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Stable laws and Beurling Kernels
by
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Abstract. We identify a close relation between stable distributions and the limiting homomorphisms central to the theory of regular variation. In so doing some simplifications are achieved in the direct analysis of these laws in Pitman and Pitman [PitP]; stable distributions are themselves linked to homomorphy.

Keywords. Stable laws, Beurling regular variation, quantifier weakening, homomorphism, Goldie equation, Gołąb-Schinzel equation, Levi-Civita equation.
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1 Introduction

This note takes its inspiration from Pitman and Pitman's approach [PitP], in this volume, to the characterization of stable laws *directly* from their characteristic functional equation [PitP, (2.2)], ($ChFE$) below, which they complement with the derivation of parameter restrictions by an appeal to *Karamata* (classical) regular variation (rather than *indirectly* as a special case of the Lévy-Khintchine characterization of infinitely decomposable laws – cf. [PitP, §4]). We take up their functional-equation tactic with three aims in mind. The first and primary one is to extract a hidden connection with the more general theory of *Beurling regular variation*, which embraces the original Karamata theory and its later 'Bojanić-Karamata-de Haan' variants. (This has received renewed attention: [BinO1,4], [Ost1]). The connection is made via another functional equation, the *Goldie equation*

$$\kappa(x+y)-\kappa(x)=\gamma(x)\kappa(y) \qquad (x,y\in\mathbb{R}_+), \qquad (GFE)$$

with *vanishing side condition* $\kappa(0)=0$ and *auxiliary function* γ, or more properly with its multiplicative equivalent:

$$K(st)-K(s)=G(s)K(t) \qquad (s,t\in\mathbb{R}_+), \qquad (GFE_\times)$$

with corresponding side condition $K(1)=0$; the additive variant arises first in [BinG] (see also [BinGT, Lemma 3.2.1 and Th. 3.2.5]), but has only latterly been so named in recognition of its key role both there and in the recent developments [BinO2,3], inspired both by *Beurling slow variation* ([BinGT, §2.11]) and by its generalizations [BinO1,4] and [Ost1]. This equation describes the family of *Beurling kernels* (the asymptotic homomorphisms of Beurling regular variation), that is, the functions K_F arising as locally uniform limits of the form

$$K_F(t):=\lim_{x\to\infty}[F(x+t\varphi(x))-F(x)], \qquad (BKer)$$

for $\varphi(.)$ ranging over *self-neglecting* functions (SN). (See [Ost1, 2] for the larger family of kernels arising when $\varphi(.)$ ranges over the *self-equivarying* functions SE, both classes recalled in the complements section §4.1.)

A secondary aim is achieved in the omission of extensive special-case arguments for the limiting cases in the Pitman analysis (especially the case of characteristic exponent $\alpha = 1$ in [PitP, §5.2 – affecting parts of §8]), employing here instead the more natural approach of interpreting the 'generic' case 'in the limit' via the L'Hospital rule. A final general objective of further streamlining is achieved, en passant, by telescoping various cases into one, simple, group-theoretic argument; this helps clarify the 'group' aspects as distinct from 'asymptotics', which relate parameter restrictions to tail balance – see the Remark in §3(a).

A random variable X has a *stable law* if for each $n \in \mathbb{N}$ the law of the random walk $S_n := X_1 + ... + X_n$, where the n steps are independent and with law identical to X, is of the same type, i.e. the same in distribution up to scale and location:

$$S_n \sim a_n X + b_n,$$

for some real constants a_n, b_n with $a_n > 0$ – cf. [Fel, VI.1] and [PitP, (1.1)]. These laws may be characterized by the *characteristic functional equation* (of the characteristic function of X, $\varphi(t) = \mathbb{E}[e^{\mathrm{i}tX}]$), as in [PitP, (2.2)]:

$$\varphi(t)^n = \varphi(a_n t)\exp(\mathrm{i}b_n t) \qquad (n \in \mathbb{N},\ t \in \mathbb{R}_+). \qquad (ChFE)$$

The standard way of solving ($ChFE$) is to deduce the equations satisfied by the functions $a : n \mapsto a_n$ and $b : n \mapsto b_n$. Pitman and Pitman [PitP] proceed directly by proving the map a *injective,* then extending the map b to $\mathbb{R}_+ := (0, \infty)$, and exploiting the classical Cauchy (or Hamel) exponential functional equation (for which see [AczD] and [Kuc]):

$$K(xy) = K(x)K(y) \qquad (x, y \in \mathbb{R}_+); \qquad (CEE)$$

(CEE) is satisfied by $K(.) = a(.)$ on the smaller domain $\mathbb{N}$, as a consequence of ($ChFE$). See [RamL] for a similar, but less self-contained account. For other applications see the recent [GupJTS], which characterizes 'generalized stable laws'.

We show in §2 the surprising equivalence of ($ChFE$) with the fundamental equation (GFE) of the recently established theory of *Beurling regular variation.* There is thus a one-to-one relation between Beurling kernels arising through ($BKer$) and the solutions of ($ChFE$), amongst which are the one-dimensional stable distributions. This involves passage from discrete to continuous, a normal feature of the theory of regular variation (see [BinGT,1.9]) which, rather than unquestioningly adopt, we track carefully via the Lemma and Corollary of §2: the ultimate justification here is the extension of a to $\mathbb{R}_+$ (Ger's extension theorem [Kuc, §18.7] being thematic here), and the continuity of characteristic functions.

The emergence of a particular kind of functional equation, one interpretable as a *group* homomorphism (see §4.3), is linked to the simpler than usual form here of ‘probabilistic associativity’ (as in [Bin]) in the incrementation process of the stable random walk; in more general walks, functional equations (and integrated functional equations – see [RamL]) arise over an associated *hypergroup*, as with the Kingman-Bessel hypergroup and Bingham-ultraspherical hypergroup (see [Bin] and [BloH]). We return to these matters, and connections with the theory of flows, elsewhere – [Ost3].

The material is organized as follows. Below we identify the solutions to (GFE) and in §2 we prove equivalence of (GFE) and $(ChFE)$; our proof is self-contained modulo the (elementary) result that for φ a characteristic function $(ChFE)$ implies $a_n = n^k$ for some $k > 0$. Then in §3 we read off the form of the characteristic functions of the stable laws. We conclude in §4 with complements describing the families SN and SE mentioned above, and identifying the group structure implied, or ‘encoded’, by $(GFE_\times)$ to be $(\mathbb{R}_+, \times)$, the multiplicative positive reals.

The following result, which has antecedents in several settings (some cited below), is key; on account of its significance, this has recently received further attention in [BinO3, esp. Th. 3], [Ost2, esp. Th. 1], to which we refer for background – cf. §4.2.

Theorem GFE ([BinO3, Th. 1], [BojK, (2.2)], [BinGT, Lemma 3.2.1]; cf. [AczG]). *For $\mathbb{C}$-valued functions κ and γ with γ locally bounded at 0 with $\gamma(0) = 1$ and $\gamma \neq 1$ except at 0, if $\kappa \not\equiv 0$ satisfies (GFE), subject to the side condition $\kappa(0) = 0$,*
– then for some $\gamma_0, \kappa_0 \in \mathbb{C}$:

$$\gamma(u) = e^{\gamma_0 u} \text{ and } \kappa(x) \equiv \kappa_0 H_{\gamma_0}(x) := \kappa_0 \int_0^x \gamma(u)du = \kappa_0(e^{\gamma_0 x} - 1)/\gamma_0,$$

under the usual L’Hospital convention for interpreting $\gamma_0 = 0$.

Remarks. 1. The cited proof is ostensibly for $\mathbb{R}$-valued $\kappa(.)$ but immediately extends to $\mathbb{C}$-valued κ. Indeed, in brief, the proof rests on symmetry:

$$\gamma(v)\kappa(u) + \kappa(v) = \kappa(u+v) = \kappa(v+u) = \gamma(u)\kappa(v) + \kappa(u).$$

So, for u, v not in $\{x : \gamma(x) = 1\}$, an additive subgroup,

$$\kappa(u)[\gamma(v) - 1] = \kappa(v)[\gamma(u) - 1] : \qquad \frac{\kappa(u)}{\gamma(u) - 1} = \frac{\kappa(v)}{\gamma(v) - 1} = \kappa_0,$$

as in [BinGT, Lemma 3.2.1]. If $\kappa(.)$ is to satisfy (GFE), then $\gamma(.)$ needs to satisfy (CEE).

The notation H_ρ (originating in [BojK]) is from [BinGT, Ch. 3: de Haan theory] and, modulo exponentiation, links to the ‘inverse’ functions $\eta_\rho(t) = 1+\rho t$ (see §4.3) which permeate regular variation (albeit long undetected), a testament to the underlying *flow* and *group* structure, for which see especially [BinO3,4].

The Goldie equation is a special case of the *Levi-Civita equations*; for a text-book treatment of their solutions for domain a semigroup and range $\mathbb{C}$ see [Ste,

Ch. 5].
2. We denote the constants γ_0 and κ_0 more simply by γ and κ, whenever context permits. To prevent conflict with the γ of [PitP, §5.1] we denote that here by $\gamma_{\mathrm{P}}(k)$, showing also dependence on the index of growth of a_n: see §3(b).
3. To solve $(GFE_\times)$ write $s = e^u$ and $t = e^v$ obtaining (GFE); then

$$\begin{aligned} G(e^u) &= \gamma(u) = e^{\gamma u}: \qquad G(s) = s^\gamma \\ K(e^u) &= \kappa(u) = \kappa \cdot (e^{\gamma u} - 1)/\gamma: \qquad K(s) = \kappa \cdot (s^\gamma - 1)/\gamma. \end{aligned}$$

4. Alternative regularity conditions, yielding continuity and the same H_γ conclusion, include in [BinO3, Th. 2] the case of $\mathbb{R}$-valued functions with $\kappa(.)$ and $\gamma(.)$ both non-negative on $\mathbb{R}_+$ with $\gamma \neq 1$ except at 0 (as then either $\kappa \equiv 0$, or both are continuous).

2 Reduction to the Goldie Equation

In this section we establish a Proposition connecting $(ChFE)$ with $(GFE_\times)$ and so stable laws with Beurling kernels. In the interests of brevity[1], this makes use of a well-known result, concerning the norming constants (cf. [Fel, Th. VI.1.], [PitP, Lemma 5.3]), that $a : n \mapsto a_n$ satisfies $a_n = n^k$ for some $k > 0$, so is extendable to a continuous surjection onto $\mathbb{R}_+ := (0, \infty)$:

$$\tilde{a}(\nu) = \nu^k \qquad (\nu > 0);$$

this is used below to justify the validity of the definition

$$f(t) := \log \varphi(t) \qquad (t > 0),$$

with log here the principal logarithm, a tacit step in [PitP, §5.1], albeit based on [PitP, Lemma 5.2]. We write $a_{m/n} = \tilde{a}_{m/n} = a_m/a_n$ and put

$$\mathbb{A}_{\mathbb{N}} := \{a_n : n \in \mathbb{N}\}, \qquad \mathbb{A}_{\mathbb{Q}} := \{a_{m/n} : m, n \in \mathbb{N}\}.$$

The Lemma below reproves an assertion from [PitP, Lemma 5.2], but without assuming that φ is a characteristic function. Its Corollary needs no explicit formula for $b_{m/n}$, since the term will eventually be eliminated.

Lemma. *For continuous $\varphi \not\equiv 0$ satisfying $(ChFE)$ with $a_n = n^k$, φ has no zeros on $\mathbb{R}_+$.*

Proof. If $\varphi(\tau) = 0$ for some $\tau > 0$, then $\varphi(a_m \tau) = 0$ for all m, by $(ChFE)$. Again by $(ChFE)$, $|\varphi(\tau a_m/a_n)|^n = |\varphi(a_m \tau)| = 0$, so φ is zero on the dense subset of points $\tau a_m/a_n$; then, by continuity, $\varphi \equiv 0$ on $\mathbb{R}_+$, a contradiction. □

[1] On this point, see the fuller arXiv version.

Corollary. *The equation* $(ChFE)$ *holds on the dense subgroup* $\mathbb{A}_{\mathbb{Q}}$*: there are constants* $\{b_{m/n}\}_{m,n\in\mathbb{N}}$ *with*

$$\varphi(t)^{m/n} = \varphi(a_{m/n}t)\exp(\mathrm{i}b_{m/n}t).$$

Proof. Taking t/a_n for t in $(ChFE)$:

$$\varphi(t/a_n)^n = \varphi(t)\exp(\mathrm{i}b_n/a_n t),$$

so by the Lemma, using principal values,

$$\varphi(t)^{1/n} = \varphi(t/a_n)\exp(-\mathrm{i}tb_n/na_n).$$

From here, as $a_{m/n} = a_m/a_n$,

$$\begin{aligned}\varphi(t)^{m/n} &= \varphi(t/a_n)^m \exp(-\mathrm{i}tmb_n/na_n) = \varphi(a_{m/n}t)\exp(\mathrm{i}t[b_m - m/nb_n]/a_n) \\ &= \varphi(a_{m/n}t)\exp(\mathrm{i}b_{m/n}t),\end{aligned}$$

the last term, like the remaining ones, depending only on m/n. □

Our main result below, on equational equivalence, uses a condition $(G_{\mathbb{A},\mathbb{R}_+})$ applied to the dense subgroup $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$. This is a *quantifier weakening* relative to (GFE) and is similar to a condition with all variables ranging over $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ denoted $(G_{\mathbb{A}})$ in [BinO3], to which we refer for background on quantifier weakening. In the Proposition below we may also impose just $(G_{\mathbb{A}_{\mathbb{Q}}})$, granted continuity of φ.

Proposition. *For* φ *continuous and* $a_n = n^k$*, the functional equation (ChFE) is equivalent to*

$$K(st) - K(s) = K(t)G(s) \qquad (s \in \mathbb{A}, t \in \mathbb{R}_+), \qquad (G_{\mathbb{A},\mathbb{R}_+})$$

for either of $\mathbb{A} = \mathbb{A}_{\mathbb{N}}$ *or* $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$*, both with side condition* $K(0) = 1$*; the latter directly implies* $(GFE_\times)$*. The correspondence is given by:*

$$K(t) = \begin{cases} f(t)/t, & \text{if } f(1) = 0, \\ (f(t)/tf(1)) - 1, & \text{if } f(1) \neq 0. \end{cases}$$

Proof. By the Lemma, using principal values, $(ChFE)$ may be re-written as

$$\varphi(t)^{n/t} = \varphi(a_n t)^{1/t}\exp(\mathrm{i}b_n) \qquad (n \in \mathbb{N},\ t \in \mathbb{R}_+).$$

From here, on taking principal logarithms and adjusting notation ($f := \log\varphi$, $h(n) = -\mathrm{i}b_n$, and $g(n) := a_n \in \mathbb{R}_+$), pass first to the form

$$f(g(n)t)/t = nf(t)/t + h(n) \qquad (n \in \mathbb{N}, t \in \mathbb{R}_+);$$

here the last term does not depend on t, and is defined for each n so as to achieve equality. Then, with $s := g(n) \in \mathbb{R}_+$, replacement of n by $g^{-1}(s)$, valid by injectivity, gives, on cross-multiplying by t,

$$f(st) = g^{-1}(s)f(t) + h(g^{-1}(s))t.$$

As $s, t \in \mathbb{R}_+$, take $F(t) := f(t)/t$, $G(s) := g^{-1}(s)/s$, $H(s) := h(g^{-1}(s))/s$; then

$$F(st) = F(t)G(s) + H(s) \qquad (s \in \mathbb{A}_{\mathbb{N}}, t \in \mathbb{R}_+). \tag{$\dagger$}$$

This equation contains *three* unknown functions: F, G, H (cf. the Pexider-like formats considered in [BinO2, §5]), but we may reduce the number of unknown functions to *two* by entirely eliminating[2] H. The elimination argument splits according as $F(1) = f(1)$ is zero or not.
Case 1: $f(1) = 0$ (i.e. $\varphi(1) = 1$); taking $t = 1$ in ($\dagger$) yields $F(s) = H(s)$, and so $(G_{\mathbb{A}_{\mathbb{N}},\mathbb{R}_+})$ holds for $K = F$, with side condition $K(1) = 0$ $(= F(1))$.
Case 2: $f(1) \neq 0$; then, with $\tilde{F} := F/F(1)$ and $\tilde{H} := H/F(1)$ in ($\dagger$),

$$\tilde{F}(st) = \tilde{F}(t)G(s) + \tilde{H}(s), \text{ and } \tilde{F}(1) = 1,$$

for $s \in \mathbb{A}, t \in \mathbb{R}_+$. Taking again $t = 1$ gives

$$\tilde{F}(s) = G(s) + \tilde{H}(s).$$

Setting

$$K(t) := \tilde{F}(t) - 1 = F(t)/F(1) - 1 \tag{$\dagger\dagger$}$$

(so that $K(1) = 0$), and using $\tilde{H} = \tilde{F} - G$ in ($\dagger$) gives

$$\begin{aligned} \tilde{F}(st) &= \tilde{F}(t)G(s) + \tilde{F}(s) - G(s), \\ (\tilde{F}(st) - 1) - (\tilde{F}(s) - 1) &= (\tilde{F}(t) - 1)G(s), \\ K(st) - K(s) &= K(t)G(s). \end{aligned}$$

That is, K satisfies $(G_{\mathbb{A},\mathbb{R}_+})$ with side condition $K(1) = 0$.

In summary: in both cases elimination of H yields $(G_{\mathbb{A},\mathbb{R}_+})$ and the side condition of vanishing at the identity.

So far, in $(G_{\mathbb{A},\mathbb{R}_+})$ above, t ranges over $\mathbb{R}_+$ whereas s ranges over $\mathbb{A}_{\mathbb{N}} = \{a_n : n \in \mathbb{N}\}$, but s may as well range over over $\{a_{m/n} : m, n \in \mathbb{N}\}$, by the Corollary. As before, since $a : n \mapsto a_n$ has $\tilde{a}$ as its continuous extension to a bijection onto $\mathbb{R}_+$ and φ is continuous, we conclude that s may as well range over $\mathbb{R}_+$, yielding the multiplicative form of the Goldie equation $(GFE_\times)$ with the side-condition of vanishing at the identity. □

Remarks. 1. As in [PitP, §5], we consider only *non-degenerate* stable distributions, consequently 'Case 1' will not figure below (as this case yields an arithmetic distribution – cf. [Fel, XVI.1 Lemma 4], so here concentrated on 0).
2. In 'Case 2' above, $\tilde{H}(st) - \tilde{H}(s) = \tilde{H}(t)G(s)$, since $G(st) = G(s)G(t)$, by Remark 3 of §1. So $\tilde{H}(e^u) = \kappa H_\gamma(u) = \kappa \cdot (e^{\gamma u} - 1)/\gamma$. We use this in §3.

[2] This loses the "affine action": $K \mapsto G(t)K + H(t)$.

3 Stable laws: their form

This section demonstrates how to 'telescope' several cases of the [PitP] analysis into one, and to make L'Hospital's Rule carry the burden of the 'limiting' case $\alpha = 1$. At little cost, we also deduce the form of the location constants b_n, without needing the separate analysis conducted in [PitP, §5.2].

We break up the material into steps, beginning with a statement of the result.

(a) *Form of the law.* The *form* of φ for a non-degenerate stable distribution is an immediate corollary of Theorem GFE (§1) applied to (††) above. For some $\gamma \in \mathbb{R}$, $\kappa \in \mathbb{C}$ and with $A := \kappa/\gamma$ and $B := 1 - A$,

$$f(t) = \log\varphi(t) = \begin{cases} f(1)(At^{\gamma+1} + Bt), & \text{for } \gamma \neq 0, \\ f(1)(t + \kappa t \log t), & \text{with } \gamma = 0, \end{cases} \qquad (t > 0).$$

Here $\alpha := \gamma + 1$ is the *characteristic exponent*. From this follows a formula for $t < 0$ (by complex conjugation– see below). The connection with [PitP, §5 at end] is given by:
(i) $f(1) := \log\varphi(1) = -c + \mathrm{i}y$ (with $c > 0$, as $|\varphi(t)| < 1$ for $t > 0$);
(ii) $f(1)\kappa = -\mathrm{i}\lambda$. So $f(1)B = -c + \mathrm{i}(y + \lambda/\gamma)$, and $\kappa = \lambda(-y + \mathrm{i}c)/(c^2 + y^2)$.

Remark. We note, for the sake of completeness, that restrictions on the two parameters α and κ (equivalently γ and κ) follow from asymptotic analysis of the 'initial' behaviour of the characteristic function φ (i.e. near the origin). This is equivalent to the 'final' or tail behaviour (i.e. at infinity) of the corresponding distribution function. Specifically, the 'dominance ratio' of the imaginary part of the *dominant* behaviour in $f(t)$ to the value c (as in (i) above) relates to the 'tail balance' ratio β of [PitP, (6.10)], i.e. the asymptotic ratio of the distribution's tail difference to its tail sum – cf. [PitP, §8]. Technical arguments, based on Fourier inversion, exploit the regularly varying behaviour as $t \downarrow 0$ (with index of variation α – see above) in the real and imaginary parts of $1 - \varphi(t)$ to yield the not unexpected result [PitP, Th. 6.2] that, for $\alpha \neq 1$, the dominance ratio is proportional to the tail-balance ratio β by a factor equal to the ratio of the sine and cosine variants of Euler's Gamma integral[3] (on account of the dominant power function) – compare [BinGT, Th. 4.10.3].

(b) *On notation.* The parameter $\gamma := \alpha - 1$ is linked to the auxiliary function G of (GFE); this usage of γ conflicts with [PitP], where two letters are used for the constant ratio $b_n/n : \lambda$ for the 'case $\alpha = 1$', and otherwise γ (following Feller [Fel, VI.1 Footnote 2]). The latter we denote by $\gamma_{\mathrm{P}}(k)$, reflecting the k value in the 'case $\alpha = 1/k \neq 1$'. In (d) below it emerges that $\gamma_{\mathrm{P}}(1+) = \lambda$.

(c) *Verification of* (a). By Remark 1 of §2, only the second case of the Proposition applies: the function $K(t) = \tilde{F}(t) - 1 = f(t)/tf(1) - 1$ solves $(GFE_\times)$

[3] In view of that factor's key role, a quick and elementary derivation is offered in the Appendix of the fuller arXiv version of this note (for $0 < \alpha < 1$).

with side-condition $K(1) = 0$. Writing $t = e^u$ (as in §1 Remark 2) yields

$$f(t)/tf(1) = f(e^u)e^{-u}/f(1) = 1 + K(e^u) = \kappa(u) = 1 + \kappa(e^{\gamma u} - 1)/\gamma,$$

for some complex κ and $\gamma \neq 0$ (with passage to $\gamma = 0$, in the limit, to follow). So, for $t > 0$, with $A := \kappa/\gamma$ and $B := 1 - A$, as above,

$$f(t) = \log \varphi(t) = f(1)t\,(1 + \kappa\,(t^\gamma - 1)\,/\gamma) = f(1)(At^\alpha + Bt), \text{ with } \alpha = \gamma + 1.$$

On the domain $t > 0$, this agrees with [PitP, (5.5)]; for $t < 0$ the appropriate formula is immediate via complex conjugation, verbatim as in the derivation of [PitP, (5.5)], save for the γ usage. To cover the case $\gamma = 0$, apply the L'Hospital convention; as in [PitP, (5.8)], for $t > 0$ and $u > 0$ and some $\kappa \in \mathbb{C}$,

$$\kappa(t) := f(e^t)e^{-t}/f(1) = 1 + \kappa t : \qquad f(u) = f(1)(u + \kappa u \log u).$$

(d) *Location parameters: general case* $\alpha \neq 1$. Here $\gamma = \alpha - 1 \neq 0$. From the proof of the Proposition, $G(t) := g^{-1}(e^t)e^{-t}$; so

$$g^{-1}(e^t) = e^t e^{\gamma t} = e^{\alpha t}.$$

Put $k = 1/\alpha$; then

$$v = g^{-1}(u) = u^\alpha : \qquad u = g(v) = v^{1/\alpha} = v^k,$$

confirming $a_n = g(n) = n^k$, as in [PitP, Lemma 5.3]. (Here $k > 0$, as *strict* monotonicity was assumed in the Proposition). Furthermore, as in Remark 2 of §2, $\kappa \cdot (e^{\gamma t} - 1)/\gamma = \tilde{H}(e^t) = h(g^{-1}(e^t))e^{-t}/f(1)$; so

$$\begin{aligned} h(g^{-1}(e^t)) &= f(1)\kappa \cdot (e^{\alpha t} - e^t)/\gamma \\ h(u) &= f(1)\kappa \cdot (u - u^{1/\alpha})/\gamma = f(1)(\kappa/\gamma)(u - u^k), \end{aligned}$$

where $\gamma = \alpha - 1 = (1 - k)/k$. So $b_n = \mathrm{i}h(n) = \mathrm{i}f(1) \cdot (\kappa/\gamma) \cdot (n - n^k)$, as in the Pitman analysis: see [PitP, §5.1]. Here b_n is real, since $f(1)\kappa = -\mathrm{i}\lambda$, according to (a) above and conforming with [PitP, §5.1]. So as $b_n/n = \gamma_{\mathrm{P}}(k)$, similarly to [PitP, end of proof of Lemma 4.1], again as $f(1)\kappa = -\mathrm{i}\lambda$,

$$\lim_{k\to1} \gamma_{\mathrm{P}}(k) = \mathrm{i}f(1)\kappa \cdot \lim_{k\to1} k\frac{1 - n^{k-1}}{k - 1} = \lambda \log n.$$

(e) *Location parameters: special case* $\alpha = 1$. Here $\gamma = 0$. In (c) above the form of g specializes to

$$g^{-1}(e^t) = e^t : \qquad g(u) = u.$$

Applying the L'Hospital convention yields the form of h: for $t > 0$ and $u > 0$,

$$h(g^{-1}(e^t)) = h(e^t) = f(1)\kappa t e^t : \qquad h(u) = f(1)\kappa \cdot u \log u;$$

so, as in [PitP, (5.8)], $b_n = \lambda n \log n$ (since $b_n = \mathrm{i}h(n)$ and again $\lambda = \mathrm{i}f(1)\kappa$).

4 Complements

1. Self-neglecting and self-equivarying functions. Recall (cf. [BinGT, §2.11]) that a self-map φ of $\mathbb{R}_+$ is *self-neglecting* ($\varphi \in SN$) if

$$\varphi(x + t\varphi(x))/\varphi(x) \to 1 \text{ locally uniformly in } t \text{ for all } t \in \mathbb{R}_+, \qquad (SN)$$

and $\varphi(x) = o(x)$. This traditional restriction may be usefully relaxed in two ways, as in [Ost1]: firstly, in imposing the weaker order condition $\varphi(x) = O(x)$, and secondly by replacing the limit 1 by a general limit function η, so that

$$\varphi(x + t\varphi(x))/\varphi(x) \to \eta(t) \text{ locally uniformly at } t \text{ for all } t \in \mathbb{R}_+. \qquad (SE)$$

Such a φ is called *self-equivarying* in [Ost1], and the limit function $\eta = \eta^\varphi$ necessarily satisfies the equation

$$\eta(u + v\eta(u)) = \eta(u)\eta(v) \text{ for } u, v \in \mathbb{R}_+ \qquad (BFE)$$

(this is a special case of the Gołąb-Schinzel equation – see also e.g. [Brz], or [BinO2], where the equation above is termed the *Beurling functional equation*). As $\eta \geq 0$, imposing the natural condition $\eta > 0$ (on $\mathbb{R}_+$) implies that it is continuous and of the form

$$\eta(t) = 1 + \rho t, \qquad \text{for some } \rho \geq 0$$

(see [BinO2]); the case $\rho = 0$ recovers SN. A function $\varphi \in SE$ has the representation

$$\varphi(t) \sim \eta^\varphi(t) \int_1^t e(u)du \text{ for some continuous } e \to 0$$

(where $f \sim g$ if $f(x)/g(x) \to 1$, as $x \to \infty$) and the second factor is in SN (see [BinO1, Th. 9], [Ost1]).

2. Theorem GFE. This theorem has antecedents in [Acz] and [Chu], [Ost 2, Th. 1], and is generalized in [BinO2, Th.3]). It is also studied in [BinO3] and [Ost2].

3. Homomorphisms and random walks. In the context of a ring, the 'Gołąb-Schinzel functions' $\eta_\rho(t) = 1 + \rho t$, as above, were used by Popa and Javor (see [Ost2] for references) to define associated (generalized) *circle operations*:

$$a \circ_\rho b = a + \eta_\rho(a)b = a + (1 + \rho a)b = a + b + \rho ab.$$

(Note that $a \circ_1 b = a + b + ab$ is the familiar circle operation, and $a \circ_0 b = a + b$.) These were studied in the context of $\mathbb{R}$ in [Ost2, §3.1]; it is straightforward to lift that analysis to the present context of the ring $\mathbb{C}$, yielding the *complex circle groups*

$$\mathbb{C}_\rho := \{x \in \mathbb{C} : 1 + \rho x \neq 0\} = \mathbb{C}\backslash\{\rho^{-1}\} \qquad (\rho \neq 0).$$

Since

$$\begin{aligned} (1 + \rho a)(1 + \rho b) &= 1 + \rho a + \rho b + \rho^2 ab = 1 + \rho[a + b + \rho ab], \\ \eta_\rho(a)\eta_\rho(a) &= \eta_\rho(a \circ_\rho b), \end{aligned}$$

$\eta_\rho : (\mathbb{C}_\rho, \circ_\rho) \to (\mathbb{C}^*, \cdot) = (\mathbb{C}\backslash\{0\}, \times)$ is an isomorphism ('from $\mathbb{C}_\rho$ to $\mathbb{C}_\infty$').

We may recast $(GFE_\times)$ along the lines of (†) so that $G(s) = s^\gamma$ with $\gamma \neq 0$, and

$$K(t) = (t^\gamma - 1)\rho^{-1}, \text{ for } \rho = \gamma/\kappa = (1-k)/(k\kappa).$$

Then, as $\eta_\rho(x) = 1 + \rho x = G(K^{-1}(x))$,

$$K(st) = K(s) \circ_\rho K(t) = K(s) + \eta_\rho(K(s))K(t) = K(s) + G(s)K(t).$$

For $\gamma \neq 0$, K is a homomorphism from the multiplicative reals $\mathbb{R}_+$ into $\mathbb{C}_\rho$; more precisely, it is an isomorphism between $\mathbb{R}_+$ and the conjugate subgroup

$$(\mathbb{R}_+ - 1)\rho^{-1}.$$

In the case $\gamma = 0$ ($k = 1$), $\mathbb{C}_0 = \mathbb{C}$ is the additive group of complex numbers; from $(GFE_\times)$ it is immediate that K maps logarithmically into $(\mathbb{R}, +)$, 'the additive reals'.

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References

[Acz] J. Aczél, Extension of a generalized Pexider equation. *Proc. Amer. Math. Soc.* **133** (2005), 3227–3233.

[AczD] J. Aczél, J. Dhombres, *Functional equations in several variables. With applications to mathematics, information theory and to the natural and social sciences.* Encyclopedia of Math. and its App. **31**, CUP, 1989

[AczG] J. Aczél and S. Gołąb, Remarks on one-parameter subsemigroups of the affine group and their homo- and isomorphisms, *Aequationes Math.*, **4** (1970), 1-10.

[Bin] N. H. Bingham, Random walk on spheres. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **22** (1972), 169–192.

[BinG] N. H. Bingham, C. M. Goldie, Extensions of regular variation: I. Uniformity and quantifiers, *Proc. London Math. Soc.* (3) **44** (1982), 473-496.

[BinGT] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular variation*, 2nd ed., Cambridge University Press, 1989 (1st ed. 1987).

[BinO1] N. H. Bingham and A. J. Ostaszewski, Beurling slow and regular variation, *Trans. London Math. Soc.* **1** (2014), 29-56.

[BinO2] N. H. Bingham and A. J. Ostaszewski, Cauchy's functional equation and extensions: Goldie's equation and inequality, the Gołąb-Schinzel equation and Beurling's equation, *Aequationes Math.* **89** (2015), 1293-1310.

[BinO3] N. H. Bingham and A. J. Ostaszewski: Additivity, subadditivity and linearity: automatic continuity and quantifier weakening, arXiv.1405.3948.

[BinO4] N. H. Bingham and A. J. Ostaszewski, Beurling moving averages and approximate homomorphisms, *Indagationes Math.,* online 2015 (fuller version:

arXiv: 1407.4093).
[BloH] W. R. Bloom and H. Heyer, *Harmonic analysis of probability measures on hypergroups*, de Gruyter Studies in Math. **20**, 1995.
[BojK] R. Bojanić and J. Karamata, *On a class of functions of regular asymptotic behavior*, Math. Research Center Tech. Report 436, Madison, Wis. 1963; reprinted in *Selected papers of Jovan Karamata* (ed. V. Marić, Zevod Udžbenike, Beograd, 2009), 545-569.
[Brz] J. Brzdęk, The Gołąb-Schinzel equation and its generalizations, *Aequationes Math.* **70** (2005), 14-24.
[Chu] J. Chudziak, Semigroup-valued solutions of the Gołąb-Schinzel type functional equation, *Abh. Math. Sem. Univ. Hamburg* **76** (2006), 91-98.
[Fel] W. Feller, *An introduction to Probability Theory and its Applications*, Vol. 2, 2^{nd} ed. 1971 Wiley [1^{st} ed. 1966].
[GupJTS] A. K. Gupta et al. Characterization of stable laws via functional equations, *Math. Nachr.* **279** (2006), 571-580.
[Kuc] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality.* 2nd ed., Birkhäuser, 2009 [1st ed. PWN, Warszawa, 1985].
[Ost1] A. J. Ostaszewski, Beurling regular variation, Bloom dichotomy, and the Gołąb-Schinzel functional equation, *Aequationes Math.* **89** (2015), 725-744
[Ost2] A. J. Ostaszewski, Homomorphisms from functional equations: the Goldie equation, *Aequationes Math.*, **90** (2016) 427-448 (fuller version: arXiv: 1407.4089 (v2: 7.11.2014)).
[Ost3] A. J. Ostaszewski, Homomorphisms from Functional Equations in Probability, in: *Developments in Functional Equations and Related Topics*, eds. J. Brzdęk, K. Ciepliński, Th. M. Rassias, Springer, to appear.
[PitP] E. J. G. Pitman and J. Pitman, A direct approach to the stable distributions, this volume.
[RamL] B. Ramachandran, Ka-Sing Lau, *Functional Equations in Probability Theory*, Academic Press, 1991.
[Ste] H. Stetkær, *Functional equations on groups*, World Scientific, 2013.

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