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THE ALGEBRAIC METHOD IN TREE PERCOLATION*

FATEMEH MOHAMMADI[†], EDUARDO SÁENZ-DE-CABEZÓN[‡], AND HENRY P. WYNN[§]

Abstract. We apply the methods of algebraic reliability to the study of percolation on trees. To a complete k -ary tree $T_{k,n}$ of depth n we assign a monomial ideal $I_{k,n}$ on $\sum_{i=1}^n k^i$ variables and k^n minimal monomial generators. We give explicit recursive formulae for the Betti numbers of $I_{k,n}$ and their Hilbert series, which allow us to study explicitly percolation on $T_{k,n}$. We study bounds on this percolation and study its asymptotical behavior with the mentioned commutative algebra techniques.

Key words. percolation, Betti numbers, monomial ideals, Hilbert series

AMS subject classifications. 13D02, 60K35, 05C05

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1. Introduction. The study of monomial ideals has experienced much growth in the last couple of decades, not only from a theoretical point of view [8] but also from the point of view of applications and algorithms [1]. Of particular interest are the relations between the algebra of monomial ideals and the combinatorics of graphs and networks [22, 15, 21]. In relation with these lines of research, the authors have developed an algebraic theory of system reliability which can be applied to industrial, biological, and communication systems, among others [4, 16, 19, 20]. In this theory, a monomial ideal is associated to a coherent system, and the study of the reliability of the system is performed by studying algebraic invariants of the ideal, such as the Hilbert series and Betti numbers. This algebraic approach to system reliability analysis is an example of enumerative methods for reliability evaluation. In particular, it is an improvement of the inclusion-exclusion method, which is the most general one for coherent systems [4, 16].

A main difficulty and the first step in the use of monomial ideals to study the reliability of coherent systems is the enumeration of the working and failure states of the system. This made the authors focus on several widely used and structured systems, like k -out-of- n systems [16], series-parallel systems [18], all-terminal networks [13, 14, 12], and the more general category of two-terminal networks [17, 12]. The present paper follows this line extending the application of the algebraic approach to reliability analysis to a more general situation, which allows us to introduce these techniques in percolation theory, a branch of probability theory.

In the setting of two-terminal networks the situation is the following. Consider a network as a simple connected graph $G = (V, E)$, where V is the set of vertices (nodes) and E is the set of edges (connections). To have a two-terminal network, we select two special vertices in the graph, s (source) and t (target), and study the connections between s and t in the network. We consider that vertices are reliable but

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edges may fail. The network fails to communicate between s and t whenever there is a set of failing (removed) edges such that there is no path connecting s and t using only the remaining edges. Such a set of edges is called a *cut* in this context. On the other hand, a *path* is a set of working edges that connect s and t . We say that the network is working whenever there is a path of working edges between s and t . In the algebraic approach we consider a polynomial ring on n variables, where n is the number of edges of G , i.e., $n = |E|$. We associate a variable x_e , $e \in \{1, \dots, n\}$, to each edge x_e in E . To a set of edges we associate the product of their corresponding variables. The main observation in the algebraic approach to network reliability is that the monomials corresponding to the set of cuts (respectively, paths) of a network G generate a monomial ideal, which we call the cut ideal of G , J_G (respectively, the path ideal of G , I_G). The evaluation of the (numerator of the) Hilbert series of either the cut ideal or the path ideal of G , using the probabilities of failure or function of each edge and their combinations, gives us the reliability of G . Furthermore, if we consider the form of the Hilbert series given by a free resolution of the ideal, we can obtain bounds for the reliability of G , which are tighter than the usual Bonferroni bounds [16].

The outline of the paper is the following. In section 2 we generalize this setting to any situation in which a cut and a path are defined in opposition to each other, in an obvious way: a cut separates a designated set of pairs of vertices and a path connects all such pairs. This allows us to study the problem of *all-terminal reliability* and *multisource multiterminal reliability*. These more general situations include the setting of *percolation theory*. In section 3 we apply this method to study percolation on complete trees. This is a new and relevant application of the algebraic method in reliability. We describe the path and cut ideals in this case and compute exact Hilbert series and Betti numbers. We also give and compute recursive formulae for them. With these results in hand, we study in section 4 path and cut bounds for percolation in trees, recover some classical results on critical values, and study the asymptotic behavior of percolation on trees and their corresponding Betti numbers.

2. Monomial ideals, Betti numbers, and tight inclusion-exclusion bounds.

DEFINITION 1. *Given two disjoint nonempty subsets A, B of $V(G)$ we define*

$$E(A, B) = \{e \in E(G) : e \cap A \neq \emptyset \text{ and } e \cap B \neq \emptyset\}.$$

For a nonempty subset A of $V(G)$, $E(A, A^c)$ is called a cut of G . A cut $E(A, A^c)$ is called connected if $G[A]$ and $G[A^c]$ are connected, where $G[A]$ denotes the induced subgraph of G with the vertex set A . A cut which is minimal with respect to inclusion is called minimal.

Fix a vertex q of G as a source, and fix a subset $L \subseteq V(G) \setminus \{q\}$ as targets. Let $\mathcal{S}_{L,q}$ be the set containing all connected cuts $E(A, A^c)$ of G , with $L \subset A$ and $q \in A^c$, and let $\mathcal{D}_{L,q}$ be the set containing all paths between q and one of the vertices of L .

Let K be a field and let $S = K[\mathbf{x}]$ be the polynomial ring in the $n = |E(G)|$ variables $\{x_e : e \in E(G)\}$. We associate the monomial $m_C = \prod_{e \in C} x_e$ to each cut $C = E(A, A^c)$, and the monomial $m_P = \prod_{e \in P} x_e$ to each path P . We will be concerned with the following ideals in R :

$$\mathcal{C}_{L,q} = \langle m_C : C \in \mathcal{S}_{L,q} \rangle \quad \text{and} \quad \mathcal{P}_{L,q} = \langle m_P : P \in \mathcal{D}_{L,q} \rangle.$$

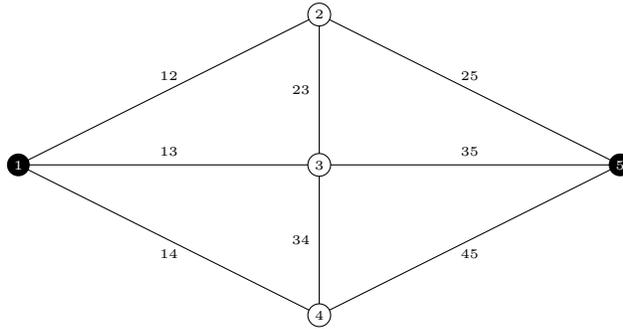


FIG. 1. Double bridge network.

Example 2. Consider the two-terminal network G depicted in Figure 1, known as the *double bridge* network. We have $E_G = \{12, 13, 14, 23, 25, 34, 35, 45\}$. Consider vertex $q = 1$ as the source and let $L = \{5\}$ be the set of targets. Following the notation in Definition 1 we obtain the following table of cuts:

A	A^c	$E(A, A^c)$
$\{5\}$	$\{1, 2, 3, 4\}$	$\{24, 35, 45\}$
$\{2, 5\}$	$\{1, 3, 4\}$	$\{12, 23, 35, 45\}$
$\{3, 5\}$	$\{1, 2, 4\}$	$\{13, 23, 34, 25, 45\}$
$\{4, 5\}$	$\{1, 2, 3\}$	$\{14, 25, 34, 35\}$
$\{2, 3, 5\}$	$\{1, 4\}$	$\{12, 13, 34, 45\}$
$\{2, 4, 5\}$	$\{1, 3\}$	$\{12, 23, 14, 34, 35\}$
$\{3, 4, 5\}$	$\{1, 2\}$	$\{13, 23, 14, 25\}$
$\{2, 3, 4, 5\}$	$\{1\}$	$\{12, 13, 14\}$

Hence $\mathcal{C}_{L,q} = \langle x_{25}x_{35}x_{45}, x_{12}x_{23}x_{35}x_{45}, x_{13}x_{23}x_{25}x_{34}x_{45}, x_{14}x_{25}x_{34}x_{35}, x_{12}x_{13}x_{34}x_{45}, x_{12}x_{14}x_{23}x_{34}x_{35}, x_{13}x_{14}x_{23}x_{25}, x_{12}x_{13}x_{14} \rangle$, which is exactly the cut ideal of G in the two-terminal setting.

Remark 3. Alexander duality will be very useful in this context (see [11, Def. 5.20]). We recall that the square-free Alexander dual of $I = \langle \mathbf{x}^{a_1}, \dots, \mathbf{x}^{a_r} \rangle$ is the ideal $I^* = \underline{m}^{a_1} \cap \dots \cap \underline{m}^{a_r}$, where $\mathbf{x}^{a_i} = \prod_{a_i^j \neq 0} x_j$ and $\underline{m}^a = \langle x_j : a_i^j \neq 0 \rangle$ for each vector $a_i = (a_i^1, \dots, a_i^n) \in \mathbb{N}^n$, which is a zero-one vector.

In this setting, we show that the path ideal is the Alexander dual of the cut ideal. Let us give a brief reminder adapted to our setting. Let Σ_G denote the associated simplicial complex to $\mathcal{P}_{L,q}$ on the vertices $\{x_e : e \in E(G)\}$. The following result is a slight generalization of [14, Prop. 8.1].

PROPOSITION 4. *The number of facets of Σ_G is the same as the number of minimal cuts of G . For each cut C , the corresponding facet τ_C is $\tau_C = \{x_e : e \in E(G) \setminus C\}$. The minimal prime decomposition of $\mathcal{P}_{L,q}$ is*

$$\mathcal{P}_{L,q} = \bigcap_{C \in \mathcal{S}_{L,q}} \langle x_e : e \in C \rangle,$$

the intersection being over all minimal cuts of G . In particular, $\mathcal{P}_{L,q}$ is the Alexander dual of $\mathcal{C}_{L,q}$.

Proof. The ideal $\mathcal{P}_{L,q}$ is generated by monomials $\prod_{e \in P} x_e$, where P is a path from q to one of the vertices in L . First we show that for each cut C , the monomial $m_{\bar{C}} := \prod_{e \in E(G) \setminus C} x_e$ does not belong to $\mathcal{P}_{L,q}$. Clearly $m_{\bar{C}} \in \mathcal{P}_{L,q}$ if and only if $m_{\bar{C}}$ is divisible by one of the given generators $\prod_{e \in P} x_e$. But

$$\prod_{e \in P} x_e \mid \prod_{e \in E(G) \setminus C} x_e \iff P \subseteq (E(G) \setminus C).$$

However, it follows from the definition of cuts that $E(G) \setminus C$ does not contain any path from q to any element of L . This shows that $\tau_C = \{x_e : e \in E(G) \setminus C\}$ is a face in the simplicial complex Σ_G . Next we show that τ_C must be a facet; for $f \in C$, because C is a minimal cut of G , $G[C \setminus \{f\}]$ still has a path between q and some element of L . Then the monomial $m_C \cdot x_f$ is divisible by $\prod_{e \in P} x_e$.

It remains to show that for any monomial $m = \prod_{e \in F} x_e$ that does not belong to $\mathcal{P}_{L,q}$ we have $F \subseteq (E(G) \setminus C)$ for some cut C . To show this, we repeatedly use the fact that m is not divisible by generators of the form $\prod_{e \in P} x_e$ for various P , and we construct a cut C . Note that if $\prod_{x \in F} x_e$ is not divisible by $\prod_{e \in P} x_e$, then there exists an $e \in P$ such that $e \notin F$. We consider the set consisting of all such edges which is clearly a (not necessarily minimal) cut. The proof now is complete by [11, Thm. 1.7]. \square

Let us now step into probability theory. In order to apply monomial algebra to network reliability, we assign a working probability to each of the connections (edges) of our network (graph). We shall consider that each edge e operates with independent probability p_e and fails with probability $q_e = 1 - p_e$. Our task is then to compute the probability $\mathcal{P}(p_e)$ that the system operates (at least one path) or fails (at least one cut), which is $1 - \mathcal{P}(p_e)$. For these computations, we use the numerator of the Hilbert series of the path or cut ideals. Note that we could also consider dependent probabilities for each edge. This would need more complicated computations but not different methods.

The multigraded Hilbert series of S/I for an ideal I can be expressed in terms of the multidegrees of the modules in any multigraded resolution of S/I , as

$$\mathcal{H}_{S/I}(x, t) = \frac{1 + \sum_{i=1}^d (-1)^i x^i (\sum_{\alpha \in \mathbb{N}^n} \gamma_{i,\alpha} t^\alpha)}{\prod_{j=1}^n (1 - t_j)},$$

where the $\gamma_{i,\alpha}$ are the ranks of the multigraded modules in the resolution. If the resolution is minimal, then

$$(1) \quad \mathcal{H}_{S/I}(x, t) = \frac{1 + \sum_{i=1}^d (-1)^i x^i (\sum_{\alpha \in \mathbb{N}^n} \beta_{i,\alpha} t^\alpha)}{\prod_{j=1}^n (1 - t_j)},$$

where the $\beta_{i,\alpha}$ depend only on I and are known as the *multigraded Betti numbers* of S/I . Observe that the minimality of the resolution means that

$$\beta_{i,\alpha} \leq \gamma_{i,\alpha} \quad \forall \alpha, i.$$

To simplify our notation, we set

$$(2) \quad \mathcal{H}_I(x, t) = - \sum_{i=1}^d (-1)^i x^i \left(\sum_{\alpha \in \mathbb{N}^n} \beta_{i,\alpha} t^\alpha \right),$$

and we refer to this as the numerator of the Hilbert series of I , which can be seen as a special kind of inclusion-exclusion formula for counting the monomials in the union of the ideals based on each individual minimal generator. By truncating at different homological degrees, or “depths,” i we obtain successive upper and lower bounds for the indicator function of this union. A key point here is that the bounds given by the minimal resolution are tighter, or at least as tight as, those given by the classical Bonferroni (inclusion-exclusion) bounds. In the algebraic setting the Bonferroni bounds correspond to the computation of the Hilbert series of S/I using the Taylor resolution; see [16] for a full explanation.

Observe that if we have that each edge e of G has a different operating probability p_e , then we need the multigraded version of the Hilbert series of S/I to obtain the bounds and probability formulae for the reliability of the network. However, if all the edges operate (do not fail) independently with the same probability p , i.e., if $p_e = p$ for each edge e of G , then we only need the graded Betti numbers to obtain the bounds. Each of the graded Betti numbers sums up all the multigraded ones of the same total degree j for each homological degree i , i.e., $\beta_{i,j}(I) = \sum_{\text{deg}(\mu)=j} \beta_{i,\mu}(I)$.

A good way to relate the information of the Hilbert series and Betti numbers is the use of generating functions. We first fix some notation and then express the bounds in terms of the Betti numbers $\beta_{i,j}$. First, consider a variable x as a placeholder for the homological degree i . Thus we define the graded Betti number generating function of an ideal I as

$$G_I(x, t) = \sum_{i,j} \beta_{i,j}(S/I) x^i t^j \text{ for } i, j > 0.$$

The numerator of the graded Hilbert series can now be expressed as

$$(3) \quad \mathcal{H}_I(x, t) = -G_I(-x, t).$$

NOTATION 5. In our setting, to simplify the notation, $\mathcal{H}(x, t)$ denotes the numerator of the Hilbert series of the path ideal $\mathcal{P}_{L,q}$ and $\tilde{\mathcal{H}}(x, t)$ denotes the numerator of the Hilbert series of the cut ideal $\mathcal{C}_{L,q}$. Similarly, we denote $G(x, t)$, $\tilde{G}(x, t)$, $\beta_{i,j}$, and $\tilde{\beta}_{i,j}$ for $G_{\mathcal{C}_{L,q}}(x, t)$, $G_{\mathcal{P}_{L,q}}(x, t)$, $\beta_{i,j}(S/\mathcal{P}_{L,q})$, and $\beta_{i,j}(S/\mathcal{C}_{L,q})$.

Remark 6. Using the path ideal, the path probability (percolation) is given by

$$(4) \quad \mathcal{P}(p) = \mathcal{H}(1, p).$$

Observe that this expression comes from the fact that the numerator of the Hilbert series represents the full inclusion-exclusion of the orthants with “corner” at the minimal paths, and then replacing t by p translates this into the reliability function; see [16].

On the other hand, as mentioned, the cut probability is given by

$$(5) \quad 1 - \mathcal{P}(p) = 1 - \mathcal{H}(1, p).$$

Put briefly, the full Hilbert series gives the full operating set. We also have the cut ideal and use a tilde for the cut quantities. So we have for the cut ideal

$$\tilde{G}(x, t) = \sum_{i,j} \tilde{\beta}_{i,j} x^i t^j \text{ for } i, j > 0,$$

$$\tilde{\mathcal{H}}(x, t) = -\tilde{G}(-x, t).$$

The probability of system failure is

$$\tilde{\mathcal{P}}(q) = \tilde{\mathcal{H}}(1, q).$$

So combining the formulae, we have two ways of expressing the probability $\mathcal{P}(p)$:

$$(6) \quad \mathcal{P}(p) = 1 - \tilde{\mathcal{P}}(1 - p).$$

This is a manifestation of the Alexander duality in terms of probability.

To complete the notation, consider any power series in x , say $f(x) = \sum_{i \geq 0} c_i x^i$, and let $T_m(f(x)) = \sum_{i=0}^m c_i x^i$ be the truncated version at $i = m$. Thus, powers of the “dummy” variable x can be used to pick out the depth at which we truncate to get bounds:

$$T_{2r+2}(\mathcal{H}(x, p))|_{x=1} \leq \mathcal{P}(p) \leq T_{2r+1}(\mathcal{H}(x, p))|_{x=1}, \quad r = 0, 1, \dots,$$

$$T_{2r+2}(\tilde{\mathcal{H}}(x, q))|_{x=1} \leq \tilde{\mathcal{P}}(q) \leq T_{2r+1}(\tilde{\mathcal{H}}(x, q))|_{x=1}, \quad r = 0, 1, \dots$$

As we will see in detail in section 4, the path bounds are accurate for small p , and the cut bounds for small q , or $p = 1 - q$ close to 1.

3. Tree percolation. Let us apply these techniques to a prominent example, namely, percolation in complete k -ary trees. A complete k -ary tree $T_{k,n}$ of depth n is a tree with n levels in which each node (except the leaves) has exactly k children. Each edge between nodes is called a bond. See Figure 2 for $k = 2$ and $n = 3$. We are interested in standard tree bond percolation on $T_{k,n}$. Each bond has an independent probability p of being operative. A *percolation* is a path of bonds from the first generation (root) to the last (a leaf). If we consider the unique minimal ways to connect each of the leaves with the root as minimal connecting events, then we want to find the probability of the union of events that contain at least one minimal connecting path from the root to a leaf. We will use an algebraic approach to solve this problem.

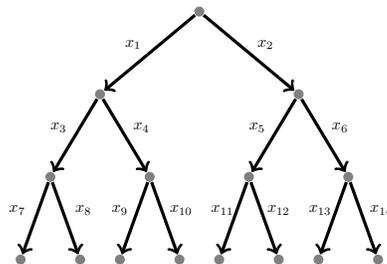


FIG. 2. $T_{2,3}$.

Much of the theory of percolation is about critical values. For the bond percolation on $T_{k,n}$, there is a critical value, denoted p_c , such that for $0 \leq p \leq p_c$, as $n \rightarrow \infty$, the probability of a percolation tends to zero, whereas for $p_c \leq p \leq 1$ the probability tends to a positive probability. This critical value is known to be $p_c = \frac{1}{k}$. This is a classical result and is often covered in the theory of branching processes, where the positive probability is referred to as the probability of survival of a branching process, in which in every generation each individual has k offspring. For the general theory of percolation see [5], and for work on percolation on trees see [9] and Chapter 5 of [10].

For the study of critical values in our algebraic setting, we use recurrence relationships for the Hilbert series, giving $\mathcal{H}_{k,n}(x, t)$ in terms of $\mathcal{H}_{k,n-1}(x, t)$. In the final section we use these recurrences to study the behavior of the bounds, as $n \rightarrow \infty$, including a notion of asymptotic Betti numbers.

3.1. The path ideal of $T_{k,n}$. Let us consider $T_{k,n}$ as a rooted graph with the edge set E (the edges are oriented going away from the root) as in Figure 2. We label each node with increasing integers, starting from the root, which has label 0, and within the same level from left to right. Each edge will be labeled with x_i , where i is the head of the edge, i.e., the edge is directed toward i .

Let K be a field and let $S = K[\mathbf{x}]$ be the polynomial ring in the $m = |E| = \sum_{j=1}^n k^j$ variables $\{x_e : e \in E\}$. The path ideal $I_{k,n}$ is the monomial ideal in R generated by the monomials $x_{i_1} \cdots x_{i_n}$, where $0, i_1, \dots, i_n$ is a unique path from the root to the leaf i_n . The ideal $I_{k,n}$ then has k^n minimal generators (one for each leaf).

NOTATION 7. For fixed integers k and n , we fix 0 as a source, and the set of leaves as targets instead of q and L from section 2. The ideals $I_{k,n}$ and $J_{k,n}$ denote the corresponding path ideal $\mathcal{P}_{L,q}$ and the corresponding cut ideal $\mathcal{C}_{L,q}$ from section 2.

Remark 8. Let $R = K[\mathbf{y}]$ be the polynomial ring over a field K on $|V(T_{k,n})|$ variables. The path ideal of length n associated to $T_{k,n}$ is the monomial ideal $I_{n+1} \subset S$ generated by monomials $y_{i_0} y_{i_1} \cdots y_{i_n}$, where i_0, i_1, \dots, i_n is a path in $T_{k,n}$. Such ideals are studied in [2, 7]. Note that if n is the depth of the tree, then y_{i_0} is the variable corresponding to the root. In particular I_{n+1} is isomorphic to our ideal $I_{k,n}$ under the induced isomorphism

$$\varphi: \mathcal{G}(I_{n+1}) \rightarrow \mathcal{G}(I_{k,n}) \text{ with } y_{i_0} y_{i_1} \cdots y_{i_n} \mapsto x_{i_1} \cdots x_{i_n},$$

where $\mathcal{G}(I)$ denotes the generating set of I , and x_{i_ℓ} is the variable corresponding to the edge between the vertices $y_{i_{\ell-1}}$ and y_{i_ℓ} .

LEMMA 9. Let $T_{k,n}$ be a k -ary tree and $I_{k,n}$ its path ideal. Then we have that

- (i) the Taylor resolution of $I_{k,n}$ is minimal for all k and n ,
- (ii) the Betti numbers are given by $\beta_i(I_{k,n}) = \binom{k^n}{i}$,
- (iii) the length of the resolution, i.e., the projective dimension of $I_{k,n}$, is k^n .

Proof. Each minimal generator m_α of $I_{k,n}$ has a variable that appears only in it, namely, the one corresponding to the edge ij , where j is the leaf in m_α . Hence the monomials corresponding to the least common multiple of any two different sets of minimal generators are different; hence the multidegrees of the generators of the Taylor resolution of $I_{k,n}$ are all different, and thus the Taylor resolution of $I_{k,n}$ is minimal. The Betti numbers of $I_{k,n}$ are then the ranks of the modules in the Taylor resolution. \square

3.2. Path formulae. Here we read the resolution of $I_{k,n}$ as a tensor product of the resolutions of ideals arising from $I_{k,n-1}$ to obtain the generating function and a recursive formula for the ideal's Betti numbers (see Appendix A for some basic facts about tensor products of chain complexes).

THEOREM 10. The total Betti numbers of $I_{k,n}$ are $\beta_i(I_{k,n}) = \binom{k^n}{i}$, and the graded Betti numbers $\beta_{i,j}$ can be determined recursively as

$$\beta_{i-1,j}(I_{k,n}) = \beta_{i,j}(S/I_{k,n}) = \sum_{s=1}^k \sum_{\substack{(i_1, \dots, i_s) \in A_{i,s} \\ (j_1, \dots, j_s) \in B_{j,s}}} \binom{k}{s} \beta_{i_1, j_1}(S/I_{k,n-1}) \cdots \beta_{i_s, j_s}(S/I_{k,n-1}),$$

where

$$A_{i,s} = \{(i_1, \dots, i_s) : i_1 + \dots + i_s = i, i_1, \dots, i_s > 0\} \text{ and}$$

$$B_{j,s} = \{(j_1, \dots, j_s) : j_1 + \dots + j_s = j - s\}.$$

Proof. Assume that x_1, x_2, \dots, x_k are the variables corresponding to the edges of the tree connected to the root. Then

$$I_{k,n} = x_1 I_{k,n-1}^{(1)} + x_2 I_{k,n-1}^{(2)} + \dots + x_k I_{k,n-1}^{(k)},$$

where each $I_{k,n-1}^{(i)}$ is a tree ideal associated to a k -ary tree of depth $n - 1$. Note that their corresponding trees are disjoint, so the ideals $I_{k,n-1}^{(i)}$ live in disjoint polynomial rings. Therefore, the resolution of $I_{k,n}$ is the tensor product of the resolutions of the ideals $x_i I_{k,n-1}^{(i)}$. On the other hand, $\beta_{i,j}(x_i I_{k,n-1}^{(i)}) = \beta_{i,j+1}(I_{k,n-1})$ for all k and n . Thus the statement is an immediate consequence of Lemma 25(iii). \square

Remark 11. Let us denote by $G_{k,n} = \sum_{i,j} \beta_{ij}(I_{k,n}) x^i t^j$ the generating function for the Betti numbers of the ideal $I_{k,n}$. We also denote the numerator of the graded Hilbert series of $I_{k,n}$ by $\mathcal{H}_{k,n}$. Note that by Remark 3 we have

$$\mathcal{H}_{k,n}(x, t) = -G_{k,n}(-x, t).$$

We recall that the ideal $I_{k,1} = \langle x_1, x_2, \dots, x_k \rangle$ is generated by k variables. Thus $\beta_{i,j} = \binom{k}{i}$ if $i = j$, and it is zero otherwise. Therefore

$$G_{k,1}(x, t) = (1 + tx)^k - 1.$$

From Theorem 10 and the above argument we obtain the following compact result.

THEOREM 12. *The generating function for the Betti numbers of $I_{k,n}$ for all k and n is equal to*

$$(7) \quad G_{k,n}(x, t) = (1 + tG_{k,n-1}(x, t))^k - 1.$$

3.3. Cut ideal and cut formulae. As explained in Proposition 4 the cut ideal is the Alexander dual ideal of the path ideal $I_{k,n}$. We consider the following problem: Given a probability p_i for each edge i in $T_{k,n}$ to be operative, we want to find the probability of *disconnecting* the root with all leaves of graph. If we consider all minimal possible ways to disconnect the leaves with the root as minimal connecting events, then what we want to find is the probability of the union of events that does not contain any path connecting the root to a leaf. As before, we consider $p_i = p$ for all i .

Here we read the ideal $J_{k,n}$ as the Alexander dual of the tree ideal $I_{k,n}$ studied in section 3.2 to obtain the generating function and a recursive formula for its Hilbert series.

For all k and n , the generating function for the Betti numbers of the ideal $J_{k,n}$ is denoted by $\tilde{G}_{k,n}$. We recall that the ideal $I_{k,1} = \langle x_1, x_2, \dots, x_k \rangle$ is generated by k variables, and its dual is $J_{k,1} = \langle x_1 x_2 \cdots x_k \rangle$. Thus $\beta_{0,0}(S/J_{k,1}) = 1$, $\beta_{1,k}(S/J_{k,1}) = 1$, and it is zero otherwise. Therefore $\tilde{G}_{k,1}(x, t) = t^k x$.

THEOREM 13. *The generating function of the Betti numbers of $J_{k,n}$ for all k and all $n > 1$ is equal to*

$$(8) \quad \tilde{G}_{k,n}(x, t) = x^{-(k-1)} ((1 + tx)(1 + \tilde{G}_{k,n-1}(x, t)) - 1)^k.$$

Proof. Assume that x_1, x_2, \dots, x_k are the variables corresponding to the edges of a tree connected to the root, and $I_{k,n-1}^{(i)}$ is a tree ideal associated to a k -ary tree of depth $n - 1$. We denote $J_{k,n-1}^{(i)}$ for the Alexander dual of the ideal $I_{k,n-1}^{(i)}$. Thus the Alexander dual of the ideal $x_i I_{k,n-1}^{(i)}$ is equal to $\langle x_i \rangle + J_{k,n-1}^{(i)}$, because x_i doesn't appear in the support of any monomial from the generating set of $I_{k,n-1}^{(i)}$. Thus the numerator of the graded Hilbert series (1) of $\langle x_i \rangle + J_{k,n-1}^{(i)}$ is equal to

$$(1 - tx)(1 + \tilde{G}_{k,n-1}(-x, t)).$$

On the other hand, $J_{k,n}$ can be written as the multiplication of the ideals $\langle x_i \rangle + J_{k,n-1}^{(i)}$ living in the polynomial rings on disjoint variables:

$$J_{k,n} = (\langle x_1 \rangle + J_{k,n-1}^{(1)}) \cdots (\langle x_k \rangle + J_{k,n-1}^{(k)}).$$

Now applying Lemma 26(iii), the same argument as Theorem 12 implies that the minimal free resolution of the ideal $J_{k,n}$ is the tensor product of that of $x_i + J_{k,n-1}^{(i)}$, and so we have

$$\mathcal{H}_{J_{k,n}}(x, t) = \frac{1 + (-x)^{-(k-1)}((1 - tx)(1 + \tilde{G}_{k,n-1}(-x, t)) - 1)^k}{(1 - t)^d}.$$

Therefore

$$(9) \quad \mathcal{H}_{J_{k,n}}(x, t) = -(-x)^{-(k-1)}((1 - tx)(1 + \tilde{G}_{k,n-1}(-x, t)) - 1)^k$$

and

$$\tilde{G}(x, n) = x^{-(k-1)}((1 + tx)(1 + \tilde{G}_{k,n-1}(x, t)) - 1)^k. \quad \square$$

4. Bounds and critical values. Using the same notation as in Remark 11 and applying Remark 6 the percolation probability of the path ideal $I_{k,n}$ is then given by

$$\mathcal{P}_{k,n}(p) = \mathcal{H}_{k,n}(1, p).$$

Similarly we have

$$\tilde{\mathcal{P}}_{k,n}(q) = \tilde{\mathcal{H}}_{k,n}(1, q).$$

From (6) and Theorem 12 we have the iterative formula

$$(10) \quad \mathcal{H}_{k,n}(x, t) = 1 - (1 - t(\mathcal{H}_{k,n-1}(x, t)))^k.$$

This gives the formula for $\mathcal{P}_{k,n}(p)$:

$$(11) \quad \mathcal{P}_{k,n}(p) = 1 - (1 - p \mathcal{P}_{k,n-1}(p))^k,$$

which is well known from the theory of branching processes [6]. We will have in mind the classical asymptotic form for $\mathcal{P}_{k,n}(p)$. As $n \rightarrow \infty$, and for fixed k , $\mathcal{P}_{k,n}(p)$ converges to the function

$$\mathcal{P}_{k,\infty}(p) = \max(0, 1 - u),$$

where u is the solution of

$$u = (1 - p(1 - u))^k.$$

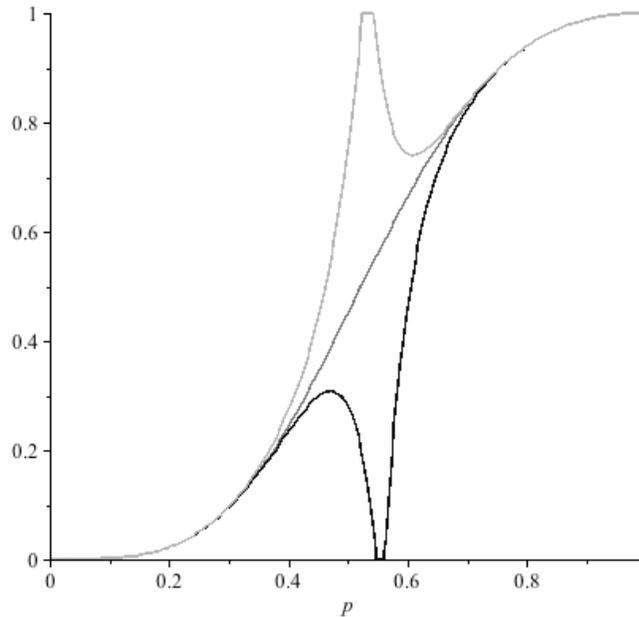


FIG. 3. Upper and lower path and cut bounds for $k = 2$, $n = 4$, and $m = 3, 4$.

The value $p_c = \frac{1}{k}$ is the maximum value of p for which $\mathcal{P}_{k,\infty}(p) = 0$.

For the bounds given in section 2 we write the path and cut bounds, respectively, as

$$B_{k,n,m}(p) = T_m(\mathcal{H}(x, p))|_{x=1},$$

$$C_{k,n,m}(q) = T_m(\tilde{\mathcal{H}}(x, q))|_{x=1}.$$

We now discuss how the bounds for percolation based on $B_{k,n,m}$ and $C_{k,n,m}$ behave. As a brief guide to a quite technical section, the following is an informal list of the main features found by the authors:

1. The path bounds, $B_{k,n,m}$, are accurate as $p \rightarrow 0$.
2. The cut bounds, $C_{k,n,m}$, are accurate as $p \rightarrow 1$ ($q = 1 - p \rightarrow 0$).
3. The path bounds display critical behavior at the critical value $p_c = \frac{1}{k}$ in that they diverge from the true probability, as n increases, and can only be controlled by taking higher depth m .
4. The cut bounds reveal a new type of critical value $p_k^* = 1 - q_k^* > p_c$.

All these results are consequences of having the iterative formulae (7) and (8). It should also be noted that the path bounds are easier to handle than the cut bounds, which is a consequence of the Taylor resolution (standard inclusion-exclusion) being the minimal free resolution in the path case, which is not true in the cut case. By working on the first few bounds, we can obtain exact formula and limits in some cases.

Example 14. Figure 3 gives an example combining the path and the cut bounds for $k = 2$, $n = 4$ and with depth $m = 3, 4$. Observe that together with the curve showing the true probability of percolation there are four curves plotted together in this figure, two on the left side of the figure, i.e., probability p closer to 0, and two on the right side, i.e., probability p closer to 1. To cope with the divergence near the critical value, the upper and lower bounds are truncated, respectively, at 1 and 0.

The upper bounds are shown in light gray: $m = 3$ for the path bound on the left and $m = 4$ for the cut bound on the right. The lower bounds are shown in black: $m = 4$ for the path bound on the left, and $m = 3$ for the cut bound on the right. The central gray curve is the true probability of percolation.

We begin with some formulae for the path case. Multiplying the bounds by a truncated version of the product $(1 - kp)(1 - k^2p^2)(1 - k^3p^3) \dots$ leads to tractable formulae. Interestingly, the inverse of this infinite product is the generating function for integer partitions. For the path bounds we have the following for $n = 1, 2, 3$:

$$\begin{aligned}
 B_{k,n,1}(p) &= p^n k^n, \\
 B_{k,n,2}(p)(1 - kp) &= p^n k^n \left(1 - \frac{1}{2}(3k - 1)p + \frac{1}{2}(k - 1)k^n p^{n+1} \right), \\
 B_{k,n,3}(p)(1 - kp)(1 - k^2p^2) &= p^n k^n \left(1 - \frac{1}{2}(3k - 1)p - \frac{1}{6}(k + 1)(5k - 2)p^2 \right. \\
 &\quad + \frac{1}{6}k(11k^2 - 6k + 1)p^3 + \frac{1}{2}k^n(k - 1)p^{n+1} \\
 &\quad - \frac{1}{2}k^n(k - 1)^2 p^{n+2} - \frac{1}{2}k^{n+1}(2k - 1)(k - 1)p^{n+3} \\
 &\quad + \frac{1}{6}k^{2n}(2k - 1)(k - 1)p^{2n+2} \\
 &\quad \left. + \frac{1}{6}k^{2n+1}(k - 1)(k - 2)p^{2n+3} \right).
 \end{aligned}$$

The general formula, whose proof is omitted, is

$$B_{k,n,m} \prod_{i=1}^{m-1} (1 - k^i p^i) = p^n k^n (Q_{k,p,m}(k, p) + O(p^{n+1})),$$

where $Q_{k,p,m}(k, p)$ is a polynomial in p , the degree of which depends only on k and m . This gives some asymptotics as $n \rightarrow \infty$. To aid this we set $p = \frac{R}{k}$, having in mind that $\frac{1}{k}$ is the critical value.

After a little algebra we have the following formulae:

$$\begin{aligned}
 B_{k,n,1}(p) &= R^n, \\
 B_{k,n,2}(p) &= R^n \left(1 - \frac{1}{2} \frac{k - 1}{k} \frac{R}{1 - R} \right) + O(R^{2n+1}), \\
 B_{k,n,3}(p) &= R^n \left(1 - \frac{1}{6} \frac{k - 1}{k^2} \frac{R(-5R^2k + R^2 - Rk + 2R + 3k)}{(1 - R)(1 - R^2)} \right) + O(R^{2n+1}).
 \end{aligned}$$

For fixed $R < 1$ the first bound $B_{k,n,1} \rightarrow 0$, as expected. It is instructive to let $k \rightarrow \infty$, again while keeping $R < 1$ fixed. Combining the first two bounds ($m = 2, 3$) asymptotically, we obtain

$$R^n \left(1 - \frac{1}{2} \frac{R}{(1 - R)} \right) \leq B_{\infty,n,\infty} \leq R^n \left(1 - \frac{1}{6} \frac{R(3 - R - 5R^2)}{(1 - R)(1 - R^2)} \right).$$

The bounds agree to order R^{n+1} , the left bound reaches zero at $R = \frac{1}{2}$, and the bounds diverge to $\pm\infty$ as $R \rightarrow 1$. It has a pole at $R = 1$ for all the bounds except the first, from which we claim that $p = p_c = \frac{1}{k}$ is a critical value for the path bounds, albeit buried under a basic R^n convergence rate for $R < 1$.

Now let us consider the cut bounds. We start with a basic form of the iteration for the cut generating function from (9):

$$(12) \quad g(x, t, k, u) = (-1)^k \frac{1}{x^{k-1}} ((1-tx)(1-u)-1)^k.$$

Then the successive values of $\mathcal{H}_{k,n}(x, t)$ are given by the recurrence relation:

$$\mathcal{H}_{k,n}(x, t) = g(x, t, k, \mathcal{H}_{k,n-1}(x, t)).$$

Note that

$$\mathcal{H}_{k,1}(x, t) = t^k x.$$

The main difficulty is that although we have recurrence for $\mathcal{H}_{k,n+1}(x, y)$ there is not in general such a nice formula for the truncated version, which is given by extracting the Taylor expansion in x up to degree m .

However, there is one simple case, namely, the first upper cut bound, i.e., $m = 2$, which we denote by $C_{k,n,2}(q)$. We have for all $n \geq 1$

$$(13) \quad C_{k,n,2}(q) = (C_{k,n-1,2}(q) + q)^k.$$

At any fixed k, q the value $C_{k,n,2}(q)$ increases with n . There is a critical value $q = q_{k,n,2}^*$. For any q above this value $C_{k,n,2}(q) \rightarrow \infty$ as $n \rightarrow \infty$. In the interval $0 \leq q \leq q_{k,n,2}^*$, $C_{k,n,2}$ tends from below to the solution to

$$z = (z + q)^k.$$

We can solve this explicitly for q , giving $q = z^{\frac{1}{k}} - 1$. The critical values of q and z are found by solving $\frac{dz}{dq} = 0$, and we obtain

$$q_k^* = \frac{k-1}{k^2} k^{\frac{k-2}{k-1}},$$

which plays a key role in the cut theory.

Example 15. For $n = 2, 3$ and $k = 2$ we obtain

$$\begin{aligned} \mathcal{H}_{2,2}(x, t) &= t^2(t+1)^2x - 2t^4(t+1)x^2 + t^6x^3 \\ \mathcal{H}_{2,3}(x, t) &= t^2(t^3 + 2t^2 + t + 1)^2x - 2t^4(t^3 + 2t^2 + t + 1)(t+1)(3t+1)x^2 \\ &\quad + t^6(15t^4 + 40t^3 + 36t^2 + 18t + 5)x^3 - 2t^8(10t^3 + 20t^2 + 12t + 3)x^4 \\ &\quad + t^{10}(15t^2 + 20t + 6)x^5 - 2t^{12}(3t+2)x^6 + t^{14}x^7. \end{aligned}$$

Remark 16. Note that for all i and j , we can read off the Betti numbers $\beta_{i,j}(S/J_{k,n})$ as the coefficients of $x^i t^j$ in the polynomial $\mathcal{H}_{k,n}(x, t)$.

The limiting behavior of $C_{k,n,m}(q)$ as $n \rightarrow \infty$ will be considered below.

LEMMA 17. For fixed q and $k = m = 2$, the lower asymptotic bound of $C_{k,n,m}(q)$ as $n \rightarrow \infty$ is

$$C_{2,\infty,2}(q) = \frac{1}{2} - q^2 + \frac{1}{2} \frac{6q^2 + 2q - 1}{\sqrt{1-4q}}.$$

Proof. We consider the solution in u of $u - g(x, t, k, u) = 0$. As $k = 2$, we use the solution

$$(14) \quad u^*(x, t) = \frac{1}{2} \frac{(2t^2x - 2t + 1 + \sqrt{-4t + 1 + 4t^2x})x}{(1-xt)^2}.$$

If we expand in powers of x , we obtain

$$\left(\frac{1}{2}\sqrt{1-4t}\right)x - \left(\frac{t}{2}\sqrt{1-4t} + t^2 - t\sqrt{1-4t}\right)x^2 + O(x^3).$$

The cut bounds are obtained by truncating such expansions and setting $t = q$ and $x = 1$. Thus the above expansion gives the $k = 2, m = 2$ lower asymptotic bound:

$$C_{2,\infty,2}(q) = \frac{1}{2} - q^2 + \frac{1}{2} \frac{6q^2 + 2q - 1}{\sqrt{1-4q}}. \quad \square$$

Further expansion in t gives asymptotic Betti numbers, which we will cover in the next section. We collect these informal results into the following.

THEOREM 18. *For fixed k, m , and q , the cut bounds $C_{k,n,m}(q)$ converge to the function which is the truncated Taylor expansion of the smallest solution, in u , of the equation $u - g(x, q, k, u) = 0$, provided $0 \leq q \leq q_k^* = \frac{k-1}{k^2} k^{\frac{k-2}{k-1}}$. For $q_k^* < q \leq 1$, $C_{k,n,m}(q) \rightarrow \infty$ when m is odd and $-\infty$ when m is even.*

Proof. The case $m = 1$ above is a good guide, because it gives the first term in the Taylor expansions, and further analysis shows that it also gives a pole governing the expansion for any $m > 1$. Although we cannot get closed forms for the solutions of $u - g(x, q, k, u) = 0$ for $k > 3$, nonetheless we can show the presence of a pole at q_k^* .

We use the shorthand $g(u) = g(x, t, k, u)$. Then $g(u)$ is convex and increasing in u in the region $0 \leq x, t \leq 1$ and $g(0) = xt^k$. Moreover $u_1 = xt^k$, where u_1 is the starting value in the iteration $u_{n+1} = g(u_n)$. Suppose first that, under such suitable conditions on x, t , the equation $u = g(u)$ has at least one solution, and let u^* be the smallest (there can be no more than two by the convexity of $g(u)$). Then, for the dynamic system $u_{n+1} = g(u_n)$, the iterate u_n converge upward to u^* . The complication is that the existence of the solution x^* depends on x and q . We know that x^* does not exist if and only if $g(u) > u$ for all $u > 0$.

We consider the case $k = 2$, for which $q_2^* = \frac{1}{4}$. The function $g(u) - u$ has roots

$$\frac{1}{2} \frac{(2t^2x - 2t + 1 + \sqrt{4t^2x - 4t + 1})x}{1 + t^2x^2 - 2tx}, \quad -\frac{1}{2} \frac{(-2t^2x + 2t - 1 + \sqrt{4t^2x - 4t + 1})x}{1 + t^2x^2 - 2tx}.$$

In the region $0 \leq q \leq \frac{1}{4}$ these roots exist for all $x > 0$, except when $t = \frac{1}{x}$, which can be eliminated by taking x sufficiently small. When $q > \frac{1}{4}$, however, the roots are complex for x sufficiently small, noting that the smallest root is

$$\left(-t + \frac{1}{2} - \frac{1}{2}\sqrt{1-4t}\right)x + O(x^2).$$

In that case $g(u) > u$ and u_n diverges to infinity. The case of general k proceeds along the same lines. The fact that the bounds, which are achieved at $x = 1$, converge in the “good” region, $0 \leq q \leq q_k^*$, follows by standard analysis on the uniform convergence of power series. For $q_k^* < q \leq 1$, the divergence of $C_{k,n,m}(q)$ follows immediately from the divergence of u_n , with sign dependent on m □

Figure 4 shows an example of the behavior of the cut bounds for $k = 2$ and relatively modest value $n = 6$. The cut upper bound for $m = 3$ (dashed line) and the cut lower bound for $m = 4$ (solid line) are already approaching the vertical line at the critical value $q = \frac{1}{4}$.

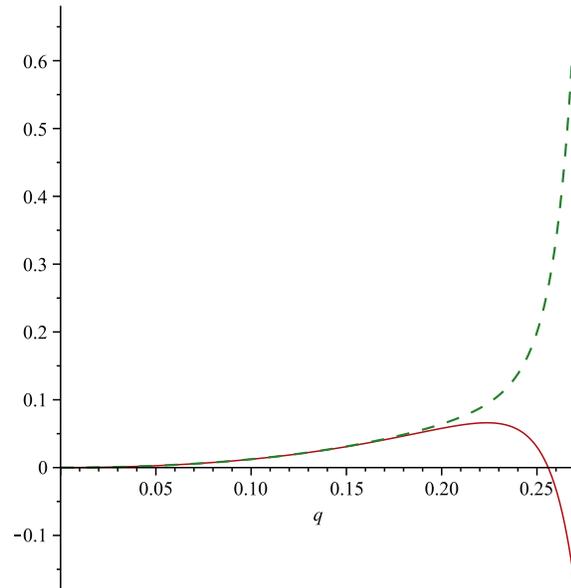


FIG. 4. Approaching the $q^* = \frac{1}{4}$ critical cut bounds for $n = 6$, $k = 2$, $m = 3$ (cut upper bound, dashed line) and $m = 4$ (cut lower bound, solid line).

4.1. Asymptotic Betti number: Cut case. For the case $k = 2$ we use the appropriate generating function in (8) with the discussion in the last section to give the graded Betti number generating function:

$$(15) \quad G_{2,\infty}(x, t) = -\frac{1}{2} \frac{2t - 1 + 2t^2x + (\sqrt{1 - 4t - 4t^2x})x}{(1 + xt)^2}.$$

There are more complex formulae for $k > 2$.

This generating function enables us to derive a combinatorial formula for the coefficients, which we shall call the *asymptotic graded Betti numbers*.

THEOREM 19. *The asymptotic graded Betti numbers for the cut ideal on a binary ($k = 2$) tree are given by*

$$\beta_{i,j}(J_{2,\infty}) = \frac{[2(j-i)]!}{(j-i+1)(j-i)(j-i)!(j-2i)!(i-1)!} \text{ for } j \geq 2i \text{ and zero otherwise.}$$

Proof. We temporarily make the transformation $t = \frac{z}{1+y}$, $x = \frac{y(1+y)}{z}$ in the generating function (15), giving

$$(16) \quad G_{2,\infty}(x, t) = \frac{1 - \sqrt{1 - 4z}}{2z} \frac{y}{1+y} - \frac{y}{1+y}.$$

We recognize the first term on the right-hand side as the generating function of the Catalan numbers $c_r = \frac{1}{r+1} \binom{2r}{r} z^r$,

$$\frac{1 - \sqrt{1 - 4z}}{2z} = \sum_{r=0}^{\infty} c_r z^r.$$

We now back substitute $y = xt$, $z = t(1 + xt)$ so that the generating function for the c_r becomes $\sum_{r=0}^{\infty} c_r t^r (1 + xt)^r$. Expanding each term $(1 + xt)^r$ into binomial terms and incorporating the other terms y , we find the Betti numbers $\beta_{i,j}(J_{2,\infty})$ as the coefficient of $x^i x^j$ for $j \geq 2i$:

$$\begin{aligned} \beta_{i,j}(J_{2,\infty}) &= c_{j-i} \sum_{r=0}^{i-1} (-1)^{i+j-1} \binom{j-i}{r} \\ &= c_{j-i} \frac{(j-i-1)!}{(i-1)!(j-2i)!} \end{aligned}$$

for $j \geq 2i$ and zero otherwise. Using the form of c_{j-i} , we obtain the result. □

THEOREM 20. *In the region $0 \leq q \leq q_k^*$, for each i there is a maximal integer $N(n, i)$ such that for any $j < N(n, j)$*

$$\beta_{i,j}(J_{k,n}) = \beta_{i,j}(J_{k,\infty}).$$

Furthermore $N(n, j)$ is increasing in n for fixed i .

Proof. This follows from the uniform convergence of the power series derived from u_n and the fact that the coefficients are integers. □

The following tables show the graded Betti numbers $\beta_{i,j}$ of the cut ideal for $k = 2$, $n = 2, \dots, 5$, and ranges of values of $i = 1, \dots, 7$ and $j = 2, \dots, 14$. Note that its (i, j) -entry is simply $\beta_{i,i+j}(S/J_{k,n})$. The tables for $n = 2, 3$ are complete. The last table gives the asymptotic Betti numbers.

$i \setminus j$	0	1	2	3
Total	1	4	4	1
0	1			
1		1		
2		2	2	
3		1	2	1

$n = 2$

$i \setminus j$	0	1	2	3	4	5	6	7
Total	1	25	80	114	90	41	10	1
0	1							
1		1						
2		2	2					
3		5	10	5				
4		6	18	18	6			
5		6	24	36	24	6		
6		4	20	40	40	20	4	
7		1	6	15	20	15	6	1

$n = 3$

$i \setminus j$	0	1	2	3	4	5	6	7
Total	1	676	5460	21113	51348	87288	109314	103726
0	1							
1		1						
2			2					
3			5	10	5			
4			14	42	42	14		
5			26	104	156	104	26	
6			44	220	440	440	220	44
7			69	414	1035	1380	1035	414
8			94	658	1974	3290	3290	1974
9			114	912	3192	6384	7980	6384
10			116	1044	4176	9744	14616	14616
11			94	940	4230	11280	19740	23688
12			60	660	3300	9900	19800	27720
13			28	336	1848	6160	13860	22176
14			8	104	624	2288	5720	10296

 $n = 4$

$i \setminus j$	0	1	2	3	4	5	6	7
Total	1	458329	8308144	73630338	424216050	1783078865	5818552406	15319701281
0	1							
1		1						
2			2					
3			5	10	5			
4			14	42	42	14		
5			42	168	252	168	42	
6			100	500	1000	1000	500	100
7			221	1326	3315	4420	3315	1326
8			470	3290	9870	16450	16450	9870
9			958	7664	26824	53648	67060	53648
10			1860	16740	66960	156240	234360	234360
11			3434	34340	154530	412080	721140	865368
12			6036	66396	331980	995940	1991880	2788632
13			10068	120816	664488	2214960	4983660	7973856
14			15864	206232	1237392	4537104	11342760	20416968

 $n = 5$

$i \setminus j$	0	1	2	3	4	5	6	7
0	1							
1		1						
2			2					
3			5	10	5			
4			14	42	42	14		
5			42	168	252	168	42	
6			132	660	1320	1320	660	132
7			429	2574	6435	8580	6435	2574
8			1430	10010	30030	50050	50050	30030
9			4862	38896	136136	272272	340340	272272
10			16796	151164	604656	1410864	2116296	2116296
11			58786	587860	2645370	7054320	12345060	14814072
12			208012	2288132	11440660	34321980	68643960	96101544
13			742900	8914800	49031400	163438000	367735500	588376800
14			2674440	34767720	208606320	764889840	1912224600	3442004280

 $n = \infty$

Let us make a final remark on the relation of the Betti numbers of $J_{2,n}$ and $J_{2,\infty}$ with Mandelbrot and Catalan numbers. This will make evident the interplay of algebra, combinatorics, and asymptotics that permeates this paper.

Mandelbrot numbers are defined as follows: the Mandelbrot set is a fractal formed by iterating the following polynomials:

$$\begin{aligned} z_0(q) &= 0, \\ z_n(q) &= z_{n-1}(q)^2 + q. \end{aligned}$$

The first few polynomials in the sequence are

$$\begin{aligned} z_0(q) &= 0, \\ z_1(q) &= q, \\ z_2(q) &= q^2 + q, \\ z_3(q) &= (q^2 + q)^2 + q = q + q^2 + 2q^3 + q^4, \\ z_4(q) &= (q + q^2 + 2q^3 + q^4)^2 + q = q + q^2 + 2q^3 + 5q^4 + 6q^5 + 6q^6 + 4q^7 + q^8. \end{aligned}$$

We see that the coefficient of q^j in $z_i(q)$ is the (i, j) -Mandelbrot number, and we denote it by $M_{i,j}$.

LEMMA 21. $C_{1,n,2}(q) = z_{n+1}(q) - q$.

Proof. The proof goes by induction. For $n = 1$ we have that $C_{1,1,2}(q) = q^2$ and $z_2(q) = q^2 + q$. Now, $C_{1,n+1,2}(q) = (C_{1,n,2}(q) + q)^2$, and by the induction step this is $(z_{n+1}(q) - q + q)^2 = z_{n+1}(q)^2$. On the other hand, we have that, by definition, $z_{n+2}(q) = z_{n+1}(q)^2 + q$. \square

This identification of polynomials gives us an expression of the Betti numbers of $J_{2,n}$ in terms of Mandelbrot numbers.

COROLLARY 22. $\beta_{i,i+j}(J_{2,n}) = M_{n+1,j+1} \binom{j-1}{i-1}$.

Proof. We have that $\beta_{1,j+1}(J_{2,n})$ is the coefficient of q^j in $C_{1,n,2}(q)$. Since $C_{1,n,2}(q) = z_{n+1}(q) - q$, the result holds for $i = 1$. For $i > 1$ we only need to multiply by the binomial coefficient, which comes from the recursive expression of the Betti numbers of $J_{2,n}$. \square

In the asymptotic case $n = \infty$ we have a similar expression, where Mandelbrot numbers are substituted by Catalan numbers.

COROLLARY 23. $\beta_{i,i+j}(J_{2,\infty}) = c_j \binom{j-1}{i-1}$.

Proof. The proof is obtained simply by substituting j by $i+j$ in the last expression of the proof of Theorem 19. \square

The last two corollaries together with the asymptotic study of the Betti numbers imply the following.

COROLLARY 24. $\lim_{n \rightarrow \infty} M_{n,j} = c_j$.

Appendix A. Tensor product of complexes. To keep the paper self-contained, we review here some basic and relevant notions of the tensor product of resolutions. We begin by recalling the tensor product of resolutions from [3]. The tensor product of two chain complexes (A, d_1) and (B, d_2) , say $A \otimes B$, is formed by taking all products $A_i \otimes B_j$ and letting $(A \otimes B)_k = \bigoplus_{i+j=k} A_i \otimes B_j$. The differential maps are defined as $\partial(a \otimes b) = d_1 a \otimes b + (-1)^i a \otimes d_2 b$ when $a \in A_i$. Then we have $\partial^2 = 0$ and ∂ induces a natural map $\partial : H(A) \otimes H(B) \rightarrow H(A \otimes B)$ such that $\partial(a \otimes b) = a \otimes b$. If $a = d_1 c$ is a boundary and b is a cycle, then $a \otimes b = \partial(c \otimes b)$ is again a boundary, which shows that ∂ is well-defined.

LEMMA 25. Let $I_i \subseteq S_i$ be a monomial ideal in the polynomial ring S_i for $i = 1, \dots, r$, and let $I = I_1 + \dots + I_r$ be the ideal in the polynomial ring $S = S_1 \otimes \dots \otimes S_r$. Assume that \mathcal{F}_i is the minimal free resolution of S/I_i for all i . Then the minimal free resolution of S/I is obtained by $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \dots \otimes \mathcal{F}_r$. In particular, the following hold:

$$(i) \beta_{i,j}(S/I) = \sum_{\substack{(i_1, \dots, i_r) \in A_{i,r} \\ (j_1, \dots, j_r) \in B_{j,r}}} \beta_{i_1, j_1}(S/I_1) \cdots \beta_{i_r, j_r}(S/I_r), \text{ where}$$

$$A_{i,r} = \{(i_1, \dots, i_r) : i_1 + \dots + i_r = i\} \text{ and } B_{j,r} = \{(j_1, \dots, j_r) : j_1 + \dots + j_r = j\}.$$

$$(ii) \text{ The Hilbert series of } S/I \text{ is } \mathcal{H}_{S/I}(t) = \frac{\prod_{i=1}^r (1 + Q_{S/I_i}(t))}{(1-t)^d}, \text{ where } \mathcal{H}_{S/I_i}(t) = \frac{1 + Q_{S/I_i}(t)}{(1-t)^d} \text{ and } d \text{ is the number of variables of the ring } S. \text{ Note that here we are looking at the ideals } I_i \text{ in the polynomial ring } S.$$

In particular, if all the ideals are the same (but in polynomial rings on disjoint set of variables), then we have $\mathcal{H}_{S/I}(x, t) = \frac{(1 + Q_{S/I_i}(t))^r}{(1-t)^d}$ and

$$(iii) \beta_{i,j}(S/I) = \sum_{s=1}^r \sum_{\substack{(i_1, \dots, i_s) \in A_{i,s} \\ (j_1, \dots, j_s) \in B_{j,s}}} \binom{r}{s} \beta_{i_1, j_1}(S/I_1) \cdots \beta_{i_s, j_s}(S/I_1), \text{ where}$$

$$A_{i,s} = \{(i_1, \dots, i_s) : i_1 + \dots + i_s = i, i_1, \dots, i_s > 0\} \text{ and}$$

$$B_{j,s} = \{(j_1, \dots, j_s) : j_1 + \dots + j_s = j\}.$$

Proof. The proof is by induction on r . Assume that $r > 1$. Since differential maps of the tensor complex are defined in terms of differential maps of \mathcal{F}_ℓ 's, the minimality of the tensor complex follows by the minimality of the resolutions of all components. On the other hand, these ideals live in rings with disjoint variables, which implies that $\text{Tor}_i(S/(I_1 + \dots + I_{r-1}), S/I_r) = 0$ for $i > 0$, and so the constructed complex is indeed a minimal free resolution for S/I . \square

In Lemma 25 if we replace $I_1 + \dots + I_r$ and S/I by $I_1 I_2 \cdots I_r$ and I , then an analogous statement holds. The original statement appeared in the Habilitationsschrift of Jürgen Herzog in 1974, and the proof is similar to the proof of the above lemma.

LEMMA 26. Let $I_i \subseteq S_i$ be a monomial ideal in the polynomial ring S_i for $i = 1, \dots, r$, and let $I = I_1 I_2 \cdots I_r$ be the ideal in the polynomial ring $S = S_1 \otimes \dots \otimes S_r$. Assume that \mathcal{F}_i is the minimal free resolution of I_i for all i . Then the minimal free resolution of I is obtained by $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \dots \otimes \mathcal{F}_r$. In particular, the following hold:

$$(i) \beta_{i,j}(I) = \sum_{\substack{(i_1, \dots, i_r) \in A_{i,r} \\ (j_1, \dots, j_r) \in B_{j,r}}} \beta_{i_1, j_1}(I_1) \cdots \beta_{i_r, j_r}(I_r), \text{ where}$$

$$A_{i,r} = \{(i_1, \dots, i_r) : i_1 + \dots + i_r = i\} \text{ and}$$

$$B_{j,r} = \{(j_1, \dots, j_r) : j_1 + \dots + j_r = j\}.$$

$$(ii) \text{ The Hilbert series of } S/I \text{ is } \mathcal{H}_{S/I}(x, t) = \frac{1 + (-x)^{-(r-1)} \prod_{i=1}^r Q_{S/I_i}(t)}{(1-t)^d}, \text{ where } \mathcal{H}_{S/I_i}(t) = \frac{1 + Q_{S/I_i}(t)}{(1-t)^d} \text{ and } d \text{ is the number of variables of the ring } S. \text{ Note that here we are looking at the ideals } I_i \text{ in the polynomial ring } S.$$

(iii) In particular, if all the ideals are the same (but in polynomial rings on disjoint set of variables), then we have

$$\mathcal{H}_{S/I}(x, t) = \frac{1 + (-x)^{-(r-1)} (Q_{S/I_i}(t))^r}{(1-t)^d}.$$

Example 27. Let $I = \langle x_1x_2x_3, x_2x_4, x_1x_5x_6 \rangle$ and $J = \langle y_1y_2y_3, y_2y_4y_5 \rangle$. The Betti tables of S/I , S/J , $S/(I + J)$, and S/IJ are as follows:

$i \setminus j$	0	1	2	3
Total	1	3	3	1
0	1			
1		1		
2		2	1	
3			2	1

$i \setminus j$	0	1	2
Total	1	2	1
0	1		
1			
2		2	
3			1

$i \setminus j$	0	1	2	3	4	5
Total	1	5	10	10	5	1
0	1					
1		1				
2		4				
3			1			
4			5	1		
5			4	3		
6				6	3	
7					2	1

$i \setminus j$	0	1	2	3	4
Total	1	6	9	5	1
0	1				
1					
2					
3					
4		2			
5		4	3		
6			6	3	
7				2	1

Note that the (i, j) -entry of the table corresponding to S/I is simply $\beta_{i,i+j}(S/I) = \beta_{i-1,i+j}(I)$, and we have only listed the entries corresponding to the nonzero Betti numbers. Since

$$\mathcal{H}_{S/I}(x, t) = \frac{1 - x(t^2 + 2t^3) + x^2(t^4 + 2t^5) - x^3t^6}{(1 - t)^{12}} \quad \text{and} \quad \mathcal{H}_{S/J}(x, t) = \frac{1 - 2xt^3 + x^2t^5}{(1 - t)^{12}},$$

we have that

$$\begin{aligned} &\mathcal{H}_{S_{I+J}}(x, t) \\ &= \frac{(1 - x(t^2 + 2t^3) + x^2(t^4 + 2t^5) - x^3t^6)(1 - 2xt^3 + x^2t^5)}{(1 - t)^{12}} \\ &= \frac{1 - x(t^2 + 4t^3) + x^2(t^4 + 5t^5 + 4t^6) - x^3(t^6 + 3t^7 + 6t^8) + x^4(3t^9 + 2t^{10}) - x^5t^{11}}{(1 - t)^{12}}. \end{aligned}$$

Note that the above formula includes the graded Betti numbers. For example,

$$\begin{aligned} \beta_1(S/(I + J)) &= \beta_{1,3}(S/(I + J)) + \beta_{1,2}(S/(I + J)) \\ &= (\beta_0(S/I)\beta_{1,3}(S/J) + \beta_{1,3}(S/I)\beta_0(S/J)) + \beta_{1,2}(S/I)\beta_0(S/J) \\ &= (2 + 2) + 1 = 5, \end{aligned}$$

which is encoded as the coefficient of $-x$ (for $t = 1$) in the above formula.

Similarly, we have that

$$\begin{aligned} \mathcal{H}_{S_{IJ}}(x, t) &= \frac{1 - x^{-1}(-x(t^2 + 2t^3) + x^2(t^4 + 2t^5) - x^3t^6)(-2xt^3 + x^2t^5)}{(1 - t)^{12}} \\ &= \frac{1 - x(2t^5 + 4t^6) + x^2(3t^7 + 6t^8) - x^3(3t^9 + 2t^{10}) + x^4t^{11}}{(1 - t)^{12}}. \end{aligned}$$

From the above formula we obtain the graded Betti numbers. For example,

$$\beta_0(IJ) = \beta_{0,5}(IJ) + \beta_{0,6}(IJ) = \beta_{0,2}(I)\beta_{0,3}(J) + \beta_{0,3}(I)\beta_{0,3}(J) = 2 + 4 = 6,$$

which is encoded as the coefficient of $-x$ in the above formula.

$$\begin{aligned}\beta_1(IJ) &= \beta_{1,7}(IJ) + \beta_{1,8}(IJ) \\ &= (\beta_{0,2}(I)\beta_{1,5}(J) + \beta_{1,4}(I)\beta_{0,3}(J)) + (\beta_{1,5}(I)\beta_{0,3}(J) + \beta_{0,3}(I)\beta_{1,5}(J)) \\ &= (1 + 2) + (4 + 2) = 9.\end{aligned}$$

The term $x^2(3t^7 + 6t^8)$ in the above formula shows that $\beta_{1,7} = 3$ and $\beta_{1,8} = 6$.

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