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# The Dependency Diagram of a Mixed Integer Linear Programme

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## Abstract

The Dependency Diagram of a Linear Programme (LP) shows how the successive inequalities of an LP depend on former inequalities, when variables are projected out by Fourier-Motzkin Elimination. This is explained in a paper referenced below. The paper, given here, extends the results to the Mixed Integer case (MILP). It is shown how projection of a MILP leads to a *finite* disjunction of polytopes. This is expressed as a set of inequalities (mirroring those in the LP case) augmented by correction terms with finite domains which are subject to linear congruences.

## 1 Introduction

The Dependency Diagram of an LP, and associated theorems, is explained in Williams[9]. In this paper we extend those results to give the Dependency Diagram for a MILP.The Dependency Diagram is a tree structure showing how, when projection is used, which inequalites depend on which previous inequalities. In the LP case this is valuable indemonstating which of the original constraints are binding for each dual feasible solution and what are the associated dual values. In the MILP case the Dependency Diagram has the added value of demonstrating how the optimal solution arises from both the original inequality constraints and a lattice of integer points derived from the original inequalities.

The major purpose of this paper is to give insight into the structure of MILPs and contend that they are naturally thought of as a series of inequalities and a lattice of points in non-negativebounded integers defined by a set of linear congruences.In section 2 we repeat the results for the LP case. In section 3 we show how these can be extended to deal with the elimination of integer variables.

## 2 The Dependency Diagram of an LP

The projection (elimination) of a variable, from an LP, relies on the following theorem (using logical terminology as applied, for example, by Langford in terms of eliminating an $\exists$ quantifier).

**Theorem 1** $\exists x_j \; \{a_{ij}x_j \geq f_i \;\; i \in I, \; -a_{kj}x_j \geq g_k \;\; k \in K\} \Longleftrightarrow 0 \geq a_{kj}f_i + a_{ij}g_k \;\;\; i \in I, k \in K$ *where* $a_{ij} > 0, i \in I \cup K, x_j \in \mathcal{R}$.

***Proof.*** *(i)* $\Rightarrow$ *This is obtained by adding each inequality, in the form* $x_j \geq f_i/a_{ij}$ *to each inequality, in the form* $-x_j \geq g_k/a_{kj}$ *respectively to give* $f_i/a_{ij} \leq$ -$g_k/a_{kj}$, $i \in I, k \in K$ *ie* $0 \geq a_{kj}f_i + a_{ij}g_k$ $i \in I, k \in K$.

*(ii)*$\Leftarrow$*Suppose* $0 \geq a_{kj}f_i + a_{ij}g_k$ *ie* $-a_{ij}g_k \geq a_{kj}f_i$. *This can expressed as* $-g_k/a_{kj} \geq f_i/a_{ij}$. *Let* $x_j = \max_i\{f_i/a_{ij}\}$ *(or* $\min_k\{-g_k/a_{kj}\}$*). Then* $a_{ij}x_j \geq f_i$ *and* $-a_{kj}x_j \geq g_k$ $i \in I, k \in K$ ■

Note that if either $I$ or $K$ (or both) are empty then the conclusion is tautologically true and the variable $x_j$ (and all inequalities containing it) can be removed with no resultant inequalities. We will refer to such an elimination as 'trivial'.

For illustration we apply this result to the following numerical example. To give it greater generality we take general RHS coefficients.

**M1:**

$$Minimize \;\; x_1 + 2x_2$$
$$subject\ to:$$

$$\begin{aligned} 2x_1 + x_2 &\geq b_1 \\ 5x_1 + 2x_2 &\leq b_2 \\ -4x_1 + 5x_2 &\geq b_3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Expressing this model in standard inequality form, with $z$ representing the objective, we have:

**M2:**

$$\begin{aligned} -x_1 - 2x_2 + z &\geq 0: \; C0 \\ 2x_1 + x_2 &\geq b_1: \; C1 \\ -5x_1 - 2x_2 &\leq -b_2: \; C2 \\ -4x_1 + 5x_2 &\geq b_3: \; C3 \\ x_1 &\geq 0: \; C4 \\ x_2 &\geq 0: \; C5 \end{aligned}$$

Eliminating $x_1$, using theorem 1, gives the Dependency Diagram in figure 1.

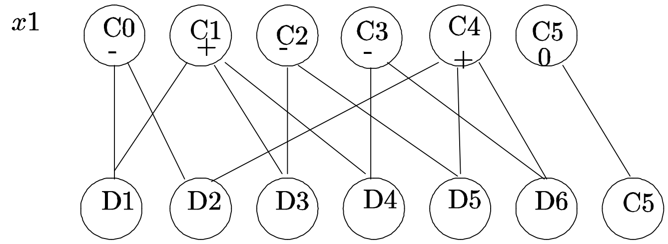


Figure 1: Dependency Diagram after the elimination of x1

The resultant inequalities are:

**M3:**

$$
\begin{aligned}
-3x_2 + 2z &\geq b_1 : \ D1 \\
-2x_2 + z &\geq 0 : \ D2 \\
x_2 &\geq 5b_1 - 2b_2 : \ D3 \\
7x_2 &\geq 2b_1 + b_3 : \ D4 \\
-2x_2 &\geq -b_2 : \ D5 \\
5x_2 &\geq 2b_3 \ : \ D6 \\
x_2 &\geq 0 : \ C5
\end{aligned}
$$

We refer to the two inequalities, from which each new inequality is derived, as the parents.Hence D0 has C0 and C1 as parents. That with a positive coefficient, for the eliminated variable, will be referred to as the **father** and that with a **negative** coefficient as the **mother**. Note that the result of carrying out successive eliminations of variables will be to produce inequalites which are **positive** combinations of some of the original inequalities (which we will refer to as the '**ancestors**').

In order to reduce the number of derived constraints, we can rely on the following theorem (attributed to Kohler[3] and Chernikov[1]).

**Theorem 2** *If an inequality depends on a proper, or the same, subset of the inequalities which give rise to another inequality then this latter inequality is redundant.*

The proofs of this, and the following two theorems and corollary are given in [9] and not repeated here.

**Theorem 3** *If, after eliminating* $n$ *variables by Fourier-Motzkin Elimination, an inequality depends on more than* $n+1$ *of the original inequalities it is redundant.*

This theorem can be strengthened by the following corollary.

**Corollary 4** *Any non-redundant inequality, after the non-trivial elimination of* $n$ *variables depends on* ***exactly*** $n+1$ *of the original inequalities.*

By "non-trivial' we mean each elimination of a variable is between an inequality in which it has a negative coefficient and an inequality in which it has a positive coefficient. A 'trivial' elimination is that remarked on after theorem 1 where the variable has all coefficients zero,or of the same sign, resulting in the removal of the variable and all inequalities in which it occurs.

We now proceed to the elimination of $x_2$ from the example using the results of the foregoing theorems to avoid generating redundant inequalities. The result is given in figure 2.

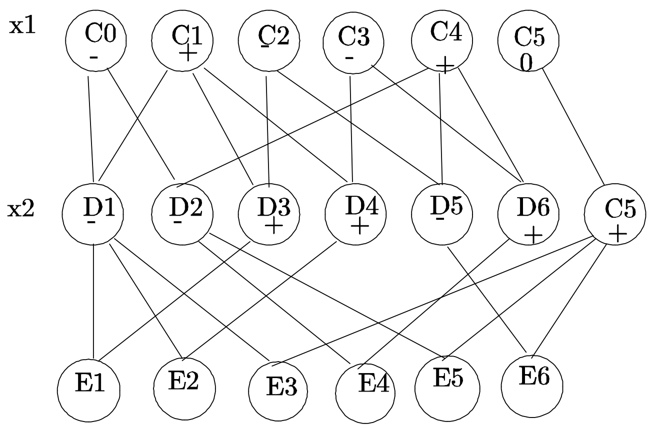


Figure2: Dependency Diagram after the elimination of x1 and x2.

The derived inequalities, after eliminating $x_2$ are:

**M4:**

$$
\begin{aligned}
z &\geq 16b_1 - 6b_2 : E1 \\
14z &\geq 13b_1 + 3b_3 : E2 \\
2z &\geq b_1 : E3 \\
5z &\geq b_3 : E4 \\
z &\geq 0 : E5 \\
0 &\geq -b_2 : E6
\end{aligned}
$$

This gives the value function of the original LP model as $z = \max(16b_1-6b_2, (13b_1+3b_3)/14, b_1/2, b_3/5, 0)$. The model is feasible if $b_2 \geq 0$.

We now consider the IP model, of which the above model is the LP relaxation. Before doing that we extend theorem 1 to deal with the elimination of integer variables.

# 3 The Dependency Diagram of a MILP

The extension of Fourier-Motzkin Elimination to IP models was explained by Williams[7].

In order to eliminate *integer* variables between inequalities we make use of the following theorems.

**Theorem 5** $\exists x_j \ \{a_{ij}x_j \geq f_i \quad i \in I, \ -a_{kj}x_j \geq g_k \ k \in K\} \iff 0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i$, $f_i + u_i \equiv 0(\operatorname{mod} a_{ij})$, $u_i \in \{0,1,2,...,a_{ij}-1\}$, $i \in I, k \in K$,*where* $a_{ij}, a_{kj} > 0, i \in I \cup K, x_j \in \mathcal{Z}$.

***Proof.*** *(i)* $\Rightarrow$ *We can write the inequalities in the form* $a_{kj}f_i \leq a_{kj}a_{ij}x_j \leq -a_{ij}g_k$ *implying that a multiple of* $a_{kj}a_{ij}$ *lies between the left and rightmost terms. If we apply a non-negative 'correction term'* $a_{kj}u_i$ *to the left side we have* $a_{kj}f_i + a_{kj}u_i \leq a_{kj}a_{ij}x_j \leq -a_{ij}g_k$ *so long as* , $f_i + u_i \equiv 0(\operatorname{mod} a_{ij})$.*This implies* $0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i$.*Whatsmore there is no loss of generality in restricting* $u_i$ *to the domain*$\{0,1,2,...,a_{ij}-1\}$,*for if* $u_i \geq a_{ij}$ *the resultant inequality and congruence would still be satisfied by reducing its value* $(\operatorname{mod} a_{ij})$ *to lie within the domain. Note that we could alternatively apply (different) correction terms to the right side.*

*(ii)*$\Leftarrow$*Suppose* $0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i$ *and* $f_i + u_i \equiv 0(\operatorname{mod} a_{ij})$ *where* $u_i \in \{0,1,2,...,a_{ij}-1\}$. *This can expressed as* $-g_k/a_{kj} \geq f_i/a_{ij} + u_i/a_{ij}$. *Let* $x_j = \max_i\{f_i/a_{ij} + u_i/a_{ij}\}$ *which is integral by virtue of the congruence. Then* $a_{ij}x_j \geq f_i$*and* $-a_{kj}x_j \geq g_k$ , $i \in I, k \in K$. ■

When we project out an integer variable we, in general, produce congruence relations as well as inequalities. These must be taken account of in the elimination of subsequent variables.

Before doing this it is convenient to eliminate the next variable, to be projected out, from all except one of the current set of congruence relations. This may be done by means of the Generalised Chinese Remainder Theorem (GCRT). This result is encapsulated in the following theorem.

**Theorem 6** $ex \equiv d_l(\operatorname{mod} m_l)\ l \in L \iff ex \equiv \sum_l \lambda_l m_l' d_l (\operatorname{mod} M)\ , 0 \equiv d_l - d_s(\operatorname{mod} \gcd(m_l, m_s))$ $l, s \in L$ *where* $M = \operatorname{lcm}_l(m_l)$ ,$m_l m_l' = M$, $l \in L$ *and* $\sum_l \lambda_l m_l' = 1$

***Proof.*** *(i)* $\Longrightarrow$ *The result that there exist* $\lambda_l$ *such that* $\sum_l \lambda_l m_l' = \gcd_l(m_l') = 1$ *is well known and proved using the Euclidean Algorithm. We do not repeat the proof here. Multiplying each of the original congruences by* $\lambda_l m_l'$ *we obtain* $ex \equiv \sum_l \lambda_l m_l' d_l (\operatorname{mod} M)$.*Subtracting the congruences* *in pairs we obtain* $0 \equiv d_l - d_s(\bmod \gcd(m_l, m_s))$.*(ii)*$\Longleftarrow$ *If* $0 \equiv d_l - d_s(\bmod \gcd(m_l, m_s))$ $l, s \ \epsilon$ $L$ *then* $\sum_l \lambda_l m_l' d_l \equiv d_s \sum_l \lambda_l m_l' (\bmod \gcd_l(\lambda_l M, m_s \sum_l \lambda_l m_l')$ $s \ \epsilon \ L$.*Since* $ex \equiv \sum_l \lambda_l m_l' d_l (\bmod M)$ *and* $\sum_l \lambda_l m\prime = 1$ *this implies* $ex \equiv d_s(\bmod m_s)$ $s \in L$. ■

Having aggregated all the congruences, involving the variable to be eliminated, into one congruence (together with congruences involving the other variables) we are in a position to eliminate a variable between a set of inequalities and this congruence. However two cases need to be distinguished, depending on whether the new variable to be eliminated is integer or real. We consider the two cases in the following two theorems.

**Theorem 7** $\exists x_j \ \{a_{ij}x_j \geq f_i \ \ i \in I, -a_{kj}x_j \geq g_k \ k \in K \ , ex_j \equiv d(\bmod m)\} \Longleftrightarrow 0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i, 0 \equiv d(\bmod \gcd(e, m)), f_i - \lambda_m a_{ij}d/\gcd(e, m) + u_i \equiv 0(\bmod a_{ij}m/\gcd(e, m))$ *where* $a_{ij}, a_{kj} > 0, i \in I \cup K, x_j \in \mathcal{Z}$ , $\lambda_e m/\gcd(e, m) + \lambda_m e/\gcd(e, m) = 1$ *and* $u_i \in \{0, 1, 2, ..., a_{ij}m/\gcd(e, m) - 1\}$

***Proof.*** *(i)* $\Rightarrow$*We can write the inequalities in the form* $a_{kj}ef_i \leq a_{ij}a_{kj}ex_j \leq -a_{ij}eg_k$. *From the congruence,*$a_{ij}a_{kj}ex_j \equiv a_{ij}a_{kj}d(\bmod\ a_{ij}a_{kj}m)$.*Let* $y = a_{ij}a_{kj}ex_j$.*Then* $y \equiv 0(\bmod a_{ij}a_{kj}e)$ *and* $y \equiv a_{ij}a_{kj}d(\bmod a_{ij}a_{kj}m)$. *Applying the GCRT gives* $0 \equiv d(\bmod \gcd(e, m))$ *and* $y \equiv \lambda_m a_{ij}a_{kj}ed/\gcd(e, m)$ $\bmod(a_{ij}a_{kj} \operatorname{lcm}(e, m))$. *Therefore* $a_{kj}ef_i - \lambda_m a_{ij}a_{kj}ed/\gcd(e, m) \leq$*a multiple of* $a_{ij}a_{kj} \operatorname{lcm}(e, m) \leq -a_{ij}eg_k - \lambda_m a_{ij}a_{kj}ed/\gcd(e, m)$. *Since* $(e, m)$ *divides* $d$*, by the congruence, the leftmost expression, in the above inequality, is a multiple of* $a_{kj}e$. *Hence we can apply a non-negative 'correction term'* $a_{kj}eu_i$ *to the left side giving* $a_{kj}ef_i - \lambda_m a_{ij}a_{kj}ed/\gcd(e, m) + a_{kj}eu_i \equiv 0(\bmod a_{ij}a_{kj} \operatorname{lcm}(e, m))$. *ie* $f_i - \lambda_m a_{ij}d/\gcd(e, m) + u_i \equiv 0(\bmod a_{ij}m/\gcd(e, m))$. $u_i$ *can be restricted to the domain*$\{0, 1, 2, ..., a_{ij}m/\gcd(e, m) - 1\}$.*The resultant inequalities are* $0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i$.

*(ii)*$\Leftarrow$*Suppose* $0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i, 0 \equiv d(\bmod \gcd(e, m))$, $f_i - \lambda_m a_{ij}d/\gcd(e, m) + u_i \equiv 0(\bmod a_{ij}m/\gcd(e, m))$,*where* $a_{ij}, a_{kj} > 0, i \in I \cup K$, *and* $u_i \in \{0, 1, 2, ..., a_{ij}m/\gcd(e, m) - 1\}$ *The inequality can be expressed as* $-a_{ij}g_k \geq a_{kj}f_i + a_{kj}u_i$.*But* $a_{kj}f_i + a_{kj}u_i \equiv \lambda_m a_{ij}a_{kj}d/\gcd(e, m)$ $(\bmod\ a_{ij}a_{kj}m/\gcd(e, m))$ *by the second congruence. Let* $a_{ij}a_{kj}x_j = \max_i\{a_{kj}f_i + a_{kj}u_i\}$ *giving* $x_j \in \mathcal{Z}$ . *Then* $a_{ij}x_j \geq f_i$ *and* $-a_{kj}x_j \geq g_k, i \in I, k \in K$ . *Also* $\exists i$ *such that* $a_{ij}x_j = f_i + u_i$. *Combining this with the above congruence gives* $x_j \equiv \lambda_m d/\gcd(e, m)$ $(\bmod m/\gcd(e, m))$ *ie* $x_j \equiv (1 - \lambda_e m)d/\gcd(e, m)$ $(\bmod m/\gcd(e, m))$ . *This implies* $ex_j \equiv d(\bmod m)\}$. ■

**Theorem 8** $\exists x_j \ \{a_{ij}x_j \geq f_i \ \ i \in I, -a_{kj}x_j \geq g_k \ k \in K \ , ex_j \equiv d(\bmod m)\} \Longleftrightarrow 0 \geq a_{kj}ef_i + a_{ij}eg_k + a_{kj}eu_i, ef_i - a_{ij}d + eu_i \equiv 0(\bmod a_{ij}m)$ $i \in I, k \in K$
*where* $a_{ij}, a_{kj} > 0, i \in I \cup K, x_j \in \mathcal{R}$ *and* $u_i \in [0, 1, 2, ..., a_{ij}m/e)$.

***Proof.*** *(i)* $\Rightarrow$*We can write the inequalities in the form* $ef_i/a_{ij} \leq ex_j \leq -eg_k/a_{kj}$ *implying that* $ef_i/a_{ij} - d \leq ex_j - d \leq -eg_k/a_{kj} - d$ *ie a multiple of* $m$ *lies between the left and rightmost expressions. We apply a non-negative 'correction term' to the left side. This correction term is from the continuum of the rationals, so may be scaled. To maintain correspondence with Theorem 4 it is convenient to denote it by* $eu_i/a_{ij}$ *giving* $ef_i/a_{ij} - d + eu_i/a_{ij} \equiv (\bmod m)$. *ie* $ef_i - a_{ij}d + eu_i \equiv$

$(\bmod a_{ij}m)$. *$u_i$ can be restricted to the interval* $[0,1,2,...,a_{ij}m/e)$.*The resultant inequalities are* $0 \geq a_{kj}ef_i + a_{ij}eg_k + a_{kj}eu_i$.

*(ii)⇐Suppose* $0 \geq a_{kj}ef_i + a_{ij}eg_k + a_{kj}eu_i$ *and* $ef_i - a_{ij}d + eu_i \equiv 0(\bmod a_{ij}m)$ *where* $u_i \in [0,1,2,...,a_{ij}m/e)$. *The inequalities expressed as* $-eg_{kj}/a_{kj} \geq ef_i/a_{ij}+eu_i/a_{ij}$. *Let* $ex_j = \max_i\{ef_i/a_{ij}+eu_i/a_{ij}\}$ *ie Then* $a_{ij}x_j \geq f_i$ *and* $-a_{kj}x_j \geq g_k$ *,* $i \in I, k \in K$ *. Also* $\exists i$ *such that* $ex_j = ef_i/a_{ij} + eu_i/a_{ij}$ *ie* $ef_i + eu_i = a_{ij}ex_j$.*Combining this with the above congruence gives* $ex_j \equiv d(\bmod m)$. ■

Theorems 5 and 7 demonstrate how the elimination of an integer variable, from a pair of inequalities, results in the same inequality, as in the LP case, but strengthened by the addition of a correction term. The correction term has a finite domain of possible values and is subject to a linear congruence relation involving the remaining variables

It will be shown (theorem 9) below that theorems 2 and 3 still apply in the IP case .

From the theorems above it can be seen that the congruence relation can be derived from either the father or the mother inequality. For the purpose of this paper we will always derive the congruence from the father. Suppose, therefore, that the father and mother inequalities are respectively $a_1x + f \geq b_1$ and $-a_2x + g \geq b_2$ (where $a_1, a_2 > 0$) and (after aggregating) the congruence involving $x$ is $ex \equiv d(\bmod m)$. The derived relations are then:

**R:**

$$\begin{aligned} a_2f + a_1g &\geq a_2b_1 + a_1b_2 + a_2u \\ d &\equiv 0(\bmod (e,m)) \\ (e,m)f + \lambda a_1 d - (e,m)u &\equiv (e,m)b_1(\bmod a_1m) \\ & \quad u \,\epsilon\, \{0,1,...,\frac{a_1m}{(e,m)} - 1\} \end{aligned}$$

where $(e,m)$ denotes the greatest common divisor of $e$ and $m$. $\lambda$ is the inverse of $\frac{e}{(e,m)}$ $\bmod(\frac{m}{(e,m)})$.

Note that the derived inequality is of the same form as in the LP case, but strengthened by the term $a_2u$. In addition two congruences are generated and the domain of the correction term defined. Hence once the Dependency Diagram has been generated for the LP relaxation correction terms can be added to the derived inequalities and linear congruences generated, based on the fathers of the derived constraints at each stage. We illustrate this by the numerical example. Before doing this we prove that it is still valid to apply theorems 2 and 3, in order to remove redundant inequalities, even though we are now dealing with IPs.

The fact that more than the original, LP binding, consraints are needed to determine the optimal IP solution arises from the fact that they are needed in the generation of 'binding' congruences.

**Theorem 9** *Theorems 2 and 3 still apply in the IP case.*

***Proof.*** *Suppose we ignore theorems 2 and 3 and create an inequality that depends on more than* $n+1$ *ancestors, after the elimination of* $n$ *variables. Then, at some stage in the Dependency Diagram, we must have the situation shown in figure 3*

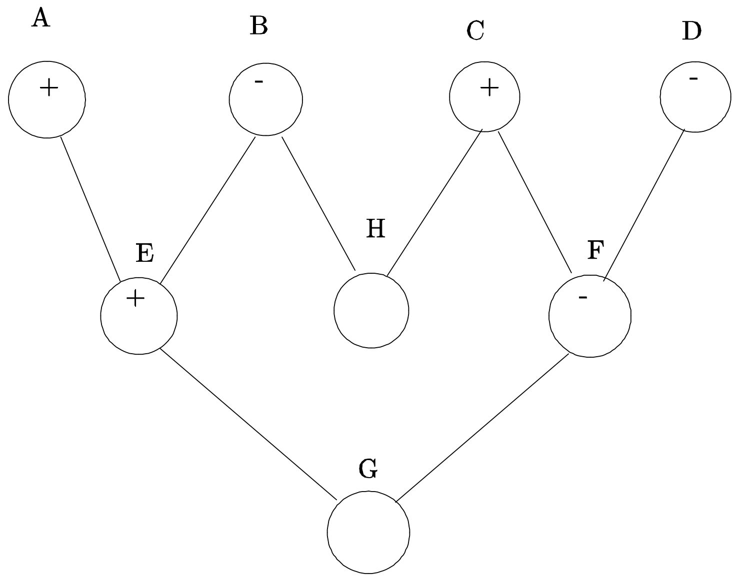


*Figure 3: A Redundant IP Constraint*

■

*Congruences and correction terms will be generated, at the second level, from the fathers A and C, and at the third level from the father E, or alternatively from the mother F. These correction terms will apply to inequality G. But we will also generate an inequality H at the second level, from B and C. If this inequality has a negative sign, for the next variable to be eliminated, it can be combined with E in order to create a new inequality at level 3, which has ancestors A, B and C at level 1. If the new inequality, at level 2, has a positive sign, for the next variable to be eliminated, then it can be combined with F in order to create a new inequality, at level 3, which has ancestors B, C and D at level 1. The new inequality at level 3 will have the same correction terms, subject to the same congruences (depending on whether the father or mother inequalities are used to generate the congruences at level 2) as G Whatsmore the ancestors of G, at level 1, are a superset of those for the new inequality at level 1. Ultimately values will be be determined for the correction terms. Then the inequalities, at level 3, can be treated as in the LP case, demonstrating that G is redundant. If the new inequality, at level 2, has a zero coefficient, for the next variable to be eliminated, then this inequality renders G redundant by the above argument.*

We now return to the numerical example.

At the top level, in figure 2, we have no congruences. The only non-trivial congruence is generated from the father inequality $C1$, which has a non-unit positive coefficient.

This gives $x_2 \equiv b_1 + u_1 (\bmod 2)$, $u_1 \epsilon \ \{0, 1\}$. Hence the inequalities, after the elimination of $x_1$ , are amended to:

**M5:**

$$\begin{aligned}
-3x_2 + 2z &\geq b_1 + u_1 : \ D1 \\
-2x_2 + z &\geq 0 : \ D2 \\
x_2 &\geq 5b_1 - 2b_2 + 5u_1 : \ D3 \\
7x_2 &\geq 2b_1 + b_3 + 2u_1 : \ D4 \\
-2x_2 &\geq -b_2 : \ D5 \\
5x_2 &\geq b_3 \ : \ D6 \\
x_2 &\geq 0 : \ C5 \\
x_2 &\equiv b_1 + u_1 (\bmod 2) : J1, u_1 \epsilon \{0, 1\}
\end{aligned}$$

We name the correction terms which apply to C$_i$ as $u_i$ and those which apply to D$_i$ as $v_i$ .

We now eliminate $x_2$ ,from the above set of relations, using the result of theorem 5, illustrated in **T**. This results in the amended set of inequalites **M5** with congruences generated from the father inequalities D3, D4,D6,C5, combined with the congruence in $x_2$.

**M6:**

$$\begin{aligned}
2z &\geq 16b_1 - 6b_2 + 16u_1 + v_3 : E1 \\
14z &\geq 13b_1 + 3b_3 + 13u_1 + 3v_4 : E2 \\
2z &\geq b_1 + u_1 + 3u_5 : E3 \\
5z &\geq 2b_3 + 2v_6 : E4 \\
z &\geq 2u_5 : E5 \\
0 &\geq -b_2 + u_5 : E6 \\
v_3 &\equiv 0 (\bmod 2) : K1 \\
9u_1 + v_4 &\equiv 5b_1 + 13b_3 (\bmod 14) : K2 \\
5u_1 + v_6 &\equiv 5b_1 + 9b_3 (\bmod 10) : K3 \\
u_1 + u_5 &\equiv b_1 (\bmod 2) : K4 \\
& \quad u_1 \epsilon \ \{0, 1\}, v_3 \epsilon \ \{0, 1\}, v_4 \epsilon \ \{0, 1, ..., 13\}, v_6 \epsilon \ \{0, 1, ..., 9\}, u_5 \epsilon \ \{0, 1\},
\end{aligned}$$

To make this example specific we will take $b_1 = 13, b_2 = 30, b_3 = 27$.
This results in:

**M7:**

$$
\begin{aligned}
2z &\geq 28 + 16u_1 + v_3 : E1 \\
14z &\geq 250 + 13u_1 + 3v_4 : E2 \\
2z &\geq 13 + u_1 + u_5 : E3 \\
5z &\geq 54 + 2v_6 : E4 \\
z &\geq 0 + u_5 : E5 \\
0 &\geq -30 + u_5 : E6 \\
v_3 &\equiv 0 (\bmod 2) : K1 \\
9u_1 + v_4 &\equiv 10 (\bmod 14) : K2 \\
5u_1 + v_6 &\equiv 8 (\bmod 10) : K3 \\
u_1 + u_5 &\equiv 1 (\bmod 2) : K4 \\
& u_1 \epsilon \ \{0,1\}, v_3 \epsilon \ \{0,1\}, v_4 \epsilon \ \{0,1,...,13\}, v_6 \epsilon \ \{0,1,...,9\}, u_5 \epsilon \ \{0,1\},
\end{aligned}
$$

The optimal solution occurs when $u_1 = v_3 = 0, v_4 = 10, v_6 = 2, u_5 = 1$ giving $z = 20$.
The values of the variables can be obtained by backtracking through earlier inequalities and congruences in the Dependency Diagram giving $x_2 = 9, x_1 = 2$.

In contrast the solution of the LP relaxation is obtained by dropping the congruences and correction terms and backtracking giving $z = 17\frac{6}{7}, x_2 = 7\frac{4}{7}, x_1 = 2\frac{5}{7}$.

A number of observations are worth making regarding the method described in this paper.

1. The correction terms are not nessessarily the same as the surplus variables, but have finite domains requiring the final solution to be obtained by solving linear congruences as well as inequalities.

2. The correction terms and congruences are not unique. There will be alternative (and sometimes more economical representations) obtained by using a mixture of mother, as well as father, inequalities to obtain congruences and correction terms.

3. As is well known (and the numerical example demonstrates) the optimal solution to an IP may not be the same as that obtained by solving the IP subject only to the constraints binding at the LP optimum (an IP over a cone). In the example while the optimal IP solution arises from the same final inequality as the optimal LP solution it also depends on correction terms and congruences arising from inequalities not binding at the LP optimum. In this example constraints $C0, C1, C2, C3$ are all binding at the IP optimum although only $C0, C1$ and $C3$ are binding at the LP optimum. If we were to solve, using the cone constraints $C0, C1, C3$, we would only obtain the final constraint $E2$ and congruence $K2$ allowing the (infeasible) solution $x_1 = 3, x_2 = 8, z = 19$.

4. The inequalities and congruence systems are 'ı́ndependent' in the sense that (a) the final inequality system takes the same form as that for the LP relaxation but augmented by correction terms and (b) the correction terms and congruences and domains, applying to them arise from the original matrix independently of which inequalities are finally used.

5. The (M)IP over a cone is simpler to solve by this method than a general MIP and forms the subject of Williams**??**..An analytic solution is given for a model with general coefficients.

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