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A note on the differences of computably enumerable reals

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Abstract. We show that given any non-computable left-c.e. real α there exists a left-c.e. real β such that $\alpha \neq \beta + \gamma$ for all left-c.e. reals and all right-c.e. reals γ. The proof is non-uniform, the dichotomy being whether the given real α is Martin-Löf random or not. It follows that given any universal machine U, there is another universal machine V such that the halting probability Ω_U of U is not a translation of the halting probability Ω_V of V by a left-c.e. real. We do not know if there is a uniform proof of this fact.

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1 Introduction

The reals which have a computably enumerable left or right Dedekind cut, also known as c.e. reals, play a ubiquitous role in computable analysis and algorithmic randomness. The differences of c.e. reals, also known as d.c.e. reals, form a field under the usual addition and multiplication, as was demonstrated by Ambos-Spies, Weihrauch, and Zheng [ASWZ00]. Raichev [Rai05] and Ng [Ng06] showed that this field is real-closed. Downey, Wu and Zheng [DWZ04] studied the Turing degrees of d.c.e. reals. Clearly d.c.e. reals are Δ_2^0 since they can be computably approximated. Downey, Wu and Zheng [DWZ04] showed that every real which is truth-table reducible to the halting problem is Turing equivalent to a d.c.e. real. However they also showed that there are Δ_2^0 degrees which do not contain any d.c.e. reals. In this strong sense, d.c.e. reals form a strict subclass of the Δ_2^0 reals.

Despite this considerable body of work on d.c.e. reals, the following rather basic question does not have an answer in the current literature. Given a non-computable c.e. real α, is there a c.e. real β such that $\alpha - \beta$ is not a c.e. real? The answer is, perhaps unsurprisingly, positive. We say that a real is left-c.e. or right-c.e. if its left or right Dedekind cut respectively is computably enumerable.

Theorem 1.1. *If α is a non-computable left-c.e. real there exists a left-c.e. real β such that $\alpha \neq \beta + \gamma$ for all left-c.e. and all right-c.e. reals γ.*

An interesting aspect of Theorem 1.1 is that its proof depends crucially on the well-developed theory of Martin-Löf random left-c.e. reals, and in particular the methodology developed by Downey, Hirschfeldt and Nies in [DHN02]. The proof is nonuniform and one has to consider separately the case where α is Martin-Löf random and the case where it is not. We do not know if there is a uniform proof of Theorem 1.1, in the sense that from a left-c.e. approximation to a non-computable real α we can compute a left-c.e. approximation to a real β such that $\alpha \neq \beta + \gamma$ for all left-c.e. and all right-c.e. reals γ.

Let us focus on the connection with the theory of Martin-Löf random left-c.e. reals, as it is crucial in both of the two cases. It follows from the work of Downey, Hirschfeldt and Nies [DHN02] that:

> if α, β are left-c.e. reals and α is Martin-Löf random while β is not, then $\alpha - \beta$ is a Martin-Löf random left-c.e. real. $\qquad\qquad$ (1.0.1)

This, in particular, means that in Theorem 1.1, α is Martin-Löf random if and only if β is Martin-Löf random. Moreover we can use this fact in order to reduce Theorem 1.1 to the following special case, which we prove in Section 3.

Lemma 1.2. *If α is a left-c.e. real which is neither computable nor Martin-Löf random, then there exists a left-c.e. real β (also not Martin-Löf random) such that $\alpha - \beta$ is neither a left-c.e. real nor a right-c.e. real.*

Let us now see how Theorem 1.1 can be derived from this special case. First, assume that the given α is Martin-Löf random. Lemma 1.2 implies the existence of two left-c.e. reals δ_0, δ_1 which are not Martin-Löf random and such that $\delta := \delta_0 - \delta_1$ is neither a left-c.e. nor a right-c.e. real. Indeed, we can start with any non-computable left-c.e. real δ_0 which is not Martin-Löf random (such as the halting problem) and apply Lemma 1.2 in order to get δ_1 with the required properties. Note that δ_1 is necessarily not Martin-Löf random, because otherwise, given that δ_0 is not Martin-Löf random, it would follow from (1.0.1) that $\delta_0 - \delta_1$ would be a right-c.e. real. To establish Theorem 1.1 for this case, we choose $\beta = \alpha + \delta$. First note that $\alpha - \beta$ is not a left-c.e. real or a right-c.e. real, by the choice of δ. Second, $\beta = (\alpha - \delta_1) + \delta_0$ and $\alpha - \delta_1$ is Martin-Löf random by (1.0.1), since α is Martin-Löf random. Then β is a Martin-Löf random left-c.e. real as the sum of

a Martin-Löf random left-c.e. real and another left-c.e. real (a result that was originally proved by Demuth [Dem75]). The case of Theorem 1.1 when α is not Martin-Löf random is exactly Lemma 1.2. We note that, as will become apparent in Section 3, the proof of this case also makes essential use of (1.0.1).

A subclass of the left-c.e. reals are the characteristic functions of c.e. sets (viewed as binary expansions). These reals were called *strongly left-c.e. reals* by Downey, Hirschfeldt and Nies [DHN02] and are highly non-random reals. It will be clear from the discussion of Section 2 that in Theorem 1.1 we cannot (in general) choose the real β to be strongly left-c.e. as in that case, if the given α is Martin-Löf random, then $\alpha - \beta$ is a left-c.e. real. However the following can be proved using standard finite injury methods.

Proposition 1.3 (Properly d.c.e. reals). *There exist strongly left-c.e. reals α, β such that $\alpha - \beta$ is not a left-c.e. real and is not a right-c.e. real.*

We conclude this discussion with a corollary of Theorem 1.1 in terms of halting probabilities. The cumulative work of Solovay [Sol75], Calude, Hertling, Khoussainov and Wang [CHKW01] and Kučera and Slaman [KS01] has shown that the Martin-Löf random left-c.e. reals are exactly the halting probabilities of universal machines. This class remains the same whether we consider prefix-free machines or plain Turing machines. Here we consider Turing machines operating on strings, and given an effective list of all Turing machines (M_e), a Turing machine U is called *universal* if there exists a computable function $e \mapsto \sigma_e$ from numbers to strings such that $U(\sigma_e * \tau) = M_e(\tau)$ for all e and all strings τ. A similar definition applies to universal prefix-free machines, restricted to Turing machines with prefix-free domain.

Halting probabilities, or equivalently Martin-Löf random left-c.e. reals, are all similar in the sense that they all have the same degree with respect to a wide variety of degree structures (see Downey and Hirschfeldt [DH10, Chapter 9]). A number of results have been established, however, which show that halting probabilities may differ in certain ways, depending on the universal machine used. For example, Figueira, Stephan, and Wu [FSW06] showed that for each universal machine U there exists universal machine V such that Ω_U and Ω_V have incomparable truth-table degrees. Their proof consists of considering $\Omega_V = \Omega_U + X$ for a creative set X like the halting problem, and then using the fact from [Ben88, CN97] that no Martin-Löf random real truth-table computes a creative set. Recall that the use of an oracle computation of a set A from a set B is an upper bound (as a function of n) on the largest position in the oracle B queried in the computation of the first n bits of A. Frank Stephan (see [BDG10, Section 6]) showed that for each universal machine U there exists universal machine V such that Ω_U cannot compute Ω_V with use $n + c$ for any constant c. Recently Barmpalias and Lewis-Pye have improved the use-bound in this statement to $n + \log n$, while they also showed that Ω_U, Ω_V can be computed from each other with use $n + 2\log n$, for any universal machines U, V. Along these lines, we can formulate Theorem 1.1 as follows.

Corollary 1.4. *For each universal by adjunction machine U_0 there exists another universal by adjunction machine U_1 such that for all left-c.e. and all right-c.e. reals β we have $\Omega_{U_0} \neq \Omega_{U_1} + \beta$.*

This shows that halting probabilities are not always translations of the halting probability of a fixed universal machine by a left-c.e. or a right-c.e. real.

2 Overview of Martin-Löf random left-c.e. reals

Some familiarity with the basic concepts of algorithmic information theory and the basic methods of computability theory would be helpful for the reader. For such background we refer to one of the monographs

[LV97, DH10, Nie09], where the latter two are more focused on computability theory aspects of algorithmic randomness. The theory of left-c.e. reals has grown into a significant part of modern algorithmic randomness, and is best presented in [DH10, Chapters 5 and 9]. The present section is an original presentation of some facts regarding Martin-Löf random reals that stem from [Sol75, CHKW01, KS01] and are further elaborated on in [DHN02], which are essential for the proof of Theorem 1.1. Moreover, some of these facts are not given explicitly in the sources above, but can be recovered from the proofs.

The systematic study of Martin-Löf random c.e. reals started with Solovay in [Sol75], who showed that Chaitin's halting probability of a prefix-free machine (a well known Martin-Löf random left-c.e. real) has maximum degree in a degree structure that measures the hardness of approximating left-c.e. reals by increasing sequences of rationals. This result was complemented by the work of Calude, Hertling, Khoussainov and Wang [CHKW01] and Kučera and Slaman [KS01], who showed that these maximally hard to approximate left-c.e. reals are exactly the halting probabilities of universal machines, which also coincide with the Martin-Löf random left-c.e. reals. The degree structure introduced in [Sol75] is now known as the *Solovay degrees of left-c.e. reals* and was extensively studied in [DHN02]. An increasing computable sequence of rationals (α_i) that converges to a real α is called a *left-c.e. approximation to α*, denoted $(\alpha_s) \to \alpha$. The Solovay reducibility $\beta \leq_S \alpha$ between left-c.e. reals α, β can be defined equivalently by any of the following clauses:

(a) there exists a rational q such that $q\alpha - \beta$ is left-c.e.

(b) there exist a rational q and $(\alpha_s) \to \alpha$, $(\beta_s) \to \beta$ such that $\beta - \beta_s < q \cdot (\alpha - \alpha_s)$ for all s;

(c) there exist a rational q and $(\alpha_s) \to \alpha$, $(\beta_s) \to \beta$ such that $\beta_{s+1} - \beta_s < q \cdot (\alpha_{s+1} - \alpha_s)$ for all s.

Note that the set of rationals q for which one of the above clauses holds is upward closed - if the clause holds for the rational q then it also holds for all rationals $q' > q$. Although it is not explicitly stated in [DHN02], it follows from the proofs that when $\beta \leq_S \alpha$, the infimums of the rationals q for which the clauses (a), (b) and (c) hold are equal.

Kučera and Slaman [KS01] proved that:

> if (α_s), (β_s) are left-c.e. approximations to α, β respectively and if α is Martin-Löf random, then $\liminf_s [(\alpha - \alpha_s)/(\beta - \beta_s)] > 0$. (2.0.1)

In this sense, Martin-Löf random left-c.e. reals can only have slow left-c.e. approximations, compared to any other left-c.e. real and any left-c.e. approximation to it. Downey, Hirschfeldt and Nies [DHN02] showed that any left-c.e. approximation to a non-random left-c.e. real is considerably faster than every left-c.e. approximation to any Martin-Löf random real, in the sense that:

> if (α_s), (β_s) are left-c.e. approximations to α, β respectively, β is Martin-Löf random and α is not Martin-Löf random, then $\liminf_s [(\alpha - \alpha_s)/(\beta - \beta_s)] = 0$. (2.0.2)

Demuth [Dem75] showed that if α, β are left-c.e. reals and at least one of them is Martin-Löf random, then $\alpha + \beta$ is also Martin-Löf random. Downey, Hirschfeldt and Nies [DHN02] proved that the converse also holds, i.e.:

> if α, β are left-c.e. reals and $\alpha + \beta$ is Martin-Löf random then at least one of α, β is Martin-Löf random. (2.0.3)

We conclude our overview with a proof of (1.0.1) which is essential for the proof of Theorem 1.1, but which is not stated or proved in [DHN02] (although it follows easily from the arguments in that paper). We need

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the following fact which was proved in [DHN02] (but stated in a weaker form) and which is also related to the above discussion regarding clauses (a)-(c).

Lemma 2.1 (Downey, Hirschfeldt and Nies [DHN02])**.** *Suppose that α, β have left-c.e. approximations $(\alpha_s), (\beta_s)$ such that $\forall s \, (\alpha - \alpha_s < q \cdot (\beta - \beta_s))$ for some rational $q > 0$. If $p > q$ is another rational, then there exists a left-c.e. approximation (γ_s) to α such that $\forall s \, (\gamma_{s+1} - \gamma_s < p \cdot (\beta_{s+1} - \beta_s))$.*

Now for (1.0.1), assume that α is Martin-Löf random and β is not Martin-Löf random. By (2.0.2) for each left-c.e. approximation (α_s) to α there exists a left-c.e. approximation (β_s) to β such that $\beta - \beta_s < 2^{-1} \cdot (\alpha - \alpha_s)$ for all s. Then by Lemma 2.1 there exists a left-c.e. approximation $(\gamma_s) \to \beta$ such that $\gamma_{s+1} - \gamma_s < \alpha_{s+1} - \alpha_s$ for all s. This means that the approximation $(\alpha_s - \gamma_s)$ to $\alpha - \beta$ is an increasing left-c.e. approximation. So $\alpha - \beta$ is a left-c.e. real. It remains to show that $\alpha - \beta$ is Martin-Löf random. Since β is not Martin-Löf random, by (2.0.3) it suffices to show that $(\alpha - \beta) + \beta$ is Martin-Löf random. The latter follows from the hypothesis that α is Martin-Löf random.

3   Proof of Lemma 1.2

We can use a priority injury construction. Let $(\gamma_s^i), (\delta_s^i)$ be an effective list of all increasing and decreasing computable sequences of rationals in $(0, 1)$ respectively. Let γ^i be the limit of (γ_s^i) and let δ^i be the limit of (δ_s^i). Given α as in the statement of the lemma, it suffices to construct a left-c.e. real β such that the following conditions are met:

$$\mathcal{L}_i : \; \alpha - \beta \neq \gamma^i \qquad \text{and} \qquad \mathcal{R}_i : \; \alpha - \beta \neq \delta^i.$$

Given an increasing computable sequence of rationals (α_s) that coverges to α, our construction will define an increasing sequence of rationals (β_s) converging to β such that the above requirements are met. We list the requirements in order of priority as $\mathcal{L}_0, \mathcal{R}_0, \mathcal{L}_1, \dots$.

Parameters of the construction.   Let $\beta_0 = 0$. The strategy for $\mathcal{L}_i$ will use a dynamically defined parameter c_i and the strategy for $\mathcal{R}_i$ will use a similar parameter d_i. Let $c_i[0] = d_i[0] = 0$. We say that stage $s + 1$ is $\mathcal{L}_i$-expansionary if $|\alpha_{s+1} - \beta_{s+1} - \gamma_{s+1}^i| < 2^{-c_i[s]}$. Similarly, stage $s + 1$ is $\mathcal{R}_i$-expansionary if $|\alpha_{s+1} - \beta_{s+1} - \delta_{s+1}^i| < 2^{-d_i[s]}$. The strategy for each requirement $\mathcal{L}_i$ will define a left-c.e. real β^i, which will be its contribution toward the global left-c.e. real β. Formally, given the approximations (β_s^i) defined by the requirements $\mathcal{L}_i$ respectively, for each s we define:

$$\beta_s = \sum_{i \leq s} \beta_s^i.$$

If $s + 1$ is $\mathcal{L}_i$-expansionary we let $c_i[s+1] = c_i[s] + 1$, and otherwise we let $c_i[s+1] = c_i[s]$. Similarly, if $s + 1$ is $\mathcal{R}_i$-expansionary we let $d_i[s+1] = d_i[s] + 1$, and if not we let $d_i[s+1] = d_i[s]$. This completes the definition of the parameters c_i, d_i throughout the stages of the construction. At each stage $s + 1$ the strategy for $\mathcal{R}_i$ imposes an automatic restraint on the strategies for $\mathcal{L}_j$ of lower priority, which prohibits any increase of β by more than $2^{-d_i[s+1]}$. All of the strategies for the $\mathcal{L}_i$ requirements will use a fixed Martin-Löf random left-c.e. real $\eta \in (0, 1)$ and an increasing computable rational approximation (η_s) to η. The strategy for each $\mathcal{L}_i$ has an extra parameter q_i, which is updated during the stages s and which dictates the scale at which η is going to affect the growth of (β_s). At stage $s + 1$ we define $q_0[s+1] = \frac{1}{2}$, and for $i > 0$ we define $q_i[s+1]$ to be the least of all $2^{-i-d_j[s+1]-1}$ for $j < i$.

Construction of (β_s). At each stage $s + 1$ and each $i \leq s$, if $s + 1$ is $\mathcal{L}_i$-expansionary we define $\beta^i_{s+1} = \beta^i_s + q_i[s + 1] \cdot (\eta_{s+1} - \eta_t)$, where t is the largest $\mathcal{L}_i$-expansionary stage before $s + 1$ if there is such, and where $t = 0$ otherwise. If $s + 1$ is not $\mathcal{L}_i$-expansionary, we define $\beta^i_{s+1} = \beta^i_s$. This completes the definition of (β_s).

Verification. First we verify that (β_s) reaches a finite limit β. Let β^i be the limit of β^i_s as $s \to \infty$ and note that for each i:

$$\beta^i \leq 2^{-i-1} \cdot \eta < 2^{-i-1} \quad \text{so} \quad \beta = \sum_i \beta^i < 1.$$

Recall the dynamic definition of $c_i[s]$ and $d_i[s]$. It follows that if $c_i[s]$ reaches a limit, requirement $\mathcal{L}_i$ is met. Similarly, if $d_i[s]$ reaches a limit, requirement $\mathcal{R}_i$ is met. We prove both of these statements by induction. Suppose that the claim holds for all $i < n$. Also let s_0 be a stage such that $c_i[s] = c_i[s_0]$ and $d_i[s] = d_i[s_0]$ for all $i < n$ and all $s > s_0$. Then by definition $q_n[s] = q_n[s_0]$ for all $s > s_0$. Let q_n denote the limit $q_n[s_0]$ of $q_n[s]$ from now on. If $c_n[s]$ does not reach a limit, then there are infinitely many $\mathcal{L}_n$-expansionary stages, which implies that $\alpha - \beta = \gamma^n$. Moreover if (t_j) is a monotone enumeration of the $\mathcal{L}_n$-expansionary stages, then $\beta_{t_{s+1}} - \beta_{t_s} > q_n \cdot (\eta_{t_{s+1}} - \eta_{t_s})$ for all s. Since η is Martin-Löf random, this means that β is also Martin-Löf random. But by hypothesis α is not Martin-Löf random, so $\alpha - \beta$ is a Martin-Löf random right-c.e. real. This contradicts the fact that $\alpha - \beta = \gamma$ since right-c.e. reals which have a left-c.e. approximation are computable. It follows that there are only finitely many $\mathcal{L}_n$-expansionary stages, which implies that $c_n[s]$ reaches a limit. Let $s_1 > s_0$ be a stage such that $c_n[s] = c_n[s_1]$ for all $s > s_1$.

It remains to show that $d_n[s]$ reaches a limit. Towards a contradiction, suppose that this is not the case, so that there are infinitely many $\mathcal{R}_n$-expansionary stages. Then it follows that $\alpha - \beta = \delta^n$. Let (t_k) be a computable enumeration of all $\mathcal{R}_n$-expansionary stages. Then $d_n[t_k] = k$ for all k. For each $i > n$ and each k we have $\beta^i - \beta^i_{t_k} \leq 2^{-i-k-1}$ which means that for k large enough that $t_k > s_1$:

$$\beta - \beta_{t_k} < \sum_{i > n} (\beta^i - \beta^i_{t_k}) \leq \sum_{i > n} 2^{-i-k-1} \leq 2^{-k-1}.$$

This means that β is a computable real. Since $\alpha = \delta^n + \beta$ and δ^n is a right-c.e. real, it follows that α is a right-c.e. real. Since α also a left-c.e. real, it must therefore be computable, contrary to hypothesis. So we may conclude that there are finitely many $\mathcal{R}_n$-expansionary stages, which establishes that $\mathcal{R}_n$ is met and d_n reaches a limit. This concludes the induction step and the proof that the constructed real β meets the requirements $\mathcal{L}_n$ and $\mathcal{R}_n$ for all n.

Remark. The reader may wonder why a uniform argument for Theorem 1.1 might not work, i.e. why we needed to divide into two cases according to whether the given real is Martin-Löf random or not. While it is not easy to explain why some things do not work, the immediate answer is that in a construction such as the argument above, if we did not assume that the given real is not Martin-Löf random or we did not code randomness into the real we construct, we do not see a way to argue that requirements $\mathcal{L}_i$ act only finitely often. More generally, if a direct standard uniform construction worked, in our view we could use it to show that *given a left ce real α we can find a left ce real β such that $2\alpha - \beta$ is not left-c.e. and $\alpha - \beta$ not a right-c.e. real.* However we know that this is not possible by one of the results in [BLP16]. This non-uniformity seems to relate to the non-uniformities in the characterization of the halting probabilities in [Sol75, CHKW01, KS01] that we discussed in Section 1. Showing that such non-uniformities are necessary may be an interesting exercise.

4 Proof of Proposition 1.3

We can use a standard priority injury construction. Let $(\gamma^i_s), (\delta^i_s)$ be an effective list of all increasing and decreasing computable sequences of rationals in $(0, 1)$ respectively. Moreover let γ^i be the limit of (γ^i_s) and let δ^i be the limit of (δ^i_s). It suffices to satisfy the following conditions.

$$\mathcal{L}_i: \ \alpha - \beta \neq \gamma^i \quad \text{and} \quad \mathcal{R}_i: \ \alpha - \beta \neq \delta^i$$

Our construction will define increasing sequences $(\alpha_s), (\beta_s)$ of rationals which converge to α, β respectively. Let $\alpha_0 = \beta_0 = 0$. Strategies $\mathcal{L}_i$ will use a parameter c_i which takes values from $\mathbb{N}^{[2i]}$ (i.e. the even numbers) and strategies $\mathcal{R}_i$ will use a parameter d_i which takes values from $\mathbb{N}^{[2i+1]}$. We say that $\mathcal{L}_i$ requires attention at stage $s+1$ if either c_i is undefined, or $c_i[s]$ is defined and $|\alpha_s - \beta_s - \gamma^i_{s+1}| < 2^{-c_i[s]-3}$. Similarly we say that $\mathcal{R}_i$ requires attention at stage $s+1$ if either d_i is undefined, or $d_i[s]$ is defined and $|\alpha_s - \beta_s - \delta^i_{s+1}| < 2^{-d_i[s]-3}$. Strategy $\mathcal{L}_i$ will impose a restraint ℓ_i on α while strategy $\mathcal{R}_i$ will impose a restraint r_i on β. The parameters ℓ_i, r_i will be defined (and possibly redefined) dynamically during the construction, before reaching a limit. We list the requirements in order of priority as $\mathcal{L}_0, \mathcal{R}_0, \mathcal{L}_1, \dots$ and construct α, β as c.e. sets A, B with characteristic sequences the binary expansions of α, β. In this way, the restraints ℓ_i, r_i will apply to the enumerations into A and B respectively. Note that enumerating a number n into A increases $\alpha - \beta$ by 2^{-n} while enumerating n into B decreases $\alpha - \beta$ by 2^{-n}. Initializing requirement $\mathcal{L}_i$ at stage $s+1$ means to let $c_i[s+1], \ell_i[s+1]$ be undefined. Similarly, initializing $\mathcal{R}_i$ at stage $s+1$ means to let $d_i[s+1], r_i[s+1]$ be undefined. If $c_i[s]$ is defined and $\mathcal{L}_i$ is not initialized at stage $s+1$ then we automatically assume that $c_i[s] = c_i[s+1]$. Similarly, if $d_i[s]$ is defined and $\mathcal{R}_i$ is not initialized at stage $s+1$ then we automatically assume that $d_i[s] = d_i[s+1]$.

At stage $s+1$ let i be the least number $\leq s$ such that $\mathcal{L}_i$ or $\mathcal{R}_i$ requires attention. If there is no such number, go to the next stage. Otherwise, first assume that $\mathcal{L}_i$ requires attention at stage $s+1$. If $c_i[s]$ is not defined, let $c_i[s+1]$ be the least number in $\mathbb{N}^{[2i]}$ which is larger than any value of any parameter defined so far in the construction (in particular larger than all previous values of c_i and larger than any restraint r_j on β which is currently defined). If, on the other hand $c_i[s]$ is defined, then enumerate it into B, define $\ell_i[s+1] = c_i[s] + 3$ and initialize all $\mathcal{L}_{j+1}, \mathcal{R}_j$ for all $j \geq i$. In this latter case we say that $\mathcal{L}_i$ acts at stage $s+1$.

Second, assume that $\mathcal{R}_i$ requires attention at stage $s+1$. If $d_i[s]$ is not defined, let $d_i[s+1]$ be the least number in $\mathbb{N}^{[2i+1]}$ which is larger than any value of any parameter defined so far in the construction (in particular larger than all previous values of d_i and larger than any restraint ℓ_j on α which is currently defined). If, on the other hand $d_i[s]$ is defined, then enumerate it into A, define $r_i[s+1] = d_i[s] + 3$ and initialize all $\mathcal{L}_j, \mathcal{R}_j$ for all $j \geq i$. In this latter case we say that $\mathcal{R}_i$ acts at stage $s+1$.

The construction defined computable enumerations of the sets A, B which in turn define computable non-decreasing rational approximations $(\alpha_s), (\beta_s)$ to the reals α, β. Since A, B are c.e. and no c.e. set is Martin-Löf random, we immediately get that α, β are not random. It remains to show that α, β meet the requirements $\mathcal{L}_i$ and $\mathcal{R}_i$. Note that if $\mathcal{L}_i$ acts at stage $s+1$ and is not initialized at any later stage, then it will not require attention at any later stage. Indeed, in this case no higher priority requirement will act at later stages, and both $c_i[t]$ and $\ell_i[t+1]$ remain constant for all $t \geq s$. Let c_i, ℓ_i denote their final values respectively. Since $\mathcal{L}_i$ required attention at stage $s+1$ we have $|\alpha_s - \beta_s - \gamma^i_{s+1}| < 2^{-c_i-3}$. Moreover $\beta_{s+1} - \beta_s = 2^{-c_i}$ and $\alpha_s = \alpha_{s+1}$. So $\alpha_{s+1} - \beta_{s+1} < \gamma^i_{s+1} - 2^{-c_i-1}$ and since $\ell_i = c_i + 3$ we have $\alpha_t - \alpha_{s+1} < 2^{-c_i-2}$ for all $t > s$. Therefore $\alpha_t - \beta_t < \gamma^i_t - 2^{-c_i-2}$ for all $t > s$ and $\mathcal{L}_i$ will not require attention at any stage after s. Moreover we also get

that $\alpha - \beta \le \gamma^i - 2^{-c_i-2}$ which means that in this case condition $\mathcal{L}_i$ is met. We have shown that:

> If $\mathcal{L}_i$ *acts* at stage $s + 1$ and is not initialized at any later stage, then it will not require attention at any later stage and is satisfied.
> (4.0.1)

An entirely similar argument shows that:

> If $\mathcal{R}_i$ *acts* at stage $s + 1$ and is not initialized at any later stage, then it will not require attention at any later stage and is satisfied.
> (4.0.2)

It remains to use (4.0.1) and (4.0.2) inductively in order to show that $\alpha - \beta$ meets $\mathcal{L}_i, \mathcal{R}_i$ for all i. Note that $\mathcal{L}_0$ cannot be initialized. So c_0 will be defined and remain constant for the rest of the stages. If $\mathcal{L}_0$ never acts, then it does not require attention after the first time that it required (and received) attention. This means that $|\alpha_s - \beta_s - \gamma^0| \ge 2^{-c_i-3}$ for all but finitely many stages s, so $\alpha - \beta \ne \gamma^i$. If it does act at some stage, then by (4.0.1) it is satisfied and never requires attention at any later stage. Now inductively assume that the same is true for all $\mathcal{L}_i, \mathcal{R}_i, i < e$. Then consider a stage s_0 after which none of $\mathcal{L}_i, \mathcal{R}_i, i < e$ acts or requires attention. Then the same argument shows that $\mathcal{L}_e$ does not act or require attention after a certain stage, and is met. The same argument applies to $\mathcal{R}_e$ through property (4.0.2), and this concludes the induction step. We can conclude that $\alpha - \beta$ meets $\mathcal{L}_i, \mathcal{R}_i$ for all i.

Remark. The referee has pointed out that a proof of Proposition 1.3 may be given without a direct construction. Consider two c.e. sets A, B such that $A - B$ has properly d.c.e. degree, i.e. there is no c.e. set which is Turing equivalent to $A - B$. Such c.e. sets were originally constructed in Cooper [Coo71], and the standard construction gives $B \subseteq A$. Let α, β be the reals in $(0, 1)$ whose binary expansions are the characteristic sequences of A, B respectively. Then the binary expansion of $\alpha - \beta$ is the characteristic sequence of $A - B$. If $\alpha - \beta$ had a left-c.e. or a right-c.e. approximation, then $A - B$ would be Turing equivalent to the left or the right Dedekind cut of $\alpha - \beta$ which would be a c.e. set. This would contradict the choice of $A - B$. Hence α, β have the required properties.

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