d \in P$ and so $g \in P$. Consequently, $M = P$. □

Proposition 2.7 (Henriksen)**.** *Let $D \subseteq \mathbb{C}$ be a domain, and let M be a free, maximal ideal of $H(D)$. Then*

$$ki(M) = \left\{ f \in M \;\middle|\; \begin{array}{l} \text{whenever } d \in H(D) \setminus M \text{is a divisor of } f, \\ \text{say } f = q \cdot d, \text{we have } \ \mathbf{o}(q) = \infty \end{array} \right\}.$$

Moreover $ki(M)$ is the largest nonmaximal prime ideal contained in M.

Proof. Since every finitely generated ideal of $H(D)$ is principal, $ki(M)$ is easily seen to be the set of all $f \in H(D)$ such that for all $n \in \mathbb{N}$, we have a factorization $f = h_n d_n^n$, with $h_n \in H(D)$, $d_n \in M$.

Let

$$K := \left\{ f \in M \,\middle|\, \begin{array}{l} \text{whenever } d \in H(D) \setminus M \text{ is a divisor of } f, \\ \text{say } f = q \cdot d, \text{we have } \ \mathbf{o}(q) = \infty \end{array} \right\}.$$

We first prove that $ki(M) \subseteq K$. Let $f \in ki(M)$ and suppose that d is a divisor of f which does not belong to M. Say $f = dq$. We need to show that $\mathbf{o}(q) = \infty$. If not, then let $n := \mathbf{o}(q)$ and $k = 2n$. Since $f \in ki(M)$, there exists an $h_k \in H(D)$ and a $g_k \in M$ such that $dq = f = h_k g_k^k$. But then every zero of g_k must be a zero of d (disregarding multiplicities) (because each zero of q appears at most n times; on the other hand every zero of g_k appears at least $2n$ times). Thus $Z_D(g_k) \subseteq Z_D(d)$. Since $g_k \in M$, we have $d \in M$ by our Lemma 2.4, a contradiction.

Next we will show that $K \subseteq ki(M)$. Given $f \in K \subseteq M$, and $n \in \mathbb{N}$, we may factor $f \in M$ as $f = f_1 f_2$, where $Z(f_2) = \{\zeta \in Z(f) : \operatorname{ord}(f, \zeta) \geq n+1\}$ and $Z(f_1) = \{\zeta \in Z(f) : \operatorname{ord}(f, \zeta) \leq n\}$. If one of these sets is empty, we just let the associated function equal to be 1.

If $f_2 \notin M$, then we end up with $f_1 \in M$. But the definition of K now implies that $\infty = \mathbf{o}(f_1) \leq n$. Thus in our factorization $M \ni f = f_1 f_2$, we have $f_2 \in M$. Take a function h_n such that we have $Z_D(h_n) = Z_D(f_2)$, and such that h_n has only simple zeros. Then by Lemma 2.4, $h_n \in M$ because $f_2 \in M$. By construction, h_n^n divides f_2, and so $f_2 = g h_n^n$. Summarizing, $f = f_1 f_2 = f_1 g h_n^n \in M^n$. Since $n \in \mathbb{N}$ was arbitrary, it follows that $f \in ki(M)$. This completes the proof that $ki(M) = K$.

Next we show that $ki(M)$ is prime. Assume that $f = f_1 \cdot f_2 \in ki(M) \subseteq M$. Since M is prime, we have one of three possible cases:

1° $f_1 \in M$ and $f_2 \notin M$,
2° $f_1 \notin M$ and $f_2 \in M$,
3° $f_1 \in M$ and $f_2 \in M$.

Case 1°: Let $d \in H(D) \setminus M$ be a divisor of f_1. Say $f_1 = gd$. Then $f = g(df_2)$, where $df_2 \notin M$. Since $ki(M) = K$, we deduce that $g \in ki(M)$. So $f_1 \in ki(M)$. Case 2° works in the same way.

Now only the case left is when both f_1, f_2 are in M. Assuming that neither f_1 nor f_2 belongs to $ki(M)$, we proceed as follows. In this case, there exist d_i dividing f_i, with $d_i \in H(D) \setminus M$, $q_i := f_i/d_i \in M$, and $\mathbf{o}(q_i) < \infty$, $i = 1, 2$. Since M is maximal, and in particular prime, $d_1 d_2 \in H(D) \setminus M$, and $\mathbf{o}(q_1 q_2) \leq \mathbf{o}(q_1) + \mathbf{o}(q_2) < \infty$. So $f_1 f_2 \notin ki(M)$.

Consequently, $ki(M)$ is prime. Finally, we will show the following:

Claim: $ki(M)$ is the largest nonmaximal prime ideal contained in M.

First we show that $ki(M)$ is not maximal. Take any nonzero $f \in M$, and let $h \in H(\mathbb{D})$ be such that $Z(h) = Z(f)$, but $\text{ord}(h, \zeta) = 1$ for all $\zeta \in Z_D(h)$. Then by Lemma 2.4, $h \in M$ too. But with $d := 1 \in H(D) \setminus M$, and $q := h$, we have $f = qd = h \in M$, but $\mathbf{o}(q) = 1 < \infty$. Hence $f = h \notin ki(M)$. Thus $ki(M) \subsetneq M$, and so $ki(M)$ is nonmaximal.

Suppose now that P is a prime ideal such that $ki(M) \subsetneq P \subseteq M$. Let $f \in P \setminus ki(M)$. Then there exists $d \in H(D) \setminus M$ and $q \in M$ such that $f = q \cdot d$ and $\mathbf{o}(q) < \infty$. But as $d \notin M$ and hence not in P either, we have $q \in P$. By Lemma 2.6, $P = M$.

This completes the proof of Proposition 2.7. □

Example 2.8. The aim of this example is to contrast the results from Theorem 2.5 and Proposition 2.7. If we call

$$A := \{0\} \cup \Big\{ f \in M : f \neq 0 \text{ and } \lim_{\zeta \in Z_D(f)} \text{ord}(f, \zeta) = \infty \Big\}, \tag{1}$$

$$B := \{0\} \cup \Big\{ f \in M : f \neq 0 \text{ and } \sup_{\zeta \in Z_D(f)} \text{ord}(f, \zeta) = \infty \Big\}, \tag{2}$$

then in Theorem 2.5 we have shown that $A \subseteq ki(M) \subseteq B$ whenever M is a maximal free ideal in $H(D)$. We will show that

(1) there exists an element $f \in B \setminus ki(M)$, showing that $B \neq ki(M)$;
(2) there exists an element $g \in ki(M) \setminus A$, showing that $A \neq ki(M)$.

To this end, first note that A and B are not ideals. In fact, concerning A, just consider $f \in A$ and multiply f by a function with simple zeros outside $Z(f)$. Concerning B, let $f \in M$ have simple zeros (for the existence, see Lemma 2.4). Now let g_1, g_2 be in $H(D)$ with $Z(g_1) \cap Z(g_2) = \emptyset$ and $\mathbf{o}(g_j) = \infty$. Choose $a_j \in H(D)$ so that $1 = a_1 g + a_2 g_2$. Then $f a_j g_j \in B$, but

$$f a_1 g_1 + f a_2 g_2 = f \notin B.$$

Hence we conclude that $A \subset ki(M) \subset B$, the inclusions being strict.

3. Sufficient conditions for $ki(I) = I$ in uniform algebras

We recall the definition of a uniform algebra.

Definition 3.1 (Uniform algebra)**.** R is called a *uniform algebra on* X if

(1) X is a compact topological space,
(2) $R \subseteq C(X; \mathbb{C})$, the algebra of complex-valued continuous functions on X, and R separates the points of X, that is, for every $x, y \in X$ with $x \neq y$, there exists $f \in R$ such that $f(x) \neq f(y)$,
(3) the constant function $1 \in R$,
(4) R is a closed subalgebra of $C(X; \mathbb{C})$, where the latter is endowed with the usual supremum norm $\| \cdot \|_\infty$.

We also recall below the following two well-known results from the theory of uniform algebras; see [2, Lemma 1.6.3, p.72-73 and Theorem 1.6.5, p.74].

Both of these results involve the notion of an approximate identity, given below.

Definition 3.2 (Approximate identity)**.** Let R be a commutative unital Banach algebra, and M be a maximal ideal of R. We say that M has an *approximate identity* if there exists a constant K such that for every $\epsilon > 0$, and every $f_1, \cdots, f_n \in M$, there exists an $e \in M$, $\|e\| \leq K$, such that $\|ef_i - f_i\| < \epsilon$ for all $i = 1, \cdots, n$. (In other words, there exists a bounded net (e_α) in M such that $e_\alpha f \to f$ for every $f \in M$.)

Proposition 3.3. *Let R be a uniform algebra on X, and let $x \in X$. Then the following are equivalent:*

(1) (*Existence of an approximate identity.*) *The maximal ideal*
$$M := \{f \in R : f(x) = 0\}$$
has an approximate identity.

(2) (*Existence of a weak peak function.*) *There exists a function $f \in R$ with $\|f\| = 1$, $f(x) = 1$, and such that for every neighbourhood U of x, we have $|f(y)| < 1$ for all $y \in X \setminus U$.*

(3) *There exists a constant K, such that for every neighbourhood U of x, and every $\epsilon > 0$, there exists an $f \in R$ with $\|f\| < K$, $f(x) = 1$, and $|f(y)| < \epsilon$ for all $y \in X \setminus U$.*

In (2), the point x is referred to as a *weak peak point*.

Proposition 3.4 (Cohen Factorization Theorem)**.** *Let R be a commutative unital Banach algebra, M a maximal ideal of R, and suppose that M has an approximate identity. Then for every $f \in M$, there exist $f_1, f_2 \in M$ such that $f = f_1 f_2$.*

An immediate consequence of these results is the following.

Corollary 3.5. *Let R be a uniform algebra on X, and let $x \in X$. Suppose that x is a weak-peak point. Set $M := \{f \in R : f(x) = 0\}$. Then*
$$ki(M) = M.$$

Proof. Since M is maximal, by the Cohen Factorization Theorem, we have $M^2 = M$. □

Thus, for every uniform algebra R we have "many" ideals M with $M^2 = M$, namely any maximal ideal M of the form $M_x = \{f \in R : f(x) = 0\}$, where x is a weak peak point. See [2, p. 101] and also [4]. We emphasize that the set of weak peak points (sometimes called the *Choquet boundary of R*; see also [2, Definition on p.87 and Theorem 2.3.4]) is dense in the Šilov boundary of R (by [3, Corollary 4.3.7(i)]). We recall here that a closed subset F of X is called a *closed boundary* for R if
$$\sup_{x \in F} |f(x)| = \sup_{x \in X} |f(x)|.$$

The intersection of all the closed boundaries for R is called the *Šilov boundary of* R.

Example 3.6 (Disk algebra and Wiener algebra)**.** As illustrative examples, consider the disk algebra and the Wiener algebra. Let

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},$$

and set

$$\begin{aligned} A(\mathbb{D}) &:= \{f \in H(\mathbb{D}) : f \text{ has a continuous extension to } \mathbb{D} \cup \partial\mathbb{D}\}, \\ W^+(\mathbb{D}) &:= \Big\{f = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) : \|f\|_1 := \sum_{n=0}^{\infty} |a_n| < \infty\Big\}, \end{aligned}$$

with pointwise operations. $A(\mathbb{D})$ is endowed with the sup-norm $\|\cdot\|_\infty$, while $W^+(\mathbb{D})$ is endowed with the $\|\cdot\|_1$-norm defined above. The maximal ideal $M := \{f \in A(\mathbb{D}) : f(1) = 0\}$ has an approximate identity given by the sequence $(1 - p^n)_{n\in\mathbb{N}}$, where p is the peak function given by

$$p := \frac{1+z}{2}, \quad z \in \mathbb{D},$$

(for details of the proof, we refer the reader to [21, Theorem 6.6].)

Let $(r_n)_{n\in\mathbb{N}}$ be any sequence such that $r_n \searrow 1$, and

$$e_n(z) := \frac{z-1}{z-r_n}, \quad n \in \mathbb{N}.$$

Then $(e_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for

$$M := \{f \in W^+(\mathbb{D}) : f(1) = 0\}.$$

A rather lengthy proof of this result in the case when

$$r_n = 1 + \frac{1}{n}, \quad n \in \mathbb{N},$$

can be found in [14], while the result is also mentioned without proof in [6]. A short proof due to the first author of this article can be found in [17] or in [21, Theorem 6.10]. ◇

4. $ki(I)$ FOR IDEALS I IN $H^\infty(\mathbb{D})$

Let $H^\infty(\mathbb{D})$ denote the algebra of all bounded holomorphic functions in $\mathbb{D}$. We sometimes write H^∞ instead of $H^\infty(\mathbb{D})$. The spectrum (or maximal ideal space), $M(H^\infty)$ of H^∞ is the set of nonzero multiplicative linear functionals on H^∞.

Observation 4.1. Let I be an ideal in H^∞. Suppose that I is a non-free ideal; that is,

$$Z_{\mathbb{D}}(I) := \bigcap_{f\in I} Z_{\mathbb{D}}(f) \neq \emptyset.$$

Then $ki(I) = \{0\}$.

Proof. If $Z_{\mathbb{D}}(I) = \mathbb{D}$, then $I = (0)$ and so $ki(I) = \{0\}$. So suppose that there exists an isolated point $z_0 \in Z_{\mathbb{D}}(I)$. Let $f \in ki(I)$ and $n \in \mathbb{N}$ be given. Then $\text{ord}(f, z_0) \geq n$. Hence $f \equiv 0$. Again $ki(I) = \{0\}$. □

A description of the maximal ideals M in H^∞ with $ki(M) = \{0\}$ is already implicit in Kenneth Hoffman's work [12]. Recall that $\widehat{f} \in C(M(H^\infty); \mathbb{C})$ denotes the *Gelfand transform* of $f \in H^\infty$

$$m \xmapsto{\widehat{f}} m(f) =: \widehat{f}(m), \quad m \in M(H^\infty).$$

For $m \in M(H^\infty)$, and $f \in H^\infty$, let us define

$$\text{ord}(f, m) = \text{ord}(\widehat{f} \circ L_m, 0),$$

where $L_m : \mathbb{D} \to P(m)$ is the *Hoffman map* associated with m; that is

$$L_m(z) = \lim \frac{z + z_\alpha}{1 + \overline{z}_\alpha z},$$

where (z_α) is a net in $\mathbb{D}$ converging to m (all limits being taken in the weak-$*$/Gelfand topology of $M(H^\infty)$). Note that $\widehat{f} \circ L_m$ is holomorphic in $\mathbb{D}$. In particular, $\text{ord}(f, m) = \infty$ if and only if $\widehat{f} \equiv 0$ on $P(m)$, where $P(m)$ denotes the Gleason part containing m. Recall the pertinent definitions below.

Definition 4.2 (Pseudohyperbolic distance and the Gleason part)**.** The *pseudohyperbolic distance between two points* $m, \widetilde{m} \in M(H^\infty)$ is defined by

$$\rho(m, \widetilde{m}) := \sup \Big\{ |\widehat{f}(\widetilde{m})| : f \in H^\infty,\ \|f\|_\infty \leq 1,\ \widehat{f}(m) = 0 \Big\}.$$

For $m \in M(H^\infty)$, let

$$P(m) := \{\widetilde{m} \in M(H^\infty) : \rho(m, \widetilde{m}) < 1\}$$

denote the *Gleason part of* $M(H^\infty)$ *containing* m.

By [12],

$$\text{ord}(f, m) = \sup \Big\{ n \in \mathbb{N} : f = f_1 \dots f_n,\ \ \widehat{f_j}(m) = 0 \text{ for all } j = 1, \dots, n \Big\}.$$

Lemma 4.3. *Let I be an ideal in H^∞. Then* $\text{ord}(f, m) = \infty$ *for every $f \in ki(I)$ and $m \in Z(I)$.*

Here $Z(I) := Z_{M(H^\infty)}(I) := \displaystyle\bigcap_{f \in I} \Big\{ m \in M(H^\infty) : \widehat{f}(m) = 0 \Big\}$.

Proof. Let $f \in ki(I)$. Fix $n \in \mathbb{N}$. Then

$$f = \sum_{k=1}^{K} f_{k,1} \dots f_{k,n}$$

for $f_{k,\ell} \in I$. In particular, $\widehat{f_{k,\ell}}(m) = 0$ for every $m \in Z(I)$. Hence $\text{ord}(f, m) \geq n$. Since n was arbitrary, we conclude that $\text{ord}(f, m) = \infty$. □

Theorem 4.4. *Let M be a maximal ideal in H^∞ and $m \in M(H^\infty)$ with $\ker m = M$. Then the following assertions are equivalent:*

(1) $ki(M) = M$.
(2) *M does not contain an interpolating Blaschke product.*
(3) *The Gleason part $P(m)$ containing m is the singleton $\{m\}$.*

In all cases, that is for all maximal ideals in H^∞,

$$ki(M) = I\Big(\overline{P(m)}, H^\infty\Big) := \Big\{f \in H^\infty : \widehat{f} \equiv 0 \ \text{ on } \overline{P(m)}\Big\},$$

where $\overline{E}$ denotes the closure of the set $E \subseteq M(H^\infty)$. In particular, $ki(M)$ is a closed prime ideal, and

$$ki(M) = \{0\} \iff M = M_{z_0} := \{f \in H^\infty : f(z_0) = 0\} \text{ for some } z_0 \in \mathbb{D}.$$

Proof. By [12] (see also [7]), the statements (2) and (3) are equivalent. If (3) holds, then by [12], $M = M^2$ (even in the strict sense: each $f \in M$ can be written as $f = g \cdot h$, where $g, h \in M$). So $ki(M) = M$.

If b is an interpolating Blaschke product contained in M, then $b \notin M^2$, because otherwise

$$b = \sum_{k=1}^{K} f_k g_k,$$

with $f_k, g_k \in M$. Hence $\operatorname{ord}(b, m) \geq 2$, a contradiction; see [12]. This shows the equivalence of (1), (2) and (3).

To prove the rest, we note that

$$\Big\{f \in H^\infty : \widehat{f} \equiv 0 \ \text{ on } \overline{P(m)}\Big\} = \{f \in H^\infty : \operatorname{ord}(f, m) = \infty\}.$$

Moreover, for every $n \in \mathbb{N}$, any such f admits a factorization of the form $f = g_1 \cdots g_n$ with $\widehat{g_k}(m) = 0$. Hence

$$I\Big(\overline{P(m)}, H^\infty\Big) \subseteq ki(M).$$

Conversely, if $f \in ki(M)$, then f is a sum of functions in M each having order at least n at m. Thus $\operatorname{ord}(f, m) = \infty$ for every $f \in ki(M)$ and so $\widehat{f} \equiv 0$ on $P(m)$; see [12]. Thus

$$ki(M) = I\Big(\overline{P(m)}, H^\infty\Big).$$

Since

- $\overline{P(m)}$ is a proper subset of $M(H^\infty)$ if and only if $m \in M(H^\infty) \setminus \mathbb{D}$, and
- $P(z_0) = \mathbb{D}$ for every $z_0 \in \mathbb{D}$,

we conclude that $ki(M) = \{0\}$ if and only if $M = M_{z_0}$.

It is easily seen and well-known that $I\Big(\overline{P(m)}, H^\infty\Big)$ is a closed prime ideal. □

Using Izuchi's [13] extensions of Hoffman's factorization theorems, we also obtain the following result:

Proposition 4.5. *Let $E \subseteq M(H^\infty)$ be a hull-kernel closed subset of $M(H^\infty)$, that is, E is the zero-set of the ideal*

$$I(E, H^\infty) = \Big\{ f \in H^\infty : \widehat{f}|_E \equiv 0 \Big\}.$$

Suppose that E is a union of Gleason parts. Then

$$ki\big(I(E, H^\infty)\big) = I(E, H^\infty).$$

Proof. Let $f \in I(E, H^\infty)$ and $n \in \mathbb{N}$. Since we have that $\operatorname{ord}(f, m) = \infty$ for every $m \in E$ (because, by hypothesis, $m \in E$ implies $P(m) \subseteq E$), it follows that $f \in I(E, H^\infty)$ has a factorization $f = f_1 \cdots f_n$, with $f_k \in I(E, H^\infty)$; see [13]. Conversely, if $f \in ki(I(E, H^\infty))$, then $\operatorname{ord}(f, m) = \infty$ for every $m \in E$. This yields the assertion. □

Corollary 4.6. *Let I be a non-maximal closed prime ideal in H^∞. Then $ki(I) = I$.*

Proof. By [8], every non-maximal closed prime ideal in H^∞ has the form $I = I(E, H^\infty)$, where $E = Z(I)$ is a union of Gleason parts. □

We will now collect a few technical results, which will be used in the proof of Proposition 4.8 below.

Let $g \in H^\infty$ be zero-free. Suppose that $\|g\|_\infty \le 1$. Then there exists a positive measure μ on the unit circle $\mathbb{T}$ such that

$$g(z) = g_\mu(z) := \exp\left(\int_{\mathbb{T}} \frac{z + \xi}{z - \xi}\, d\mu(\xi) \right).$$

If $\xi = e^{it}$, this μ has the form

$$d\mu(\xi) = \log \frac{1}{|g(e^{it})|} dt + d\mu_s(\xi),$$

where μ_s is singular with respect to Lebesgue measure on $\mathbb{T}$.

The following result corresponds to assertion (1.1) in [16, p. 170], given there without proof.

Lemma 4.7. *Let $g = g_\mu \in H^\infty$ be zero-free and suppose that $\|g\|_\infty \le 1$. Then, for every $z \in \mathbb{D}$,*

$$|1 - g(z)| \le \frac{1 + |z|}{1 - |z|}\, \mu(\mathbb{T}).$$

Proof. First we note that for $w \in \mathbb{C}$ with $\operatorname{Re} w \le 0$,

$$|1 - e^w| = \left| \int_{[0,w]} e^\zeta\, d\zeta \right| \le |w|.$$

Since $\operatorname{Re} \dfrac{z + \xi}{z - \xi} = \dfrac{|z|^2 - 1}{|z - \xi|^2} \le 0$ and $\mu \ge 0$, we deduce that

$$|1 - g(z)| \le \left| \int_{\mathbb{T}} \frac{z + \xi}{z - \xi}\, d\mu(\xi) \right| \le \frac{1 + |z|}{1 - |z|} \mu(\mathbb{T}).$$ □

Proposition 4.8. *If P is a prime ideal in H^∞, then $ki(P) = \{0\}$ if and only if P is a maximal ideal of the form M_{z_0} for some $z_0 \in \mathbb{D}$.*

Proof. We have already seen that $ki(M_{z_0}) = \{0\}$. If $Z_{\mathbb{D}}(P) \cap \mathbb{D} \neq \emptyset$, then it easily follows that $P \subseteq (z - z_0)H^\infty$ for all $z_0 \in Z_{\mathbb{D}}(P) \cap \mathbb{D}$. Due to primeness $z - z_0 \in P$ (each $f \in P$ factors as $f = (z - z_0)^n g$, where n is the order of the zero z_0, but then $g \notin P$, so $z - z_0 \in P$), and so $P = M_{z_0}$ again. Now suppose that $Z_{\mathbb{D}}(P) = \emptyset$; that is, P is a free prime ideal. We show that $ki(P)$ contains elements different from the zero function.

Case 1° Suppose that P contains a Blaschke product B, with zero sequence (z_n) (multiplicities included). In particular,

$$\sum_{n=1}^{\infty} (1 - |z_n|^2) < \infty.$$

For each k, choose a tail of the sequence so that

$$\sum_{n=N_k}^{\infty} (1 - |z_n|^2) \leq \frac{1}{2^k}.$$

Let B_k be the Blaschke product associated with these zero sequences. Since B_k differs from B only by finitely many zeros, the freeness of P implies that $B_k \in P$ (otherwise we would have $z - z_0 \in P$, hence $P = M_{z_0}$ again). Since

$$\sum_{k=1}^{\infty} \sum_{j=N_k}^{\infty} (1 - |z_j|^2) < \infty,$$

the collection of all zeros of all B_k is a Blaschke sequence again. Hence, due to absolute convergence of the associated products, any reordering converges again, and so

$$B_* := \prod_{k=1}^{\infty} B_k$$

is a Blaschke product again. Clearly, $B_* \in ki(P)$.

Case 2° Let $Bg \in P$, where g is a zero-free function, and we may assume that $\|g\|_\infty \leq 1$. Either $B \in P$ (and we are done by the first case) or $g \in P$. Since g has roots of any order, we see that $g^{1/n} \in P$ for every n. Choose n_k going to infinity so fast that

$$\sup_{|z| \leq 1 - 1/k} |1 - g^{1/n_k}(z)| < \frac{1}{2^k},$$

which is possible by Lemma 4.7 above. Then the infinite product

$$h = \prod_{k=1}^{\infty} g^{1/n_k}$$

converges locally uniformly to a function $h \in H^\infty$. Clearly, $h \in ki(P)$. $\square$

Remark 4.9. The proof above shows the following:

(1) If I is any free ideal in H^∞ containing a Blaschke product, then $ki(I) \neq \{0\}$.
(2) If I is any free ideal in H^∞ containing a zero-free function g and all of its roots, then $ki(I) \neq \{0\}$.

Let us also remark that there do exist free ideals with $ki(I) = \{0\}$, as demonstrated below.

Observation 4.10. $ki(SH^\infty) = \{0\}$, where S is the atomic inner function

$$S(z) = \exp\Big(-\frac{1+z}{1-z}\Big), \quad z \in \mathbb{D}.$$

Proof. If $f \in ki(SH^\infty)$, $f \not\equiv 0$, then, for every n,

$$f = \sum_{k=1}^{m} \prod_{j=1}^{n} (h_{kj} S) = h_n S^n.$$

In particular, S^n divides the inner factor φ of f for every n, say $\varphi = u_n S^n$ for inner functions u_n. This is impossible though, because S^n goes to zero locally uniformly in $\mathbb{D}$, and so due to the boundedness of u_n, $\varphi = 0$. $\square$

Observation 4.11. Let I be a countably generated free prime ideal in H^∞. Then $ki(I) = I$.

Proof. By [9, 16], I is generated by $\{S_\alpha(z)^{1/n} : n \in \mathbb{N}\}$, where

$$S_\alpha(z) = \exp\left(-\frac{\alpha + z}{\alpha - z}\right)$$

for some $\alpha \in T$. But $I = \{h\ S_\alpha^{1/n} : n \in \mathbb{N}, h \in H^\infty\}$. Hence, given $n \in \mathbb{N}$, every $f = hS_\alpha^{1/p} \in I$ can be factorized as

$$f = h\,\underbrace{S_\alpha^{1/(pn)} \dots S_\alpha^{1/(pn)}}_{n-times}.$$

So $f \in I^n$. $\square$

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