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Abstract

The second-order stochastic dominance criterion for inequality analysis introduced by Atkinson (1970) covers nearly all well-known inequality indices. The same cannot be said, in respect to poverty indices, for the second-order stochastic dominance criterion for poverty analysis introduced by Atkinson (1987). Indeed, two of the best known poverty indices, the head-count ratio and the Sen index, are excluded by it. This paper introduces a more general 'mixed' dominance criterion which provides a more comprehensive coverage of poverty indices. By establishing the relationship between welfare and poverty functions, it also generalizes the proofs given by Atkinson (1987) to include non-separable as well as separable functions.

Keywords: Poverty, second-order stochastic dominance criterion; welfare; inequality analyis.
JEL Nos.: D63, I32.

1. Introduction

In his pioneering 1970 article, Atkinson showed that, if one distribution's Lorenz curve lies nowhere below and somewhere above another distribution's Lorenz curve, all inequality indices in a very wide class will register the former distribution as having no more inequality than the latter and at least one will register it as having less. This application of the second-order stochastic dominance criterion to inequality analysis is now widely used. Atkinson (1987) showed how second-order stochastic dominance could also be applied to poverty analysis in terms of the poverty deficit curve. Although this is an extremely useful extension, dominance in terms of the deficit curve covers a narrower class of poverty indices than does Lorenz dominance in relation to inequality indices. The only regularly-used inequality index not covered by the Lorenz dominance criterion is the standard deviation of logarithms. By contrast, of the three best-known poverty indices - the head-count, the poverty gap and the Sen index - only one, the poverty gap, is covered by the poverty second-order dominance criterion. In this paper, I show that all well-known poverty functions (or, rather, their negatives) can be characterized as "almost-egalitarian" functions. I then show that, whereas Atkinson's second-order stochastic dominance criterion covers only the sub-class of egalitarian functions, the new criterion of "mixed" dominance - a mixture of first- and second-order stochastic dominance criteria - covers the class of all almost-egalitarian functions and, thus, all well-known poverty indices.

In addition, as has been widely commented on (see, for example, Ravallion, 1992), the proofs used by Atkinson (1987) are valid only for additively separable functions. Yet, as Atkinson mentions (p.759), the stochastic dominance criteria cover both separable and non-separable functions. This paper also generalizes Atkinson's proofs by making no use of the assumption of separability but instead utilizing the close relationship between poverty and welfare functions. Setting out the precise nature of this relationship is also of use given the recent attention which has been paid to this issue (Ravallion, 1993).

Section 2 sets out and explains the axiomatic framework and defines the classes of poverty indices with which we will be concerned and give examples. Section 3 defines the various dominance criteria and gives and proves the key theorems. Section 4 concludes.

2. Egalitarian and almost-egalitarian opulence functions

Consider a pair of distributions, defined over y - "income" - denoted by their distribution functions, F and F^*. All F and F^* in $\mathcal{F}$ are non-decreasing and right-continuous, bounded by zero and one, and have finite means. Let p=F(y) so that p is the proportion with income less than or equal to y. F and F^* may be continuous, discrete or mixed.[2]

2.1 Axiomatic framework

Since the proofs in the next section are based on linking poverty to welfare analysis and since welfare is a good and poverty a bad, it will simplify matters and cause no loss of generality to talk in terms of "inverse-poverty" or *opulence* functions, defined to be the negatives of poverty functions. To avoid repetition of the phrase "the negative of", I will use the names of various poverty functions as the names also for the negatives of the respective functions. An opulence or welfare function will then be defined to be a function, $S:\mathcal{F}\rightarrow\mathbf{R}$, which satisfies some combination of the following assumptions.

1 **S is weakly increasing in y** (illustrated by Figure 1)
Let $F(y)-F^*(y)=c>0$ *for* $y_1\leq y<y_2$ *and* $F^*(y)=F(y)$ $\forall y<y_1$ *and* $\forall y\geq y_2$. *Then, for all F,* $S(F^*)\geq S(F)$. Note that F is generated from F^* by a *reduction in income.*

2S **S is insensitive to changes in y at or above a poverty indifference line, Z^N**
If $F(y)=F^*(y)$, $\forall y<Z^N$, *then, for all F,* $S(F)=S(F^*)$.

2W **S is insensitive to changes in y above Z^N**
If $F(y)=F^*(y)$, $\forall y \leq Z^N$, then, for all F, $S(F)=S(F^*)$.

3S **S satisfies the transfer principle** (illustrated by Figure 2)
Let the mean of F (F^*) be μ (μ^*). Let $\mu=\mu^*$, $F(y)-F^*(y)=c>0$ for $y_1 \leq y<y_2$, $F(y)-F^*(y)=d<0$ for $y_3 \leq y<y_4$, where $y_2 \leq y_3$, and $F^*(y)=F(y)$ $\forall y<y_1$, $y=y_2$, $y_2<y<y_3$ (if $y_3 \neq y_2$) and $\forall y \geq y_4$. Then, for all F, $S(F^*) \geq S(F)$. Note that F is generated from F^* by a single *mean-preserving spread* (Rothschild and Stiglitz, 1970) or rank-preserving regressive transfer (if the distributions have discrete members).

3W **S satisfies the transfer principle except possibly for crossings of Z^N**
Define F^* and F as in 3S. If in addition it holds that $F(Z^N)=F^*(Z^N)$ then, for all F, $S(F^*) \geq S(F)$.

A function which satisfies the transfer principle, 3S, will be said to be an *egalitarian* (strictly, non anti-egalitarian) function.[3] Since those functions which satisfy 3W ('W' for weak) satisfy 3S ('S' for strong), except possibly where crossings of a poverty indifference line are concerned, they will be called *almost egalitarian*. The notion of a "poverty indifference line" is explained in the next sub-section.

The types of functions with which we will be concerned can now be defined. *Welfare* functions are increasing in income and so satisfy 1; the sub-class of egalitarian welfare functions satisfies also 3S. *Opulence* functions are also increasing in income but are insensitive to changes above some level of income and so satisfy 2W as well as 1; almost-egalitarian opulence functions satisfy in addition 3W; and egalitarian in addition 3S and 2S. Note that since a function which satisfies 2S must satisfy 2W and one which satisfies 3S must satisfy 3W, all egalitarian opulence functions are also almost-egalitarian opulence functions. The relationships between these various types of welfare and opulence functions are illustrated in Figure 3.

2.2 Poverty indifference lines and poverty lines

Somewhat unusually, the axiomatic framework set out above makes reference not to poverty lines but to *poverty indifference* lines, Z^N. This turns out to make analysis of poverty dominance much easier as it means we can be concerned with functions which, however else they differ, all share such an indifference line, either fixed at some value or between an upper and lower bound. In turn this makes possible more succinct proofs and statements of results, in particular in relation to the new mixed dominance criterion. However, in the end, results must also be expressed in terms of the more widely used notion, the poverty line. The *poverty line*, Z^P, will be defined as the minimum value of Z^N for which some opulence function satisfies 2W. Note that, given a functional form, choice of Z^P uniquely defines a particular opulence function; as we will see, this is true only for those opulence functions which satisfies 3W but not 3S if only Z^N is chosen.

The following two results relates sets of opulence functions defined in relation to Z^N and sets defined in relation to Z^P and are drawn on later in the paper.

Lemma 1a *The set of all S satisfying any combination of assumptions which make no reference to Z^N and either 2W or 2S for Z^N=Z is coincident with the set satisfying the same combination of assumptions for (i) $Z^N \leq Z$ and (ii) $Z^P \leq Z$.*

Proof Say the coincidence (i) were not true. Then there would be some S with $Z^N=Z^*<Z$ which was not an S satisfying the same assumptions with $Z^N=Z$. But then one could have $F(y)=F^*(y)$, $\forall y \leq Z$ ($\forall y<Z$ if the S satisfies 2S) and not have $S(F)=S(F^*)$, which, since the former implies $F(y)=F^*(y)$, $\forall y \leq Z^*$, violates 2W (and 2S) for $Z^N=Z^*$, which is a contradiction. The coincidence (ii) then follows directly from the definition of Z^P.■

Lemma 1b *The set of all S satisfying 1, 2W and 3W but not 3S for Z^N=Z is not coincident (i) with the set satisfying the same assumptions for $Z^N \leq Z$ and is coincident (ii) with the set satisfying the same assumptions* *for* $Z^P = Z$.

Proof If S satisfies 2W and 3W in relation to $Z^N = Z^*$ but not 3S then there exists F and F^* such that F^* is generated from F by a mean-preserving spread which results in $F(Z^*) \neq F(Z)$ and $S(F) > S(F^*)$. Then by 3W S cannot be an almost-egalitarian function with $Z^N > Z^*$. The coincidence (ii) follows from this and the definition of Z^P.■

It follows from Lemma 1a that the set of (egalitarian) opulence functions with $Z^N = Z^*$ is a strict sub-set of the set of (egalitarian) opulence functions with $Z^N = Z > Z^*$. However, from Lemma 1b, this does not hold for the set of almost-egalitarian opulence functions. In terms of Figure 3, the sets of opulence and egalitarian opulence functions, but not almost-egalitarian opulence functions are increasing in Z^N. In the extreme case, if Z^N is allowed to approach infinity the sets of (egalitarian) welfare and (egalitarian) opulence functions become equivalent.

2.3 Egalitarian and almost-egalitarian opulence functions

All opulence functions, by invoking 2W, render distributional information above the poverty line irrelevant to the measurement of poverty. But 3W treats the poverty line also as a potential threshold, able to have a discrete impact on well-being. 3S by contrast regards poverty entirely as a matter of degree. If S satisfies 3S, it can never be increased by a regressive transfer. If it satisfies 3W only, the poverty line is treated as a threshold and S can be increased if the regressive transfer reduces the number below it. Formally,

Theorem 1 *If S satisfies 1, 2W, and 3W, but not 3S, then the only mean-preserving spreads which can increase S are those which reduce the proportion with income less than or equal to* Z^P.

Proof Let F be generated from F^* by a mean-preserving spread (defined in 3S). There are three possible cases. First, $F(Z^N) = F^*(Z^N)$. Then the mean-preserving spread cannot increase S by 3W. Second, $F(Z^N)>F^*(Z^N)$. Then 3W cannot be appealed to. But it must be the case that $y_1 \leq Z^N < y_2$ (see 3S). The change from F^* to F at and above y_3 can be ignored by 2W. The change below y_2 cannot increase S by 1. Third, $F(Z^N)<F^*(Z^N)$. Then $y_3 \leq Z^N < y_4$. Now the change from F^* to F at and above y_3 cannot necessarily be ignored (2W cannot be invoked) so S may rise or fall. Further, from Lemma 1b, if S satisfies 3W but not 3S, $Z^N=Z^P$.■

The restriction to satisfying 3W seems widely accepted. But whether 3S or 3W is the more appropriate assumption is a matter for debate (see especially Sen, 1982, pp.32-33 and Atkinson, 1987, p.759). On the one hand, it may be held objectionable to allow for the possibility of regressive transfers increasing S. On the other, the very suggestion that there is a poverty line does seem to carry with it the idea of a threshold, and the existence of a strong asymmetry between being just below and just above the poverty line. To capture this asymmetry, recourse to 3W may be required. For, as Sen writes, an assumption such as 3S

> ... takes no note whatever of the poverty line, and while that is quite legitimate for a general measure of economic inequality [or welfare, one might add] for the *whole* community, it is arguable that this is not so for a measure of poverty as such. (1982, p.33)[4]

Taking Sen's cue, there would be little point considering the class of almost-egalitarian welfare functions (S satisfying 1 and 3W), but there certainly is point to considering the class of almost-egalitarian opulence functions, and developing a class of dominance criteria which covers them. This is the purpose of this paper.

The requirement that egalitarian opulence functions should satisfy, as well as 3S, 2S (insensitivity to changes on or above some poverty indifference line) rather than only the weaker 2W (insensitivity to changes above some poverty indifference line) is admittedly arbitrary. The coupling is made simply because it seems to characterize the widest class of opulence functions in relation to which a second-order stochastic dominance proof can be given (see 3.2 below).

But note the loss of generality it causes is negligible. Separable opulence functions which satisfy 3S must satisfy 2S (since separability and 3S together imply concavity which implies continuity). There are no poverty indices in use which have negatives which satisfy 3S but not 2S.

2.4 Some examples

Nearly all poverty indices can be written as the negative of egalitarian or almost-egalitarian opulence functions. Examples are given in Table 1. The table is not meant to be exhaustive, but it does contain the best-known indices. Separable opulence functions can be written as

$$S(F) = \int_{-\infty}^{Z^P} s(y)dF(y) \tag{1}$$

For such functions, only s(y) for $y \leq Z^P$ is given in the table. For $y > Z^P$, $s(y)=0$. For the only non-separable function given - the Sen index - the formula is written assuming a discrete distribution with N members, q of whom have income less than or equal to Z^P.[5]

There are some opulence functions which do not conform to the above set of assumptions and so are not included in the table. The Thon (1979) variant of the Sen index, for example, is not defined over F (it is not replication invariant). The Hamada-Takayama (1977) class of indices fails to satisfy 1. However, these indices are not widely used. In addition, the assumptions they violate are non-controversial.

All the opulence functions in Table 1 are well-known - in their more conventional form as poverty indices - except for the last, the 'generalized Clark' function introduced in Howes (1993). This is included since it illustrates clearly the difference between 3S and 3W. The second term of the generalized Clark function (C) captures the 'fixed cost' of being poor, the first the 'variable cost' increasing in the ratio between one's income and Z^P. By adjusting C and α, one can vary the relative importance of these two components.[6] At the

TABLE 1
Examples of opulence functions

Name	Function
Almost-egalitarian functions (satisfying 1, 2W and 3W)	
Head count	-1
Sen (1976)	$-2/[(q+1)N]*\Sigma_{i=1}^{q}[(1-y_i/Z^P)*(q+1-i)$
Generalized Clark	$(1-C)/\alpha[(y/Z_m^P)^\alpha-1]-C$, $\alpha\leq1$, $\alpha\neq0$, $1\geq C\geq0$
Of which egalitarian functions (satisfying 1, 2S and 3S)	
Poverty gap	y/Z^P-1
Watts (1968)	$-\ln(y/Z^P)$
Clark *et al.* (1981)	$1/\alpha[(y/Z^P)^\alpha-1]$, $\alpha\leq1$, $\alpha\neq0$
Foster *et al.* (1984)	$-(1-y/Z^P)^\beta$, $\beta\geq1$

Notes: The poverty gap (Watts function) is the Clark *et al.* function with $\alpha=1$ (=0). The poverty gap (head count) is the Foster *et al.* function with $\beta=1$ (=0). It is assumed, for simplicity, that $Z^N>0$ and that $F(0)=0$.

extremes, if C=1 there is no variable cost, and one is back with the head-count ratio; if C=0, there is no fixed cost, and one has the Clark *et al.* function.

Alternative axiomatizations of poverty functions are given by Chakravarty (1990). The advantages of the new axiomatization given here are its simplicity (at most three assumptions are used to define some class of S) and its linking of poverty and welfare functions. This not only makes for very simple poverty stochastic dominance proofs, as we will see. In addition, debate has recently broken out over which class of functions - welfare or poverty - should be used to form policy recommendations (see Stern, 1987, and Ravallion, 1993, for opposing views). It is hoped that the framework established here clarifies the choices being made when one engages in either welfare or poverty analysis.

3. First-order, second-order and mixed dominance

If a *criterion*, D, (e.g., first-order stochastic dominance) ranks F above F^* then F is said to have D over F^* (or FDF^*). A criterion *covers* a class of functions if and only if its ranking of two distributions is a necessary and sufficient condition for all functions in the class to weakly prefer, and at least one to strongly prefer, the dominating distribution. In this section, the two main results of Atkinson (1987) relating to the criteria of first- and second-order stochastic dominance are generalized to cover non-separable functions. Then the new mixed dominance criterion is presented and its coverage of the class of almost-egalitarian opulence functions shown.

3.1 First-order stochastic dominance

Definition: If $F(y) \leq F^*(y)$ $\forall y \leq Z$ and $F(y) < F^*(y)$ $\exists y \leq Z$ there is *first-order stochastic dominance* (FSD) by F of F^* up to Z ($FD_1F^*(Z)$).

Theorem 2a (welfare FSD) : *Iff* $FD_1F^*(\infty)$ *then* $S(F) \geq S(F^*)$ $\forall S \in \Sigma$ *and* $S(F) > S(F^*)$ $\exists S \in \Sigma$, *where* Σ *is the set of all welfare functions (S satisfying 1).*

Theorem 2b (poverty FSD): *Iff* $FD_1F^*(Z)$ *then* $S(F) \geq S(F^*)$ $\forall S \in \Sigma$ *and* $S(F) > S(F^*)$ $\exists S \in \Sigma$, *where* Σ *is the set of all opulence functions (S satisfying 1 and 2W) with* $Z^N = Z$ *(equivalently,* $Z^P \leq Z$*).*

A ranking by FSD requires simply that the dominating distribution have, in the relevant range, a no higher and somewhere lower distribution function. The proof for welfare FSD (for which see Thistle, 1989, and the references therein, going back to Hadar and Russel, 1969) rests on the fact that if $FD_1F^*(\infty)$ then F^* can be obtained from F by a sequence of reductions in income as defined in assumption 1. The poverty FSD Theorem 2b (a generalization of 2a) corresponds to Atkinson's (1987) Condition 1A, but imposes no requirement of separability. The bracketed equivalence follows from Lemma 1a. Showing necessity (of FSD for dominance over the class of functions) is elementary. For sufficiency, the method of proof utilized here is the simple one of showing how, under a certain distributional transformation, stochastic dominance over some restricted range implies stochastic dominance over the whole income range and how indifference to such a transformation restricts one to the sub-class of opulence functions.

Necessity proof for poverty FSD We have to show that if it is not the case that $FDF^*(Z)$ then either $S(F)=S^*(F)$ $\forall S\in\Sigma$ or $S(F)<S^*(F)$ $\exists S\in\Sigma$. If it is not the case that $FDF^*(Z)$ then either $F(y)=F^*(y)$ $\forall y\leq Z$ or $F(y)>F^*(y)$ $\exists y\leq Z$. If the former, then, for all S satisfying 2W for $Z^N=Z$, $S(F)=S(F^*)$. If the latter, then $S(F)<S(F^*)$ $\exists S\in\Sigma$ since $S=-F(y)$ satisfies 1 and 2W for any $y\leq Z^N$.■

Sufficiency proof for poverty FSD Assume $FD_1F^*(Z)$, and choose $Z^N=Z$. Now generate F_1 from F and F_1^* from F^* so that, for some t>0, $F_1^{(*)}(y)=F^{(*)}(y)$, $\forall y<Z$, $F_1^{(*)}(y)=F^{(*)}(Z)$, $\forall y\in[Z,Z+t)$ and $F_1^{(*)}(y)=1$, $\forall y\geq Z+t$. (See Figure 4 for an illustration.) FSD up to Z by F then implies FSD by F_1 of F_1^* over the whole income range ($F_1D_1F_1^*(\infty)$). So for all S satisfying 1, $S(F_1)\geq S(F_1^*)$ (see Theorem 2a). If in addition S satisfies 2W for $Z^N=Z$, then $S(F)=S(F_1)$ and $S(F^*)=S(F_1^*)$. So if S satisfies 1 and 2W, $S(F)\geq S(F^*)$. The inequality $S(F)\geq S(F^*)$ will hold strictly for at least one $S=-F(y)$, $y\leq Z$.■

3.2 Second-order stochastic dominance

Definition: Let $G(y_k)=\int_{-\infty}^{y_k}F(y)dy$. If $G(y_k)\leq G^*(y_k)$ $\forall y_k\leq Z$ and $G(y_k)<G^*(y_k)$ $\exists y_k\leq Z$, there is *second-order stochastic dominance* (SSD) by F of F^* up to Z ($FD_2F^*(Z)$).

Theorem 3a (welfare SSD): *Iff $FD_2F^*(\infty)$ then $S(F)\geq S(F^*)$ $\forall S\in\Sigma$ and $S(F)>S(F^*)$ $\exists S\in\Sigma$, where Σ is the set of egalitarian welfare functions (S satisfying 1 and 3S).*

Theorem 3b (poverty SSD): *Iff $FD_2F^*(Z)$ then $S(F)\geq S(F^*)$ $\forall S\in\Sigma$ and $S(F)>S(F^*)$ $\exists S\in\Sigma$, where Σ is the set of egalitarian opulence functions (S satisfying 1,* *2S and 3S) with Z^N=Z (equivalently $Z^P \leq Z$).*

$G(y_k)$ - the integral of the distribution function - defines the poverty deficit curve. Theorem 3a is well-known. Proofs can be found in Rothschild and Stiglitz (1973, pp.192-3), Kolm (1976, pp.90-1), Shorrocks (1983) and Kakwani (1984). They rest on showing that if $FD_2F^*(\infty)$ then F^* can be generated from F by a sequence of reductions in income and mean-preserving spreads. Theorem 3b (which generalizes 3a) corresponds to Condition IIA in Atkinson (1987), but again imposes no requirement of separability. The bracketed equivalence follows from Lemma 1a. Necessity is shown as for poverty FSD: simply replace -F(y) by $-G(y_k)$, which satisfies, for any $y_k \leq Z^N$, 1, 2S and 3S. The sufficiency proof also follows a similar, though not identical, path to that for FSD.

Sufficiency proof for poverty SSD Assume $FD_2F^*(Z)$, and choose Z^N=Z. Now generate F_1 from F and F_1^* from F^* so that $F_1^{(*)}(y)=F^{(*)}(y)$, $\forall y<Z$, $F_1^{(*)}(y)=1$, $\forall y \geq Z$. (See Figure 5 for an illustration.) SSD up to Z by F then implies SSD by F_1 over the whole income range ($F_1D_2F_1^*(\infty)$). So for all S satisfying 1 and 3S, $S(F_1) \geq S(F_1^*)$ (see Theorem 3a). If in addition S satisfies 2S for Z^N=Z, then $S(F)=S(F_1)$ and $S(F^*)=S(F_1^*)$. So if S satisfies 1, 2S and 3S, $S(F) \geq S(F^*)$. The inequality $S(F) \geq S(F^*)$ will hold strictly for at least one S since $-G(y_k)$ itself satisfies assumptions 1, 2S and 3S for $y_k \leq Z$.■

Note the role played by the strong definition of the poor (assumption 2S). If 2W was assumed, one would require $F_1^{(*)}(y)=F^{(*)}(y)$, $\forall y \leq Z$ and $F_1^{(*)}(y)=1$, $\forall y>Z$, but then F_1 and F_1^* would not be distribution functions as they would not be right-continuous.

For both FSD and SSD, Z can be interpreted as the upper bound on the set of poverty lines one is prepared to consider. Even if one would be happy also to bound this set from below, no weaker dominance conditions could be given. Formally, this is because they are defined in relation to Z^N, which, whether combined with Lemma 1a, makes a lower bound on Z^P irrelevant. Intuitively, in the absence of further restrictions on functional form below the poverty line, any minimum poverty line has no force as the degree of sensitivity to changes below any such lower bound can be made arbitrarily low (cf. Atkinson, 1987, p.760). Finally, note that both welfare FSD and welfare SSD are equivalent to poverty FSD and SSD, respectively, with Z^N approaching infinity (cf. Foster and Shorrocks, 1988). This must follow given the equivalence of opulence and welfare functions in this one case (see the end of 2.2).

3.3 Mixed dominance

Poverty first-order stochastic dominance can legitimately be viewed as too demanding a criterion, since it includes in its coverage functions which violate the transfer principle even when no crossings of the poverty line occur. Poverty second-order stochastic dominance may, by the same token, be regarded as too lenient a criterion, since it covers only functions which satisfy the transfer principle everywhere. An intermediate criterion is required.

Definition: If $G(y_k)\leq G^*(y_k)\ \forall y_k\leq Z^-$ and $F(y)\leq F^*(y)\ \forall y\in[Z^-,Z]$ and either $G(y_k)<G^*(y_k)\ \exists y_k\leq Z^-$ or $F(y)<F^*(y)\ \exists y\in[Z^-,Z]$, then there is *mixed dominance* by F of F^* between Z^- and Z ($FD_mF^*(Z^-,Z)$).

Theorem 4: *Iff $FD_mF^*(Z^-,Z)$ then $S(F)\geq S(F^*)\ \forall S\in\Sigma$ and $S(F)>S(F^*)\ \exists S\in\Sigma$, where Σ is the set of almost-egalitarian opulence functions (S satisfying 1, 2W and 3W) with $Z^N\in[Z^-,Z]$ (equivalently $Z^P\leq Z$ if S satisfies 3S and $Z^P\in[Z^-,Z]$ otherwise).*

If we rule out the possibility that either $F(y)=F^*(y)\forall y\in[Z^-,Z]$ or $G(y_k)=G^*(y_k)$ $\forall y_k\leq Z^-$, this criterion is equivalent to SSD up to Z^- and a type of "restricted" FSD (no higher and somewhere lower distribution function) between Z^- and Z. Hence the name: mixed dominance requires a mixture of dominance conditions to hold. Figure 6a illustrates. Note that (unrestricted) FSD implies mixed dominance implies SSD, all up to Z. The bracketed equivalence follows from Lemmas 1a and 1b. If Z can be interpreted as the maximum poverty line one is prepared to consider, Z^- can be interpreted as the minimum if the poverty line is treated as a threshold (see 2.3). Note that this restriction on functional form means that for the mixed dominance, unlike the FSD or SSD, criterion, increasing the lower bound can make a difference, i.e., it weakens the requirement of dominance. Indeed, inability to specify a lower bound results in the mixed dominance criterion collapsing to the FSD criterion.

Necessity for the mixed dominance theorem is simply shown. If there is no mixed dominance then either (a) $S(F)<S(F^*)$ where either $S=-F(y)$ $\exists y\in[Z^-,Z]$ or $S=-G(y_k)$ $\exists y_k\leq Z$ or (b) $F(y)=F^*(y)$ $\forall y\leq Z$ which implies $S(F)=S(F^*)$ for all S satisfying 2W for any $Z^N\leq Z$. Sufficiency can be most easily proved in the case in which S is a separable and, up to $Z^N\in[Z^-,Z]$, differentiable function. Then S can be written as in (1), which gives the result, by repeated integration of parts, that

$$(2)\quad S(F)-S(F^*) = \Delta F(Z^N)s(Z^N) - \Delta G(Z^N)s'(Z^N) + \int_{-\infty}^{Z^N}\Delta G(y)s''(y)dy$$

where $\Delta F= F-F^*$ and $\Delta G=G-G^*$. $s(Z^N)$ must be non-positive and $s'(Z^N)$ non-negative by 1; $s''(y)$ must be non-positive by 3W for $y\leq Z^N$. Then mixed dominance will ensure that the first and third terms are non-negative and the second term non-positive, making the RHS overall non-negative. Allowing Z^N to vary between Z^- and Z proves sufficiency.

A more general proof of sufficiency - without the assumptions of separability and differentiability - follows the lines of the proofs given for Theorems 2b and 3b. First, we need to prove the following lemma.

Lemma 2 *If $FD_2F^*(\infty)$ and $F(y)\leq F^*(y)$, $\forall y\geq Z^N$ then F^* can be generated from F by a sequence of (a) reductions in income (as defined in assumption 1) and (b) mean-preserving spreads (as defined in assumption 3S) which leave $F(Z^N)$ unchanged.*

Proof Generate $F_1(y)$ by reductions in income at and above Z^N so that $F_1(y)=F(y)$ $\forall y<Z^N$ and $F_1(y)=F^*(y)$ $\forall y\geq Z^N$. Then either $F_1(y)=F^*(y)$ $\forall y$ or $F_1D_2F^*(\infty)$. If the former, the lemma is shown to be correct. If the latter, F^* can be generated from F_1 by a sequence of mean-preserving spreads and reductions

in income (see Rothschild and Stiglitz, 1973, pp.192-3). Since $F_1(y)=F(y)$ $\forall y \geq Z^N$, no further changes to F_1 at or above Z^N will be required, which shows the lemma to be correct in this second case as well.■

Sufficiency proof for Theorem 4 Assume $FD_m F^*(Z^-,Z)$ and choose $Z^N \in [Z^-,Z]$. Generate F_1 from F and F_1^* from F^* so that, for any t>0, $F_1^{(*)}(y)=F^{(*)}(y)$, $\forall y \leq Z^N$, $F_1^{(*)}(y)=F^{(*)}(Z^N)$, $\forall y \in [Z^N,Z^N+t)$, and $F^{(*)}(y)=1$, $\forall y \geq Z^N+t$. From Lemma 2, F_1^* can be generated from F_1 by a sequence of reductions in income and mean-preserving spreads which do not involve crossings of the chosen poverty indifference line. So for any S which satisfies 1 and 3W, $S(F_1) \geq S(F_1^*)$. For any S which satisfies 2W, $S(F)=S(F_1)$ and $S(F^*)=S(F_1^*)$. So for all S satisfying 1, 2W and 3W, $S(F) \geq S(F^*)$. Varying Z^N within $[Z^-,Z]$ will give the same result, and the inequality will hold strictly for at least one S. The assumptions used to get this result restrict the set Σ to being that of all almost-egalitarian opulence functions with poverty lines between Z^- and Z.■

The proof is illustrated in Figure 6b. Note that the requirement that $F(Z^N) \leq F^*(Z^N)$ enables assumption 2W to be used without encountering the problems described at the end of the last subsection.

Another way of understanding mixed dominance is as adding to the requirement of SSD up to the upper bound on the set of reasonable poverty lines that of "head-count dominance": a lower (or at least no higher) head-count ratio over the range of values Z^N can take on. It is striking that adding this additional requirement (identical to Atkinson's Condition I) enables one to expand greatly the class of functions over which dominance is guaranteed.[7] One can divide the almost-egalitarian class of opulence functions into three types: those which only treat the poverty line as a threshold, those which only allow for varying intensities of poverty, and those which do both. The head-count is the first type, the egalitarian class constitutes the second, and an intermediate class of functions, such as the Sen index and the generalized Clark (with 0<C<1), constitutes the third. What the mixed dominance theorem tells us is that if we ensure dominance in relation to the two extreme types of functions we also have it in relation to all the intermediate functions.[8]

The mixed dominance result can be subject to the following criticism. I have claimed to provide through it coverage of a class of functions intermediate to that covered by FSD and SSD. But in fact the set covered by mixed dominance differs from those covered by FSD and SSD not only in relation to the assumptions the functions in the respective classes adhere to but also in relation to the poverty lines in relation to which these assumptions hold. In other words, I have not separated clearly the identification aspect (choice of a poverty line) and aggregation aspect (choice of a functional form) of poverty analysis. (See Sen, 1976, for this bifurcation.) The correct response to this criticism is that the intermediate coverage of the mixed dominance criterion can be clearly seen from the three sets of opulence functions drawn in Figure 3.[9] The distinction between the identification and aggregation aspects of poverty analysis, on the other hand, is not a watertight one: a function which is first steep and then relatively flat up to a high poverty line can be made arbitrarily similar to one which is steep up to a low poverty line. But if one does want to think about poverty analysis in this way, one should think of the identification aspect as requiring choice of a poverty indifference line. Then set Z^{-}=Z, and the three criteria cover different sets of functional forms all with the same indifference line.

4. Conclusion

To summarize, the most demanding criterion, first-order stochastic dominance up to Z, covers all poverty functions (i.e., functions which are weakly decreasing in income) which are indifferent to distributional changes above Z. Mixed dominance between Z^{-} and Z covers all poverty functions in this class which are indifferent to distributional changes above some point between these two bounds and which have negatives which are almost egalitarian in relation to the same point. Second-order dominance up to Z covers those functions in this class which have negatives which are egalitarian. Z can in all cases be chosen as maximum poverty line one is prepared to consider, equivalently as the highest value up to which one is prepared to be sensitive to distributional changes. If Z^{-} is invoked, it can be chosen as providing a lower bound on the range (bounded from above by Z) of "thresholds" above which distributional changes can be ignored **and** crossings of which, even if at the expense of others worse-off, may have a net beneficial impact.

The extent to which mixed dominance will increase the number of distributions which can be ranked compared to FSD and decrease the number compared to SSD is an empirical matter. It is easily shown that if mixed dominance is to obtain where there is no FSD, it must be the case that the distribution functions being compared cross an even number of times up to $Z^{\cdot}$.[10] But whatever its relative ranking power, the mixed dominance criterion provides a more general and appropriate framework for poverty analysis than the SSD criterion. The latter's restricted coverage of poverty functions limits its relevance. It is also unsatisfactory since, as argued, functions which satisfy the transfer principle except where crossings of the poverty line are involved have a claim to our attention in poverty analysis which is absent in the cases of equality and welfare analysis.

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Endnotes

1.I would like to thank Toni Haniotis, Jenny Lanjouw and Nick Stern for their useful comments.

2.Even though assumptions 1 and 3 (S and W) below are defined only for step functions, the theorems which utilize them can also be applied to comparisons of continuous distributions since these can be "approximated arbitrarily closely by step functions" (Rothschild and Stiglitz, 1970, pp.232-234).

3.Functions which satisfy 3S are S-concave. See Sen (1973, p.64).

4.Sen (1982, pp.32-33) considers a strong and a weak transfer axiom. If we replace his "increase" by "not decrease", then 3S corresponds to his strong transfer axiom and, by Theorem 1, the combination of 3W with 1 and 2W to his weak transfer axiom.

5.The Sen index is a member of a more general class of functions, the Kakwani (1980) class which itself is a sub-class of the Blackorby-Donaldson (1980) class. Chakravarty (1990) shows that the members of this last, most general class, in our terminology, satisfy 3W but not 3S.

6.It is easily shown that for given α (C) as C (α) increases the fixed cost increases (decreases) relative to the variable cost, i.e., the absolute value of the ratio of the second to first term increases (decreases). One can use this fact to set C and α - see Howes (1993) for details.

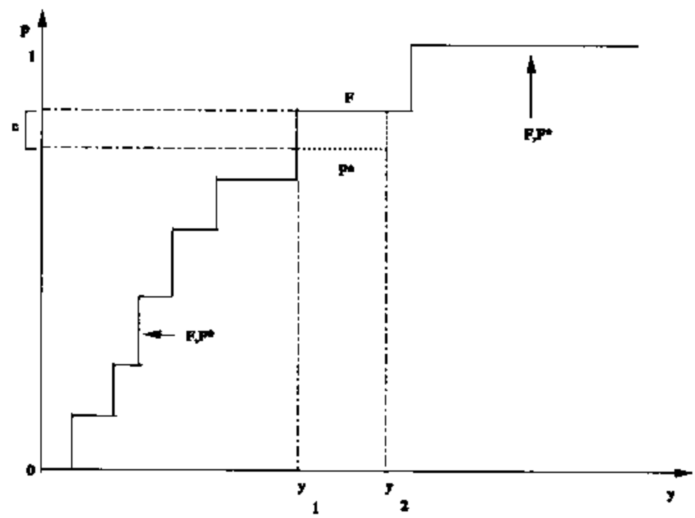
7.The importance of the head-count ratio in this regard can also be seen in the argument of Foster and Shorrocks (1987) that any "sub-group consistent" poverty function can be written as a function whose arguments are a continuous poverty function and the head-count ratio. (The generalized Clark function given earlier is precisely an example of this.) But note that Foster and Shorrocks' class of indices is quite different to that of this paper: we are not restricted to sub-group consistent functions, but do introduce assumptions of egalitarianism.

8.In fact, the mixed dominance criterion will ensure dominance over a larger set of functions than defined here. The sufficiency proof above indicates that mixed dominance will cover all opulence functions which satisfy the transfer principle except possibly in the range of Z^- to Z (rather than just at some $Z^N \in [Z^-,Z]$). However, since mixed dominance remains both necessary and sufficient for the smaller class of almost-egalitarian opulence functions and since this smaller class, unlike the larger, is of intrinsic interest, the proofs are given in terms of it.

9. The mixed dominance criterion covers the set of almost-egalitarian opulence functions in Figure 3 if $Z^-=Z$. As Z^- is reduced, the semi-circle in the figure will increase. As Z^- approaches $-\infty$ (so that mixed dominance collapses to FSD), the set of almost-egalitarian opulence functions becomes coincident with that of all opulence functions.

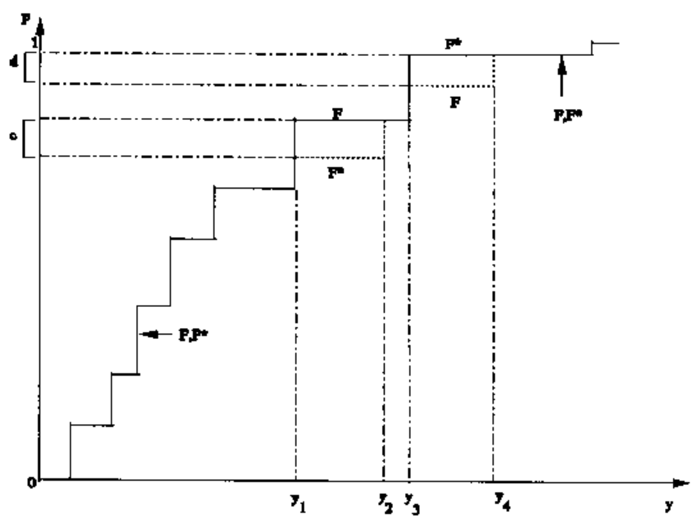
10.SSD to Z requires that the dominating distribution have "minimum dominance", which in turn implies that, assuming the distributions cross only a finite number of times, if $FD_2F^*(Z)$ then $\exists Z^*$ such that $\forall y \leq Z^*$, $F(y) \leq F^*(y)$ (Lambert, 1989, p.71).

Figure 1-Illustration of assumption 1: S is weakly increasing in y



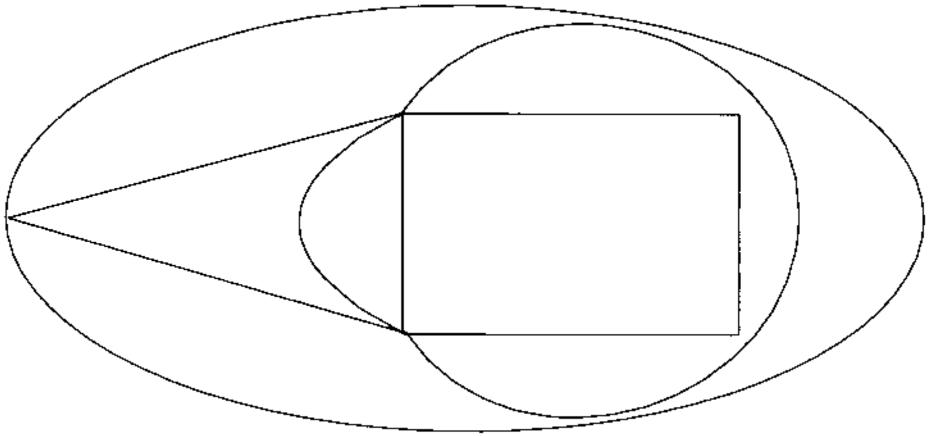
Notes: $S(F^*) \geq S(F)$; F and F^* are assumed right-continuous.

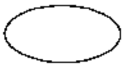
Figure 2-Illustration of assumption 3S: S satisfies the transfer principle

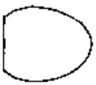


Notes: $S(F^*) \geq S(F)$; F and F^* are assumed right-continuous.

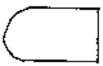
Figure 3-Schematic representation of relationship between different types of welfare and opulence functions



 **All welfare functions (1)**

 **All egalitarian welfare functions (1,3S)**

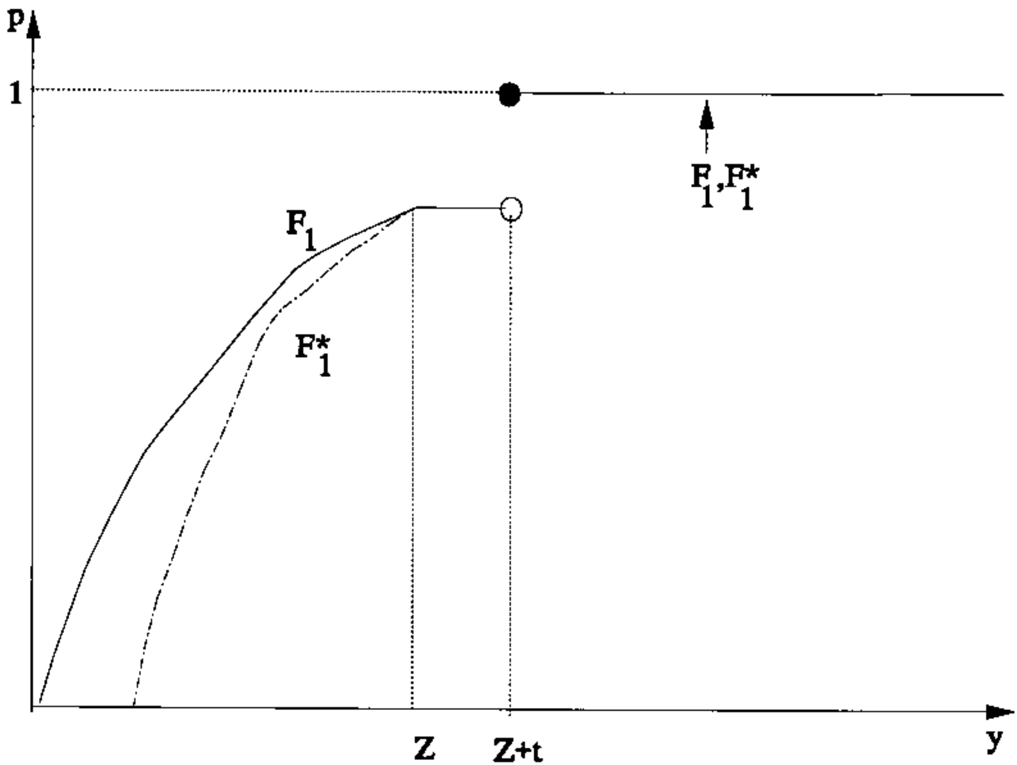
 **All opulence functions with $Z^N = Z$ (equivalently, $Z \overset{P}{\lesseqgtr} Z$) (1,2W)**

 **All almost egalitarian opulence functions with $Z^N = Z$ (equivalently $Z^P \lesseqgtr Z$ if egalitarian and $Z^P = Z$ otherwise) (1,2W,3W)**

 **All egalitarian opulence functions with $Z^N = Z$ (equivalently, $Z \overset{P}{\lesseqgtr} Z$) (1,2S,3S)**

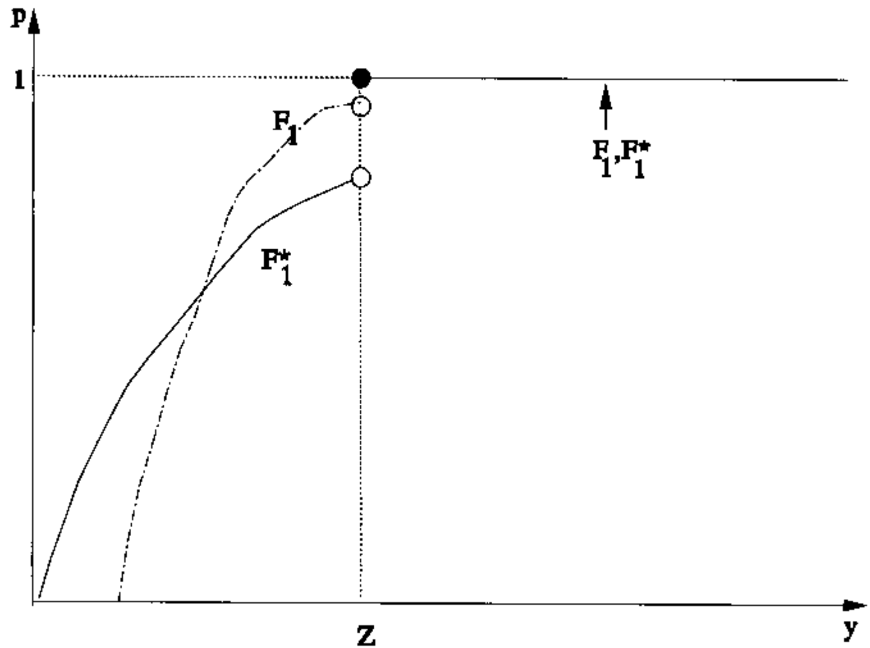
Notes: The numbered assumptions in the brackets indicate those to which the various sets of S conform: see 2.1. The bracketed equivalences follow from the lemmas of 2.2.

Figure 4-Illustration of sufficiency proof for poverty FSD



Notes: F_1 and F_1^* are generated respectively from F and F^*. The latter and the former in each pair, respectively, are identical up to Z^N=Z.

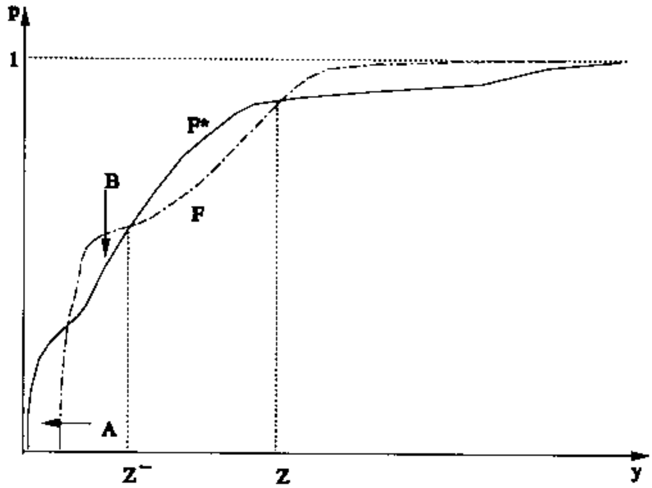
Figure 5-Illustration of sufficiency proof for poverty SSD



<u>Notes</u>: F_1 and F_1^* are generated respectively from F and F^* (where it is assumed that $FD_2F^*(Z)$. The latter and the former in each pair, respectively, are identical up to Z^N=Z.

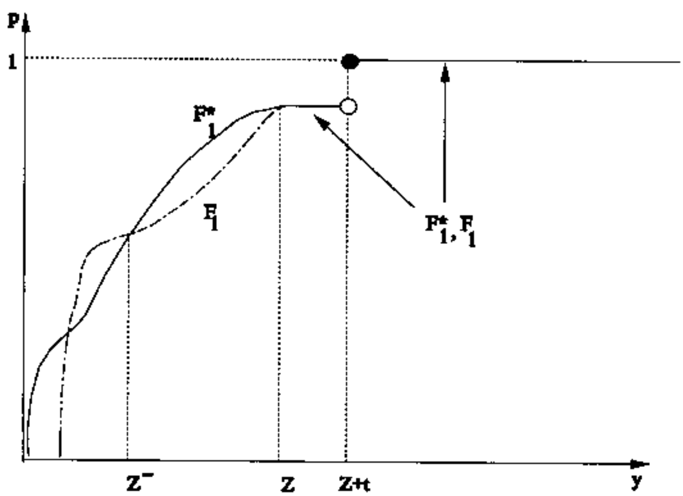
Figure 6-Illustration of Mixed Dominance and Sufficiency Proof

6a. $FD_mF^*(Z^-,Z)$



Notes: Assuming that area A is no smaller than area B, F has mixed dominance over F^* between Z^- and Z.

6b. $F_1D_2F_1^*(\infty)$



Notes: F_1 and F_1^* are generated respectively from F and F^*. The latter and the former in each pair, respectively, are identical up to Z^N=Z.