Distributional Analysis Research Programme
Discussion Paper

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October 1993

LSE STICERD Research Paper No. DARP 06
This paper can be downloaded without charge from:
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MONOTONICITY, DOMINANCE AND THE PARETO PRINCIPLE

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Discussion Paper
No.DARP/6
October 1993

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Abstract

The property of monotonicity, th criterion of (first-degree) income dominance and the Pareto principle appear frequently in the literature on the axiomatic approach to the welfare economics of income distribution. Sometimes these are regarded as almost interchangeable for practical purposes. However, as we shall show, this interchangeability arises because of other important assumptions that are also commonly invoked, but which may be questionable.

Keywords: monotonicity, Pareto principle, (first-degree) income dominance, welfare economics, income distribution, interchangeability.

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Monotonicity, Dominance and
The Pareto Principle

The property of monotonicity, the criterion of (first-degree) income dominance and the Pareto principle appear frequently in the literature on the axiomatic approach to the welfare-economics of income distribution. Sometimes these three are regarded as almost interchangeable for practical purposes. However, as we shall show, this interchangeability arises because of other important assumptions that are also commonly invoked, but which may be questionable.

Let an economy consist of n individuals who are endowed with incomes $x_i \in X$, where $X \subseteq \mathbf{R}$. The set of income distributions X^n is the n-fold cartesian product of X. Each person's well-being is determined by a utility function $U_i: X^n \to \mathbf{R}$; this specification of U_i allows for the possibility that each person's well-being may be affected by the incomes enjoyed by other people., thus:

$$u_i = U_i(x_i, \mathbf{x}_{-i}) \tag{1}$$

where $\mathbf{x}_{-i}$ denotes the (n-1)-dimensional subvector formed by dropping the ith component of the n-vector $\mathbf{x}$. Apart from the endowment x_i and the functions U_i people may be regarded as identical. The social welfare function is a function $W:\mathbf{R}^n \to \mathbf{R}$; this formulation allows for the possibility that social welfare may be specified as a function of the n-vector of incomes, in which case the welfare index w is given by:

$$w = W\left(x_1, x_2, \ldots, x_n\right) , \tag{2}$$

or as a function of the vector of utility levels, in which case w is given by

$$w = W(u_1, u_2, \ldots, u_n) , \tag{3}$$

which may in turn be written as:

$$w = W\left(U_1(x_1, \mathbf{x}_{-1}),\ \ U_2(x_2, \mathbf{x}_{-2}), \ldots, U_n(x_n, \mathbf{x}_{-n})\right) , \tag{4}$$

Using this framework we may then introduce the following three descriptions of the basic principles:

- *Monotonicity* implies that social welfare w is increasing in any income x_i (Ebert, 1988). It is used in this sense also when applied to poverty measures, although the relevant domain of the welfare function is then the subset of incomes that are less than the poverty line - see Seidl (1988), Sen (1976).

- *Respect for (first-degree) income dominance* implies that welfare is higher for $\mathbf{x} \in X^n$ than for $\mathbf{x}' \in X^n$ if $F(y;\mathbf{x}) \geq F(y;\mathbf{x}')$ for all $y \in X$ with $>$ for some $y \in X$, where $F(.;\mathbf{x})$ denotes the conventional distribution function for the discrete distribution $\mathbf{x}$. It is the concept of first-degree stochastic dominance applied specifically to the income distribution.[1]

- *The Pareto principle* means that social welfare is increasing in any utility level u_i.[2]

For a given profile of utility functions $\{U_i: \ i=1,2,...,n\}$ any income distribution $\mathbf{x} \in X^n$ can be transformed using (1) into a utility distribution $\mathbf{u}:=(u_1,u_2,...,u_n)$. Then we may naturally introduce a fourth criterion:

- *Respect for (first-degree) utility dominance* implies that welfare is higher for $\mathbf{x} \in X^n$ than for $\mathbf{x}' \in X^n$ if $F(v;\mathbf{u}) \geq F(v;\mathbf{u}')$ for all v with $>$ for some v. It is the concept of first-degree stochastic dominance applied specifically to the utility distribution.

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[1] Saposnik (1981) shows that this dominance criterion is equivalent to "rank dominance": $\mathbf{x}$ rank dominates $\mathbf{x}'$ if $x_i \geq x_i'$ for all i with $>$ for some i.

[2] Note that some authors use the term "Paretian" to mean "satisfies monotonicity" - see for example Saposnik (1981,1983).

Respect for income dominance is implied by monotonicity, but not *vice versa*. For example the distribution B=(5,3,4) would be ranked as superior to the distribution A=(1,4,2) by all welfare functions respecting the income dominance criterion, but not according to the monotonicity criterion alone. However, if we require that W also be symmetric then monotonicity and respect for income dominance will be equivalent. Similar remarks apply to the relationship between the Pareto principle and utility dominance.

Symmetry of the function W is closely related to the welfare property of anonymity (in our simplified framework they are identical). However anonymity itself may be questionable as a welfare criterion when the social-welfare function is to take into account something more than the end-state distribution of incomes. For instance - to continue the example just given - if the end state provides all the information to make a welfare judgment then by anonymity distribution B is equivalent to B′=(3,5,4), so that by monotonicity $w_A < w_{B'} = w_B$. But if "history matters" then in the process of distributional change A→B distribution B and B′ should not be regarded as equivalent; under these circumstances anonymity applied to the end-state income distributions is inappropriate and social welfare should be defined over an array of pairs ([1,5],[4,3],[2,4]).

The difference between monotonicity and the Pareto principle lies in the relationship between utility and income. If each person's utility depends *only* upon his own income, and utility is a strictly increasing function of income - i.e. if each U_i is strictly increasing in its first argument and constant with respect to its second argument - then the two principles are equivalent. If, by contrast, there are externalities in income distribution - so that person *i*'s utility may depend on person *j*'s income - then monotonicity will be neither necessary nor sufficient for the Pareto principle.

For an example of a social-welfare function that satisfies the Pareto principle, but violates the requirement of monotonicity, let W be a special case of (3):

$$w = u_1 + u_2 + \dots + u_n \ , \tag{5}$$

and consider the case where utility is:

$$u_i = x_i - \frac{1}{\#\{j: x_j > x_i\}} \sum_{x_j > x_i} x_j \tag{6}$$

In other words, each person's well-being is determined by the difference between his own income and the average income of everyone above him. Now let the richest person's income go up by $1, then if there are more than two persons in the society, social welfare falls.

For an example of a social-welfare function that satisfies monotonicity, but not the Pareto principle, consider the following. The social-welfare function is "national income" - in other words:

$$w = \sum_{i=1}^{n} x_i \quad , \tag{7}$$

and each person's utility is given by:

$$u_i = x_i + \sum_{j \neq i} \log(x_j) \tag{8}$$

Now take the case where there are two persons: a rich person with \$100 and a poor person with \$0.01: if you take \$2 off the rich person give \$1 to the poor person and throw the other \$1 away, then each person's utility will go up but social welfare will decrease. Before the transfer the rich person's utility is 100 + log(0.01) = 95.395; after the transfer it is 98 + log(1.01) = 98.010; before the transfer the poor person's utility is 0.01 + log(100) = 4.615; after the transfer it is 1.01 + log(98) = 5.595.

Notice that in neither of our examples have we had to appeal to differences in tastes within the population: the utility functions U_i happen to be the same for all i. Nor have we had to resort to assumptions that run counter to the standard criteria of the welfare analysis of distributions - in the first example the principle of transfers would be satisfied, and in the although in the second example transfers leave social welfare unchanged, the simple "national income" welfare function is one that underpins a lot of applied welfare economics.[3] Of course we have appealed to the concept of externality, and this part of the argument might at first glance appear to be akin to Alfred Hitchcock's MacGuffin that gets you out of trouble in a plot. However in the context of income distribution such externalities may not be optional extras that are bolted on to the analysis, but may rather form a fundamental basis for concern about inequality - see for example Hochman and Rodgers (1969). In fact this simple point raises an issue that is of fundamental importance in welfare economics: What is meant by an individualistic social welfare function? Is it just that it depends only on each individual's utility, or is it also that each individual's utility depends only on his own income (or some other indicator of his own resources or consumption)?

Thus we have the curious situation that monotonicity, the Pareto principle and respect for income dominance are equivalent only if history does not matter to society and income distribution does not matter to the individual members of society.

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[3] It would not be diffiult to provide a modified (but more complicated) version of this example in which the transfer principle was strictly satisfied.

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