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Inference on Nonstationary Time Series with Moving Mean

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Abstract

A semiparametric model is proposed in which a parametric filtering of a nonstationary time series, incorporating fractionally differencing with short memory correction, removes correlation but leaves a nonparametric deterministic trend. Estimates of the memory parameter and other dependence parameters are proposed, and shown to be consistent and asymptotically normally distributed with parametric rate. Tests with standard asymptotics for $I(1)$ and other hypotheses are thereby justified. Estimation of the trend function is also considered. We include a Monte Carlo study of finite-sample performance.

Keywords: fractional time series; fixed design nonparametric regression; nonstationary time series; $I(1)$ tests.

JEL Classifications: C14, C22.

Proposed running head: Nonstationary Time Series

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1 INTRODUCTION

A long-established vehicle for smoothing a deterministically-trending time series y_t, $t = 1, ..., T$, is the fixed-design nonparametric regression model given by

$$y_t = g\left(\frac{t}{T}\right) + u_t,\ t = 1, ..., T, \tag{1}$$

where $g(x),\ x \in [0,1]$, is an unknown, smooth, nonparametric function, and u_t is an unobservable sequence of random variables with zero mean. The dependence on sample size T of $g(t/T)$ in (1) is to ensure sufficient accumulation of information to enable consistent estimation of $g(\tau)$ at any $\tau \in (0,1)$. A more basic trend function is a polynomial in t of given degree, as still frequently employed in various econometric models. A more general class of models than polynomials (and having analogy with the fractional stochastic trends we will employ in the current paper) involves fractional powers, i.e.

$$y_t = \beta_0 + \beta_1 t^{\gamma_1} + ... + \beta_p t^{\gamma_p} + u_t, \quad t = 1, ..., T, \tag{2}$$

where all the β_i and γ_i are unknown and real-valued. Subject to identifiability and other restrictions, these parameters can be estimated consistently and asymptotically normally, e.g. by nonlinear least squares (Phillips (2007), Robinson (2012a)). Models such as (2) can be especially useful in modest sample sizes. However, and as with any parametric function of t, misspecification leads to inconsistent estimation, and a nonparametric treatment affords greater flexibility when T is large (recognizing that nonparametric estimates converge more slowly than parametric ones). Apart from the nonparametric/parametric aspect, the models (1) and (2) differ in that (1) entails a weak trend while (2) can describe stonger and weaker trends. (see Phillips and Sul (2007)).

With independent and identically distributed (iid) u_t, with finite variance, various kernel-type estimates of g in (1) were developed by Gasser and Mueller (1979), Priestley and Chao (1972) , with statistical properties established; in particular, under regularity conditions kernel estimates of $g(\tau)$ are consistent and asymptotically normally distributed as $T \to \infty$ (see e.g. Benedetti (1977)). A suitable choice of kernel (and bandwidth) is an important ingredient in this theory, although kernel estimates are essentially an elaboration on simple moving window averages, which have a much longer history in empirical work. More recent empirical uses of (1) include Starica and Granger (2005) in modelling stock price series.

The iid assumption on u_t is very restrictive, but similar asymptotic properties result when u_t has weak dependence, for example is a covariance stationary process, generated by a linear process or satisfying suitable mixing conditions, and having finite and positive spectral density at degree zero (see e.g. Roussas, Tran and Ioannides (1992), Tran, Roussas, Yakowitz and Truong Van (1996)). The rate of convergence of kernel estimates is unaffected by this level of serial correlation, though the asymptotic variance differs from that in the iid case (unlike in the stochastic-design model in which the argument of g in (1) is instead a weakly dependent stationary stochastic sequence).

Long-range dependence in u_t has a greater impact on large-sample inference. If u_t is a stationary and invertible fractional process, for example

$$(1-L)^{\delta_0} u_t = \varepsilon_t, \ \ |\delta_0| < 1/2, \tag{3}$$

L being the lag operator and the ε_t forming an iid sequence, or if u_t has a "semi-parametric" specification with spectral density $f(\lambda)$ having rate $\lambda^{-2\delta_0}$ as frequency λ approaches zero from above, then the convergence rate of kernel estimates of $g(\tau)$ is slower when $\delta_0 > 0$ and faster when $\delta_0 < 0$. References dealing with (1) for such u_t include Beran and Feng (2002), Csorgo and Mielniczuk (1995), Deo (1997), Guo and Koul (2007), Robinson (1997), Zhao, Zhang and Li (2014). The asymptotic variance of the kernel estimates depends on δ_0 and any other time series parameters; for the "semiparametric" specification Robinson (1997) justified studentization using local Whittle estimates of δ_0.

The restriction $\delta_0 < 1/2$ implies stationarity of u_t, so that y_t given by (1) is nonstationary only in the mean. Stochastic trends are also an important source of nonstationarity in many empirical time series. However, a nonstationary stochastic trend in y_t generated by a nonstationary u_t, for example an $I(1)$ trend, would render $g\left(t/T\right)$ undetectable, so we do not pursue this line. An alternative, semiparametric, model which both incorporates a possibly nonstationary stochastic trend and enables estimation of a nonparametric deterministic trend is

$$\Delta_t^{\delta_0} y_t = g\left(\frac{t}{T}\right) + u_t, \quad t = 1, ..., T, \tag{4}$$

where u_t is a sequence of uncorrelated, homoscedastic random variables and, for any real δ, Δ_t^{δ} is the truncated fractional differencing operator,

$$\Delta_t^{\delta} = \sum_{j=0}^{t-1} \alpha_j(\delta) L^j, \ t \geq 1, \tag{5}$$

the $\alpha_j(\delta)$ being coefficients in the (possibly formal) expansion

$$(1-z)^{\delta} = \sum_{j=0}^{\infty} \alpha_j(\delta) z^j,$$

namely

$$\alpha_j(\delta) = \frac{\Gamma(j-\delta)}{\Gamma(-\delta)\Gamma(j+1)}. \tag{6}$$

The truncation in (5) reflects non-observability of y_t when $t \leq 0$, and avoids explosion of the moving average representation of (4) when $\delta_0 \geq 1/2$, the nonstationary region for δ_0; this region may be of particular interest with economic data, but our work covers also stationary settings, where $\delta_0 < 1/2$.

One nonstationary δ_0 has assumed wide empirical importance in connection with a variety of econometric models, the $I(1)$ case $\delta_0 = 1$, when (4) becomes

$$(1-L)y_t = g\left(\frac{t}{T}\right) + u_t, \ t = 1, ..., T. \tag{7}$$

The bulk of the econometric literature nests (7) as the unit root in autoregressive structures, which suggests treating (7) as a special case of

$$(1-\rho L)y_t = g\left(\frac{t}{T}\right) + u_t, \quad t = 1, ..., T, \tag{8}$$

rather than (4). The autoregressive unit root literature suggests that estimates of ρ in (8) will have a nonstandard limit distribution under $\rho = 1$ (7), but a normal one in the "stationary" region $|\rho| < 1$. By contrast we can anticipate, for example from literature concerning (4) with $g(x)$ *a priori* constant, that estimates of δ_0 such as ones optimizing an approximate pseudo-Gaussian likelihood, and Wald and other test statistics, will enjoy standard asymptotics, with the usual parametric convergence rate, $\sqrt{T}$, whatever the value of δ_0, due essentially to smoothness properties of the fractional operator; tests are also expected to have the classical local efficiency properties. While (4) cannot, unlike (8), describe "explosive" behaviour (occurring when $|\rho| > 1$), it describes a continuum of stochastic trends indexed by δ_0. A consequence of the $T-$dependence in $g(t/T)$ is that the left side of (4) is also T-dependent, so the $y_t = y_{tT}$ in fact form a triangular array, but in common with the bulk of literature concerning versions of (1) we suppress reference to T. The model (4) (which nests (1) with iid u_t on taking $\delta_0 = 0$) supposes that the fractional filtering of y_t successfully eliminates correlation, but possibly leaves a trend which we are not prepared to parameterize.

To provide greater generality than (4), the paper in fact considers the extended model

$$\xi_t\left(L; \delta_0, \theta_0\right) y_t = g\left(\frac{t}{T}\right) + u_t, \quad t = 1, ..., T, \tag{9}$$

where θ_0 is an unknown $p-$dimensional column vector and

$$\xi_t\left(z; \delta, \theta\right) = \sum_{j=0}^{t-1} \xi_j(\delta, \theta) z^j, \ t \geq 1,$$

where the $\xi_j(\delta,\theta)$ are coefficients in the possibly formal expansion

$$\xi\left(z;\delta,\theta\right)=\sum_{j=0}^{\infty}\xi_j(\delta,\theta)z^j,$$

such that

$$\xi\left(z;\delta,\theta\right)=(1-z)^{\delta}\zeta\left(z;\theta\right),$$

where

$$\zeta\left(z;\theta\right)=\sum_{j=0}^{\infty}\zeta_j(\theta)z^j$$

is a known function of z and θ that is at least continuous, and nonzero for z on or inside the unit circle in the complex plane. When $\zeta\left(z;\theta\right)\equiv 1$, we have $\xi\left(z;\delta,\theta\right)=(1-z)^{\delta}$. Leading examples of $\zeta\left(z;\theta\right)$ are stationary and invertible autoregressive moving average operators of known degree, for example the first order autoregressive operator $\zeta\left(z;\theta\right)=1-\theta z$, with θ here a scalar such that $|\theta|<1$. In general ζ leaves the essential memory or degree of nonstationarity δ_0 unchanged but allows otherwise richer dependence structure.

It would be possible to consider a nonparametric ζ, $\zeta\left(z\right)$, satisfying smoothness assumptions only near $z=1$, and hence a "semiparametric" operator on y_t. This would lead to an estimate of δ_0 with only a nonparametric convergence rate. However, establishing the parametric, $\sqrt{T}$, rate for estimating δ_0 and θ_0 seems actually more challenging and delicate, because of the presence of the nonparametric $g\left(t/T\right)$ in (9), estimates of which converge more slowly than $\sqrt{T}$. Proving consistency requires establishing that certain (stochastic and deterministic) contributions to residuals, whose squares make up the objective function minimized by the parameter estimates, are negligible uniformly over the parameter space; these contributions are of larger order than would be the case with a parametric trend (and this fact also explains why we find ourselves unable to choose the parameter space for δ_0 as large as is possible with a parametric trend). Then, corresponding contributions to scores evaluated at δ_0, θ_0 are also of larger order than in the parametric trend case and have to be shown to be negligible after being normalized by $\sqrt{T}$, rather than by a slower, nonparametric, rate, in order to prove asymptotic normality of the parameter estimates with $\sqrt{T}$ rate. The strong dependence in y_t also impacts on the conditions, due to non-summability of certain weight sequences.

The following section proposes estimates of δ_0 and θ_0, and establishes their consistency and asymptotic normality, the proofs appearing in Appendices A and B. Section 3 develops $I(1)$ tests based on Wald, pseudo log likelihood ratio and Lagrange multiplier principles. Section 4 proposes estimates of $g(x)$ and establishes their asymptotic properties. A small Monte Carlo study of finite-sample performance is contained in Section 5. Section 6 concludes by describing further issues that might be considered.

2. ESTIMATION OF DEPENDENCE PARAMETERS

Were $g(x) \equiv 0$ in (9) *a priori*, a natural method of estimating δ_0 and θ_0 would be conditional-sum-of-squares, which approximates Gaussian pseudo-maximum-likelihood estimation. We modify this method by employing residuals, which requires preliminary estimation of $g(x)$. Note that under the conditions imposed below, $g(t/T) - g((t-1)/T) = O(T^{-1})$, $2 \leq t \leq T$, so we might instead first difference (9) as this would effectively eliminate the deterministic trend; however this also induces a non-invertible moving average $u_t - u_{t-1}$.

Let $k(x), x \in \mathcal{R}$, be a user-chosen kernel function and h a user-chosen positive bandwidth number. For any δ, θ write $\xi_{t\delta\theta}(z) = \xi_t(z; \delta, \theta)$ and introduce

$$\hat{g}_{\delta\theta}(x) = \sum_{s=1}^{T} \xi_{s\delta\theta}(L)\, y_s k\left(\frac{x - s/T}{h}\right) / \sum_{s=1}^{T} k\left(\frac{x - s/T}{h}\right), \tag{10}$$

for any $x \in [0,1]$. The corresponding estimate of Priestley and Chao (1972) type replaces the denominator by Th, but we prefer to use weights (of the $\xi_{s\delta\theta}(L)\, y_s$) that exactly sum to 1 for all x. Define residuals

$$\hat{u}_t(\delta,\theta) = \xi_{t\delta\theta}(L)\, y_t - \hat{g}_{\delta\theta}(t/T) = \frac{\sum_{s=1}^{T} \left(\xi_{t\delta\theta}(L)\, y_t - \xi_{s\delta\theta}(L)\, y_s\right) k_{ts}}{\sum_{s=1}^{T} k_{ts}}, \tag{11}$$

where $k_{ts} = k\left((t-s)/Th\right)$. Denote by ∇_1, ∇_2 chosen real numbers such that $\nabla_1 < \nabla_2$, write $\nabla = [\nabla_1, \nabla_2]$, and let Θ be a suitably chosen compact subset of $\mathcal{R}^p$. We estimate δ_0 and θ_0 by

$$\left(\widehat{\delta}, \widehat{\theta}\right) = \arg \min_{\delta\in\nabla;\theta\in\Theta} Q(\delta,\theta), \tag{12}$$

where

$$Q(\delta,\theta) = \sum_{t=1}^{T} \hat{u}_t^2(\delta,\theta). \tag{13}$$

We first establish consistency of $\widehat{\delta}$, $\widehat{\theta}$, under the following regularity conditons.

Assumption 1

The u_t are stationary and ergodic with finite fourth moment, and

$$E\left(u_t|\, \mathcal{F}_{t-1}\right) = 0, \quad E\left(u_t^2\big|\, \mathcal{F}_{t-1}\right) = \sigma^2$$

almost surely, where $\mathcal{F}_t$ is the σ-field of events generated by ε_s, $s \le t$, and conditional on $\mathcal{F}_{t-1}$ 3rd and 4th moments of u_t equal corresponding unconditional moments.

Assumption 2

(i) $\theta_0 \in \Theta$;

(ii) $|\zeta(z;\theta)| \neq |\zeta(z;\theta_0)|$ *,for all $\theta \neq \theta_0$, $\theta \in \Theta$, on a $z-$set of positive Lebesgue measure;*

(iii) *for all $\theta \in \Theta$ and real λ, $\zeta\left(e^{i\lambda};\theta\right)$ is differentiable in λ with derivative in $Lip(\varsigma)$, $\varsigma > 1/2$;*

(iv) *for all λ, $\zeta\left(e^{i\lambda};\theta\right)$ is continuous in θ;*

(v) *for all $\theta \in \Theta$, $|\zeta(z;\theta)| \neq 0$, $|z| \le 1$.*

Assumption 1 is weaker than imposing independence and identity of distribution of u_t, and Assumption 2 is standard from the literature on parametric short memory models since Hannan (1973), ensuring identifiability of θ_0 and easily covering stationary and invertible moving averages. These assumptions correspond to ones of Hualde and Robinson (2011), who established consistency of the same kind of estimates when $g(x) \equiv 0$ in (9) *a priori.* In that setting they were able to choose the set of admissible memory parameters (our ∇) arbitrarily large, to simultaneously cover stationary, nonstationary, invertible and non-invertible values. This seems impossible to achieve in the presence of the unknown, nonparametric g in (9), which can only be estimated with a slow rate of convergence, and we impose:

Assumption 3

$$\nabla_2 - \nabla_1 < 1/2, \tag{14}$$

$$\delta_0 \in \nabla. \tag{15}$$

As can be inferred from the proof of Theorem 1 below, for consistency the weaker condition $\nabla_2 - \delta_0 < 1/2$ suffices, but since δ_0 is known from (15) only to be no less than ∇_1 and $\nabla_2 - \delta_0 \leq \nabla_2 - \nabla_1$ the restriction (14) is appropriate. Condition (15) can be satisfied by stationary or nonstationary values of δ_0, and ∇ could be chosen to include potential ones in both categories, for example with $\nabla_1 > 1/4$, $\nabla_2 < 3/4$, or to cover the $I(1)$ case $\delta_0 = 1$ one might pick $\nabla_1 > 3/4$, $\nabla_2 < 5/4$. Condition (15) is also consistent with $\nabla_1 > 0$, $\nabla_2 < 1/2$ employed in the early stationary long memory literature with $g(.) \equiv 0$ *a priori* (e.g. Fox and Taqqu (1986)). But by more recent standards (14) is highly restrictive. It arises in showing that the effect of estimating the nonparametric function g has a negligible effect in proving consistency of the parameter estimates $\left(\widehat{\delta}, \widehat{\theta}\right)$, which entails establishing uniform convergence to zero on ∇ of the difference between $T^{-1}Q(\delta, \theta)$ and the corresponding function with $g(.) \equiv 0$ *a priori.* The restriction (14) is not a consequence of the uniformity aspect of the proof, but arises even in establishing pointwise convergence, as demonsrated in a lemma stated and proved after the proof of Theorem 1in Appendix A.

We also need conditions on g, k and h.

Assumption 4

The function $g(x)$ is twice boundedly differentiable on $[0,1]$ and $(d/dx)\, g(0) = 0$.

Assumption 5

The function $k(x)$ is even, differentiable at all but possibly finitely many x, with derivative $k'(x)$, and

$$\int_{\mathcal{R}} k(x)dx = 1,$$

$$k(x) = O((1 + x^{2+\eta})^{-1}),\ k'(x) = O((1 + |x|^{1+\eta})^{-1}),\ \textit{some}\ \eta > 0.$$

Assumption 6

As $T \to \infty$, the positive-valued sequence $h = h_T$ satisfies:

$$(Th)^{-1} + T^{2(\nabla_2 - \nabla_1)} h^3 \to 0. \tag{16}$$

Assumption 5 is virtually costless, covering many of the usual kernel choices. Assumption 6, however, represents a trade-off with Assumption 3: in the latter, $\nabla_2 - \nabla_1$ is desirably as close to 1/2 as possible, but as it approaches 1/2 from below the range of h satisfying Assumption 6 reduces to $(Th)^{-1} + Th^3 \to 0$.

Theorem 1

Let (9) and Assumptions 1-6 hold. Then as $T \to \infty$,

$$\widehat{\delta} \to_p \delta_0,\ \widehat{\theta} \to_p \theta_0.$$

The proof is in Appendix A. Asymptotic normality entails two further assumptions.

Assumption 7

(i) $\delta_0 \in (\nabla_1, \nabla_2)$; *$\theta_0$ is an interior point of Θ.*

(ii) *for all real* λ, $\zeta\left(e^{i\lambda};\theta\right)$ *is twice continuously differentiable in* θ *on a closed neighbourhood of radius* $< 1/2$ *about* θ_0;

(iii) *the matrix*

$$\Omega = \begin{pmatrix} \pi^2/6 & -\sum_{j=1}^{\infty} \psi_j'\left(\theta_0\right)/j \\ -\sum_{j=1}^{\infty} \psi_j\left(\theta_0\right)/j & \sum_{j=1}^{\infty} \psi_j\left(\theta_0\right)\psi_j'\left(\theta_0\right) \end{pmatrix}$$

is non-singular, where

$$\psi_j\left(\theta\right) = \sum_{k=0}^{j-1} \phi_k\left(\theta\right) \frac{\partial \zeta_{j-k}\left(\theta\right)}{\partial \theta},$$

the $\phi_j\left(\theta\right)$ *being coefficients in the expansion*

$$\phi_j\left(z;\theta\right) = \zeta\left(z;\theta\right)^{-1} = \sum\nolimits_{j=0}^{\infty} \phi_j\left(\theta\right) z^j.$$

This condition again is based on one of Hualde and Robinson (2011), but is similar to others in the literature, and practically unrestrictive. However we have to strengthen the first component of Assumption 6 on h.

Assumption 8

As $T \to \infty$, $Th^2/\left(\log T\right)^2 \to \infty$.

Recall that Assumption 6 requires h to decay faster than $T^{-2(\nabla_2 - \nabla_1)/3}$, where the latter rate is slower than $T^{-1/3}$ in view of Assumption 3, whereas Assumption 8 prevents h decaying as fast as $T^{-1/2}$.

Theorem 2

Let (9) and Assumptions 1-8 hold. Then as $T \to \infty$

$$T^{1/2}\begin{pmatrix} \widehat{\delta} - \delta_0 \\ \widehat{\theta} - \theta_0 \end{pmatrix} \to_d \mathcal{N}(0, \Omega^{-1}).$$

The proof is in Appendix B. Note that the same limit distributions results when g is known or replaced by a parametric function. In the special case (4) of (9), we deduce that as $T \to \infty$

$$T^{1/2}\left(\widehat{\delta} - \delta_0\right) \to_d \mathcal{N}(0, 6/\pi^2).$$

3. $I(1)$ TESTING

We first econsider $t-$tests (based on the square root of the Wald staistic) for $\delta_0 = 1$ in (9), based on Theorem 2. Define

$$\Omega(\theta) = \begin{pmatrix} \pi^2/6 & -\sum_{j=1}^{\infty} \psi_j'(\theta)/j \\ -\sum_{j=1}^{\infty} \psi_j(\theta)/j & \sum_{j=1}^{\infty} \psi_j(\theta)\psi_j'(\theta) \end{pmatrix}$$

and denote by $\widehat{\Omega}^{(1,1)}$ the element in the top left hand corner of $\Omega\left(\widehat{\theta}\right)^{-1}$. Put

$$W = T^{1/2}\left(\widehat{\delta} - 1\right)/\widehat{\Omega}^{(1,1)1/2}.$$

Pseudo-log likelihood ratio tests can also be constructed. Define

$$\widetilde{\theta} = \arg\min_{\theta\in\Theta} Q_T(1, \theta), \tag{17}$$

and

$$LR = \log \frac{Q_T(1, \widetilde{\theta})}{Q_T(\widehat{\delta}, \widehat{\theta})}.$$

Though it of course does not use $\widehat{\delta}$, $\widehat{\theta}$, for completeness we also present a Lagrange multiplier-type test, as it and the Wald and pseudo-log likelihood tests are expected to have equal local power. Robinson (1994) developed Lagrange multiplier tests for $I(1)$ and other hypotheses against fractional alternatives for the disturbances in multiple linear regression models. The stress there was on frequency-domain tests, but starting from an initial time-domain statistic, and to avoid introducing considerable additional notation we stay in the time domain here. Writing $\partial = \partial/\partial(\delta, \theta')'$, from (13)

$$\partial Q(\delta, \theta) = \frac{2}{T}\sum_{t=1}^{T} \hat{u}_t(\delta, \theta)\partial\hat{u}_t(\delta, \theta), \tag{18}$$

where

$$\partial\hat{u}_t(\delta, \theta) = \xi_{t\delta\theta}(L)\, y_t - \partial\hat{g}_{\delta\theta}(t/T) = \frac{\sum_{s=1}^{T} \left(\partial\xi_{t\delta\theta}(L)\, y_t - \partial\xi_{s\delta\theta}(L)\, y_s\right) k_{ts}}{\sum_{s=1}^{T} k_{ts}}, \tag{19}$$

in which

$$\partial\xi_{t\delta\theta}(z) = \sum_{j=0}^{t-1} \partial\xi_j(\delta,\theta) z^j, \ \partial\xi_j(\delta,\theta) = \sum_{l=0}^{j} \left(\begin{array}{c} \left(\partial\alpha_l(\delta)/\partial\delta\right) \zeta_{j-l}(\theta)\right) \\ \alpha_l(\delta)\partial\zeta_{j-l}(\theta)/\partial\theta \end{array} \right). \tag{20}$$

In fact

$$\partial\xi_{t\delta\theta}(1,\theta) = \left(\begin{array}{c} \sum_{j=0}^{t-1} \left(\sum_{l=1}^{j} \frac{\zeta_{j-l}(\theta)}{l} \right) (y_{t-j} - y_{t-j-1}) \\ \sum_{j=0}^{t-1} \left(\partial\zeta_j(\theta)/\partial\theta\right) (y_{t-j} - y_{t-j-1}) \end{array} \right).$$

Define

$$LM = \frac{T}{4} \partial Q(1,\widetilde{\theta})' \Omega\left(\widetilde{\theta}\right)^{-1} \partial Q(1,\widetilde{\theta}),$$

with $\widetilde{\theta}$ given by (17).

Theorem 3

Let $\delta_0 = 1$ in (9) and let Assumptions 1-8 hold. Then as $T \to \infty$,

$$W \to_d \mathcal{N}(0,1), \ LR \to_d \chi_1^2, \ LM \to_d \chi_1^2.$$

For W, the theorem follows from Theorem 2 and $\Omega\left(\widehat{\theta}\right) \to_p \Omega$, where the latter is implied by the proof of Theorem 2. We can reject the $I(1)$ null against more nonstationary alternatives when W falls in the appropriate upper tail of the standard normal density, and reject against less nonstationary alternatives when it falls in the appropriate lower tail. For LR, the proof is standard, given Theorem 2 and a central limit theorem for $\widetilde{\theta}$ (see e.g. Hannan (1973), or implied by Hualde and Robinso n(2011)). For LM the proof is likewise straightforward.

4. NONPARAMETRIC REGRESSION ESTIMATION

We can base estimation of $g(x)$ on our estimates of $\widehat{\delta}$, $\widehat{\theta}$ and (10), but in view of the stringent conditions on the bandwidth h in Theorems 1 and 2 we allow use of a possibly different bandwidth, b, in

$$\widetilde{g}_{\delta\theta}(x) = \sum_{s=1}^{T} \xi_{s\delta\theta}(L)\, y_s k\left(\frac{x - s/T}{b}\right) / \sum_{s=1}^{T} k\left(\frac{x - s/T}{b}\right), \tag{21}$$

We provide a multivariate central limit theorem for $\widetilde{g}_{\widehat{\delta}\widehat{\theta}}(\tau_i)$, $i = 1, 2, .., q$, where τ_i, $i = 1, 2, .., q$, are distinct fixed points, imposing also:

Assumption 9

As $T \to \infty$, $(bT)^{-1} + b^5 T \to 0$.

The proof of the following theorem is omitted as univariate and multivariate central limit theorems for the $\widetilde{g}_{\delta_0\theta_0}(x_i)$ are already in the literature (see e.g. Benedetti (1977), Robinson (1997)) and from Theorem 2 it is readily shown that $\widetilde{g}_{\widehat{\delta}\widehat{\theta}}(x) - \widetilde{g}_{\delta_0\theta_0}(x) = O_p(T^{-1/2})$ for all x.

Theorem 4

Let (9) and Assumptions 1-9 hold. Then as $T \to \infty$, the $(bT)^{1/2}\left(\widetilde{g}_{\widehat{\delta}\widehat{\theta}}(\tau_i) - g(\tau_i)\right)$, $i = 1, 2, ..., q$, converge in distribution to independent $N\left(0, \sigma^2 \int_{\mathcal{R}} k(x)^2 dx\right)$ random variables, where σ^2 is consistently estimated by

$$\widehat{\sigma}^2 = Q\left(\widehat{\delta}, \widehat{\theta}\right).$$

This is the same limit distribution as results if δ_0 and θ_0 are known, i.e. the same as in the model *(1)* with iid u_t.

5. FINITE-SAMPLE PERFORMANCE

A small Monte Carlo study was carried out to investigate the finite-sample behaviour of our parameter estimates, and of one of our unit root tests. To generate data, we took $g(x) = \sin(2\pi x)$, $p = 1$, $\zeta(z; \theta) = 1 - \theta z$ (so y_t was a FARIMA$(1, \delta_0, 0)$), for various values of δ_0 and θ_0, and ε_t standard normally distributed. Throughout, parameter estimates were obtained taking k to be the standard normal kernel.

Tables 1-3 contain Monte Carlo biases b_δ and b_θ of $\widehat{\delta}$ and $\widehat{\theta}$ and corresponding Monte Carlo standard deviations s_δ and s_θ. In Table 1 $(\delta_0, \theta_0) = \left(1, \frac{1}{2}\right)$ and $(\delta_0, \theta_0) = (1, 0)$; in Table 2 $(\delta_0, \theta_0) = \left(\frac{7}{8}, \frac{1}{2}\right)$ and $(\delta_0, \theta_0) = \left(\frac{7}{8}, 0\right)$; in Table 3 $(\delta_0, \theta_0) = \left(\frac{9}{8}, \frac{1}{2}\right)$ and $(\delta_0, \theta_0) = \left(\frac{9}{8}, 0\right)$. We took $\nabla = [0.76, 1.24]$ and $\Theta = [-0.99, 0.99]$. Note that in the cases $\theta_0 = 0$, one of which is included in each table, y_t reduces to a FARIMA$(0, \delta_0, 0)$, but we suppose that the practitioner does not know this. Three different (T, Th) combinations were employed: $(250, 30)$, $(600, 60)$ and $(1000, 90)$. The results are based on 1000 replications. In the tables, somewhat surprisingly the biases b_δ are greater for $\theta = 0$ than for $\theta = \frac{1}{2}$, though for b_θ the pattern is opposite. Always $b_\delta > 0$ and $b_\delta < 0$. The standard deviations s_δ and s_θ are predominately greater for $\theta = \frac{1}{2}$ than for $\theta = 0$. Biases and standard deviations diminish with increasing T, while there is reasonable stability across corresponding elements of the three tables.

Table 1: Bias and standard deviation of $\left(\widehat{\delta}, \widehat{\theta}\right)$, $(\delta_0, \theta_0) = \left(1, \frac{1}{2}\right)$, $(1, 0)$.

(δ_0, θ_0)	$\left(1, \frac{1}{2}\right)$				$(1, 0)$			
T	b_δ	s_δ	b_δ	s_θ	b_δ	s_δ	b_θ	s_θ
250	0.0956	0.1540	-0.0832	0.1756	0.1105	0.0526	-0.0751	0.0833
600	0.0853	0.0598	-0.0770	0.0808	0.0905	0.0320	-0.0670	0.0519
1000	0.0678	0.0449	-0.0653	0.0610	0.0710	0.0223	-0.0563	0.0430

Table 2: Bias and standard deviation of $\left(\widehat{\delta}, \widehat{\theta}\right)$, $(\delta_0, \theta_0) = \left(\frac{7}{8}, \frac{1}{2}\right)$, $\left(\frac{7}{8}, 0\right)$.

(δ_0, θ_0)	$\left(\frac{7}{8}, \frac{1}{2}\right)$				$\left(\frac{7}{8}, 0\right)$			
T	b_δ	s_δ	b_θ	s_θ	b_δ	s_δ	b_θ	s_θ
250	0.0940	0.1385	-0.1090	0.1116	0.1106	0.0524	-0.0966	0.1028
600	0.0866	0.0592	-0.0785	0.0801	0.0900	0.0319	-0.0679	0.0851
1000	0.0685	0.0447	-0.0662	0.0608	0.0713	0.0255	-0.0568	0.0712

Table 3: Bias and standard deviation of $\left(\widehat{\delta}, \widehat{\theta}\right)$, $(\delta_0, \theta_0) = \left(\frac{9}{8}, \frac{1}{2}\right)$, $\left(\frac{9}{8}, 0\right)$.

(δ_0, θ_0)	$\left(\frac{9}{8}, \frac{1}{2}\right)$				$\left(\frac{9}{8}, 0\right)$			
T	b_δ	s_δ	b_θ	s_θ	b_δ	s_δ	b_θ	s_θ
250	0.0981	0.1171	-0.1071	0.1065	0.1064	0.0528	-0.0934	0.0837
600	0.0838	0.0607	-0.0753	0.0817	0.0871	0.0420	-0.0763	0.0521
1000	0.0670	0.0451	-0.0644	0.0612	0.0707	0.0255	-0.0559	0.0432

Table 4 contains Monte Carlo sizes and powers for the LR $I(1)$ test described in Section 3, based on nominal 1% and 5% levels. Sizes were obtained using $(\delta_0, \theta_0) = \left(1, \frac{1}{2}\right)$ and powers using $(\delta_0, \theta_0) = \left(\frac{7}{8}, \frac{1}{2}\right)$ and $(\delta_0, \theta_0) = \left(\frac{9}{8}, \frac{1}{2}\right)$, with the same T and h as before, but now on the basis of 10,000 replications. The Monte Carlo sizes seem quite satisfactory, and given that the alternative δ_0 are both close to 1 the differences between powers and corresponding sizes seem quite satisfactory.

Table 4: Sizes and powers of LR test at nominal 1% and 5% levels

	Size, $(\delta_0, \theta_0) = \left(1, \frac{1}{2}\right)$			
T	1%		5%	
250	0.0111		0.0487	
600	0.0095		0.0508	
1000	0.0102		0.0498	
	Power, $(\delta_0, \theta_0) = \left(\frac{7}{8}, \frac{1}{2}\right)$		Power, $(\delta_0, \theta_0) = \left(\frac{9}{8}, \frac{1}{2}\right)$	
T	1%	5%	1%	5%
250	0.1189	0.2855	0.1260	0.2946
600	0.3070	0.5089	0.2972	0.4998
1000	0.4961	0.6998	0.5017	0.7023

Table 5 contains corresponding results for the LM $I(1)$ test described in Section 3, but for data generated with $g(.) \equiv 0$ in (9), and this assumed in the parameter estimation, so that no nonparametric estimation was involved, and the setting is like tht in Robinson (1994). It might be expected that the sizes would be better than in Table 4 (though the LR and LM tests are not strictly comparable), but they are actually almost the same, though the powers are significantly better.

Table 5:Sizes and powers of LR test at nominal 1% and 5% levels

	Size, $(\delta_0, \theta_0) = \left(1, \frac{1}{2}\right)$			
T	1%		5%	
250	0.0109		0.0511	
600	0.0097		0.0497	
1000	0.0101		0.0507	
	Power, $(\delta_0, \theta_0) = \left(\frac{7}{8}, \frac{1}{2}\right)$		Power, $(\delta_0, \theta_0) = \left(\frac{9}{8}, \frac{1}{2}\right)$	
T	1%	5%	1%	5%
250	0.3305	0.4841	0.3215	0.4785
600	0.5249	0.6760	0.5337	0.6830
1000	0.7179	0.8696	0.7203	0.8705

6. FINAL REMARKS

The paper has justified large sample inference on the fractional and short memory parameters and nonparametric regression function in a semiparametric model incorporating nonstationary stochastic and deterministic trends. For parametric inference, the restrictions on the admissible memory parameter interval and the range of bandwidths are relatively strong, due to the presence of the nonparametric function and the extent of the time series dependence. As always when nonparametric estimation is involved, bandwidth choice is a practical issue, though as in other semiparametric settings one might expect parameter estimates to be less sensitive than nonparametric estimates, and the problem, in our fixed nonparametric design setting, may also be less acute than in the stochastic design setting in which the density of explanatory variables varies over their support. In our Monte Carlo study only one value of h was used for each T, but sensitivity of estimates and tests to h, and b, can be gauged by carrying out the computations over a range of choices. With respect to automatic rules, in the model (1) a cross–validation choice of b is known to minimize mean integrated squared error (MISE), and we can extend this property to our setting, using $\widehat{\delta}$, $\widehat{\theta}$, though as usual the minimum-MSE rate does not quite satisfy conditions (our Assumption 9) for asymptotic normality about g; for h, as is familiar in the semiparametric literature the minimum-MISE rate is clearly excluded by the conditions (our Assumption 8) for asymptotic normality of parameter estimates, and a more appropriate goal may be to make a selection that matches the orders of the next two terms after the normal distribution function in an Edgeworth expansion for distribution function of $\widehat{\delta}$, $\widehat{\theta}$, and thereby minimizes the departure from the normal limit and leads to better-sized tests and more accurate interval estimates; in some settings this problem has a neat solution, but we do not know whether this is the case in ours. Bootstrapping is also likely to improve finite-sample properties. Inference issues that might be investigated include testing constancy or other parametric restrictions on $g(x)$. Possible model extensions that require non-trivial further work include adding a nonparametric function of explanatory variables to $g(t/T)$ in (9), and allowing for unconditional or conditional heteroscedasticity in u_t. Our work might also be extended to a panel setting, including individual effects and possible cross–sectional dependence.

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APPENDIX A

Proof of Theorem 1

For $t = 1, ..., T$

$$y_t = \xi_t \left(L; \delta_0, \theta_0\right)^{-1} \left(g\left(\frac{t}{T}\right) + u_t\right) = \xi \left(L; \delta_0, \theta_0\right)^{-1} \left\{\left(g\left(\frac{t}{T}\right) + u_t\right) 1(t > 0)\right\},$$

so

$$\xi_t \left(L; \delta, \theta\right) y_t = (1 - L)^{\delta - \delta_0} \tau \left(z; \theta\right) \left\{\left(g\left(\frac{t}{T}\right) + u_t\right) 1(t > 0)\right\},$$

where

$$\tau \left(z; \theta\right) = \zeta \left(z; \theta\right) \zeta \left(z; \theta_0\right)^{-1} = \sum_{j=0}^{\infty} \tau_j(\theta) z^j.$$

From Zygmund (1977, p. 46), Assumption 2 implies that the Fourier coefficients $\tau_j(\theta)$ satisfy

$$\sup_{\Theta} |\tau_j(\theta)| = O(j^{-1-\varsigma}). \tag{A.1}$$

The Fourier coeficients $\chi_j(\delta, \theta)$ of

$$(1 - z)^{\delta - \delta_0} \tau \left(z; \theta\right) = \sum_{j=0}^{\infty} \chi_j(\delta, \theta) z^j$$

are given by

$$\chi_j(\delta, \theta) = \sum_{l=0}^{j} \tau_l(\theta) \alpha_{j-l}(\delta - \delta_0).$$

(Note that $\alpha_j(0) = \tau_j(\theta_0) = \chi_j(\delta_0, \theta_0) \equiv 0,\ j \geq 1$). From (6), uniformly in $\delta \in \nabla \setminus \{\delta_0\}$

$$\alpha_j(\delta - \delta_0) = O(j^{\delta_0 - \delta - 1}), \text{ as } j \to \infty, \tag{A.2}$$

and so, using also (A.1)

$$\begin{aligned}\left|\chi_j(\delta,\theta)\right| &\leq \left|\sum_{l=0}^{[j/2]}\tau_l(\theta)\alpha_{j-l}(\delta-\delta_0)\right| + \left|\sum_{l=[j/2]+1}^{j}\tau_l(\theta)\alpha_{j-l}(\delta-\delta_0)\right| \\ &\leq Kj^{\delta_0-\delta-1}\sum_{l=0}^{\infty}|\tau_l(\theta)| + Kj^{-1-\varsigma}\left|\sum_{l=0}^{j}\alpha_l(\delta-\delta_0)\right| \\ &\leq Kj^{\delta_0-\delta-1} + Kj^{\delta_0-\delta-1-\varsigma} \leq Kj^{\delta_0-\delta-1} \qquad (A.3)\end{aligned}$$

uniformly in δ,θ, where K throughout denotes a generic finite, positive constant. Also for future use note that from (6), uniformly in $\delta \in \nabla \setminus \{\delta_0\}$, $\theta \in \Theta$,

$$|\alpha_j(\delta-\delta_0) - \alpha_{j+1}(\delta-\delta_0)| = O(j^{\delta_0-\delta-2}), \text{ as } j \to \infty, \qquad (A.4)$$

$$\begin{aligned}\left|\chi_j(\delta,\theta) - \chi_{j+1}(\delta,\theta)\right| &\leq \left|\sum_{l=0}^{j}\tau_l(\theta)\left(\alpha_{j-l}(\delta-\delta_0) - \alpha_{j+1-l}(\delta-\delta_0)\right)\right| + |\tau_{j+1}(\theta)| \\ &\leq Kj^{\delta_0-\delta-2}\sum_{l=0}^{\infty}|\tau_l(\theta)| + Kj^{-1-\varsigma}\sum_{l=1}^{\infty}l^{\delta_0-\delta-2} + Kj^{-1-\varsigma} \\ &\leq Kj^{\max(\delta_0-\delta,1-\varsigma)-2}. \qquad (A.5)\end{aligned}$$

With the abbreviations

$$\chi_{t\delta\theta} = \sum_{j=0}^{t-1}\chi_j(\delta,\theta)L^j,\ g_t = g\left(\frac{t}{T}\right),\ k_t = \frac{1}{Th}\sum_{s=1}^{T}k_{ts}$$

we have from (11)

$$\begin{aligned}\hat{u}_t(\delta,\theta) &= \frac{1}{Th}\sum_{s=1}^{T}\left(\chi_{t\delta\theta}g_t - \chi_{s\delta\theta}g_s\right)k_{ts}/k_t + \frac{1}{Th}\sum_{s=1}^{T}\left(\chi_{t\delta\theta}u_t - \chi_{s\delta\theta}u_s\right)k_{ts}/k_t \\ &= \chi_{t\delta\theta}u_t + D_{t\delta\theta} - S_{t\delta\theta},\end{aligned}$$

where

$$D_{t\delta\theta} = \frac{1}{Th}\sum_{s=1}^{T}\left(\chi_{t\delta\theta}g_t - \chi_{s\delta\theta}g_s\right)k_{ts}/k_t$$

and

$$S_{t\delta\theta} = \frac{1}{Th}\sum_{s=1}^{T} \chi_{s\delta\theta} u_s k_{ts}/k_t$$

are respectively the deterministic and stochastic errors contributing to the residual, that are absent when $g(t/T) \equiv 0$ in (9). Thus

$$\begin{aligned}
Q(\delta,\theta) &= \frac{1}{T}\sum_{t=1}^{T} \left(\chi_{t\delta\theta} u_t + D_{t\delta\theta} - S_{t\delta\theta}\right)^2 \\
&= \frac{1}{T}\sum_{t=1}^{T} \left(\chi_{t\delta\theta} u_t\right)^2 + \frac{1}{T}\sum_{t=1}^{T} D_{t\delta\theta}^2 + \frac{1}{T}\sum_{t=1}^{T} S_{t\delta\theta}^2 \\
&\quad + \frac{2}{T}\sum_{t=1}^{T} \left(\chi_{t\delta\theta} u_t\right) D_{t\delta\theta} - \frac{2}{T}\sum_{t=1}^{T} \left(\chi_{t\delta\theta} u_t\right) S_{t\delta\theta} \\
&\quad - \frac{2}{T}\sum_{t=1}^{T} D_{t\delta\theta} S_{t\delta\theta}. \qquad \text{(A.6)}
\end{aligned}$$

Hualde and Robinson (2011) show that the estimates minimizing

$$\frac{1}{T}\sum_{t=1}^{T} \left(\chi_{t\delta\theta} u_t\right)^2 \qquad \text{(A.7)}$$

are consistent for δ_0, θ_0. From their proof it suffices to show that as $T \to \infty$,

$$\sup \frac{1}{T}\sum_{t=1}^{T} D_{t\delta\theta}^2 \;\to\; 0, \qquad \text{(A.8)}$$

$$\sup \frac{1}{T}\sum_{t=1}^{T} S_{t\delta\theta}^2 \;\to_p\; 0, \qquad \text{(A.9)}$$

$$\sup \left| \frac{1}{T}\sum_{t=1}^{T} \left(\chi_{t\delta\theta} u_t\right) D_{t\delta\theta} \right| \;\to_p\; 0, \qquad \text{(A.10)}$$

$$\sup \left| \frac{1}{T}\sum_{t=1}^{T} \left(\chi_{t\delta\theta} u_t\right) S_{t\delta\theta} \right| \;\to_p\; 0, \qquad \text{(A.11)}$$

$$\sup \left| \frac{1}{T}\sum_{t=1}^{T} D_{t\delta\theta} S_{t\delta\theta} \right| \;\to_p\; 0, \qquad \text{(A.12)}$$

where the suprema here and subsequently are over $\delta \in \nabla$, $\theta \in \Theta$. Given (A.8) and (A.9), and using the Cauchy inequality, (A.10)-(A.12) follow from the fact, implied by the proof of Hualde and Robinson (2011), that (A.7) is uniformly $O_p(1)$.

To prove (A.8) note first that Lemma 3 of Robinson (2012b) gives, for all sufficiently large T,

$$\inf_t |k_t| \geq \frac{1}{8}. \tag{A.13}$$

Suppressing reference to δ, θ in $\chi_j = \chi_j(\delta, \theta)$,

$$\sum_{s=1}^{T} \left(\chi_{t\delta\theta} g_t - \chi_{s\delta\theta} g_s\right) k_{ts} = \sum_{s=1}^{T} \left(\sum_{j=0}^{t-1} \chi_j g_{t-j} - \sum_{j=0}^{s-1} \chi_j g_{s-j} \right) k_{ts}.$$

Defining $g(x) = g(0)$, $x \in (-1, 0)$, this is

$$\sum_{j=0}^{T-1} \chi_j \sum_{s=1}^{T} \left(g_{t-j} - g_{s-j}\right) k_{ts} - g(0) \sum_{s=1}^{T} \left(\sum_{j=t}^{T} \chi_j - \sum_{j=s}^{T} \chi_j \right) k_{ts}. \tag{A.14}$$

Considering the first term in (A.14),

$$\sup \left| \sum_{s=1}^{T} \left(\chi_{t\delta\theta} g_t - \chi_{s\delta\theta} g_s\right) k_{ts} \right| \leq \sum_{j=0}^{T-1} \sup \left|\chi_j\right| \left| \sum_{s=1}^{T} \left(g_{t-j} - g_{s-j}\right) k_{ts} \right|. \tag{A.15}$$

From (A.3) and Assumption 3

$$\sup \left|\chi_j\right| \leq K j^{\nabla_2 - \nabla_1 - 1}. \tag{A.16}$$

Applying Assumption 4 and with $\dot{g}_t$ denoting the derivative of $g(x)$ at $x = t/T$,

$$\sum_{s=1}^{T} \left(g_{t-j} - g_{s-j}\right) k_{ts} = \dot{g}_{t-j} \sum_{s=1}^{T} \left(\frac{t-s}{T}\right) k_{ts} + O\left(\sum_{s=1}^{T} \left(\frac{t-s}{T}\right)^2 |k_{ts}| \right), \tag{A.17}$$

where $\dot{g}_t = 0$, $t \leq 0$. By, e.g., Lemma 2 of Robinson (2012b)

$$\sum_{s=1}^{T} \left(\frac{t-s}{T}\right) k_{ts} = T h^2 \left(\frac{1}{Th} \sum_{s=1}^{T} \left(\frac{t-s}{Th}\right) k_{ts} - \int u k(u) du \right) = O(h) \tag{A.18}$$

uniformly in $t \in (Th, T - Th)$. Uniformly in $t \leq Th$, $t \geq T - Th$,

$$\sum_{s=1}^{T} \left(\frac{t-s}{T}\right) k_{ts} = O(Th^2) \tag{A.19}$$

from, e.g., Lemma 1 of Robinson (2012b). By the same lemma,

$$\sum_{s=1}^{T}(\frac{t-s}{T})^2 \left|k_{ts}\right| = O(Th^3) \tag{A.20}$$

uniformly in t. Thus from (A.17)-(A.20),

$$\begin{aligned} \max_j \left|\sum_{s=1}^{T} \left(g_{t-j} - g_{s-j}\right) k_{ts}\right| &= O\left(h + Th^3\right), \ t \in (Th, T-Th) \\ &= O\left(Th^2\right), \ t \leq Th, t \geq T - Th, \end{aligned} \tag{A.21}$$

uniformly.

Next, considering the second term in (A.14),

$$\begin{aligned} \sum_{s=1}^{T}\left(\sum_{j=t}^{T}\chi_j - \sum_{j=s}^{T}\chi_j\right) k_{ts} &= \sum_{s=t+1}^{T}\left(\sum_{j=t}^{s-1}\chi_j\right) k_{ts} - \sum_{s=1}^{t-1}\left(\sum_{j=s}^{t-1}\chi_j\right) k_{ts} \\ &= \sum_{r=1}^{T-t}\sum_{j=t}^{t+r-1}\chi_j k_{t,t+r} - \sum_{r=1}^{t-1}\sum_{j=t-r}^{t-1}\chi_j k_{t,t+r}. \end{aligned}$$

For $t \leq (T+1)/2$ this is

$$\sum_{r=1}^{t-1}\left(\sum_{j=t}^{t+r-1}\chi_j - \sum_{j=t-r}^{t-1}\chi_j\right) k_{t,t+r} + \sum_{r=t}^{T-t}\sum_{j=t-r}^{t+r-1}\chi_j k_{t,t+r}.$$

by Assumption 5. Now

$$\begin{aligned} \left|\sum_{j=t}^{t+r-1}\chi_j - \sum_{j=t-r}^{t-1}\chi_j\right| &= \left|\sum_{j=0}^{r-1}\left(\chi_{t+j} - \chi_{t-j-1}\right)\right| \leq K\sum_{j=0}^{r-1}\left((t-j-1)^{\delta-\delta_0-1} - (t+j)^{\delta-\delta_0-1}\right) \\ &\leq Kt^{\delta-\delta_0-2}\sum_{j=0}^{r-1} j \leq Kt^{\delta-\delta_0-2} r^2, \end{aligned}$$

so from Assumption 5

$$\begin{aligned} \sup\left|\sum_{r=1}^{t-1}\left(\sum_{j=t}^{t+r-1}\chi_j - \sum_{j=t-r}^{t-1}\chi_j\right) k_{t,t+r}\right| &\leq Kt^{\nabla_2-\nabla_1-2}\sum_{r=1}^{t-1} r^2\left(1+r^{2+\eta}\right)^{-1} \\ &\leq Kt^{\nabla_2-\nabla_1-2}\sum_{r=1}^{t-1} r^{-\eta} \\ &\leq Kt^{\nabla_2-\nabla_1-1-\eta}. \end{aligned}$$

For $(T+1)/2$ we have

$$\sum_{r=1}^{T-t}\sum_{j=t}^{t+r-1}\chi_j k_{t,t+r}-\sum_{r=1}^{t-1}\sum_{j=t-r}^{t-1}\chi_j k_{t,t+r}=\sum_{r=1}^{T-t}\left(\sum_{j=t}^{t+r-1}\chi_j-\sum_{j=t-r}^{t-1}\chi_j\right)k_{t,t+r}-\sum_{r=T-t+1}^{t-1}\sum_{j=t-r}^{t-1}\chi_j k_{t,t+r},$$

where much as before

$$\left|\sum_{j=t}^{t+r-1}\chi_j-\sum_{j=t-r}^{t-1}\chi_j\right|\le Kt^{\delta-\delta_0-2}r^2$$

and thence

$$\sup\left|\sum_{r=1}^{T-t}\left(\sum_{j=t}^{t+r-1}\chi_j-\sum_{j=t-r}^{t-1}\chi_j\right)k_{t,t+r}\right|\le Kt^{\nabla_2-\nabla_1-1-\eta}.$$

From these results and also (A.13), (A.15) and (A.16),

$$\begin{aligned}\sup_{\delta,\theta}|D_{t\delta\theta}| &\le K(Th)^{-1}\left(\left(h+Th^3\right)\sum_{j=0}^{T-1}(1+j)^{\delta_0-\nabla_1-1}+t^{\nabla_2-\nabla_1-1-\eta}\right)\\ &\le K(T^{\nabla_2-\nabla_1-1}+T^{\nabla_2-\nabla_1}h^2),\ t\in(Th,T-Th),\end{aligned}$$

and

$$\begin{aligned}\sup_{\delta,\theta}|D_{t\delta\theta}| &\le K(Th)^{-1}\left(Th^2\sum_{j=0}^{T-1}(1+j)^{\nabla_2-\nabla_1-1}+t^{\nabla_2-\nabla_1-1-\eta}\right)\\ &\le KT^{\nabla_2-\nabla_1}h,\ t\le Th, t\ge T-Th,\end{aligned}$$

uniformly over the stated ranges of t. Thus

$$\sup\frac{1}{T}\sum_{t=1}^{T}D^2_{t\delta\theta}\le K(T^{2(\nabla_2-\nabla_1-1)}+T^{2(\nabla_2-\nabla_1)}h^4+T^{2(\nabla_2-\nabla_1)}h^3)\to 0$$

by Assumption 6, verifying (A.8).

To prove (A.9), we have

$$\sum_{s=1}^{T}\chi_{s\delta\theta}u_s k_{ts}=\sum_{j=0}^{T-1}\chi_j c_{tj}=\sum_{j=0}^{[Th]}\chi_j c_{tj}+\sum_{j=[Th]+1}^{T-1}\chi_j c_{tj}$$

where

$$c_{tj} = \sum_{r=1}^{T-j} u_r k_{t,r+j},$$

so, using (A.3),

$$\sup \left| \sum_{j=0}^{[Th]} \chi_j c_{tj} \right| \leq \sum_{j=0}^{[Th]} \left(\sup \left| \chi_j \right| \right) |c_{tj}| \leq K \sum_{j=1}^{[Th]} j^{\nabla_2 - \nabla_1 - 1} |c_{tj}|$$

and thus

$$E \left(\sup \left| \sum_{j=0}^{[Th]} \chi_j c_{tj} \right| \right)^2 \leq K \sum_{j=1}^{[Th]} \sum_{l=1}^{[Th]} j^{\nabla_2 - \nabla_1 - 1} l^{\nabla_2 - \nabla_1 - 1} \left(E c_{tj}^2 E c_{tl}^2 \right)^{1/2}. \tag{A.22}$$

Now

$$E c_{tj}^2 = \sigma^2 \sum_{r=1}^{T-j} k_{t,r+j}^2 = O(Th)$$

by Assumption 6, so (A.22) is $O((Th)^{2(\nabla_2 - \nabla_1)+1}) = o((Th)^2)$ uniformly in t, by Assumption 3.

By summation-by-parts

$$\sum_{j=[Th]+1}^{T-1} \chi_j c_{tj} = \sum_{j=[Th]+1}^{T-2} \left(\chi_j - \chi_{j+1} \right) d_{tj} + \chi_{T-1} d_{t,T-1}. \tag{A.23}$$

where

$$d_{tj} = \sum_{l=0}^{j} c_{tl}.$$

Now (A.23) is bounded uniformly by

$$\begin{aligned} & \sum_{j=[Th]+1}^{T-2} \left(\sup \left| \chi_j - \chi_{j+1} \right| \right) |d_{tj}| + \left(\sup \left| \chi_{T-1} \right| \right) |d_{t,T-1}| \\ \leq \ & K \sum_{j=[Th]+1}^{T-2} j^{\gamma - 1} |d_{tj}| + K T^{\gamma} |d_{t,T-1}| \end{aligned} \tag{A.24}$$

using (A.3) and (A.5) and writing $\gamma = \max(\nabla_2 - \nabla_1, 1 - \varsigma) - 1$. Rearranging,

$$d_{tj} = \sum_{r=1}^{T} u_r \left(\sum_{s=r}^{\min(r+j,T)} k_{ts} \right),$$

so

$$Ed_{tj}^2 = \sigma^2 \sum_{r=1}^{T} \left(\sum_{s=r}^{\min(r+j,T)} k_{ts} \right)^2 \leq Kj \left(\sum_{s=1}^{T} |k_{ts}| \right)^2 \leq Kj \left(Th\right)^2$$

and (A.24) has second moment bounded by

$$\begin{aligned} & K \left(Th\right)^2 \sum_{j=[Th]+1}^{T-2} \sum_{l=[Th]+1}^{T-2} j^{\gamma-1/2} l^{\gamma-1/2} + K \left(Th\right)^2 T^{2\gamma+1} \\ \leq \ & K \left(Th\right)^{2\gamma+3} = o((Th)^2) \end{aligned}$$

uniformly in t, since $\gamma < -1/2$. We have established that $E \sup S_{t\delta\theta}^2 = o(1)$ uniformly in t, whence follows (A.9), to complete the proof of the theorem.

In the preceding proof it was shown that $\sup_{\nabla} T^{-1} \sum_{t=1}^{T} S_{t\delta\theta}^2 = O_p\left((Th)^{2\gamma+1} \right)$, and since $\gamma \geq \nabla_2 - \nabla_2 - 1$, (14) bites in order to establish (A.9). In a more specialized setting the following Lemma implies that $\sup_{\nabla} T^{-1} \sum_{t=1}^{T} E\left(S_{t\delta\theta}^2\right) > c\left(Th\right)^{2(\nabla_2 - \delta_0)-1}$, and when as indicated after Assumption 3 and elsewhere in the proofs we replace $\nabla_2 - \delta_0$ by $\nabla_2 - \nabla_1$ here it appears that (14) may be sharp.

Lemma *Under (4) with $\delta - \delta_0 > 0$ and $k\left(u\right) = 1\left(|u| \leq 1\right)/2$, for a generic arbitrarily small $c > 0$,*

$$\frac{1}{T} \sum_{t=1}^{T} E\left(S_{t\delta\theta}^2\right) > c\left(Th\right)^{2(\delta - \delta_0)+1}.$$

Proof of Lemma

With the above definitions, and since for given δ χ_j is either always positive or always negative,

$$
\begin{aligned}
(Th)^2 \frac{1}{T} \sum_{t=1}^{T} E\left(S_{t\delta\theta}^2\right) &= \frac{1}{T} \sum_{t=1}^{T} E\left(\sum_{j=0}^{T-1} \chi_j c_{tj}\right)^2 \\
&= \frac{1}{T} \sum_{j=0}^{T-1} \sum_{\ell=0}^{T-1} \chi_j \chi_\ell \sum_{r=1}^{T-j} \sum_{s=1}^{T-\ell} E\left(u_r u_s\right) \sum_{t=1}^{T} k_{t,r+j} k_{t,s+\ell} \\
&= \frac{\sigma^2}{T} \sum_{j=0}^{T-1} \sum_{\ell=0}^{T-1} \chi_j \chi_\ell \sum_{r=1}^{T-\max(j,\ell)} \sum_{t=1}^{T} k_{t,r+j} k_{t,r+\ell} \\
&> \frac{\sigma^2}{4T} \sum_{j=0}^{T-1} \sum_{\ell=j+1}^{T-1} \chi_j \chi_\ell \sum_{t=1}^{T} \sum_{r=1}^{T-\ell} 1\left(|r+\ell-t| \leq Th\right)\ 1\left(|r+j-t| \leq Th\right) \\
&> \frac{c}{T} \sum_{j=0}^{[Th/4]} \sum_{\ell=j+1}^{[Th/2]} \chi_j \chi_\ell \sum_{t=1}^{T} \sum_{r=1}^{T-[Th/2]} 1\left(|r-t| \leq Th/2\right) \\
&> cTh \sum_{j=0}^{[Th/4]} j^{\delta-\delta_0-1} \sum_{\ell=[Th/4]+1}^{[Th/2]} \ell^{\delta-\delta_0-1} \\
&> c\,(Th)^{2(\delta-\delta_0)+1}
\end{aligned}
$$

to complete the proof.

APPENDIX B

Proof of Theorem 2

Writing $\partial = \partial/\partial(\delta,\theta')$, by the mean value theorem

$$
0 = \partial Q(\widehat{\delta},\widehat{\theta})/2 = \partial Q(\delta_0,\theta_0)/2 + \overline{\Omega} \begin{pmatrix} \widehat{\delta}-\delta_0 \\ \widehat{\theta}-\theta_0 \end{pmatrix},
$$

where $\partial Q(\delta,\theta)$ is given by (18)-(20) and $\overline{\Omega}$ is obtained from the matrix $\partial^2 Q(\delta,\theta)/2 = \partial\partial' Q(\delta,\theta)/2$ by evaluating each row at a generally different $\widetilde{\delta},\widetilde{\theta}$ such that $\left\|\widetilde{\delta}-\delta_0, \widetilde{\theta}'-\theta_0'\right\| \leq \left\|\widehat{\delta}-\delta_0, \widehat{\theta}'-\theta_0'\right\|$. The theorem follows if

$$
\frac{T^{1/2}}{2} \partial Q(\delta_0,\theta_0) \rightarrow_d \mathcal{N}(0,\Omega), \tag{B.1}
$$

$$
\overline{\Omega} \rightarrow_p \Omega. \tag{B.2}
$$

From (11)

$$\begin{aligned}\hat{u}_t(\delta_0,\theta_0) &= u_t + D_t - S_t \\ &= u_t + \sum_{s=1}^{T} (g_t - g_s)\, k_{ts} - \sum_{s=1}^{T} u_s k_{ts}\end{aligned}$$

where

$$D_t = \sum_{s=1}^{T} (g_t - g_s)\, k_{ts}/k_t, \quad S_t = \sum_{s=1}^{T} u_s k_{ts}/k_t.$$

From (19), (20)

$$\partial \hat{u}_t(\delta,\theta) = \partial \chi_{t\delta\theta} u_t + \partial D_{t\delta\theta} - \partial S_{t\delta\theta}.$$

Write

$$\pi_j = \partial \chi_j \,|_{\delta_0,\theta_0} = \sum_{l=0}^{j} \begin{pmatrix} \tau_l(\theta_0)\partial\alpha_{j-l}(\delta_0)/\partial\delta \\ \alpha_{j-l}(\delta_0)\partial\tau_l(\theta_0)/\partial\theta \end{pmatrix} = \begin{pmatrix} j^{-1} \\ \psi_j(\theta_0) \end{pmatrix}, \; j \geq 1$$

Then

$$\partial \hat{u}_t(\delta_0,\theta_0) = v_t + \partial D_t - \partial S_t,$$

where

$$\begin{aligned}v_t &= \partial \chi_{t\delta\theta} u_t \,|_{\delta_0,\theta_0} = \sum_{j=1}^{t-1} \pi_j u_{t-j}, \\ \partial D_t &= \partial D_{t\delta\theta} \,|_{\delta_0,\theta_0} = \sum_{s=1}^{T} \left(\sum_{j=1}^{t-1} g_{t-j}\pi_j - \sum_{j=1}^{s-1} g_{s-j}\pi_j \right) k_{ts}/k_t, \\ \partial S_t &= \partial S_{t\delta\theta} \,|_{\delta_0,\theta_0} = \sum_{s=1}^{T} \sum_{j=1}^{s-1} u_{s-j}\pi_j k_{ts}/k_t.\end{aligned}$$

Thus from (18)

$$\partial Q(\delta_0,\theta_0) = \frac{2}{T} \sum_{t=1}^{T} (u_t + D_t - S_t)(v_t + \partial D_t - \partial S_t).$$

Hualde and Robinson (2011) show that

$$T^{-1/2} \sum_{t=1}^{T} u_t v_t \;\to_d\; \mathcal{N}(0,\Omega).$$

Thus (B.1) holds if all the following are $o_p(1)$:

$$T^{-1/2}\sum_{t=1}^{T}u_t\partial D_t,\ T^{-1/2}\sum_{t=1}^{T}u_t\partial S_t,\ T^{-1/2}\sum_{t=1}^{T}D_tv_t,\ T^{-1/2}\sum_{t=1}^{T}S_tv_t,$$
$$T^{-1/2}\sum_{t=1}^{T}D_t\partial D_t,\ T^{-1/2}\sum_{t=1}^{T}D_t\partial S_t,\ T^{-1/2}\sum_{t=1}^{T}S_t\partial D_t,\ T^{-1/2}\sum_{t=1}^{T}S_t\partial S_t. \quad \text{(B.3)}$$

Note first that, as in (A.14), and using (A.13), (A.21) and $\|\pi_j\| = O\left(j^{-1}\right)$,

$$\begin{aligned}
\|\partial D_t\| &\leq \frac{K}{Th}\left\|\sum_{j=1}^{T}\pi_j \sum_{s=1}^{T}\left(g_{t-j}-g_{s-j}\right)k_{ts}\right\| \\
&\leq \frac{K}{Th}\max_{t,j}\left|\sum_{s=1}^{T}\left(g_{t-j}-g_{s-j}\right)k_{ts}\right|\sum_{j=1}^{T}j^{-1} \\
&= O\left(\left(T^{-1}+h^2\right)\log T\right),\quad t\in(Th,T-Th) \\
&= O\left(h\log T\right),\ t\leq Th, t\geq T-Th, \qquad \text{(B.4)}
\end{aligned}$$

uniformly over the stated ranges of t. Similarly but more easily, we derive, uniformly,

$$\begin{aligned}
D_t &= O\left(\left(T^{-1}+h^2\right)\right),\quad t\in(Th,T-Th) \\
&= O\left(h\right),\ t\leq Th, t\geq T-Th. \qquad \text{(B.5)}
\end{aligned}$$

We check each claim of (B.3) in turn; for notational convenience, when $j\leq 0$ we take $\pi_j = 0$ and interpret $1/j$ to be 0.

First, using (B.4) and Assumption 6,

$$\begin{aligned}
E\left\|T^{-1/2}\sum_{t=1}^{T}u_t\partial D_t\right\|^2 &\leq \frac{K}{T}\sum_{t=1}^{T}\|\partial D_t\|^2 \\
&\leq \frac{K}{T}\left(T\left(T^{-1}+h^2\right)^2\log^2 T + Th^3\log^2 T\right) \\
&= O\left(T^{-2}\log^2 T + h^4\log^2 T + h^3\log^2 T\right) = o(1).
\end{aligned}$$

Next,

$$\begin{aligned}
\left\| T^{-1/2}\sum_{t=1}^{T} u_t \partial S_t \right\|^2 &= \frac{1}{T} E\left\{ \sum_{t=1}^{T} u_t \sum_{r=1}^{T} u_r \sum_{s=1}^{T}\sum_{j=1}^{s-1} u_{s-j}\pi_j' \frac{k_{ts}}{k_t} \sum_{q=1}^{T}\sum_{l=1}^{q-1} u_{q-l}\pi_l \frac{k_{rq}}{k_r} \right\} \\
&= \frac{\sigma^4}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{j=1}^{s-1}\sum_{q=1}^{T} \pi_j' \pi_{q-s+j} \frac{k_{ts}k_{tq}}{k_t k_r} \\
&\quad + \frac{\sigma^4}{T}\sum_{t=1}^{T}\sum_{r=1}^{T}\sum_{s=1}^{T}\sum_{q=1}^{T} \pi_{s-t}' \pi_{q-r} \frac{k_{ts}}{k_t}\frac{k_{rq}}{k_r} \\
&\quad + \frac{\sigma^4}{T}\sum_{t=1}^{T}\sum_{r=1}^{T}\sum_{s=1}^{T}\sum_{q=1}^{T} \pi_{s-r}' \pi_{q-t} \frac{k_{ts}k_{rq}}{k_t k_r} \\
&\quad + \frac{Eu_t^4}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{q=1}^{T} \pi_{s-t}' \pi_{q-t} \frac{k_{ts}k_{tq}}{k_t k_r}.
\end{aligned}$$

By boundedness of k, the final term in the last displayed expression is bounded by

$$\frac{K}{T^3h^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{q=1}^{T}\frac{1}{s-t}\frac{1}{q-t} = O\left(\frac{\log^2 T}{T^2h^2}\right),$$

while the other terms are bounded by

$$\begin{aligned}
& \frac{K}{T^3h^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{j=1}^{s-1}\sum_{q=1}^{T}\frac{|k_{ts}|}{j}\frac{|k_{tq}|}{q-s+j} + \frac{K}{T^3h^2}\sum_{t=1}^{T}\sum_{r=1}^{T}\sum_{s=1}^{T}\sum_{q=1}^{T}\frac{|k_{ts}|}{s-r}\frac{|k_{rq}|}{q-t} \\
\le\ & \frac{K}{T^3h^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{j=1}^{s-1}\sum_{q=1}^{T}\frac{|k_{ts}|}{j}\frac{1}{q-s+j} + \frac{K}{T^3h^2}\sum_{t=1}^{T}\sum_{r=1}^{T}\sum_{s=1}^{T}\sum_{q=1}^{T}\frac{1}{s-r}\frac{|k_{rq}|}{q-t} \\
\le\ & \frac{KT^2h\log^2 T}{T^3h^2} = O\left(\frac{\log^2 T}{Th}\right) = o(1),
\end{aligned}$$

by Assumption 8.

Next, noting that

$$\sum_{t=1}^{T} D_t v_t = \sum_{t=1}^{T} u_t \sum_{s=t+1}^{T} D_s \pi_{s-t},$$

from (B.5)

$$
\begin{aligned}
E\left\|T^{-1/2}\sum_{t=1}^{T}D_t v_t\right\|^2 &= \sigma^2 T^{-1} E\sum_{t=1}^{T}\left\|\sum_{s=t+1}^{T}D_s\pi_{s-t}\right\|^2 \\
&\leq KT^{-1}h^2\sum_{t=1}^{T}\left(\sum_{s=t+1}^{T}(s-t)^{-1}\right)^2 \\
&\leq Kh^2\log^2 T = o(1),
\end{aligned}
$$

by Assumption 6.

Next, using (B.4) and (B.5),

$$
\begin{aligned}
T^{-1/2}\sum_{t=1}^{T}D_t\partial D_t &= O\left(T^{-1/2}T\left(T^{-1}+h^2\right)^2\log T + T^{-1/2}Th^3\log T\right) \\
&= O\left(T^{-3/2}\log T + T^{1/2}h^4\log T + T^{1/2}h^3\log T\right) = o(1),
\end{aligned}
$$

by Assumption 6.

Next,

$$
E\left\|T^{-1/2}\sum_{t=1}^{T}S_t v_t\right\|^2 = T^{-1}E\left\{\sum_{t=1}^{T}\sum_{s=1}^{T}u_s\frac{k_{ts}}{k_t}\sum_{j=1}^{t-1}\pi_j' u_{t-j}\sum_{r=1}^{T}\sum_{q=1}^{T}u_q\frac{k_{rq}}{k_r}\sum_{l=1}^{r-1}\pi_l u_{r-l}\right\}
$$

which equals

$$
\begin{aligned}
&\sigma^4 T^{-1}\left(\sum_{t=1}^{T}\sum_{s=1}^{t-1}\frac{k_{ts}}{k_t}\pi_{t-s}'\right)^2 \\
&+\sigma^4 T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}\frac{k_{ts}}{k_t}\frac{k_{rs}}{k_r}\sum_{j=1}^{t-1}\pi_j'\pi_{r-t+j} \\
&+\sigma^4 T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}\sum_{q=1}^{T}\frac{k_{ts}}{k_t}\frac{k_{rq}}{k_r}\pi_{t-q}'\pi_{r-s} \\
&+Eu_t^4\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}\frac{k_{ts}}{k_t}\frac{k_{rs}}{k_r}\pi_{t-s}'\pi_{r-s},
\end{aligned}
$$

and using boundedness of k this is bounded by $KT^{-1}(Th)^{-2}T^2\log^2 T + KT^{-1}(Th)^{-2}T\log^2 T \leq K\log^2 T/(Th^2) = o(1)$, by Assumption 8.

Next,

$$\begin{aligned}
E\left\|T^{-1/2}\sum_{t=1}^{T}D_t\partial S_t\right\|^2 &= \sigma^2 T^{-1}\sum_{t=1}^{T}\sum_{r=1}^{T}D_tD_r\sum_{s=1}^{T}\sum_{q=1}^{T}\sum_{j=1}^{s-1}\pi_j'\pi_{q-s+j}\frac{k_{ts}}{k_t}\frac{k_{rq}}{k_r} \\
&\leq KT^{-1}(Th)^{-2}h^2\sum_{t=1}^{T}\sum_{r=1}^{T}\sum_{s=1}^{T}|k_{ts}|\sum_{q=1}^{T}\|\pi_{q-s+j}\|\sum_{j=1}^{s-1}\|\pi_j\| \\
&\leq KT^{-1}(Th)^{-2}h^2T^2(Th)\log^2 T \\
&\leq Kh\log^2 T = o(1),
\end{aligned}$$

by (B.5) and Assumption 6.

Next,

$$\begin{aligned}
E\left\|T^{-1/2}\sum_{t=1}^{T}S_t\partial D_t\right\|^2 &= E\left\|T^{-1/2}\sum_{t=1}^{T}\partial D_t\sum_{s=1}^{T}u_s\frac{k_{ts}}{k_t}\right\|^2 \\
&= \sigma^2T^{-1}\sum_{t=1}^{T}\sum_{r=1}^{T}\partial D_t'\partial D_r\sum_{s=1}^{T}\frac{k_{ts}k_{rs}}{k_tk_r} \\
&\leq KT^{-1}(Th)^{-2}\;h^2\log^2 T\sum_{t=1}^{T}\sum_{r=1}^{T}\sum_{s=1}^{T}|k_{ts}k_{rs}| \\
&\leq KT^{-1}(Th)^{-2}h^2\left(\log^2 T\right)T(Th)^2 \leq Kh^2\log^2 T = o(1),
\end{aligned}$$

by Assumption 6.

Finally,

$$\begin{aligned}
&E\left\|T^{-1/2}\sum_{t=1}^{T}S_t\partial S_t\right\|^2 \\
&= T^{-1}E\left(\sum_{t=1}^{T}\sum_{s=1}^{T}u_s\frac{k_{ts}}{k_t}\sum_{r=1}^{T}\sum_{j=1}^{r-1}u_{r-j}\pi_j'\frac{k_{tr}}{k_t}\sum_{q=1}^{T}\sum_{p=1}^{T}u_p\frac{k_{qp}}{k_q}\sum_{n=1}^{T}\sum_{l=1}^{n-1}u_{n-l}\pi_l\frac{k_{qn}}{k_q}\right),
\end{aligned}$$

which equals

$$
\begin{aligned}
&\sigma^4 T^{-1} \sum_{t=1}^{T} \sum_{s=1}^{T} \frac{k_{ts}}{k_t} \sum_{r=1}^{T} \pi'_{r-s} \frac{k_{tr}}{k_t} \sum_{q=1}^{T} \sum_{p=1}^{T} \frac{k_{qp}}{k_q} \sum_{n=1}^{T} \pi_{n-p} \frac{k_{qn}}{k_q} \\
&+\sigma^4 T^{-1} \sum_{t=1}^{T} \sum_{s=1}^{T} \frac{k_{ts}}{k_t} \sum_{r=1}^{T} \sum_{j=1}^{r-1} \pi'_j \frac{k_{tr}}{k_t} \sum_{q=1}^{T} \frac{k_{qs}}{k_q} \sum_{n=1}^{T} \pi_{n-r+j} \frac{k_{qn}}{k_q} \\
&+\sigma^4 T^{-1} \sum_{t=1}^{T} \sum_{s=1}^{T} \frac{k_{ts}}{k_t} \sum_{r=1}^{T} \sum_{p=1}^{T} \pi'_{p-r} \frac{k_{tr}}{k_t} \sum_{q=1}^{T} \frac{k_{qp}}{k_q} \sum_{n=1}^{T} \pi_{n-s} \frac{k_{qn}}{k_q} \\
&+E u_t^4 T^{-1} \sum_{t=1}^{T} \sum_{s=1}^{T} \frac{k_{ts}}{k_t} \sum_{r=1}^{T} \pi'_{r-s} \frac{k_{tr}}{k_t} \sum_{q=1}^{T} \frac{k_{qs}}{k_q} \sum_{n=1}^{T} \pi_{n-s} \frac{k_{qn}}{k_q},
\end{aligned}
$$

which is bounded by

$$
\begin{aligned}
& KT^{-1} (Th)^{-4} \left(\sum_{t=1}^{T} \sum_{s=1}^{T} k_{ts} \sum_{r=1}^{T} \|\pi_{r-s}\| \right)^2 \\
& +KT^{-1} (Th)^{-4} \sum_{t=1}^{T} \sum_{s=1}^{T} |k_{ts}| \sum_{r=1}^{T} \sum_{j=1}^{r-1} \|\pi_j\| \sum_{q=1}^{T} |k_{qs}| \sum_{n=1}^{T} \|\pi_{n-r+j}\| \\
& +KT^{-1} \ (Th)^{-4} \sum_{t=1}^{T} \sum_{s=1}^{T} |k_{ts}| \sum_{r=1}^{T} \sum_{p=1}^{T} \|\pi_{p-r}\| \sum_{q=1}^{T} |k_{qp}| \sum_{n=1}^{T} \|\pi_{n-s}\| \\
& +KT^{-1} (Th)^{-4} \sum_{t=1}^{T} \sum_{s=1}^{T} |k_{ts}| \sum_{r=1}^{T} \|\pi_{r-s}\| \sum_{q=1}^{T} |k_{qs}| \sum_{n=1}^{T} \|\pi_{n-s}\| \\
\leq\ & KT^{-1} (Th)^{-4} \left(T^2 h \log T\right)^2 + KT^{-1} (Th)^{-4} T \left(Th \log T\right)^2 \\
\leq\ & K \left(Th^2\right)^{-1} \log^2 T = o(1),
\end{aligned}
$$

by Assumption 8.

This completes the proof of (B.3), and thus of (B.1).

Finally (B.2) follows if

$$
\frac{1}{2} \partial^2 Q(\delta_0, \theta_0) \to_p \Omega \tag{B.6}
$$

and, given Theorem 1, for a neighbourhood $\mathcal{N}$ of δ_0, θ_0,

$$
\sup_{\mathcal{N}} \left\| \partial^2 Q(\delta, \theta) - \partial^2 Q(\delta_0, \theta_0) \right\| \to_p 0. \tag{B.7}
$$

The proof of (B.6) partly employs Theorem 2.2 of Hualde and Robinson (2011) and partly methods used above to deal with contributions from deterministic and stochastic errors, where these are less delicate than in Theorem 1's proof because $\nabla_2 - \nabla_1$ is replaced by an arbitrarily small positive number, and less delicate than the proof of Theorem 1 because $T^{-1/2}$ norming is replaced by T^{-1} norming. The proof of (B.7) uses that of Theorem 2.2 of Hualde and Robinson (2011) and standard techniques. The full details of the proofs of (B.6) and (B.7) are very lengthy but straightforward relative to what has gone before and are thus omitted.

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